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# Generalized $C_{F_1F_2}$-integrals: From Choquet-like aggregation to ordered directionally monotone functions

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**Abstract**

This paper introduces the theoretical framework for a generalization of $C_{F_1F_2}$-integrals, a family of Choquet-like integrals used successfully in the aggregation process of the fuzzy reasoning mechanisms of fuzzy rule based classification systems. The proposed generalization, called by $gC_{F_1F_2}$-integrals, is based on the so-called pseudo pre-aggregation function pairs $(F_1, F_2)$, which are pairs of fusion functions satisfying a minimal set of requirements in order to guarantee that the $gC_{F_1F_2}$-integrals to be either an aggregation function or just an ordered directionally increasing function satisfying the appropriate boundary conditions. We propose a dimension reduction of the input space, in order to deal with repeated elements in the input, avoiding ambiguities in the definition of $gC_{F_1F_2}$-integrals. We study several properties of $gC_{F_1F_2}$-integrals, considering different constraints for the functions $F_1$ and $F_2$, and state under which conditions $gC_{F_1F_2}$-integrals present or not averaging behaviors. Several examples of $gC_{F_1F_2}$-integrals are presented, considering different pseudo pre-aggregation function pairs, defined on, e.g., t-norms, overlap functions, copulas that are neither t-norms nor overlap functions and other functions that are not even pre-aggregation functions.


*Keywords:* Aggregation functions; Pre-aggregation functions; Choquet integral; $C_{F_1F_2}$-integral; Ordered directionally monotonicity; Pseudo pre-aggregation function pair

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## 1. Introduction

In 2016, Lucca et al. [1] introduced the notion of pre-aggregation function (PAF), which fulfills the boundary conditions as any aggregation function, but, instead of being an increasing function, it is just directional increasing [2]. That is, it increases along some specific ray (direction). Furthermore, the authors presented some methods to produce PAFs [3,4]. One of them is by generalizing the Choquet integral [5] replacing the product operator by a t-norm, obtaining, under some constraints, idempotent and averaging PAFs. This approach was used in Fuzzy Rule-Based Classification System (FRBCS) [6], presenting excellent results, when the Hamacher t-norm [7] is used for the generalization, overcoming the Choquet integral and classical averaging operators in classification systems.

Those excellent results motivated us to explore a more general method for constructing PAFs based on the Choquet integral. For that, instead of using just a t-norm, we replace the product operator by a fusion function $F$ that is left 0-absorbent (i.e., $F(0, x) = 0$, for all $x \in [0, 1]$), obtaining the $C_F$-integrals [8]. $C_F$-integrals are pre-aggregation functions, which, under certain conditions, may be idempotent and/or averaging functions. This allowed to analyze sub-families of $C_F$-integrals having or not the averaging behavior, showing that a $C_F$-integral does not need to be an averaging function when used in FRBCSs, since the non-averaging obtained more accurate results than the averaging ones.

In the same line of this research, Lucca et al. [9] investigated another kind of Choquet integral that leads to aggregation functions, instead of just PAFs. For that, the product operation of the standard Choquet integral was first distributed and, then, replaced by a copula [10], obtaining the CC-integrals, which happen to be averaging aggregation functions [11,12,14]. This approach presented excellent results in classification, in particular, when the minimun t-norm was the considered copula, in which case it was called CMin-integral [13]. See also the application in [37].

Recently, Luca et al. [15] developed the concept of $C_{F_1F_2}$-integral, which is a specific generalization of CC-integrals, based on two possibly different fusion functions $F_1$ and $F_2$ (instead of a copula $C$) satisfying some appropriate conditions, obtaining non-averaging Choquet-like integrals that were successfully used in the aggregation process of the fuzzy reasoning mechanisms of fuzzy rule based classification systems. Their performance was proved to be statistically equivalent to FURIA [16].

The general aim of this paper is to generalize the concept of $C_{F_1F_2}$-integrals, obtaining the so-called $gC_{F_1F_2}$-integrals, presenting a solid theoretical framework that gives the basis for applications. We shall define the $gC_{F_1F_2}$-integrals by distributing the product operation of the Choquet integral and, then, generalizing the two instances of the product operation by a pair of fusion functions $(F_1, F_2)$. For that, we face two main problems:

- Which properties/constraints should be imposed on $(F_1, F_2)$ in order to guarantee a well defined concept, satisfying the boundary conditions and some kind of increasingness (increasingness, directional increasingness or ordered directional (OD) increasing)? This leads us to the concept of pseudo pre-aggregation function pair $(F_1, F_2)$, that is, a pair of fusion functions satisfying some kind of boundary conditions, directional increasingness and $F_1$-dominance property.
- How can we deal with the problem of repeated elements in the input, which may cause ambiguity in the results (that is, the same input may produce different results when we change the order of the elements)? To solve this problem, we propose to collapse those repeated elements into one representant of the class, and to proceed to a problem dimension reduction.

Then, the specific objectives of this paper are stated as:

1. To introduce the notion of pseudo pre-aggregation function pair $(F_1, F_2)$;
2. To define a problem dimension reduction;
3. Using dimension reduction, to introduce the notion of Choquet-like integral based on pseudo pre-aggregation function pairs, called $gC_{F_1F_2}$-integrals;
4. To show under which conditions $gC_{F_1F_2}$-integrals based on pseudo pre-aggregation function pairs $(F_1, F_2)$ are (pre) aggregation functions;
5. To show under which conditions $gC_{F_1F_2}$-integrals based on pseudo pre-aggregation function pairs $(F_1, F_2)$ are ordered directional increasing functions [17] and satisfy the desirable boundary conditions;
6. To study when $gC_{F_1F_2}$-integrals are averaging [18];

7. To analyze several types of pseudo pre-aggregation function pairs $(F_1, F_2)$, built from t-norms [7], overlap functions [19–22], copulas, and other functions that are not even PAFs, showing examples of different $gC_{F_1F_2}$-integrals.

The paper is organized as follows. In Section 2, we present the basic concepts required to understand the paper. In Section 3, we introduce the concept of pseudo pre-aggregation pairs and analyze several properties. The concept of $gC_{F_1F_2}$-integrals is introduced in Section 4, where we also define the dimension reduction. In Section 5, we discuss when $gC_{F_1F_2}$-integrals are (pre) aggregation functions, and the related properties. Section 6 studies when $gC_{F_1F_2}$-integrals are not (pre) aggregation functions, but OD monotone functions. Section 7 is the Conclusion.

## 2. Preliminaries

In this paper, we call any n-ary function $F : [0, 1]^n \to [0, 1]$ by a fusion function.

**Definition 2.1.** [23,24] A function $A : [0, 1]^n \to [0, 1]$ is an aggregation function whenever the following conditions hold:

**(A1)** $A$ is increasing[1] in each argument: for each $i \in \{1, \dots, n\}$, if $x_i \leq y$, then

$$A(x_1, \dots, x_n) \leq A(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n);$$

**(A2)** $A$ satisfies the boundary conditions: **(i)** $A(0, \dots, 0) = 0$ and **(ii)** $A(1, \dots, 1) = 1$.

An aggregation function $A : [0, 1]^n \to [0, 1]$ is said to be idempotent if and only if:

**(ID)** $\forall x \in [0, 1] : A(x, \dots, x) = x$, and

it is said to be averaging if and only if:

**(AV)** $\forall (x_1, \dots, x_n) \in [0, 1]^n : \min\{x_1, \dots, x_n\} \leq A(x_1, \dots, x_n) \leq \max\{x_1, \dots, x_n\}$.

Observe that, since aggregation functions are increasing, the idempotent and averaging behaviors are equivalent in the context of aggregation functions.

**Definition 2.2.** [10] A bivariate function $C : [0, 1]^2 \to [0, 1]$ is a copula if it satisfies the following conditions, for all $x, x', y, y' \in [0, 1]$ with $x \leq x'$ and $y \leq y'$:

**(C1)** $C(x, y) + C(x', y') \geq C(x, y') + C(x', y)$;
**(C2)** $C(x, 0) = C(0, x) = 0$;
**(C3)** $C(x, 1) = C(1, x) = x$.

**Definition 2.3.** [2] Let $\vec{r} = (r_1, \dots, r_n)$ be a real $n$-dimensional vector, $\vec{r} \neq \vec{0} = (0, \dots, 0)$. A function $F : [0, 1]^n \to [0, 1]$ is said to be $\vec{r}$-increasing if for all $\vec{x} = (x_1, \dots, x_n) \in [0, 1]^n$ and for all $c > 0$ such that $\vec{x} + c\vec{r} = (x_1 + cr_1, \dots, x_n + cr_n) \in [0, 1]^n$ it holds

$$F(\vec{x} + c\vec{r}) \geq F(\vec{x}).$$

Similarly, one defines an $\vec{r}$-decreasing function.

**Definition 2.4.** [1,4] A function $PA : [0, 1]^n \to [0, 1]$ is said to be an n-ary pre-aggregation function (PAF) if the following conditions hold:

[1] For an increasing (decreasing) function we do not mean a strictly increasing (decreasing) function.

**(PA1)** Directional Increasingness: there exists $\vec{r} = (r_1, \ldots, r_n) \in [0,1]^n$, $\vec{r} \neq \vec{0}$, such that $PA$ is $\vec{r}$-increasing;
**(PA2)** Boundary conditions: (i) $PA(0, \ldots, 0) = 0$ and (ii) $PA(1, \ldots, 1) = 1$.

If $F$ is a pre-aggregation function with respect to a vector $\vec{r}$ we just say that $F$ is an $\vec{r}$-pre-aggregation function.

Another important concept used in this paper is the one of ordered directional (OD) monotonicity, introduced in [17]. Observe that, when one considers directional monotonicity, the direction along which monotonicity is required is the same for all $\vec{x} \in [0,1]^n$. On the contrary, OD monotone functions are functions that allow monotonicity along different directions depending on the ordinal size of the coordinates of each input $\vec{x} \in [0,1]^n$. First, we take a permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ to reorder the input $\vec{x} \in [0,1]^n$ in a decreasing order, obtaining $\vec{x}_\sigma \in [0,1]^n$. Then, a fusion function $F : [0,1]^n \to [0,1]$ is OD $\vec{r}$-increasing, for a real vector $\vec{r} = (r_1, \ldots, r_n)$, with $\vec{r} \neq \vec{0}$, whenever $F(\vec{x})$ is less than or equal to the values of $F$ when applied to

$$(\vec{x}_\sigma + c\vec{r})_{\sigma^{-1}} = \vec{x} + c\vec{r}_{\sigma^{-1}}, \tag{1}$$

under the assumption that $\vec{x}_\sigma$ and $\vec{x}_\sigma + c\vec{r}$ are comonotone (i.e., either they increase or decrease at the same time).

**Definition 2.5.** [17] Consider a function $F : [0,1]^n \to [0;1]$ and let $\vec{r} = (r_1, \ldots, r_n)$ be a real $n$-dimensional vector, $\vec{r} \neq \vec{0}$. $F$ is said to be ordered directionally (OD) $\vec{r}$-increasing if, for each $\vec{x} \in [0,1]^n$, any permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ with $x_{\sigma(1)} \geq \ldots \geq x_{\sigma(n)}$, and $c > 0$ such that $1 \geq x_{\sigma(1)} + cr_1 \geq \ldots \geq x_{\sigma(n)} + cr_n$, it holds that

$$F(\vec{x} + c\vec{r}_{\sigma^{-1}}) \geq F(\vec{x}),$$

where $\vec{r}_{\sigma^{-1}} = (r_{\sigma^{-1}(1)}, \ldots, r_{\sigma^{-1}(n)})$. Similarly, one defines an ordered directionally (OD) $\vec{r}$-decreasing function.

In what follows, denote $N = \{1, \ldots, n\}$, for $n > 0$.

**Definition 2.6.** [5,25] A function $\mathfrak{m} : 2^N \to [0,1]$ is said to be a fuzzy measure if, for all $X, Y \subseteq N$, the following conditions hold:

**($\mathfrak{m}$1)** Increasingness: if $X \subseteq Y$, then $\mathfrak{m}(X) \leq \mathfrak{m}(Y)$;
**($\mathfrak{m}$2)** Boundary conditions: $\mathfrak{m}(\emptyset) = 0$ and $\mathfrak{m}(N) = 1$.

**Definition 2.7.** [5] The discrete Choquet integral with respect to a fuzzy measure $\mathfrak{m}$ is the function $\mathfrak{C}_\mathfrak{m} : [0,1]^n \to [0,1]$, defined, for all of $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$, by:

$$\mathfrak{C}_\mathfrak{m}(\vec{x}) = \sum_{i=1}^{n} \left(x_{(i)} - x_{(i-1)}\right) \cdot \mathfrak{m}\left(A_{(i)}\right), \tag{2}$$

where $\left(x_{(1)}, \ldots, x_{(n)}\right)$ is an increasing permutation on the input $\vec{x}$, that is, $0 \leq x_{(1)} \leq \ldots \leq x_{(n)}$, where $x_{(0)} = 0$ and $A_{(i)} = \{(i), \ldots, (n)\}$ is the subset of indices corresponding to the $n - i + 1$ largest components of $\vec{x}$.

Whenever one distribute the product operation in Equation (2), we obtain the Choquet Integral in its expanded form:

$$\mathfrak{C}_\mathfrak{m}(\vec{x}) = \sum_{i=1}^{n} \left(x_{(i)} \cdot \mathfrak{m}\left(A_{(i)}\right) - x_{(i-1)} \cdot \mathfrak{m}\left(A_{(i)}\right)\right). \tag{3}$$

Substituting the product operation in Equation (2) by a copula $C$, Lucca et al. [9] introduced the CC-integrals, which are averaging aggregation functions (see also [26]):

**Definition 2.8.** Let $\mathfrak{m} : 2^N \to [0,1]$ be a fuzzy measure and $C : [0,1]^2 \to [0,1]$ be a bivariate copula. The Choquet-like copula-based integral with respect to $\mathfrak{m}$ is defined as a function $\mathfrak{C}_\mathfrak{m}^C : [0,1]^n \to [0,1]$, given, for all $x \in [0,1]^n$, by

$$\mathfrak{C}^C_{\mathfrak{m}}(\vec{x}) = \sum_{i=1}^{n} \left(C\left(x_{(i)}, \mathfrak{m}\left(A_{(i)}\right)\right) - C\left(x_{(i-1)}, \mathfrak{m}\left(A_{(i)}\right)\right)\right), \tag{4}$$

where $(x_{(1)}, \ldots, x_{(n)})$ is an increasing permutation on the input $x$, that is, $0 \leq x_{(1)} \leq \ldots \leq x_{(n)}$, with the convention that $x_{(0)} = 0$, and $A_{(i)} = \{(i), \ldots, (n)\}$ is the subset of indices of $n - i + 1$ largest components of $\vec{x}$.

