Arch. Rational Mech. Anal. 238 (2020) 489–540
Digital Object Identifier (DOI) https://doi.org/10.1007/s00205-020-01547-x

Derivation of von Kármán Plate Theory in the Framework of Three-Dimensional Viscoelasticity

Manuel Friedrich & Martin Kružík

Communicated by I. Fonseca

Abstract

We apply a quasistatic nonlinear model for nonsimple viscoelastic materials at a finite-strain setting in Kelvin's-Voigt's rheology to derive a viscoelastic plate model of von Kármán type. We start from time-discrete solutions to a model of three-dimensional viscoelasticity considered in Friedrich and Kružík (SIAM J Math Anal 50:4426–4456, 2018) where the viscosity stress tensor complies with the principle of time-continuous frame-indifference. Combining the derivation of nonlinear plate theory by Friesecke, James and Müller (Commun Pure Appl Math 55:1461–1506, 2002; Arch Ration Mech Anal 180:183–236, 2006), and the abstract theory of gradient flows in metric spaces by Sandier and Serfaty (Commun Pure Appl Math 57:1627–1672, 2004), we perform a dimension-reduction from three dimensions to two dimensions and identify weak solutions of viscoelastic form of von Kármán plates.

1. Introduction

Dimension-reduction problems play a significant role in nonlinear analysis and numerics because they allow for simpler computational approaches, still preserving the main features of the bulk system. In this context, it is important that a clear relationship between the full three-dimensional problem and its lower-dimensional counterpart is made rigorous. The last decades have witnessed remarkable progress in this direction through the use of variational methods, particularly by Γ-convergence [18] together with quantitative rigidity estimates [23]. Among the large body of results, we mention here only the rigorous justification of membrane theory [29,30], bending theory [16,23,37], and von Kármán theory [24,28] for plates as variational limits of nonlinear three-dimensional elasticity for vanishing thickness. In particular, we refer to [24] for the derivation of a hierarchy of different plate models and for a thorough literature review.

In the present work, we apply a similar scenario of deriving plate theories to problems in nonlinear viscoelasticity: starting from a three-dimensional model of a nonsimple viscoelastic material at a finite strain setting in Kelvin's-Voigt's rheology (that is, a spring and a damper coupled in parallel), recently treated by the authors in [22], we derive a model of von Kármán (vK) viscoelastic plates.

In [22], the existence of weak solutions for such a three-dimensional model of nonsimple viscoelastic materials was established with the help of gradient flows in metric spaces developed in [3,40]. The notion of a nonsimple (or second-grade) material refers to the fact that the elastic energy depends also on the second gradient of the deformation. This concept, first suggested by Toupin [42,43], has proved to be useful in modern mathematical elasticity; see for example [7,8,10,26,34,35,39]. We also refer to [15,20] where thermodynamical consistency of such models has been shown. We point out that this approach seems to be currently unavoidable in order to obtain the existence of solutions in the nonlinear *viscoelastic* setting; see [22,35] and [33] for a general discussion about the interplay between the elastic energy and viscous dissipation. Nevertheless, a main justification is the observation in [22] that, in the small strain limit, the problem leads to the standard system for linearized viscoelasticity without second gradient.

In the present work, we consider a thin plate of thickness h and pass to the dimension-reduction limit $h \to 0$ in the vK energy regime. We show that this gives rise to effective equations in terms of suitably rescaled in-plane and out-of-plane displacements which feature membrane and bending terms *both* in the elastic and the viscous stress. This represents a dissipative counterpart of the purely elastic vK theory which was first formulated more than hundred years ago [44]. Besides identifying the correct two-dimensional limiting equations of viscous vK plates, which to the best of our knowledge has not been done in previous literature, the main goals of this contribution are twofold: (1) we show the existence of solutions to the effective two-dimensional system, and (2) we prove rigorously that these solutions are in a certain sense the limits of solutions to the three-dimensional equations as $h \to 0$; see Theorem 2.2 and Theorem 2.3 for details.

Let us mention that there are previous works on viscoelastic plates [11,38], some even including inertial effects [12,13]. Their starting point, however, is already a plate model. Our model, derived rigorously from three-dimensional viscoelasticity, is new and, if viscosity is dropped, it reduces to the well-known model of elastic plates [24]. Let us emphasize that for purely elastic models neglecting viscosity various existence results were obtained for Föppl-von Kármán plates without resorting to a second-grade material; see for example [9,24,28,31,32]. We refer to [23] for a numerical study of vK viscoelastic plates, and mention that in that paper we have also proved the existence of solutions to the viscoelastic vK plate equations by means of converging numerical discretizations. This work, however, *already relies* on the plate model derived in the present paper. For the derivation of a plate model without viscosity but with inertia we refer to [1].