Another integral that is related to fuzzy measure is the Sugeno Integral:

**Definition 2.9.** The discrete Sugeno integral with respect to a fuzzy measure $\mathfrak{m}$ is the function $S_{\mathfrak{m}} : [0, 1]^n \to [0, 1]$, defined, for all of $\vec{x} = (x_1, \ldots, x_n) \in [0, 1]^n$, by:

$$S_{\mathfrak{m}}(\vec{x}) = \max_{i=1}^{n} \left\{\min\left\{x_{(i)}, \mathfrak{m}\left(A_{(i)}\right)\right\}\right\}, \tag{5}$$

where $\left(x_{(1)}, \ldots, x_{(n)}\right)$ is an increasing permutation on the input $\vec{x}$, that is, $0 \leq x_{(1)} \leq \ldots \leq x_{(n)}$, $A_{(i)} = \{(i), \ldots, (n)\}$ is the subset of indices corresponding to the $n - i + 1$ largest components of $\vec{x}$.

## 3. Pseudo pre-aggregation function pairs $(F_1, F_2)$

In this section, we introduce the concept of pseudo pre-aggregation function pair and study some properties. In the following, consider $N = \{1, \ldots, n\}$.

**Definition 3.1.** Consider two bivariate functions $F_1, F_2 : [0, 1]^2 \to [0, 1]$. The pair $(F_1, F_2)$ is said to be a pseudo pre-aggregation function pair whenever the following conditions hold, for all $y \in [0, 1]$:

**(DI)** Directional Increasingness: $F_1$ is $(1, 0)$-increasing, that is, it is increasing in the first coordinate;
**(BC0)** Boundary Conditions for 0:
  **(i)** $F_1(0, y) = F_2(0, y)$ and
  **(ii)** $F_1(0, 1) = 0$;
**(BC1)** Boundary Condition for 1: $F_1(1, 1) = 1$;
**(DM)** $F_1$-Dominance (or, equivalently, $F_2$-Subordination): $F_1 \geq F_2$.

**Remark 3.1.** Observe that, for any pseudo pre-aggregation function pair $(F_1, F_2)$, by ***(i)*** and ***(ii)***, it holds that $F_2(0, 1) = 0$.

We present in Table 1 examples of functions $F : [0, 1]^2 \to [0, 1]$ satisfying **(DI)**, **(BC0)(ii)** and **(BC1)**. Those functions are, then, candidates to be combined in order to build pseudo pre-aggregation function pairs. In Table 2, we present an analysis of the Dominance property **(DM)**, taking into account the functions presented in Table 1, all of them obviously satisfying **(BC0)(i)**. In this table, considering that functions $F_1$ and $F_2$ are represented, respectively, in the lines and columns of the table, the pairs marked with "yes" satisfy **(DM)** or the $F_1$-dominance (equivalently, the $F_2$-subordination). Thus, those pairs are pseudo pre-aggregation function pairs. The pairs marked with "no" are not pseudo pre-aggregation function pairs since they do not satisfy **(DM)**.

**Example 3.1.** According Tables 1 and 2, examples of pseudo pre-aggregation function pairs are: $(T_P, T_Ł)$, $(T_M, O_\alpha)$, $(T_M, F_{NA})$, $(T_{HP}, C_F)$, $(O_B, F_{mM})$, $(F_{BPC}, C_L)$, $(T_P, F_{BPC})$ (see Example 5.2), $(F_{IP}, F_{IP})$ (see Example 5.3), $(F_{BPC}, F_{BPC})$ (see Example 5.4), $(T_M, T_M)$ (see Example 5.5).

**Remark 3.2.** Whenever $(F_1, F_2)$ is a pseudo pre-aggregation function pair then, for any $F_3 : [0, 1]^2 \to [0, 1]$ such that $F_2 \leq F_3 \leq F_1$, we have that $(F_1, F_3)$ is also a pseudo pre-aggregation function pair. In particular, $(F_1, F_1)$ is a pseudo pre-aggregation function pair.

**Definition 3.2.** A pseudo pre-aggregation function pair $(F_1, F_2)$ is pairwise increasing if, for all $x, y_1, y_2 \in [0, 1]$ and $h > 0$ such that $x + h \in [0, 1]$, the following condition holds:

**(PI)** If $y_2 \leq y_1$ then $F_1(x, y_1) - F_2(x, y_2) \leq F_1(x + h, y_1) - F_2(x + h, y_2)$.

Table 1
$F_i : [0,1]^2 \to [0,1]$, $i = 1, 2$, satisfying **(DI)**, **(BC0)(ii)**, **(BC1)**, for building pseudo pre-aggregation function pairs.

| Definition | Name/Reference |
|---|---|
| **(I) T-norms [7]** | |
| $T_M(x,y) = \min\{x,y\}$ | Minimum |
| $T_P(x,y) = xy$ | Algebraic Product |
| $T_Ł(x,y) = \max\{0, x+y-1\}$ | Łukasiewicz |
| $T_{HP}(x,y) = \begin{cases} 0 & \text{if } x = y = 0 \\ \frac{xy}{x+y-xy} & \text{otherwise} \end{cases}$ | Hamacher Product |
| $T_{DP}(x,y) = \begin{cases} x & \text{if } y = 1 \\ y & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$ | Drastic Product |
| **(II) Overlap functions [19–21,27]** | |
| $O_B(x,y) = \min\{x\sqrt{y}, y\sqrt{x}\}$ | [19, Theorem 8], Cuadras–Augé family of copulas [28] |
| $O_{mM}(x,y) = \min\{x,y\}\max\{x^2,y^2\}$ | [29, Ex. 3.1.(i)], [30, Ex. 4], [31, Ex. 3.1] |
| $O_\alpha(x,y) = xy(1+\alpha(1-x)(1-y))$, $\alpha \in [-1,0[ \cup ]0,1]$ | [10, Appendix A (A.2.1)], [37], Farlie–Gumbel–Morgenstern copula family |
| $O_{Div}(x,y) = \frac{xy+\min\{x,y\}}{2}$ | [10, Ap. A (A.8.7)], [9, Table 1] |
| $GM(x,y) = \sqrt{xy}$ | Geometric Mean [32, Ex. 1] |
| $HM(x,y) = \begin{cases} 0 & \text{if } x = 0 \text{ or } y = 0 \\ \frac{2}{\frac{1}{x}+\frac{1}{y}} & \text{otherwise} \end{cases}$ | Harmonic Mean [32, Ex. 1] |
| $S(x,y) = \sin\left(\frac{\pi}{2}(xy)^{\frac{1}{4}}\right)$ | Sine [32, Ex. 1] |
| **(III) Copulas that are neither t-norms nor overlap functions [10]** | |
| $C_F(x,y) = xy + x^2y(1-x)(1-y)$ | [7, Ex. 9.5 (v)], [9, Table 1] |
| $C_L(x,y) = \max\{\min\{x, \frac{y}{2}\}, x+y-1\}$ | [10, Ap. A (A.5.3a)], [9, Table 1] |
| **(IV) Aggregation functions other than (I)–(III)** | |
| $AVG(x,y) = \frac{x+y}{2}$ | Arithmetic Mean |
| $F_{RS}(x,y) = \min\left\{\frac{(x+1)\sqrt{y}}{2}, y\sqrt{x}\right\}$ | |
| $F_{GL}(x,y) = \sqrt{\frac{x(y+1)}{2}}$ | |
| $F_{BPC}(x,y) = xy^2$ | [23, Ex. 1.80] |
| **(V) (1, 0)-Pre-Aggregation functions** | |
| $F_{NA}(x,y) = \begin{cases} x & \text{if } x \le y \\ \min\{\frac{x}{2}, y\} & \text{otherwise} \end{cases}$ | |
| $F_{NA2}(x,y) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{x+y}{2} & \text{if } 0 < x \le y \\ \min\{\frac{x}{2}, y\} & \text{otherwise} \end{cases}$ | |
| $F_\alpha(x,y) = \begin{cases} \alpha x & \text{if } x < y \\ \max\{\alpha x, y\} & \text{otherwise} \end{cases}$, $0 < \alpha < 1$ | |
| **(VI) Non Pre-Aggregation functions** | |
| $F_{IM}(x,y) = \max\{1-y, x\}$ | |
| $F_{IP}(x,y) = 1 - y + xy$ | |

Table 2
Analysis of the property **(DM)** for different candidates to pseudo pre-aggregation function pairs $(F_1, F_2)$, satisfying **(BC0)(i)**, constructed from Table 1.

| | $T_P$ | $T_M$ | $T_Ł$ | $T_{DP}$ | $T_{HP}$ | $O_B$ | $O_{mM}$ | $O_\alpha$ | $O_{Div}$ | $GM$ | $HM$ | $S$ | $F_{RS}$ | $C_F$ | $C_L$ | $F_{GL}$ | $F_{BPC}$ | $F_{NA}$ | $F_\alpha$ | $F_{NA2}$ | AVG | $F_{IM}$ | $F_{IP}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_P$ | yes | no | yes | yes | no | no | yes | no | no | no | no | no | no | no | no | no | yes | no | no | no | no | no | no |
| $T_M$ | yes | yes | yes | yes | yes | yes | yes | yes | yes | no | no | no | no | yes | yes | no | yes | yes | no | no | no | no | no |
| $T_Ł$ | no | no | yes | yes | no | no | no | no | no | no | no | no | no | no | no | no | no | no | no | no | no | no | no |
| $T_{DP}$ | no | no | no | yes | no | no | no | no | no | no | no | no | no | no | no | no | no | no | no | no | no | no | no |
| $T_{HP}$ | yes | no | yes | yes | yes | no | yes | yes | no | no | no | no | no | yes | no | no | yes | no | no | no | no | no | no |
| $O_B$ | yes | no | yes | yes | no | yes | yes | yes | no | no | no | no | no | no | no | no | yes | no | no | no | no | no | no |
| $O_{mM}$ | no | no | no | yes | no | no | yes | no | no | no | no | no | no | no | no | no | yes | no | no | no | no | no | no |
| $O_\alpha$ | yes | no | yes | yes | no | no | yes | yes | no | no | no | no | no | no | no | no | yes | no | no | no | no | no | no |
| $O_{Div}$ | yes | no | yes | yes | no | yes | yes | yes | yes | no | no | no | no | no | no | no | yes | no | no | no | no | no | no |
| $GM$ | yes | yes | yes | yes | yes | yes | yes | yes | yes | yes | yes | no | yes | yes | yes | no | yes | yes | no | no | no | no | no |
| $HM$ | yes | yes | yes | yes | yes | yes | yes | yes | yes | no | yes | no | no | yes | yes | no | yes | yes | no | no | no | no | no |
| $S$ | yes | yes | yes | yes | yes | yes | yes | yes | yes | yes | yes | yes | yes | yes | yes | yes | yes | yes | yes | yes | yes | yes | yes |
| $F_{RS}$ | yes | no | yes | yes | no | yes | yes | yes | no | no | no | no | yes | no | no | no | yes | no | no | no | no | no | no |
| $C_F$ | yes | no | yes | yes | no | no | yes | no | no | no | no | no | no | yes | no | no | yes | no | no | no | no | no | no |
| $C_L$ | no | no | yes | yes | no | no | no | no | no | no | no | no | no | no | yes | no | yes | no | no | no | no | no | no |
| $F_{GL}$ | yes | yes | yes | yes | yes | yes | yes | yes | yes | yes | yes | no | yes | yes | yes | yes | yes | yes | yes | no | no | no | no |
| $F_{BPC}$ | no | no | no | yes | no | no | no | no | no | no | no | no | no | no | yes | no | yes | no | no | no | no | no | no |
| $F_{NA}$ | no | no | no | yes | no | no | no | no | no | no | no | no | no | no | yes | no | no | yes | no | no | no | no | no |
| F$\alpha$ | no | no | no | yes | no | no | no | no | no | no | no | no | no | no | yes | no | no | no | yes | no | no | no | no |
| $F_{NA2}$ | no | no | no | yes | no | no | no | no | no | no | no | no | no | no | yes | no | no | yes | no | yes | no | no | no |
| AVG | yes | yes | yes | yes | yes | yes | yes | yes | yes | yes | yes | no | yes | yes | yes | no | yes | yes | yes | yes | yes | no | no |
| $F_{IM}$ | yes | yes | yes | yes | yes | yes | yes | yes | yes | no | no | no | no | yes | yes | no | yes | yes | yes | no | no | yes | no |
| $F_{IP}$ | yes | yes | yes | yes | yes | yes | yes | yes | yes | no | no | no | no | yes | yes | no | yes | yes | yes | no | no | yes | yes |

**Proposition 3.1.** *Let $(F_1, F_2)$ be a pseudo pre-aggregation function pair. If $F_2$ is $(1, 0)$-decreasing, then the pair $(F_1, F_2)$ satisfies **(PI)**.*

**Proof.** Since $F_1$ is $(1, 0)$-increasing, then, for any $h > 0$ and $x, y_1, y_2 \in [0, 1]$ such that $x + h \in [0, 1]$, it holds that $F_1(x + h, y_1) \geq F_1(x, y_1)$. On the other hand, since $F_2$ is $(1, 0)$-decreasing, then, for any $h > 0$ and $x, y_1, y_2 \in [0, 1]$ such that $x + h \in [0, 1]$, it holds that $-F_2(x + h, y_2) \geq -F_2(x, y_2)$. Thus, one has that $F_1(x, y_1) - F_2(x, y_2) \leq F_1(x + h, y_1) - F_2(x + h, y_2)$. □

**Proposition 3.2.** *Let $F : [0, 0]^2 \to [0, 1]$ be a $(1, 0)$-increasing function such that $F(1, 1) = 1$ and, for all $y \in [0, 1]$, $F(0, y) = 0$. If $F$ is 2-increasing (i.e., $F$ satisfies **(C1)**), then the following statements hold:*

(i) *The pair $(F, kF)$, for any $k \in ]0, 1]$, is a pseudo pre-aggregation function pair satisfying **(PI)**.*
(ii) *For any increasing 1-Lipschitz function $f : [0, 1] \to [0, 1]$, such that $f(x) \leq x$, the pair $(F, f(F))$ is a pseudo pre-aggregation function pair satisfying **(PI)**.*

**Proof.** One has that:

**(i)** It is immediate that $(F, kF)$ satisfies **(DI)**, **(BC0)**, **(BC1)** and **(DM)**. Thus, $(F, kF)$ is a pseudo pre-aggregation function pair. From **(C1)**, it is immediate that $(F, kF)$ satisfies **(PI)**.
**(ii)** It follows from **(i)**. □

**Corollary 3.1.** *For a copula $C$, $(C, C)$ is a pseudo pre-aggregation function pair satisfying **(PI)**.*

**Proof.** It follows from Proposition 3.2 **(i)**, taking $F = C$ and $k = 1$, since it is immediate that any copula $C$ is $(1, 0)$-increasing and satisfies **(C1)**. □

**Remark 3.3.** Observe that **(PI)** is a generalization of the 2-increasing property **(C1)**. In fact, for any fusion function $F$, $(F, F)$ satisfies **(PI)** if and only if $F$ satisfies **(C1)**. Note also that the class of functions $F$ such that $(F, F)$ is a pseudo pre-aggregation function pair satisfying **(PI)** is a convex class.

**Remark 3.4.** For any function $F$ such that $(F, F)$ is a pseudo-aggregation function pair satisfying **(PI)** and for any increasing functions $f, g : [0, 1] \to [0, 1]$ such that $f(0) = 0$ and $f(1) = g(1) = 1$, the pair $(G, G)$, where $G : [0, 1]^2 \to [0, 1]$ is given by

$$G(x, y) = F(f(x), g(y)), \tag{6}$$

is a pseudo-aggregation function pair satisfying **(PI)**. So, for example, for $F(x, y) = xy$ (that is, $F$ is the product copula), $f(x) = x^2$ and $g(x) = \frac{y+1}{2}$, we have that

$$G(x, y) = \frac{x^2(y + 1)}{2}$$

and $(G, G)$ is a pseudo-aggregation pair satisfying **(PI)**. Considering again Equation (6) and the product copula $F$, it is possible to observe that the same happens for the aggregation functions

$$G'(x, y) = F_{BPC(x,y)} = xy^2 \text{ for } f(x) = x \text{ and } g(x) = y^2,$$
$$G''(x, y) = F_{GL} = \sqrt{\frac{x(y + 1)}{2}} \text{ for } f(x) = \sqrt{x} \text{ and } g(x) = \sqrt{\frac{y + 1}{2}},$$

both from Table 1, where $(G', G')$ and $(G'', G'')$ are pseudo-aggregation function pairs satisfying **(PI)**.

## 4. Constructing Choquet-like integrals based on pseudo pre-aggregation function pairs $(F_1, F_2)$

In this section, we introduce a method for constructing Choquet-like integrals defined by combining the discrete Choquet integral in its expanded form (Equation (3)) with pseudo pre-aggregation function pairs $(F_1, F_2)$, just replacing the product operation in Equation (3) by $(F_1, F_2)$. Such Choquet-like integrals, which generalize the concept of $C_{F_1F_2}$-integrals introduced in [15], are called $gC_{F_1F_2}$-integrals.

Consider $N = \{1, \dots, n\}$, where $n$ is the dimension of the input vectors $\vec{x}$, that is, $\vec{x} = (x_1, \dots, x_n) \in [0, 1]^n$. First, in order to handle repetitive elements in any input $\vec{x}$, which would lead to an ambiguous definition, we proceed to a dimension reduction of such $\vec{x}$, from $n$ to $k$, such that $k \leq n$ is the cardinality of the set $\{x_1, \dots, x_n\}$ composed by the components of the vector $\vec{x}$.

For that, we introduce the following auxiliary definition:

**Definition 4.1.** The dimension reduction function is defined by the function $R : [0, 1]^n \to \cup_{k=1}^n [0, 1]^k$, given, for all input $\vec{x} = (x_1, \dots, x_n) \in [0, 1]^n$, by:

$$R(x_1, \dots, x_n) = \vec{y} = (y_1, \dots, y_k), \tag{7}$$

such that:

**(R1)** $k = |\{x_1, \dots, x_n\}|$ is the cardinality of the set $\{x_1, \dots, x_n\}$,
**(R2)** $y_1 < \dots < y_k$, and
**(R3)** $\{x_1, \dots, x_n\} = \{y_1, \dots, y_k\}$.

Note that the function $R$ is well defined, and if it is the case that some components of an input $\vec{x}$ are repeated, then those repeated elements collapse into one single value. Also, in the case that all components of an input $\vec{x}$ are the same, then they all collapse into a single value $y_1 = x_1$.

Then, denote, for each $\vec{x} \in [0, 1]^n$ and for each $j \in K = \{1, \dots, k\}$:

$$B_j^R(\vec{x}) = \{i \in N \mid x_i = y_j\}. \tag{8}$$

Observe that, for every $\vec{x} \in [0, 1]^n$, it holds that $\cup_{j=1}^k B_j^R(\vec{x})) = N$.

**Definition 4.2.** Let $F_1, F_2 : [0,1]^2 \to [0,1]$ be a pair of functions such that $F_1 \geq F_2$ (i.e., $F_1$ dominates $F_2$) and $F_1$ is $(1,0)$-increasing. Let $\mathfrak{m} : 2^N \to [0,1]$ be a fuzzy measure and $R : [0,1]^n \to \cup_{k=1}^n [0,1]^k$ be the dimension reduction function given in Definition 4.1. The generalized $C_{F_1F_2}$-integral based on $(F_1, F_2)$ with respect to $\mathfrak{m}$ is defined as a function $g\mathfrak{C}_{\mathfrak{m}}^{(F_1,F_2)} : [0,1]^n \to [0,1]$, given, for all $\vec{x} \in [0,1]^n$, by

$$g\mathfrak{C}_{\mathfrak{m}}^{(F_1,F_2)}(\vec{x}) = \min\left\{1, \sum_{j=1}^{k} F_1\left(y_j, \mathfrak{m}\left(\cup_{p=j}^k B_p^R(\vec{x})\right)\right) - F_2\left(y_{j-1}, \mathfrak{m}\left(\cup_{p=j}^k B_p^R(\vec{x})\right)\right)\right\}, \tag{9}$$

with the convention that $y_0 = 0$ and $B_j^R$ is as defined in Equation (8).

**Proposition 4.1.** *Under the conditions of Definition 4.2, $g\mathfrak{C}_{\mathfrak{m}}^{(F_1,F_2)}$ is well defined, for any pair $F_1, F_2 : [0,1]^2 \to [0,1]$ and fuzzy measure $\mathfrak{m}$.*

**Proof.** It is immediate that, for all $\vec{x}, \vec{x}' \in [0,1]^n$, whenever $g\mathfrak{C}_{\mathfrak{m}}^{(F_1,F_2)}(\vec{x}) \neq g\mathfrak{C}_{\mathfrak{m}}^{(F_1,F_2)}(\vec{x}')$, then $\vec{x} \neq \vec{x}'$. Now, consider $R$ and $B_j^R$ as defined in equations (7) and (8), respectively. Then, since $F_1$ is $(1,0)$-increasing and $F_1 \geq F_2$, one has that:

$$\begin{aligned}
&F_1\left(y_j, \mathfrak{m}\left(\cup_{p=j}^k B_p^R(\vec{x})\right)\right) - F_2\left(y_{j-1}, \mathfrak{m}\left(\cup_{p=j}^k B_p^R(\vec{x})\right)\right)\\
&\geq F_1\left(y_j, \mathfrak{m}\left(\cup_{p=j}^k B_p^R(\vec{x})\right)\right) - F_1\left(y_{j-1}, \mathfrak{m}\left(\cup_{p=j}^k B_p^R(\vec{x})\right)\right)\\
&\geq 0.
\end{aligned}$$

Therefore, it holds that $g\mathfrak{C}_{\mathfrak{m}}^{(F_1,F_2)}(\vec{x}) \geq 0$, for all $\vec{x} \in [0,1]^n$. On the other hand, it is immediate that $g\mathfrak{C}_{\mathfrak{m}}^{(F_1,F_2)}(\vec{x}) \leq 1$, for all $\vec{x} \in [0,1]^n$. Thus, $g\mathfrak{C}_{\mathfrak{m}}^{(F_1,F_2)}$ is well defined. □

**Lemma 4.1.** *Consider $R$ and $B_j^R$ as defined in equations (7) and (8), respectively. Then, for all $\vec{x} = (x, \ldots, x) \in [0,1]^n$, it holds that $k = 1$, $R(\vec{x}) = x$ and $B_1^R(\vec{x}) = N$.*

**Proof.** It is immediate that the cardinality of any $\{x, \ldots, x\}$ is $k = 1$. It follows that $R(\vec{x}) = x$, since $\{x, \ldots, x\} = \{y_1\}$ implies that $y_1 = x$. Also, one has that $B_1^R(\vec{x}) = \{i \in N \mid x_i = y_1 = x\} = \{1, \ldots, n\} = N$. □

**Proposition 4.2.** *Under the conditions of Definition 4.2, for any fuzzy measure $\mathfrak{m} : 2^N \to [0,1]$ and pseudo pre-aggregation function pair $(F_1, F_2)$, if $F_1(x,1) = x$, for all $x \in [0,1]$, then $g\mathfrak{C}_{\mathfrak{m}}^{(F_1,F_2)}$ is idempotent.*

**Proof.** Consider $R$ and $B_j^R$ as defined in equations (7) and (8), respectively. Then, one has that:

$$\begin{aligned}
g\mathfrak{C}_{\mathfrak{m}}^{(F_1,F_2)}(x, \ldots, x) &= \min\left\{1, F_1(y_1, \mathfrak{m}(B_1^R(\vec{x}))) - F_2(0, \mathfrak{m}(B_1^R(\vec{x})))\right\} \text{ by Eq. (9)}\\
&= \min\{1, F_1(x, \mathfrak{m}(N) - F_2(0, \mathfrak{m}(N))\} \text{ by Lemma 4.1}\\
&= \min\{1, F_1(x,1) - F_2(0,1)\}\\
&= x \text{ by } \mathbf{(BC0)}. \quad \square
\end{aligned}$$

**Proposition 4.3.** *Under the conditions of Definition 4.2, for any fuzzy measure $\mathfrak{m} : 2^N \to [0,1]$ and pair of functions $F_1, F_2 : [0,1]^2 \to [0,1]$, if $F_2(0,1) = 0$ and $F_1(x,1) \geq x$, for all $x \in [0,1]$, then $g\mathfrak{C}_{\mathfrak{m}}^{(F_1,F_2)} \geq \min$.*

**Proof.** Consider $R$ and $B_j^R$ as defined in equations (7) and (8), respectively. Since $F_1 \geq F_2$, for all $\vec{x} \in [0,1]^n$, one has that:

$$g\mathfrak{C}_{\mathfrak{m}}^{(F_1,F_2)}(\vec{x}) = \min\Big\{1, F_1(y_1, \mathfrak{m}(\cup_{p=1}^k B_p^R(\vec{x}))) - F_2(y_0, \mathfrak{m}(\cup_{p=1}^k B_p^R(\vec{x}))) +$$

$$\left.\sum_{j=2}^{k} F_1(y_j, \mathfrak{m}(\cup_{p=j}^{k} B_p^R(\vec{x}))) - F_2(y_{j-1}, \mathfrak{m}(\cup_{p=j}^{k} B_p^R(\vec{x})))\right\}$$
$$\geq \min\{1, F_1(y_1, \mathfrak{m}(N)) - F_2(y_0, \mathfrak{m}(N))\} \text{ by } \textbf{(DM)}$$
$$= \min\{1, F_1(y_1, 1) - F_2(0, 1)\}$$
$$\geq \min\{1, y_1 - 0\}$$
$$= y_1$$
$$= \min \vec{x}. \quad \square$$

Observe that, although we have stated sufficient conditions to have $g\mathfrak{C}_{\mathfrak{m}}^{(F_1,F_2)} \geq \min$, we do have such conditions for $g\mathfrak{C}_{\mathfrak{m}}^{(F_1,F_2)} \leq \max$. In general, a $gC_{F_1F_2}$-integral is neither an aggregation function nor averaging. In the next section, we discuss such concepts for $gC_{F_1F_2}$-integrals.

## 5. $gC_{F_1F_2}$-integrals as (pre) aggregation functions

In this section, we show that a $gC_{F_1F_2}$-integral is an aggregation function whenever $F_1 = F_2 = F$ and $(F, F)$ is a pre-aggregation function pair satisfying an additional condition, namely, the pairwise increasingness property **(PI)**. We also show the necessary and sufficient condition to have averaging $gC_{FF}$-integrals (like $CC$-integrals). Additionally, we show that, whenever one has a $gC_{FF}$-integral that is an aggregation function, then the $gC_{F(wF)}$-integral, for $w \in [0, 1]$, is a pre-aggregation function that is $(1, \ldots, 1)$-increasing (or weakly increasing [33]).

**Proposition 5.1.** *Under the conditions of Definition 4.2, for any fuzzy measure* $\mathfrak{m} : 2^N \to [0, 1]$ *and pseudo pre-aggregation function pair* $(F_1, F_2)$*,* $g\mathfrak{C}_{\mathfrak{m}}^{(F_1,F_2)}$ *satisfies the boundary conditions* **(A2)**.

**Proof.** Consider $R$ and $B_j^R$ as defined in equations (7) and (8). If $\vec{0} = (0, \ldots, 0) \in [0, 1]^n$, then $k = 1$ and it follows that:

$$g\mathfrak{C}_{\mathfrak{m}}^{(F_1,F_2)}(\vec{0}) = \min\left\{1, F_1(0, \mathfrak{m}(B_1^R(\vec{0})) - F_2(0, \mathfrak{m}(B_1^R(\vec{0}))\right\} \text{ by Eq. (9)}$$
$$= \min\{1, F_1(0, \mathfrak{m}(N) - F_2(0, \mathfrak{m}(N)\} \text{ by Lemma 4.1}$$
$$= \min\{1, F_1(0, 1) - F_2(0, 1)\}$$
$$= 0 \quad \text{by } \textbf{(BC0)}.$$

Consider $\vec{1} = (1, \ldots, 1) \in [0, 1]^n$. Then $k = 1$ and one has that:

$$g\mathfrak{C}_{\mathfrak{m}}^{(F_1,F_2)}(\vec{1}) = \min\left\{1, F_1(1, \mathfrak{m}(B_1^R(\vec{1}))) - F_2(0, \mathfrak{m}(B_1^R(\vec{1})))\right\} \text{ by Eq. (9)}$$
$$= \min\{1, F_1(1, \mathfrak{m}(N) - F_2(1, \mathfrak{m}(N)\} \text{ by Lemma 4.1}$$
$$= \min\{1, F_1(1, 1) - F_2(0, 1)$$
$$= 1 \quad \text{by } \textbf{(BC1)}, \textbf{(BC0)} \quad \square$$

**Lemma 5.1.** *Let* $\mathfrak{m} : 2^N \to [0, 1]$ *be a fuzzy measure and* $F_1, F_2 : [0, 1]^2 \to [0, 1]$ *a pair of functions satisfying the conditions of Definition 4.2. Then*

$$g\mathfrak{C}_{\mathfrak{m}}^{(F_1,F_2)}(\vec{x}) \leq g\mathfrak{C}_{\mathfrak{m}}^{(F_1,F_2)}(\vec{z}),$$

*for every* $\vec{x} = (x_1, \ldots, x_n), \vec{z} = (z_1, \ldots, z_n) \in [0, 1]^n$ *such that* $x_{(n)} \leq z_{(n)}$ *and* $x_i = z_i$*, for all* $i \in \{(1), \ldots, (n-1)\}$*, where* $(x_{(1)}, \ldots, x_{(n)})$ *is any increasing permutation of* $\vec{x}$*.*

**Proof.** Consider $\vec{x} = (x_1, \ldots, x_n), \vec{z} = (z_1, \ldots, z_n) \in [0, 1]^n$ such that $x_{(n)} < z_{(n)}$ and $x_i = z_i$, for all $i \in \{(1), \ldots, (n-1)\}$. Then, according to equations (7) and (8), for each $\vec{x} \in [0, 1]^n$, we have that:

**(i)** $R(\vec{x}) = (y_1, \ldots, y_k)$ such that $\{x_1, \ldots, x_n\} = \{y_1, \ldots, y_k\}$, with $k \leq n$, and $y_1 < \ldots < y_k$;
**(ii)** $B_j^R(\vec{x}) = \{i \in N \mid x_i = y_j\}$, for $j \in K = \{1, \ldots, k\}$;

**(iii)** $R(\vec{z}) = (h_1, \ldots, h_w)$ such that $\{z_1, \ldots, z_n\} = \{h_1, \ldots, h_w\}$, with $w \leq n$, and $h_1 < \ldots < h_w$;
**(iv)** $B_j^R(\vec{z}) = \{i \in N \mid z_i = h_j\}$, for $j \in W = \{1, \ldots, w\}$.