We now describe our setting in more detail. Without inertia, a nonlinear viscoelastic material in Kelvin's-Voigt's rheology satisfies the system of equations

$$-\mathrm{div}\Big(\partial_F W(\nabla w) + \partial_{\dot F} R(\nabla w, \partial_t \nabla w)\Big) = f e_3 \quad \text{in } [0,T] \times \Omega_h. \tag{1.1}$$

Here, $[0, T]$ is a process time interval with $T > 0$, $\Omega_h := S \times (-\frac{h}{2}, \frac{h}{2}) \subset \mathbb{R}^3$ is a smooth, thin, bounded domain representing the reference configuration, and $w : [0, T] \times \Omega_h \to \mathbb{R}^3$ is a deformation mapping with deformation gradient ∇w. Moreover, $W : \mathbb{R}^{3\times 3} \to [0, \infty]$ is a stored energy density representing a potential of the first Piola-Kirchhoff stress tensor $\partial_F W := \partial W/\partial F$ and $F \in \mathbb{R}^{3\times 3}$ is the placeholder of ∇w. The function $R : \mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3} \to [0, \infty)$ denotes a (pseudo)potential of dissipative forces, where $\dot{F} \in \mathbb{R}^{3\times 3}$ is the placeholder of $\partial_t \nabla w$. Finally, $f : \Omega_h \to \mathbb{R}$ is a volume density of external forces, which is considered independent of time and the deformation y, and for simplicity acting on Ω_h only in normal direction e_3.

We assume that W is a frame-indifferent function, that is, $W(F) = W(QF)$ for $Q \in \mathrm{SO}(3)$ and $F \in \mathbb{R}^{3\times 3}$. This implies that W depends on the right Cauchy-Green strain tensor $C := F^\top F$, see for example [17]. The second term on the left-hand side of (1.1) is the stress tensor $S(F, \dot{F}) := \partial_{\dot{F}} R(F, \dot{F})$ which has its origin in viscous dissipative mechanisms of the material. We point out that its potential R plays an analogous role as W in the case of purely elastic, that is, non-dissipative processes. Naturally, we require that $R(F, \dot{F}) \geqq R(F, 0) = 0$. The viscous stress tensor must comply with the time-continuous frame-indifference principle meaning that $S(F, \dot{F}) = F\tilde{S}(C, \dot{C})$, where $\tilde{S}$ is a symmetric matrix-valued function and $\dot{C}$ denotes the time derivative of the right Cauchy-Green strain tensor C. This condition constraints R so that $R(F, \dot{F}) = \tilde{R}(C, \dot{C})$ for some nonnegative function $\tilde{R}$, see [4,5,33]. In what follows, we suppose that the material is homogeneous, that is, neither the elastic stored energy density nor the dissipation depend on material points. Moreover, for technical reasons, we will restrict our analysis to the case of zero Poisson's ratio in the out-of-plane direction, see (2.17)–(2.18) and Remark 5.11 for some details in that direction. Such an assumption, also present in other works (see for example [9]), simplifies the analysis.

Following the study in [22], we consider a version of (1.1) for second-grade materials where the elastic stored energy density (and the first Piola-Kirchhoff stress tensor, too) depends also on the second gradient of w. In this case, we get

$$-\mathrm{div}\Big(\partial_F W(\nabla w) + \varepsilon \mathcal{L}_P(\nabla^2 w) + \partial_{\dot{F}} R(\nabla w, \partial_t \nabla w)\Big) = f e_3 \quad \text{in } [0, T] \times \Omega_h, \tag{1.2}$$

where $\varepsilon > 0$ is small and $\mathcal{L}_P$ is a first order differential operator which corresponds to an additional term $\int_{\Omega_h} P(\nabla^2 w)$ in the stored elastic energy, associated to a convex and frame-indifferent density P. (We refer to (2.9) for more details.) As already mentioned, this idea by Toupin [42,43] has proved to be useful in mathematical elasticity because it brings additional compactness to the problem. For example, concerning existence theory for second-grade materials, no convexity properties of W are needed, in particular, we do not have to assume that W is polyconvex [6,17]. Moreover, it is shown in [25] that, if W satisfies suitable and physically relevant growth conditions (as $W(F) \to \infty$ if $\det F \to 0$), then every minimizer of the elastic energy is a weak solution to the corresponding Euler-Lagrange equations.

In [24], it has been shown that for forces scaling like $\sim h^3$, which corresponds to an energy per thickness of $\sim h^4$, the nonlinear elastic energy can be related rigorously by Γ-convergence as $h \to 0$ to the so-called *von Kármán functional*. This functional is given in terms of rescaled in-plane displacements u and out-of-plane displacements v. The corresponding Euler-Lagrange equations take the form

$$\mathrm{div}\Big(\mathbb{C}_W\big(e(u) + \tfrac{1}{2}\nabla v \otimes \nabla v\big)\Big) = 0,$$
$$-\mathrm{div}\Big(\mathbb{C}_W\Big(e(u) + \frac{1}{2}\nabla v \otimes \nabla v\Big)\nabla v\Big) + \frac{1}{12}\mathrm{div}\,\mathrm{div}\big(\mathbb{C}_W \nabla^2 v\big) = f \quad \text{in } S,$$

where $e(u) := (\nabla u + (\nabla u)^\top)/2$ denotes the linear strain tensor, and $\mathbb{C}_W$ is the tensor of elastic constants, derived suitably from W (see (2.17)–(2.19) below for details). The first equation corresponds to the *membrane strain*, which was used already earlier in FÖPPL's work [21] and leads to a nonlinearity in the vK equations. The second equation includes also a *bending contribution*. In the present context, we will see that the passage from the nonlinear elastic energy to the vK functional by Γ-convergence remains true if the nonlinear energy is enhanced by a second gradient term $\varepsilon \int_{\Omega_h} P(\nabla^2 w)$ for certain scalings of ε, see Theorem 5.6.