Observe that either $w = k$ or $w = k + 1$, and $h_i = y_i$, for each $i = 1, \ldots, w - 1$. Then, one has the following cases:

$k = w$**:** In this case it holds that $y_1 = h_1 < \ldots < y_{k-1} = h_{w-1} < y_k < h_w = z_{(n)}$ and $B_j^R(\vec{x}) = B_j^R(\vec{z})$, for all $j \in K = W$. Since $F_1$ is $(1, 0)$-increasing, it follows that:

$$
\begin{aligned}
g\mathfrak{C}_{\mathfrak{m}}^{(F_1,F_2)}(\vec{x}) &= \min\left\{1, \sum_{j=1}^{k-1}\Big(F_1(y_j, \mathfrak{m}(\cup_{p=j}^{k} B_p^R(\vec{x}))) - F_2(y_{j-1}, \mathfrak{m}(\cup_{p=j}^{k} B_p^R(\vec{x})))\Big)\right.\\
&\qquad \left. + F_1(y_k, \mathfrak{m}(B_k^R(\vec{x}))) - F_2(y_{k-1}, \mathfrak{m}(B_k^R(\vec{x})))\right\}\\
&\leq \min\left\{1, \sum_{j=1}^{k-1}\Big(F_1(y_j, \mathfrak{m}(\cup_{p=j}^{k} B_p^R(\vec{x}))) - F_2(y_{j-1}, \mathfrak{m}(\cup_{p=j}^{k} B_p^R(\vec{x})))\Big)\right.\\
&\qquad \left. + F_1(h_w, \mathfrak{m}(B_k^R(\vec{x}))) - F_2(y_{k-1}, \mathfrak{m}(B_k^R(\vec{x})))\right\}\\
&= \min\left\{1, \sum_{j=1}^{w-1}\Big(F_1(h_j, \mathfrak{m}(\cup_{p=j}^{w} B_p^R(\vec{z}))) - F_2(h_{j-1}, \mathfrak{m}(\cup_{p=j}^{w} B_p^R(\vec{z})))\Big)\right.\\
&\qquad \left. + F_1(h_w, \mathfrak{m}(B_w^R(\vec{z}))) - F_2(h_{w-1}, \mathfrak{m}(B_w^R(\vec{z})))\right\}\\
&= g\mathfrak{C}_{\mathfrak{m}}^{(F_1,F_2)}(\vec{z}).
\end{aligned}
$$

$w = k + 1$**:** In this case it holds that $x_{(n)} = x_{(n-1)} = z_{(n-1)}$, $y_1 = h_1 < \ldots < y_k = h_{w-1} < h_w$, $B_j^R(\vec{x}) = B_j^R(\vec{z})$, for all $j \leq k - 1$, $B_{w-1}^R(\vec{z}) = B_k^R(\vec{x}) - \{(n)\}$ and $B_w^R(\vec{z}) = \{(n)\}$ (that is, $B_{w-1}^R(\vec{z}) \cup B_w^R(\vec{z}) = B_k^R(\vec{x})$). Since $F_1$ is $(1, 0)$-increasing, it follows that:

$$
\begin{aligned}
g\mathfrak{C}_{\mathfrak{m}}^{(F_1,F_2)}(\vec{x}) &= \min\left\{1, \sum_{j=1}^{k-1}\Big(F_1(y_j, \mathfrak{m}(\cup_{p=j}^{k} B_p^R(\vec{x}))) - F_2(y_{j-1}, \mathfrak{m}(\cup_{p=j}^{k} B_p^R(\vec{x})))\Big)\right.\\
&\qquad \left. + F_1(y_k, \mathfrak{m}(B_k^R(\vec{x}))) - F_2(y_{k-1}, \mathfrak{m}(B_k^R(\vec{x})))\right\}\\
&= \min\left\{1, \sum_{j=1}^{k-1}\Big(F_1(y_j, \mathfrak{m}(\cup_{p=j}^{k} B_p^R(\vec{x}))) - F_2(y_{j-1}, \mathfrak{m}(\cup_{p=j}^{k} B_p^R(\vec{x})))\Big)\right.\\
&\qquad \left. + F_1(y_k, \mathfrak{m}(B_{w-1}^R(\vec{z}) \cup B_w^R(\vec{z}))) - F_2(y_{k-1}, \mathfrak{m}(B_{w-1}^R(\vec{z}) \cup B_w^R(\vec{z})))\right\}\\
&\leq \min\left\{1, \sum_{j=1}^{w-2}\Big(F_1(h_j, \mathfrak{m}(\cup_{p=j}^{w} B_p^R(\vec{z}))) - F_2(h_{j-1}, \mathfrak{m}(\cup_{p=j}^{w} B_p^R(\vec{z}))))\Big)\right.\\
&\qquad + F_1(h_{w-1}, \mathfrak{m}(B_{w-1}^R(\vec{z}) \cup B_w^R(\vec{z}))) - F_2(h_{w-2}, \mathfrak{m}(B_{w-1}^R(\vec{z}) \cup B_w^R(\vec{z}))))\\
&\qquad \left. + F_1(h_w, \mathfrak{m}(B_w^R(\vec{z}))) - F_2(h_{w-1}, \mathfrak{m}(B_w^R(\vec{z})))\right\}\\
&= g\mathfrak{C}_{\mathfrak{m}}^{(F_1,F_2)}(\vec{z}). \quad \square
\end{aligned}
$$

**Theorem 5.1.** *Let $F : [0, 1]^2 \to [0, 1]$ such that $(F, F)$ is a pair of functions satisfying the conditions of Definition 4.2. The pair $(F, F)$ satisfies **(PI)** if and only if $g\mathfrak{C}_{\mathfrak{m}}^{(F,F)}$ is increasing for each fuzzy measure $\mathfrak{m} : 2^N \to [0, 1]$.*

**Proof.** $(\Rightarrow)$ Suppose that $(F, F)$ satisfies **(PI)** and consider $\vec{x} = (x_1, \ldots, x_{t-1}, x_t, x_{t+1} \ldots, x_n) \in [0, 1]^n$, with $t \in \{1, \ldots, n\}$. By convention, given $\vec{x} \in [0, 1]^n$, we state that $x_0 = x_{(0)} = 0$ and $x_{n+1} = x_{(n+1)} = 1$. We have the following cases:

**(i)** $\forall i \in N : i \neq t \to x_t \neq x_i$. In this case, considering equations (7) and (8), for each $\vec{x} \in [0, 1]^n$, one has that:

- $R(\vec{x}) = (y_1, \ldots, y_{l-1}, y_l = x_t, y_{l+1}, \ldots, y_k)$ (with $l \in K = \{1, \ldots, k\}$, $k \leq n$, $y_0 = 0$ and $y_{k+1} = 1$) such that $\{x_1, \ldots, x_{t-1}, x_t, x_{t+1}, \ldots, x_n\} = \{y_1, \ldots, y_{l-1}, y_l = x_t, y_{l+1}, \ldots y_k\}$ and $y_1 < \ldots < y_{l-1} < y_l = x_t < y_{l+1} < \ldots < y_k$.
- $B_j^R(\vec{x}) = \{i \in N \mid x_i = y_j\}$, for $j \in K = \{1, \ldots, k\}$. In particular, one has that $B_l^R(\vec{x}) = \{t\}$.

Now, consider the following cases:

**(ia)** Suppose that there exists $\vec{z} \in [0, 1]^n$, with $\vec{x} < \vec{z}$, such that $\vec{z} = (z_1 = x_1, \ldots, z_{t-1} = x_{t-1}, z_t, z_{t+1} = x_{t+1}, \ldots, z_t = x_n) \in [0, 1]^n$ with $y_1 < \ldots < y_{l-1} < y_l = x_t < z_t < y_{l+1} < \ldots < y_k$. If $t = 1$ or $t = n$ then define $\vec{z} = (z, z_2, \ldots, z_n) \in [0, 1]^n$ or $\vec{z} = (z_1, \ldots, z_{n-1}, z) \in [0, 1]^n$, respectively. In this case, considering equations (7) and (8), one has that:

- $R(\vec{z}) = (h_1 = y_1, \ldots, h_{l-1} = y_{l-1}, h_l = z_t, h_{l+1} = y_{l+1}, \ldots, h_w = y_k)$, with $w = k \leq n$, where $h_1 = y_1 < \ldots < h_{l-1} = y_{l-1} < y_l = x_t < h_l = z_t < h_{l+1} = y_{l+1} < \ldots < h_w = y_k$.
- $B_j^R(\vec{z}) = \{i \in N \mid z_i = h_j\} = \{i \in N \mid x_i = y_j\} = B_j^R(\vec{x})$, for $j \in W = K = \{1, \ldots, w = k\}$. In particular, one has that $B_l^R(\vec{z}) = \{t\} = B_l^R(\vec{x})$.

Since the pair $(F, F)$ satisfies **(PI)** and by Lemma 5.1, it follows that:

$$
\begin{aligned}
& g\mathfrak{C}_{\mathfrak{m}}^{(F,F)}(\vec{x}) \\
&= \min\Bigg\{1, \sum_{j=1}^{l-1}\Big(F(y_j, \mathfrak{m}(\cup_{p=j}^k B_p^R(\vec{x}))) - F(y_{j-1}, \mathfrak{m}(\cup_{p=j}^k B_p^R(\vec{x})))\Big) + F(y_l, \mathfrak{m}(\cup_{p=l}^k B_p^R(\vec{x}))) \\
&\quad - F(y_{l-1}, \mathfrak{m}(\cup_{p=l}^k B_p^R(\vec{x}))) + F(y_{l+1}, \mathfrak{m}(\cup_{p=l+1}^k B_p^R(\vec{x}))) - F(y_l, \mathfrak{m}(\cup_{p=l+1}^k B_p^R(\vec{x}))) \\
&\quad + \sum_{j=l+2}^{k}\Big(F(y_j, \mathfrak{m}(\cup_{p=j}^k B_p^R(\vec{x}))) - F(y_{j-1}, \mathfrak{m}(\cup_{p=j}^k B_p^R(\vec{x})))\Big)\Bigg\} \\
&= \min\Bigg\{1, \sum_{j=1}^{l-1}\Big(F(y_j, \mathfrak{m}(\cup_{p=j}^k B_p^R(\vec{x}))) - F(y_{j-1}, \mathfrak{m}(\cup_{p=j}^k B_p^R(\vec{x})))\Big) + F(x_t, \mathfrak{m}(\cup_{p=l}^k B_p^R(\vec{x}))) \\
&\quad - F(y_{l-1}, \mathfrak{m}(\cup_{p=l}^k B_p^R(\vec{x}))) + F(y_{l+1}, \mathfrak{m}(\cup_{p=l+1}^k B_p^R(\vec{x}))) - F(x_t, \mathfrak{m}(\cup_{p=l+1}^k B_p^R(\vec{x}))) \\
&\quad + \sum_{j=l+2}^{k}\Big(F(y_j, \mathfrak{m}(\cup_{p=j}^k B_p^R(\vec{x})) - F(y_{j-1}, \mathfrak{m}(\cup_{p=j}^k B_p^R(\vec{x})))\Big)\Bigg\} \\
&\leq \min\Bigg\{1, \sum_{j=1}^{l-1}\Big(F(h_j, \mathfrak{m}(\cup_{p=j}^w B_p^R(\vec{z}))) - F_2(h_{j-1}, \mathfrak{m}(\cup_{p=j}^w B_p^R(\vec{z})))\Big) + F(z_t, \mathfrak{m}(\cup_{p=l}^w B_p^R(\vec{z}))) \\
&\quad - F(h_{l-1}, \mathfrak{m}(\cup_{p=l}^w B_p^R(\vec{z}))) + F(h_{l+1}, \mathfrak{m}(\cup_{p=l+1}^w B_p^R(\vec{z}))) - F(z_t, \mathfrak{m}(\cup_{p=l+1}^w B_p^R(\vec{z}))) \\
&\quad + \sum_{j=l+2}^{w}\Big(F(h_j, \mathfrak{m}(\cup_{p=j}^w B_p^R(\vec{z}))) - F(h_{j-1}, \mathfrak{m}(\cup_{p=j}^w B_p^R(\vec{z})))\Big)\Bigg\} \\
&= g\mathfrak{C}_{\mathfrak{m}}^{(F,F)}(\vec{z}),
\end{aligned}
$$

since $\mathfrak{m}(\cup_{p=l+1}^{k} B_p^R(\vec{x})) = \mathfrak{m}(\cup_{p=l+1}^{w} B_p^R(\vec{z})) \leq \mathfrak{m}(\cup_{p=l}^{k} B_p^R(\vec{x})) = \mathfrak{m}(\cup_{p=l}^{w} B_p^R(\vec{z}))$, and, by **(PI)**, it holds that

$$F(x_t, \mathfrak{m}(\cup_{p=l}^{k} B_p^R(\vec{x}))) - F(x_t, \mathfrak{m}(\cup_{p=l+1}^{k} B_p^R(\vec{x}))) \leq F(z_t, \mathfrak{m}(\cup_{p=l}^{w} B_p^R(\vec{z}))) - F(z_t, \mathfrak{m}(\cup_{p=l+1}^{w} B_p^R(\vec{z}))).$$

**(ib)** Now, consider that $n \geq 2$ and $\vec{z} \in [0,1]^n$, with $\vec{x} < \vec{z}$, such that $\vec{z} = (z_1 = x_1, \ldots, z_{t-1} = x_{t-1}, z_t, z_{t+1} = x_{t+1}, \ldots, z_t = x_n) \in [0,1]^n$ with $y_1 < \ldots < y_{l-1} < y_l = x_t < z_t = y_{l+1} < \ldots < y_k$. If $t = 1$ or $t = n$ then define $\vec{z} = (z, z_2, \ldots, z_n) \in [0,1]^n$ or $\vec{z} = (z_1, \ldots, z_{n-1}, z) \in [0,1]^n$, respectively. In this case, considering equations (7) and (8), one has that:

- $R(\vec{z}) = (h_1 = y_l, \ldots, h_{l-1} = y_{l-1}, h_l = z_t = y_{l+1}, h_{l+1} = y_{l+2}, \ldots, h_w = y_k)$, with $w = k-1 \leq n$, where $h_1 = y_l < \ldots < h_{l-1} = y_{l-1} < y_l = x_t < h_l = z_t = y_{l+1} < h_{l+1} = y_{l+2} < \ldots < h_w = y_k$.
- $B_j^R(\vec{z}) = \{i \in N \mid z_i = h_j\}$.

Observe that, since $h_l = z_t = y_{l+1}$, with $l \in W$, then it holds that:

- $\forall j \in W : j < l \rightarrow B_j^R(\vec{z}) = B_j^R(\vec{x})$.
- $\mid B_l^R(\vec{z}) \mid = \mid B_{l+1}^R(\vec{x}) \mid +1$.
- $\forall j \in W : j > l \rightarrow B_j^R(\vec{z}) = B_{j+1}^R(\vec{x})$.
- $\mid \cup_{p=l}^{k} B_p^R(\vec{x})) \mid = \mid \cup_{p=l}^{w} B_p^R(\vec{z})) \mid$.

Since the pair $(F, F)$ satisfies **(PI)** and by Lemma 5.1, it follows that:

$$\begin{aligned}
&g\mathfrak{C}_{\mathfrak{m}}^{(F,F)}(\vec{x}) \\
&= \min\Bigg\{1, \sum_{j=1}^{l-1}\Big(F(y_j, \mathfrak{m}(\cup_{p=j}^{k} B_p^R(\vec{x}))) - F(y_{j-1}, \mathfrak{m}(\cup_{p=j}^{k} B_p^R(\vec{x})))\Big) \\
&\quad + F(y_l, \mathfrak{m}(\cup_{p=l}^{k} B_p^R(\vec{x}))) - F(y_{l-1}, \mathfrak{m}(\cup_{p=l}^{k} B_p^R(\vec{x}))) \\
&\quad + F(y_{l+1}, \mathfrak{m}(\cup_{p=l+1}^{k} B_p^R(\vec{x}))) - F(y_l, \mathfrak{m}(\cup_{p=l+1}^{k} B_p^R(\vec{x}))) \\
&\quad + F(y_{l+2}, \mathfrak{m}(\cup_{p=l+2}^{k} B_p^R(\vec{x}))) - F(y_{l+1}, \mathfrak{m}(\cup_{p=l+2}^{k} B_p^R(\vec{x}))) \\
&\quad + \sum_{j=l+3}^{k}\Big(F(y_j, \mathfrak{m}(\cup_{p=j}^{k} B_p^R(\vec{x}))) - F(y_{j-1}, \mathfrak{m}(\cup_{p=j}^{k} B_p^R(\vec{x})))\Big)\Bigg\} \\
&= \min\Bigg\{1, \sum_{j=1}^{l-1}\Big(F(y_j, \mathfrak{m}(\cup_{p=j}^{k} B_p^R(\vec{x}))) - F(y_{j-1}, \mathfrak{m}(\cup_{p=j}^{k} B_p^R(\vec{x})))\Big) \\
&\quad + F(x_t, \mathfrak{m}(\cup_{p=l}^{k} B_p^R(\vec{x}))) - F(y_{l-1}, \mathfrak{m}(\cup_{p=l}^{k} B_p^R(\vec{x}))) \\
&\quad + F(y_{l+1}, \mathfrak{m}(\cup_{p=l+1}^{k} B_p^R(\vec{x}))) - F(x_t, \mathfrak{m}(\cup_{p=l+1}^{k} B_p^R(\vec{x}))) \\
&\quad + F(y_{l+2}, \mathfrak{m}(\cup_{p=l+2}^{k} B_p^R(\vec{x}))) - F(y_{l+1}, \mathfrak{m}(\cup_{p=l+2}^{k} B_p^R(\vec{x}))) \\
&\quad + \sum_{j=l+3}^{k}\Big(F(y_j, \mathfrak{m}(\cup_{p=j}^{k} B_p^R(\vec{x}))) - F(y_{j-1}, \mathfrak{m}(\cup_{p=j}^{k} B_p^R(\vec{x})))\Big)\Bigg\} \\
&\leq \min\Bigg\{1, \sum_{j=1}^{l-1}\Big(F(h_j = y_j, \mathfrak{m}(\cup_{p=j}^{w} B_p^R(\vec{z}))) - F(h_{j-1} = y_{j-1}, \mathfrak{m}(\cup_{p=j}^{w} B_p^R(\vec{z})))\Big) \\
&\quad + F(h_l = z_t = y_{l+1}, \mathfrak{m}(\cup_{p=l}^{w} B_p^R(\vec{z}))) - F(h_{l-1} = y_{l-1}, \mathfrak{m}(\cup_{p=l}^{w} B_p^R(\vec{z}))) \\
&\quad + F(h_{l+1} = y_{l+2}, \mathfrak{m}(\cup_{p=l+1}^{w} B_p^R(\vec{x}))) - F(h_l = z_t = y_{l+1}, \mathfrak{m}(\cup_{p=l+1}^{w} B_p^R(\vec{x}))) \\
&\quad + \sum_{j=l+2}^{w}\Big(F(h_j, \mathfrak{m}(\cup_{p=j}^{w} B_p^R(\vec{z}))) - F(h_{j-1}, \mathfrak{m}(\cup_{p=j}^{w} B_p^R(\vec{z})))\Big)\Bigg\} \\
&= g\mathfrak{C}_{\mathfrak{m}}^{(F,F)}(\vec{z}),
\end{aligned}$$

since $\cup_{p=l+1}^{k} B_p^R(\vec{x}) \subset \cup_{p=l}^{k} B_p^R(\vec{x}) = \cup_{p=l}^{w} B_p^R(\vec{z})$, and, then, by **(PI)**, it holds that

$$\begin{aligned}&F(x_t, \mathfrak{m}(\cup_{p=l}^{k} B_p^R(\vec{x}))) - F(x_t, \mathfrak{m}(\cup_{p=l+1}^{k} B_p^R(\vec{x})))\\&\quad < F(h_l = z_t = y_{l+1}, \mathfrak{m}(\cup_{p=l}^{w} B_p^R(\vec{z}))) - F(y_{l+1}, \mathfrak{m}(\cup_{p=l+1}^{k} B_p^R(\vec{x}))).\end{aligned}$$

**(ic)** Now consider $l \in \{1, \dots, k-3\}$, and $\vec{z} = (z_1 = x_1, \dots, z_{t-1} = x_{t-1}, z_t, z_{t+1} = x_{t+1}, \dots, x_n) \in [0,1]^n$, such that $\vec{x} < \vec{z}$, with $y_1 < \dots < y_{l-1} < y_l = x_t < y_{l+1} < \dots < z_t < \dots < y_k$. If $t = 1$ or $t = n$ then define $\vec{z} = (z, z_2, \dots, z_n) \in [0,1]^n$ or $\vec{z} = (z_1, \dots, z_{n-1}, z) \in [0,1]^n$, respectively. In this case, considering equations (7) and (8), one has that:

- $R(\vec{z}) = (h_1, \dots, h_{l-1}, h_l = z_t, h_{l+1}, \dots, h_w)$, with $w \le n$, where $h_1 < \dots < h_{l-1} < h_l = z_t < h_{l+1} < \dots < h_w$ and $\{x_1 = z_1, \dots, x_{t-1} = z_{t-1}, z_t, x_{t+1} = z_{t+1}, \dots, z_n = x_n\} = \{h_1, \dots, h_{l-1}, h_l = z, h_{l+1}, \dots h_w\}$.
- $B_j^h = \{i \mid z_i = h_j\}$, for $j \in W = \{1, \dots, w\}$.

Consider $r \in \{2, \dots, w-l-1\}$. Suppose that $y_l = x_t < y_{l+1} < \dots < y_{k-r} < z_t < y_{k-r+2}$. Then, by **(ia)** and **(ib)**, it follows that:

$$g\mathfrak{C}_{\mathfrak{m}}^{(F_1,F_2)}(\vec{x}) \le g\mathfrak{C}_{\mathfrak{m}}^{(F_1,F_2)}(\vec{s}_1) \le \dots \le g\mathfrak{C}_{\mathfrak{m}}^{(F_1,F_2)}(\vec{s}_{n-r-l}) \le g\mathfrak{C}_{\mathfrak{m}}^{(F_1,F_2)}(\vec{z}),$$

where, for $i = 1, \dots, n-r-l$, $\vec{s}_i = (x_1, \dots, x_{t-1}, y_{l+i}, x_{t+1}, \dots, x_n)$.

**(id)** Suppose the same conditions of case **(ic)**, but for $\vec{z} = (z_1 = x_1, \dots, z_{t-1} = x_{t-1}, z_t, z_{t+1} = x_{t+1}, \dots, x_n) \in [0,1]^n$, such that $z_t = y_j$, for some $y_j > y_{l+1}$, that is, $y_1 < \dots < x_t = y_l < y_{l+1} < \dots < z_t = y_j < \dots < y_k$. In this case, considering equations (7) and (8), one has that:

- $R(\vec{z}) = (h_1 = y_1, \dots, h_{l-1} = y_{j-1}, h_l = z_t = y_j, h_{l+1} = y_{j+1}, \dots, h_w)$, with $w < k$, where $h_1 < \dots < h_{l-1} < h_l = z_t < h_{l+1} < \dots < h_w$ and $\{x_1 = z_1, \dots, x_{t-1} = z_{t-1}, z_t, x_{t+1} = z_{t+1}, \dots, z_n = x_n\} = \{h_1, \dots, h_{l-1}, h_l = z_t, h_{l+1}, \dots h_w\}$.
- $B_j^h = \{i \mid z_i = h_j\}$, for $j \in W = \{1, \dots, w\}$.

Consider $r \in \{2, \dots, w-l-1\}$. Suppose that $y_l = x_t < y_{l+1} < \dots < y_{k-r} < z_t = k-r+1 < y_{k-r+2}$. Then, considering **(ib)**, the proof is analogous to **(ic)**.

**(ii)** $\exists i \in N, i \neq t,\ s.t.\ x_t = x_i$. In this case, we have the same subcases **(ia)–(id)**, and the proofs are analogous.

($\Leftarrow$) We prove the contrapositive. Suppose that the pair $(F, F)$ does not satisfy **(PI)**. Then, there exist $a, b, c, d \in [0,1]$ such that $a \le b$, $c \le d$ and $F(a,d) - F(a,c) > F(b,d) - F(b,c)$. Observe that $a \neq 1$ and $c \neq 1$. Let $\mathfrak{m} : 2^N \to [0,1]$ be such that $\mathfrak{m}(\{n-1, n-2\}) = d$ and $\mathfrak{m}(\{n-1\}) = c$. Then, for $\vec{x} = (0, \dots, 0, a, 1)$ and $\vec{z} = (0, \dots, 0, b, 1)$, we have that $k = 3$ and $\vec{x} \le \vec{z}$. Consider $\vec{y} = (0, a, 1)$ and $\vec{h} = (0, b, 1)$. Then, one has that:

$$\begin{aligned}g\mathfrak{C}_{\mathfrak{m}}^{(F,F)}(\vec{x}) &= \min\Big\{1, F(0, \mathfrak{m}(\cup_{p=1}^{3} B_p^R(\vec{x}))) - F(0, \mathfrak{m}(\cup_{p=1}^{k} B_p^R(\vec{x}))) + F(a, \mathfrak{m}(\cup_{p=2}^{3} B_p^R(\vec{x})))\\&\quad - F(0, \mathfrak{m}(\cup_{p=2}^{3} B_p^R(\vec{x}))) + F(1, \mathfrak{m}(B_3^R(\vec{x}))) - F(a, \mathfrak{m}(B_3^R(\vec{x})))\Big\}\\&= \min\{1, F(a, \mathfrak{m}(\{n-2, n-1\})) - F(0, \mathfrak{m}(\{n-2, n-1\})) + F(1, \mathfrak{m}(\{n-1\}))\\&\quad - F(a, \mathfrak{m}(\{n-1\}))\}\\&= \min\{1, F(a,d) - F(0,d) + F(1,c) - F(a,c)\}\\&> \min\{1, F(b,d) - F(0,d) + F(1,c) - F(b,c)\}\\&= \min\{1, F(b, \mathfrak{m}(\{n-2, n-1\})) - F(0, \mathfrak{m}(\{n-2, n-1\})) + F(1, \mathfrak{m}(\{n-1\}))\\&\quad - F(b, \mathfrak{m}(\{n-1\}))\}\\&= \min\Big\{1, F(0, \mathfrak{m}(\cup_{p=1}^{3} B_p^R(\vec{x}))) - F(0, \mathfrak{m}(\cup_{p=1}^{k} B_p^R(\vec{x}))) + F(b, \mathfrak{m}(\cup_{p=2}^{3} B_p^R(\vec{x})))\\&\quad - F(0, \mathfrak{m}(\cup_{p=2}^{3} B_p^R(\vec{x}))) + F(1, \mathfrak{m}(B_3^R(\vec{x}))) - F(b, \mathfrak{m}(B_3^R(\vec{x})))\Big\}\\&= g\mathfrak{C}_{\mathfrak{m}}^{(F,F)}(\vec{z}).\end{aligned}$$

Therefore, $g\mathfrak{C}_{\mathfrak{m}}^{(F,F)}$ is not increasing for each fuzzy measure $\mathfrak{m} : 2^N \to [0,1]$. □

**Corollary 5.1.** *Under the conditions of Definition 4.2, for any fuzzy measure $\mathfrak{m} : 2^N \to [0,1]$ and pseudo pre-aggregation function pair $(F, F)$ satisfying **(PI)**, $g\mathfrak{C}_{\mathfrak{m}}^{(F,F)}$ is an aggregation function.*

**Proof.** It follows from Proposition 5.1 and Theorem 5.1. □

Observe that if for some pseudo pre-aggregation function pair $(F, F)$ and fuzzy measure $\mathfrak{m}$ we can have that $g\mathfrak{C}_{\mathfrak{m}}^{(F,F)}$ is not increasing (and, thus, it is not an aggregation function) then $(F, F)$ does not satisfy **(PI)**. Nevertheless, this does not mean that, for some other fuzzy measure $\mathfrak{m}'$, $g\mathfrak{C}_{\mathfrak{m}'}^{(F,F)}$ would not be an aggregation function.

**Example 5.1.** Let $F : [0,1]^2 \to [0,1]$ be the function defined by

$$F(x, y) = \begin{cases} 0 & \text{if } x = 0 \vee y = 0; \\ \frac{x+y}{2} & \text{if } 0 < x \leq y; \\ x & \text{otherwise.} \end{cases}$$

Clearly, $F$ is $(1,0)$-increasing, $F(0,1) = 0$ and $F(1,1)$, and therefore $(F, F)$ is a pseudo pre-aggregation pair (in fact, $F$ is an aggregation function). But, $(F, F)$ does not satisfy **(PI)**. In fact, one has that

$$F(0.3, 0.7) - F(0.3, 0.5) = 0.5 - 0.4 = 0.1 > 0 = 1 - 1 = F(1, 0.7) - F(1, 0.5).$$

Hence, by Theorem 5.1, for some fuzzy measure $\mathfrak{m}$, $g\mathfrak{C}_{\mathfrak{m}}^{(F,F)}$ is not increasing. In particular, by the proof of this Theorem, $g\mathfrak{C}_{\mathfrak{m}}^{(F,F)}$ is not increasing for any fuzzy measure $\mathfrak{m}$ such that $1 > \mathfrak{m}(\{n-2, n-1\}) > \mathfrak{m}(\{n-1\})$. However, for the fuzzy measure $\mathfrak{m}_\perp : 2^N \to [0,1]$, defined by:

$$\mathfrak{m}_\perp(X) = \begin{cases} 1 & \text{if } X = N; \\ 0 & \text{otherwise,} \end{cases}$$

one has that $g\mathfrak{C}_{\mathfrak{m}_\perp}^{(F,F)} : [0,1]^n \to [0,1]$ is the aggregation function, defined, for all $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$, by:

$$g\mathfrak{C}_{\mathfrak{m}_\perp}^{(F,F)}(\vec{x}) = \begin{cases} 0 & \text{if } \min\{x_1, \ldots, x_n\} = 0 \\ \frac{\min\{x_1, \ldots, x_n\} + 1}{2} & \text{otherwise.} \end{cases}$$

Notice that Theorem 5.1 requires that a pseudo pre-aggregation function pair $(F_1, F_2)$, with $F_1 = F_2$, to satisfy **(PI)** in order to guarantee that $g\mathfrak{C}_{\mathfrak{m}_\perp}^{(F_1,F_2)}$ is increasing. The following example shows that there exist pseudo pre-aggregation function pairs $(F_1, F_2)$, with $F_1 \neq F_2$, satisfying **(PI)** such that $g\mathfrak{C}_{\mathfrak{m}_\perp}^{(F_1,F_2)}$ is not increasing.

**Example 5.2.** Consider the pseudo pre-aggregation function pair $(T_P, F_{BPC})$, where $T_P$ is the product t-norm and $F_{BPC}$ is an aggregation function (which is neither a t-norm, overlap function nor a copula), as defined in Tables 1 and 2. Observe that $T_P$ dominates $F_{BPC}$. Moreover, this pair satisfies **(PI)**. In fact, for all $x, y_1, y_2 \in [0,1]$ and $h > 0$ such that $x + h \in [0,1]$, if $y_2 \leq y_1$, it holds that:

$$\begin{aligned} T_P(x+h, y_1) - F_{BPC}(x+h, y_2) &= (x+h)y_1 - (x+h)y_2^2 \text{ by Table 1} \\ &= xy_1 - xy_2^2 + h(y_1 - y_2^2) \\ &= T_P(x, y_1) - F_{BPC}(x, y_2) + h(y_1 - y_2^2) \text{ by Table 1} \\ &\geq T_P(x, y_1) - F_{BPC}(x, y_2), \end{aligned}$$

since $h(y_1 - y_2^2) \geq 0$. However, $g\mathfrak{C}_{\mathfrak{m}}^{(T_P, F_{BPC})}$ is not increasing. In fact, consider $\vec{x} = (0.6, 0.4, 0.6, 0.5, 0.4, 0.6, 0.7)$ and $\vec{z} = (0.6, 0.4, 0.6, 0.6, 0.4, 0.6, 0.7)$, that is, $\vec{x} < \vec{z}$. Then, $k = 4$ and $w = 3$, and:

- $R(\vec{x}) = (0.4, 0.5, 0.6, 0.7)$ and $R(\vec{z}) = (0.4, 0.6, 0.7)$;
- $B_1^R(\vec{x}) = \{2, 5\}$, $B_2^R(\vec{x}) = \{4\}$, $B_3^R(\vec{x}) = \{1, 3, 6\}$ and $B_4^R(\vec{x}) = \{7\}$;
- $B_1^R(\vec{z}) = \{2, 5\}$, $B_2^R(\vec{z}) = \{1, 3, 4, 6\}$ and $B_3^R(\vec{x}) = \{7\}$.

Suppose that the fuzzy measure $\mathfrak{m} : 2^N \to [0, 1]$ is such that: $\mathfrak{m}(\{1, 2, 3, 4, 5, 6, 7\}) = 1$, $\mathfrak{m}(\{1, 3, 4, 6, 7\}) = 0.8$, $\mathfrak{m}(\{1, 3, 6, 7\}) = 0.5$ and $\mathfrak{m}(\{7\}) = 0.2$. Then one has that:

$$\begin{aligned}
g\mathfrak{C}_{\mathfrak{m}}^{(T_P,F_{BPC})}(\vec{x}) &= \min\left\{1, \sum_{j=1}^{4}\left(T_P(y_j, \mathfrak{m}(\cup_{p=j}^{k} B_p^R(\vec{x}))) - F_{BPC}(y_{j-1}, \mathfrak{m}(\cup_{p=j}^{k} B_p^R(\vec{x})))\right)\right\} \\
&= \{1, T_P(0.4, \mathfrak{m}(\{1,2,3,4,5,6,7\})) - F_{BPC}(0, \mathfrak{m}(\{1,2,3,4,5,6,7\})) \\
&\quad + T_P(0.5, \mathfrak{m}(\{1,3,4,6,7\})) - F_{BPC}(0.4, \mathfrak{m}(\{1,3,4,6,7\})) \\
&\quad + T_P(0.6, \mathfrak{m}(\{1,3,6,7\})) - F_{BPC}(0.5, \mathfrak{m}(\{1,3,6,7\})) \\
&\quad + T_P(0.7, \mathfrak{m}(\{7\})) - F_{BPC}(0.6, \mathfrak{m}(\{7\}))\} \\
&= \min\{1, 0.4 \cdot 1 - 0 \cdot (1)^2 + 0.5 \cdot 0.8 - 0.4 \cdot (0.8)^2 + 0.6 \cdot 0.5 - 0.5 \cdot (0.5)^2 + 0,7 \cdot 0.2 \\
&\quad - 0.6 \cdot (0.2)^2\} \\
&= 0.835
\end{aligned}$$

and

$$\begin{aligned}
g\mathfrak{C}_{\mathfrak{m}}^{(T_P,F_{BPC})}(\vec{z}) &= \min\left\{1, \sum_{j=1}^{3}\left(T_P(y_j, \mathfrak{m}(\cup_{p=j}^{w} B_p^R(\vec{z}))) - F_{BPC}(y_{j-1}, \mathfrak{m}(\cup_{p=j}^{k} B_p^R(\vec{z})))\right)\right\} \\
&= \{1, T_P(0.4, \mathfrak{m}(\{1,2,3,4,5,6,7\})) - F_{BPC}(0, \mathfrak{m}(\{1,2,3,4,5,6,7\})) \\
&\quad + T_P(0.6, \mathfrak{m}(\{1,3,4,6,7\})) - F_{BPC}(0.4, \mathfrak{m}(\{1,3,4,6,7\})) \\
&\quad + T_P(0.7, \mathfrak{m}(\{7\})) - F_{BPC}(0.6, \mathfrak{m}(\{7\}))\} \\
&= \min\{1, 0.4 \cdot 1 - 0 \cdot (1)^2 + 0.6 \cdot 0.8 - 0.4 \cdot (0.8)^2 + 0,7 \cdot 0.2 - 0.6 \cdot (0.2)^2\} \\
&= 0.74.
\end{aligned}$$

Thus, $g\mathfrak{C}_{\mathfrak{m}}^{(T_P,F_{BPC})}(\vec{x}) > g\mathfrak{C}_{\mathfrak{m}}^{(T_P,F_{BPC})}(\vec{z})$ and $g\mathfrak{C}_{\mathfrak{m}}^{(T_P,F_{BPC})}$ is not an aggregation function, since it is not increasing.

Now we present an example of a pseudo pre-aggregation function pair $(F, F)$ satisfying **(PI)** (then, fulfilling all the requirements of Theorem 5.1), thus generating an aggregation function $g\mathfrak{C}_{\mathfrak{m}}^{(F,F)}$.

**Example 5.3.** Consider the pseudo pre-aggregation function pair $(F_{IP}, F_{IP})$, where $F_{IP}$ is not even a pre-aggregation function, as defined in Tables 1 and 2. This pair satisfies **(PI)**. In fact, for all $x, y_1, y_2 \in [0, 1]$ and $h > 0$ such that $x + h \in [0, 1]$, if $y_2 \leq y_1$, it holds that:

$$\begin{aligned}
F_{IP}(x+h, y_1) - F_{IP}(x+h, y_2) &= 1 - y_1 + (x+h)y_1 - (1 - y_2 + (x+h)y_2) \text{ by Table 1} \\
&= (1 - y_1 + xy_1) - (1 - y_2 + xy_2) + h(y_1 - y_2) \\
&= F_{IP}(x, y_1) - F_{IP}(x, y_2) + h(y_1 - y_2) \text{ by Table 1} \\
&\geq F_{IP}(x, y_1) - F_{IP}(x, y_2),
\end{aligned}$$

since $h(y_1 - y_2) \geq 0$. Thus, from Corollary 5.1, $g\mathfrak{C}_{\mathfrak{m}}^{(F_{IP},T_{IP})}$ is an aggregation function, for any fuzzy measure $\mathfrak{m} : 2^N \to [0, 1]$.

**Corollary 5.2.** *Under the conditions of Definition 4.2, for any fuzzy measure $\mathfrak{m} : 2^N \to [0, 1]$ and pseudo pre-aggregation function pair $(F, F)$ satisfying* **(PI)***, $g\mathfrak{C}_{\mathfrak{m}}^{(F,F)}$ is an averaging aggregation function if and only if $F(x, 1) = x$, for all $x \in [0, 1]$.*

**Proof.** It follows from Corollary 5.1 and Proposition 4.2. □

**Example 5.4.** Consider the pseudo pre-aggregation function pair $(F_{BPC}, F_{BPC})$, where $F_{BPC}$ is an aggregation function (which is neither a t-norm, overlap function nor a copula), as defined in Tables 1 and 2. This pair satisfies **(PI)**. In fact, for all $x, y_1, y_2 \in [0,1]$ and $h > 0$ such that $x + h \in [0,1]$, then, whenever $y_2 \leq y_1$, it holds that:

$$\begin{aligned} F_{BPC}(x+h, y_1) - F_{BPC}(x+h, y_2) &= (x+h)y_1^2 - (x+h)y_2^2 \text{ by Table 1} \\ &= xy_1^2 - xy_2^2 + h(y_1^2 - y_2^2) \\ &= F_{BPC}(x, y_1) - F_{BPC}(x, y_2) + h(y_1^2 - y_2^2) \text{ by Table 1} \\ &\geq F_{BPC}(x, y_1) - F_{BPC}(x, y_2), \end{aligned}$$

since $h(y_1^2 - y_2^2) \geq 0$. Therefore, since $F_{BPC}(x, 1) = x$, then, from Corollary 5.2, it follows that $g\mathfrak{C}_{\mathfrak{m}}^{(F_{BPC}, F_{BPC})}$ is an averaging aggregation function, for any fuzzy measure $\mathfrak{m} : 2^N \to [0,1]$.

**Corollary 5.3.** *Under the conditions of Definition 4.2, for any fuzzy measure $\mathfrak{m} : 2^N \to [0,1]$ and copula $C$, $g\mathfrak{C}_{\mathfrak{m}}^{(C,C)}$ is an averaging aggregation function.*

**Proof.** It follows from Corollary 5.2 and Corollary 3.1. □

**Remark 5.1.** Considering equations (4) and (9), by an easy calculation it is possible to check that, whenever $F_1 = F_2 = C$, for a copula $C$, for all $\vec{x} \in [0,1]^n$, one has that:

$$\begin{aligned} g\mathfrak{C}_{\mathfrak{m}}^{(C,C)}(\vec{x}) &= \min\left\{1, \sum_{j=1}^{k} C\left(y_j, \mathfrak{m}\left(\cup_{p=j}^{k} B_p^R(\vec{x})\right)\right) - C\left(y_{j-1}, \mathfrak{m}\left(\cup_{p=j}^{k} B_p^R(\vec{x})\right)\right)\right\} \\ &= \sum_{i=1}^{n} C\left(x_{(i)}, \mathfrak{m}\left(A_{(i)}\right)\right) - C\left(x_{(i-1)}, \mathfrak{m}\left(A_{(i)}\right)\right) \qquad (10) \\ &= \mathfrak{C}_{\mathfrak{m}}^{C}(\vec{x}), \end{aligned}$$

which is, in fact, the $CC$-Integral used in classification problems in [9]. In [34, Theorem 1], Mesiar and Stupňanová showed that the $CC$-Integral is a C-based universal integral $I_m^C$, for a fuzzy measure $\mathfrak{m}$ and copula $C$. Additionally, from [34, Corollary 2], for any fuzzy measure $\mathfrak{m} : 2^N \to [0,1]$ and copula $C : [0,1]^2 \to [0,1]$, one has that $g\mathfrak{C}_{\mathfrak{m}}^{(C,C)}$ is an OMA[2] operator and vice-versa.

**Remark 5.2.** Observe that, by Remark 5.2, whenever $F_1 = F_2 = C$, for a copula $C$, it is not necessary to make the dimension reduction to deal with duplicated elements.

**Example 5.5.** Consider the pseudo pre-aggregation pair $(T_M, T_M)$, where $T_M$ is the minimum t-norm. Then, for any fuzzy measure $\mathfrak{m} : 2^N \to [0,1]$, $g\mathfrak{C}_{\mathfrak{m}}^{(T_M,T_M)}$ is an averaging aggregation function, since $(T_M, T_M)$ satisfies **PI** and $T_M(x, 1) = x$. Moreover, by [34, Corollary 1], $g\mathfrak{C}_{\mathfrak{m}}^{(T_M,T_M)}$ is a Sugeno Integral [36]. Observe that, by Remark 5.2, since $F_1 = F_2 = T_M$, we do not need to worry about the duplicated components in the input $\vec{x}$, so that we can just consider that $K = N$ in Definition 4.1. In fact, consider $\vec{x} \in [0,1]^n$ and let $(x_{(1)}, \ldots, x_{(n)})$ be an increasing permutation on the input $\vec{x}$, and $A_{(i)} = \{(i), \ldots, (n)\}$ be the subset of indices of the $n - i + 1$ largest components of $\vec{x}$. It follows that:

$$\begin{aligned} g\mathfrak{C}_{\mathfrak{m}}^{(T_M,T_M)}(\vec{x}) &= \min\left\{1, \sum_{i=1}^{n} \min\left\{x_{(i)}, \mathfrak{m}\left(A_{(i)}\right)\right\} - \min\left\{x_{(i-1)}, \mathfrak{m}\left(A_{(i)}\right)\right\}\right\}, \\ &= \min\left\{1, \sum_{i=1}^{n} \begin{cases} x_{(i)} - x_{(i-1)} & \text{if } x_{(i)} \leq \mathfrak{m}\left(A_{(i)}\right) \\ \mathfrak{m}\left(A_{(i)}\right) - x_{(i-1)} & \text{if } x_{(i)} > \mathfrak{m}\left(A_{(i)}\right) \wedge x_{(i-1)} \leq \mathfrak{m}\left(A_{(i)}\right) \\ 0 & \text{otherwise.} \end{cases}\right\} \end{aligned}$$

[2] An aggregation function $A = [0,1]^n \to [0,1]$ is an Ordered Modular Average (OMA) operator if it is commutative, idempotent, and comonotone modular [35].

Suppose that for some $k \in \{1, \ldots, n\}$, it holds that $x_{(k)} > \mathfrak{m}\left(A_{(k)}\right)$, but $x_{(k-1)} \le \mathfrak{m}\left(A_{(k)}\right)$. Then it holds that:

$$
\begin{aligned}
g\mathfrak{C}_{\mathfrak{m}}^{(T_M,T_M)}(\vec{x}) &= \min\left\{1, \sum_{i=1}^{n} \begin{cases} x_{(i)} - x_{(i-1)} & \text{if } x_{(i)} \le \mathfrak{m}\left(A_{(i)}\right) \\ \mathfrak{m}\left(A_{(i)}\right) - x_{(i-1)} & \text{if } x_{(i)} > \mathfrak{m}\left(A_{(i)}\right) \wedge x_{(i-1)} \le \mathfrak{m}\left(A_{(i)}\right) \\ 0 & \text{otherwise.} \end{cases}\right\} \\
&= \min\Big\{1, (x_{(1)} - x_{(0)}) + (x_{(2)} - x_{(1)}) + \ldots + (x_{(k-1)} - x_{(k-2)}) + (\mathfrak{m}\left(A_{(k)}\right) - x_{(k-1)}) \\
&\quad + \underbrace{0 + \ldots + 0}_{n-k}\Big\} \\
&= \min\left\{1, \mathfrak{m}\left(A_{(k)}\right)\right\} \\
&= \mathfrak{m}\left(A_{(k)}\right)
\end{aligned}
$$

Otherwise, one has the following possibilities:

**(i)** For all $k \in \{1, \ldots, n\}$, it holds that $x_{(k)} \le \mathfrak{m}\left(A_{(k)}\right)$. In this case, one has that:

$$
\begin{aligned}
g\mathfrak{C}_{\mathfrak{m}}^{(T_M,T_M)}(\vec{x}) &= \min\left\{1, \sum_{i=1}^{n} \begin{cases} x_{(i)} - x_{(i-1)} & \text{if } x_{(i)} \le \mathfrak{m}\left(A_{(i)}\right) \\ \mathfrak{m}\left(A_{(i)}\right) - x_{(i-1)} & \text{if } x_{(i)} > \mathfrak{m}\left(A_{(i)}\right) \wedge x_{(i-1)} \le \mathfrak{m}\left(A_{(i)}\right) \\ 0 & \text{otherwise.} \end{cases}\right\} \\
&= \min\left\{1, (x_{(1)} - x_{(0)}) + \ldots + (x_{(n)} - x_{(n-1)})\right\} \\
&= \min\left\{1, x_{(n)}\right\} \\
&= x_{(n)}
\end{aligned}
$$

**(ii)** For all $k \in \{1, \ldots, n\}$ such that $x_{(k)} > \mathfrak{m}\left(A_{(k)}\right)$ it holds that $x_{(k-1)} > \mathfrak{m}\left(A_{(k)}\right)$. In this case, one has that:

$$
\begin{aligned}
g\mathfrak{C}_{\mathfrak{m}}^{(T_M,T_M)}(\vec{x}) &= \min\left\{1, \sum_{i=1}^{n} \begin{cases} x_{(i)} - x_{(i-1)} & \text{if } x_{(i)} \le \mathfrak{m}\left(A_{(i)}\right) \\ \mathfrak{m}\left(A_{(i)}\right) - x_{(i-1)} & \text{if } x_{(i)} > \mathfrak{m}\left(A_{(i)}\right) \wedge x_{(i-1)} \le \mathfrak{m}\left(A_{(i)}\right) \\ 0 & \text{otherwise.} \end{cases}\right\} \\
&= \min\Big\{1, (x_{(1)} - x_{(0)}) + (x_{(2)} - x_{(1)}) + \ldots + (x_{(k-1)} - x_{(k-2)}) \\
&\quad + \underbrace{0 + \ldots + 0}_{n-k+1}\Big\} \\
&= \min\left\{1, x_{(k-1)}\right\} \\
&= x_{(k-1)}.
\end{aligned}
$$

Then, it follows that:

$$
\begin{aligned}
g\mathfrak{C}_{\mathfrak{m}}^{(T_M,T_M)}(\vec{x}) &= \begin{cases} \mathfrak{m}\left(A_{(k)}\right) & \text{if } \exists k \in \{1, \ldots, n\} : x_{(k)} > \mathfrak{m}\left(A_{(k)}\right) \wedge x_{(k-1)} \le \mathfrak{m}\left(A_{(k)}\right) \\ x_{(n)} & \text{if } \forall k \in \{1, \ldots, n\} : x_{(k)} \le \mathfrak{m}\left(A_{(k)}\right) \\ x_{(k-1)} & \text{if } \forall k \in \{1, \ldots, n\} : x_{(k)} > \mathfrak{m}\left(A_{(k)}\right) \wedge x_{(k-1)} > \mathfrak{m}\left(A_{(k)}\right) \end{cases} \\
&= \max_{i=1}^{n}\left\{\min\left\{x_{(i)}, \mathfrak{m}\left(A_{(i)}\right)\right\}\right\} \\
&= S_{\mathfrak{m}}(\vec{x}),
\end{aligned}
$$

where $S_{\mathfrak{m}}$ is the Sugeno integral. Observe that $C_{T_M,T_M}$-integral is the $CMin$-integral analyzed in [13].

Finally, we show this interesting example of a $g\mathfrak{C}_{\mathfrak{m}}^{(F_1,F_2)}$ that is $\underbrace{(1, \ldots, 1)}_{n \text{ times}}$-increasing (or weakly increasing).

**Example 5.6.** Consider $F_1 = T_P$, the product t-norm, and $F_2 = wT_P$, for $w \in [0,1]$. Observe that, for $w = 1$, $g\mathfrak{C}_{\mathfrak{m}}^{(T_P,wT_P)}$ is the standard Choquet Integral. Take $n = 2$, $\vec{x} = (x_1, x_2)$ and a fuzzy measure $\mathfrak{m} : 2^N \to [0,1]$ such that $\mathfrak{m}(\{1\}) = a$ and $\mathfrak{m}(\{2\}) = b$, $a, b \in ]0,1[$. Then, we have that:

$$g\mathfrak{C}_{\mathfrak{m}}^{(T_P,wT_P)}(x_1,x_2) = \begin{cases} \min\{1, (1-wb)x_1 + bx_2\} & \text{if } x_1 < x_2 \\ x & \text{if } x_1 = x_2 = x \\ \min\{1, ax_1 + (1-wa)x_2\} & \text{if } x_1 > x_2 \end{cases}$$

which may be not an aggregation function whenever $w \neq 1$. In fact, take $x_1 = 0.9$, $x_2 = 0.94$, $a = b = 0.5$, $w = 0.1$. Then one has that

$$g\mathfrak{C}_{\mathfrak{m}}^{(T_P,0.1T_P)}(0.9, 0.94) = \min\{1, (1-0.1\cdot 0.5)\cdot 0.9 + 0.5 \cdot 0.94\} = \min\{1, 1.325\} = 1.$$

Now, consider $x_1 = x_2 = 0.95$. In this case, one has that $g\mathfrak{C}_{\mathfrak{m}}^{(T_P,0.1T_P)}(0.94, 0.94) = 0.94$, which shows that $g\mathfrak{C}_{\mathfrak{m}}^{(T_P,0.1T_P)}$ is not increasing. Now, observe that $g\mathfrak{C}_{\mathfrak{m}}^{(T_P,wT_P)}$ is $\underbrace{(1,\ldots,1)}_{n \text{ times}}$-increasing. One has the following cases:

**(i)** If $x_1 < x_2$ then, for all $c > 0$ such that $x_1 + c, x_2 + c \in [0,1]$ it holds that $x_1 + c < x_2 + c$ and

$$\begin{aligned} g\mathfrak{C}_{\mathfrak{m}}^{(T_P,wT_P)}(x_1+c, x_2+c) &= \min\{1, (1-wb)(x_1+c) + b(x_2+c)\} > \min\{1, (1-wb)x_1 + bx_2\} \\ &= g\mathfrak{C}_{\mathfrak{m}}^{(T_P,wT_P)}(x_1,x_2). \end{aligned}$$

**(ii)** If $x_1 = x_2 = x$ then, for all $c > 0$ such that $x + c \in [0,1]$ it holds that $x_1 + c = x_2 + c = x + c$ and

$$g\mathfrak{C}_{\mathfrak{m}}^{(T_P,wT_P)}(x_1+c, x_2+c) = x + c > x = g\mathfrak{C}_{\mathfrak{m}}^{(T_P,wT_P)}(x_1,x_2).$$

**(iii)** If $x_1 > x_2$ then, for all $c > 0$ such that $x_1 + c, x_2 + c \in [0,1]$ it holds that $x_1 + c > x_2 + c$ and

$$\begin{aligned} g\mathfrak{C}_{\mathfrak{m}}^{(T_P,wT_P)}(x_1+c, x_2+c) &= \min\{1, a(x_1+c) + (1-wa)(x_2+c)\} > \min\{1, ax_1 + (1-wa)x_2\} \\ &= g\mathfrak{C}_{\mathfrak{m}}^{(T_P,wT_P)}(x_1,x_2). \end{aligned}$$

Example 5.6 is justified by the following result:

**Theorem 5.2.** *Under the condition of Definition 4.2, for any fuzzy measure $\mathfrak{m} : 2^N \to [0,1]$ and pseudo pre-aggregation function pair $(F, F)$ satisfying **(PI)**, and for any $w \in [0,1]$, $g\mathfrak{C}_{\mathfrak{m}}^{(F,wF)}$ is a pre-aggregation function that is weakly increasing.*

**Proof.** By Proposition 5.1, $g\mathfrak{C}_{\mathfrak{m}}^{(F,wF)}$ satisfies the boundary conditions. It remains to prove the weak monotonicity of $g\mathfrak{C}_{\mathfrak{m}}^{(F,wF)}$. To show this, observe first that, for any $\vec{x} = (x_1,\ldots,x_n) \in [0,1]^n$ and $c > 0$ such that $(x_1+c,\ldots,x_n+c) \in [0,1]^n$, the sets $B_j^R(\vec{x})$ and $B_j^R(\vec{x}+c)$ coincide. Then, based on Theorem 5.1, it follows that:

$$\begin{aligned} &\sum_{j=1}^{k}(F(y_j + c, \mathfrak{m}(\cup_{p=j}^k B_p^R(\vec{x}+c))) - F(y_{j-1} + c, \mathfrak{m}(\cup_{p=j}^k B_p^R(\vec{x}+c)))) \\ &\quad \geq \sum_{j=1}^{k}(F(y_j, \mathfrak{m}(\cup_{p=j}^k B_p^R(\vec{x}))) - F(y_{j-1}, \mathfrak{m}(\cup_{p=j}^k B_p^R(\vec{x})))). \end{aligned}$$

Then, one has that:

$$\begin{aligned} &\sum_{j=1}^{k}(F(y_j + c, \mathfrak{m}(\cup_{p=j}^k B_p^R(\vec{x}+c))) - F(y_j, \mathfrak{m}(\cup_{p=j}^k B_p^R(\vec{x})))) \\ &\quad \geq \sum_{j=1}^{k}(F(y_{j-1} + c, \mathfrak{m}(\cup_{p=j}^k B_p^R(\vec{x}+c))) - F(y_{j-1}, \mathfrak{m}(\cup_{p=j}^k B_p^R(\vec{x})))) \geq 0, \end{aligned}$$

since $F$ is $(1,0)$-increasing. Consequently, for any $w \in [0,1]$ it follows

$$\sum_{j=1}^{k}(F(y_j+c, \mathfrak{m}(\cup_{p=j}^{k} B_p^R(\vec{x}+c))) - F(y_j, \mathfrak{m}(\cup_{p=j}^{k} B_p^R(\vec{x}))))$$
$$\geq w\sum_{j=1}^{k}(F(y_{j-1}+c, \mathfrak{m}(\cup_{p=j}^{k} B_p^R(\vec{x}+c))) - F(y_{j-1}, \mathfrak{m}(\cup_{p=j}^{k} B_p^R(\vec{x})))),$$

and thus

$$\sum_{j=1}^{k}(F(y_j+c, \mathfrak{m}(\cup_{p=j}^{k} B_p^R(\vec{x}+c))) - wF(y_{j-1}+c, \mathfrak{m}(\cup_{p=j}^{k} B_p^R(\vec{x}+c))))$$
$$\geq \sum_{j=1}^{k}(F(y_j, \mathfrak{m}(\cup_{p=j}^{k} B_p^R(\vec{x}))) - wF(y_{j-1}, \mathfrak{m}(\cup_{p=j}^{k} B_p^R(\vec{x})))).$$

Hence, evidently, it follows that

$$g\mathfrak{C}_{\mathfrak{m}}^{(F,wF)}(\vec{x}+c) \geq g\mathfrak{C}_{\mathfrak{m}}^{(F,wF)}(\vec{x}),$$

that is, $g\mathfrak{C}_{\mathfrak{m}}^{(F,wF)}$ is weakly increasing. □

## 6. $gC_{F_1F_2}$-integrals as OD monotone functions

In the previous section, we presented the requirements for $gC_{F_1F_2}$-integrals to be aggregation functions, showing that there exist pseudo pre-aggregation function pairs that do not fulfill such requirements, and, therefore, the corresponding $gC_{F_1F_2}$-integrals are not aggregation functions. However, under some constraints, $gC_{F_1F_2}$-integrals are OD increasing functions satisfying **(A2)**, presenting, thus, some desirable conditions to play the role of "aggregation operators" in applications (see, for example, [15]). In this section we prove such properties of $gC_{F_1F_2}$-integrals.

First, notice that, in order to study the directional increasingness feature of our integrals, it is necessary to compatibilize the dimension reduction process, which should be performed in both input $\vec{x}=(x_1,\dots,x_n)\in[0,1]^n$ and direction vector $\vec{r}=(r_1,\dots,r_n)\in\mathbb{R}^n$, $\vec{r}\neq\vec{0}$, reducing both vectors to the same dimension $k\leq n$. It is easy to see that this compatible dimension reduction is possible if it holds that:

$$\forall i,l\in\{1,\dots,n\}, i<l : x_i = x_l \Rightarrow r_i = r_l \ \vee r_l = 0. \tag{11}$$

**Example 6.1.** There are different vectors $\vec{r}\in\mathbb{R}^n$ that satisfy (11) for all $\vec{x}\in[0,1]^n$. For example, consider the vectors $(w,\dots,w)$ and $(w,0,\dots,0)$, with $w\neq 0$. However, the vector $(w,0,0,w',0)$, with $w,w'\neq 0$ does not satisfy (11) for some $\vec{x}\in[0,1]^n$. Take, for example, $\vec{x}=(0.2,0.3,0.5,0.5,0.6)$. Observe that $x_3=x_4=0.5$ but $r_3\neq r_4$ and $r_4=w'\neq 0$.

It follows that:

**Proposition 6.1.** *Let $\mathbb{R}^n_{\vec{x}}$ be the set of non null vectors $\vec{r}\in\mathbb{R}^n$ satisfying (11), for a given $\vec{x}\in[0,1]^n$. Then, for each $\vec{x}\in[0,1]^n$, $\vec{r}\in\mathbb{R}^n_{\vec{x}}$ if and only if $\vec{r}=(w,0,\dots,0)\in\mathbb{R}^n$ or $\vec{r}=(w,\dots,w)\in\mathbb{R}^n$, with $w\neq 0$.*

**Proof.** (⇒) Suppose that $\vec{r}\in\mathbb{R}^n_{\vec{x}}$, for all $\vec{x}=(x_1,\dots,x_n)\in[0,1]^n$, and $\vec{r}=(w_1,\dots,w_n)$, with $\vec{r}\neq\vec{0}$, such that there exist $i,j\in\{1,\dots,n\}$ with $w_i\neq w_j$ and there exists $h\in\{2,\dots,n\}$ with $w_h\neq 0$. Then, take $\vec{x}=(x_1,\dots,x_n)\in[0,1]^n$ such that $x_i=x_j=x_h$. Since $x_i=x_j$ then, by (11), considering that $w_i\neq w_j$, it holds that $w_j=0$. Now, since $x_j=x_h$, then, by (11), considering that $w_h\neq 0$, then $w_j=w_h$, which is a contradiction with $w_j=0$. Then, one concludes that either $w_i=w_j$, for all $i,j\in\{1,\dots,n\}$, or $w_h=0$, for all $h\in\{2,\dots,n\}$. (⇐) It is immediate. □

The dimension reduction of such direction vectors $\vec{r}$ can be done as follows:

**Definition 6.1.** Let $R : [0,1]^n \to \cup_{k=1}^n [0,1]^k$ be the dimension reduction function, as defined in Equation (7). The associated direction-dimension reduction function is defined as the function $S_R : \{(w,\dots,w) \in \mathbb{R}^n \mid w \neq 0\} \cup \{(w,0,\dots,0) \in \mathbb{R}^n \mid w \neq 0\} \to \bigcup_{k=1}^n (\{(w,\dots,w) \in \mathbb{R}^k \mid w \neq 0\} \cup \{(w,0,\dots,0) \in \mathbb{R}^k \mid w \neq 0\})$, given by:

$$S_R((w,0,\dots,0)) = (w, \underbrace{0,\dots,0}_{k-1})$$
$$S_R((w,\dots,w)) = (\underbrace{w,\dots,w}_{k}), \tag{12}$$

where $k = |\{x_1,\dots,x_n\}|$ is cardinality of the set $\{x_1,\dots,x_n\}$, for any input $\vec{x} \in [0,1]^n$ of $R$.

Then we have the following results:

**Lemma 6.1.** *Consider $\vec{r} = (w,0,\dots,0) \in \mathbb{R}^n$, $w \neq 0$. Let $R$ and $S_R$ be as defined in equations (7) and (12), respectively, and denote $S_R((w,0,\dots,0)) = (w,0,\dots,0) = (s_1,\dots,s_n)$. Let $\sigma_K : \{1,\dots,k\} \to \{1,\dots,k\}$ be a permutation in decreasing order defined, for all $j \in K = \{1,\dots,k\}$, as*

$$\sigma_K(j) = (k - j + 1), \tag{13}$$

*(i.e., $\sigma_K(1) = (k)$, $\sigma_K(2) = (k-1)$, …, $\sigma_K(k) = (1)$). Then, for all $c > 0$ such that $y_{\sigma_K(1)} + cw \in [0,1]$, if*

$$1 \geq y_{\sigma_K(1)} + cw > y_{\sigma_K(2)} > \dots > y_{\sigma_K(k)}, \tag{14}$$

*then, for any $\vec{z} = \vec{y} + c\vec{s}_{\sigma_K^{-1}}$, where $\vec{s}_{\sigma_K^{-1}} = (s_{\sigma_K^{-1}(1)},\dots,s_{\sigma_K^{-1}(k)})$, it holds that $z_{(j)} = y_j + cs_{k-j+1}$, that is, $z_{(k)} = y_k + cw$ and $z_{(j)} = y_j$, for all $j \in \{1,\dots,k-1\}$.*

**Proof.** For all $\vec{x} \in [0,1]^n$ and respective $\vec{y} \in [0,1]^k$, since $y_1 < \dots < y_k$, then $y_{\sigma_K(1)} = y_n > \dots > y_{\sigma_K(k)} = y_1$. Considering $\vec{r} = (w,0,\dots,0) \in \mathbb{R}^n$, with $w \neq 0$, and its respective $\vec{s} = (w,0,\dots,0) \in \mathbb{R}^k$, suppose that, for all $c > 0$ the inequality (14) holds (i.e., $\vec{y}_{\sigma_K}$ and $\vec{y}_{\sigma_K} + c\vec{s}$ are comonotone, and either they increase or decrease at the same time). Then, for any $\vec{z} = \vec{y} + c\vec{s}_{\sigma_K^{-1}}$, where $\vec{s}_{\sigma_K^{-1}} = (s_{\sigma_K^{-1}(1)},\dots,s_{\sigma_K^{-1}(k)})$, as the same as in Equation (1), it holds that $\vec{z}_{\sigma_K} = (\vec{y} + c\vec{s}_{\sigma_K^{-1}})_{\sigma_K} = \vec{y}_{\sigma_K} + c\vec{s}$, and, thus, by the inequality (14), it holds that

$$1 \geq z_{\sigma_K(1)} = y_{\sigma_K(1)} + cs_1 > \dots > z_{\sigma_K(k)} = y_{\sigma_K(k)} + cs_k,$$

that is,

$$1 \geq z_{\sigma_K(1)} = y_{\sigma_K(1)} + cw > z_{\sigma_K(2)} = y_{\sigma_K(2)} > \dots > z_{\sigma_K(k)} = y_{\sigma_K(k)}.$$

This means that $z_{\sigma_K(k)} = y_{\sigma_K(k)} + cw$ and, for all $j \in \{1,\dots,k-1\}$, $z_{\sigma_K(j)} = y_{\sigma_K(j)}$. From Equation (13), it holds that:

$$z_{(k)} = z_{\sigma_K^{-1}\sigma_K(k)} = y_{\sigma_K^{-1}\sigma_K(k)} + cs_{\sigma_K^{-1}(k)} = y_{(k)} + cs_1 = y_k + cw$$

and, for all $j \in \{1,\dots,k-1\}$,

$$z_{(j)} = z_{\sigma_K^{-1}\sigma_K(j)} = y_{\sigma_K^{-1}\sigma_K(j)} + cs_{\sigma_K^{-1}(j)} = y_{(j)} + cs_{k-j+1} = y_j + cs_{k-j+1} = y_j,$$

where $(\cdot) : \{1,\dots,k\} \to \{1,\dots,k\}$ is a permutation in an increasing order with $z_{(1)} < \dots < z_{(k)}$. □

**Theorem 6.1.** *Let $\mathfrak{m} : 2^N \to [0,1]$ be a fuzzy measure and $F_1, F_2 : [0,1]^2 \to [0,1]$ be fusion functions satisfying the conditions of Definition 4.2. Consider $\vec{r} = (w,0,\dots,0) \in \mathbb{R}^n$, with $w > 0$. Then $g\mathfrak{C}_{\mathfrak{m}}^{(F_1,F_2)}$ is OD $\vec{r}$-increasing.*

**Proof.** Let $R$, $B_j^R$ and $S_R$ be as defined in equations (7), (8) and (12), respectively. Let $\sigma_N : \{1,\dots,n\} \to \{1,\dots,n\}$ be any permutation such that, for all $\vec{x} \in [0,1]^n$, with

$$x_{\sigma_N(1)} \geq \dots \geq x_{\sigma_N(n)}, \tag{15}$$

and for all $c > 0$, such that $1 \geq x_{\sigma_N(1)} + cw \geq x_{\sigma_N(2)} \geq \ldots \geq x_{\sigma_N(n)}$, where $\vec{r}_{\sigma_N^{-1}} = (r_{\sigma_N^{-1}(1)}, \ldots, r_{\sigma_N^{-1}(n)}) \in \mathbb{R}^n$. Clearly, one can consider the permutation in the decreasing order $\sigma_N : \{1, \ldots, n\} \to \{1, \ldots, n\}$ defined in terms of the permutation in the increasing order $(\cdot) : \{1, \ldots, n\} \to \{1, \ldots, n\}$ as $\sigma_N(1) = (n)$, $\sigma_N(2) = (n-1)$, ..., $\sigma_N(n) = (1)$, that is, $\sigma_N(j) = (n - j + 1)$, with $j \in \{1, \ldots, n\}$. Then, one has that $x_{(1)} \leq \ldots \leq x_{(n)}$, $x_{\sigma_N(1)} \geq \ldots \geq x_{\sigma_N(n)}$ and $\vec{r}_{\sigma_N^{-1}} = (0, \ldots, 0, w)$.

Due to the dimension reduction, for each $\vec{x} \in [0,1]^n$, consider its respective $\vec{y} = (y_1, \ldots, y_k) \in [0,1]^k$ and $\vec{s} = (w, 0, \ldots, 0) \in \mathbb{R}^k$, obtained from $\vec{r}$. Let $\sigma_K : \{1, \ldots, k\} \to \{1, \ldots, k\}$ be the permutation such that $\{x_{\sigma_N(1)}, \ldots, x_{\sigma_N(n)}\} = \{y_{\sigma_K(1)}, \ldots, y_{\sigma_K(k)}\}$ and $y_{\sigma_K(1)} > \ldots > y_{\sigma_K(k)}$. Observe that, after the dimension reduction, for any $\vec{y} \in [0,1]^k$ with respect to a $\vec{x} \in [0,1]^n$ satisfying (15), and, for all $c > 0$, it holds that $1 \geq y_{\sigma_K(1)} + cw > y_{\sigma_K(2)} > \ldots > y_{\sigma_K(k)}$, with $\vec{s}_{\sigma_K^{-1}} = (s_{\sigma_K^{-1}(1)}, \ldots, s_{\sigma_K^{-1}(k)}) = (0, \ldots, 0, w) \in \mathbb{R}^k$.

Clearly, when considering $\sigma_K$ defined in terms of the permutation in the increasing order $(\cdot) : \{1, \ldots, k\} \to \{1, \ldots, k\}$, we have that $\sigma_K(1) = (k)$, $\sigma_K(2) = (k-1)$, ..., $\sigma_K(k) = (1)$, that is, $\sigma_K(j) = (k - j + 1)$, with $j \in \{1, \ldots, k\}$. Then, one has that $y_{(1)} = y_1 < \ldots < y_{(k)} = y_k$ and $y_{\sigma_K(1)} > \ldots > y_{\sigma_K(k)}$. Then, from Lemma 6.1, it follows that:

$$\begin{aligned}
g\mathfrak{C}_{\mathfrak{m}}^{(F_1,F_2)}(\vec{x} + c\vec{r}_{\sigma_N^{-1}}) &= \min\Bigg\{1, F_1(y_k + cw, \mathfrak{m}(B_k^R(\vec{x}))) - F_2(y_{k-1}, \mathfrak{m}(B_k^R(\vec{x}))) \\
&\quad + \sum_{j=1}^{k-1} F_1\left(y_j, \mathfrak{m}\left(\cup_{p=j}^k B_p^R(\vec{x})\right)\right) - F_2\left(y_{j-1}, \mathfrak{m}\left(\cup_{p=j}^k B_p^R(\vec{x})\right)\right)\Bigg\} \\
&\geq \min\Bigg\{1, F_1(y_k, \mathfrak{m}(B_k^R(\vec{x}))) - F_2(y_{k-1}, \mathfrak{m}(B_k^R(\vec{x}))) \\
&\quad + \sum_{j=1}^{k-1} F_1\left(y_j, \mathfrak{m}\left(\cup_{p=j}^k B_p^R(\vec{x})\right)\right) - F_2\left(y_{j-1}, \mathfrak{m}\left(\cup_{p=j}^k B_p^R(\vec{x})\right)\right)\Bigg\} \\
&= g\mathfrak{C}_{\mathfrak{m}}^{(F_1,F_2)}(\vec{x}),
\end{aligned}$$

since $F_1$ is $(1,0)$-increasing. Thus, $g\mathfrak{C}_{\mathfrak{m}}^{(F_1,F_2)}$ is OD $(w, 0, \ldots, 0)$-increasing, for $w > 0$. □

**Corollary 6.1.** *Let $\mathfrak{m} : 2^N \to [0,1]$ be a fuzzy measure and $(F_1, F_2)$ be a pseudo pre-aggregation function pair, under the conditions of Definition 4.2. Consider $\vec{r} = (w, 0, \ldots, 0) \in \mathbb{R}^n$, with $w > 0$. Then $\mathfrak{C}_{\mathfrak{m}}^{(F_1,F_2)}$ is an OD $\vec{r}$-increasing function satisfying the boundary conditions **(A2)**.*

**Proof.** It follows from Proposition 5.1 and Theorem 6.1. □

## 7. Conclusion

In this paper, we introduced the $gC_{F_1F_2}$-integrals, either (pre) aggregation or OD monotone functions based on pseudo pre-aggregation pairs for the generalization of $C_{F_1F_2}$-integrals. We have stated under which conditions $gC_{F_1F_2}$-integrals are (averaging) aggregation, pre-aggregation or OD pre-aggregation functions. In summary, the main features of $gC_{F_1F_2}$-integrals in relation to our previous approaches related to the generalizations of the Choquet integral are:

1. The pseudo pre-aggregation pairs $(F_1, F_2)$ used for building $gC_{F_1F_2}$-integrals satisfy a few number of constraint, less than, for example a pair of copulas $(C, C)$ of the CC-integrals [9], and we still have an (pre) aggregation function or, at least, an OD monotone function satisfying boundary conditions;
2. The obtained (pre) aggregation or OD monotone function need not to be neither averaging nor idempotent to present excellent results in classification (see [15,37]).

Recall that the Choquet integral is 1-Lipschitz (with respect to $L_1$-norm), and its stability under possible noise in aggregated data is guaranteed. Similarly, based on Remark 5.1, one can show that $(C, C)$-based integrals (where $C$ is a copula) are 1-Lipschitz. This need not be more true for $(F, F)$-based integrals characterized in Corollary 5.2, and thus a deeper study of stability in this case (in dependence of some other properties of $F$) is an important topic for the further study. As another topic for future work, we will study our generalizations in an interval-valued context, following the approach in [38–40].

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