In the frame of viscoelastic materials, we address the relation between the nonlinear equations (1.2) to the following equations for viscoelastic vK plates:

$$\begin{cases} 0 = \mathrm{div}\Big(\mathbb{C}_W\big(e(u) + \frac{1}{2}\nabla v \otimes \nabla v\big) + \mathbb{C}_R\big(e(\partial_t u) + \nabla \partial_t v \otimes \nabla v\big)\Big), \\ f = -\mathrm{div}\Big(\Big(\mathbb{C}_W\big(e(u) + \frac{1}{2}\nabla v \otimes \nabla v\big) + \mathbb{C}_R\big(e(\partial_t u) + \nabla \partial_t v \otimes \nabla v\big)\Big)\nabla v\Big) \\ \quad + \frac{1}{12}\mathrm{div}\,\mathrm{div}\Big(\mathbb{C}_W \nabla^2 v + \mathbb{C}_R \nabla^2 \partial_t v\Big) \qquad \text{in } [0,\infty) \times S, \end{cases} \tag{1.3}$$

where $\mathbb{C}_R$ is the tensor of viscosity coefficients which is derived from the dissipation potential R. More precisely, we prove the existence of solutions to (1.3) and make the dimension reduction rigorous, that is, we show that solutions to (1.2) converge to solutions of (1.3) in a specific sense. The solutions have to be understood in a weak sense, see (2.22) for the exact definition. We point out that the same relation is expected to hold also for the original problem of simple materials (1.1) (that is, only the first gradient of w is considered), but a proof seems unreachable (or at least rather difficult) at the moment. We emphasize that a nonsimple material model is used here because viscous phenomena are considered, too. In fact, dissipation due to viscosity leads to the loss of weak lower semicontinuity in semidiscretized variational problems; see [33] for a detailed discussion.

Our general strategy is to treat the system of quasistatic viscoelasticity in the abstract setting of metric gradient flows [3], where the underlying metric is given by a *dissipation distance* suitably related to the potential R (see (2.6) below). To the best of our knowledge, this was formulated for the first time in [33] for simple materials. In a fashion similar to [22], our starting point is the existence of time-discrete solutions to the three-dimensional equations (1.2). Here, the second gradient term allows to obtain an existence result without polyconvexity conditions [6] for the dissipation which seems to be incompatible with frame indifference. The existence of solutions to the equations (1.3) of viscous vK plates is guaranteed by identifying them as limits of solutions to the nonlinear three-dimensional equations (1.2). In this context, we follow the abstract framework of sequences of metric gradient flows, developed in [36,40,41]. In using this theory, the challenge lies in proving that the additional conditions needed to ensure convergence of gradient flows are satisfied.

More specifically, to use the abstract convergence result, lower semicontinuity of (i) the energies, (ii) the metrics, and (iii) the local slopes is needed. (i) The estimate for the energies essentially follows from [24] where we show that it still holds for nonsimple materials if the contribution of the second gradient in terms of ε (see (1.2)) is chosen sufficiently small, cf. Theorem 5.6. (ii) The lower semicontinuity of the metrics can be established in a very similar fashion. (iii) The lower semicontinuity of the local slopes, however, is very technical and the core of our argument. We briefly explain the main idea. The local slope of an energy ϕ in a metric space with metric $\mathcal{D}$ is defined by

$$|\partial\phi|_{\mathcal{D}}(y) := \limsup_{z\to y} \frac{(\phi(y)-\phi(z))^+}{\mathcal{D}(y,z)}.$$

Consider a sequence of deformations $(w^h)_h$ of thin plates converging to a limit (u, v). The first step in the proof is to show that the local slope in the limiting setting at some configuration $y = (u, v)$ can be determined by considering only variations of the form $z = (u_s, v_s) = (u, v) + s(\tilde{u}, \tilde{v})$ for $s > 0$ small. This follows from a specific representation; see Lemma 4.9, which is based on some generalized convexity properties. (Recall, however, that the vK model is actually nonconvex.) Then the crucial step is to choose sequences $(w^h_s)_{h,s}$ where $w^h_s \to w^h$ represents a competitor sequence for the local slope in the three-dimensional setting. For each $s > 0$, $(w^h_s)_h$ has to be constructed as a *mutual recovery sequence* of (u_s, v_s), that is, for both the elastic energies and the dissipations. In this context, the rate of convergence needs to be *linear in s* as $h \to 0$. The realization is in fact quite technical and the details are contained in Theorem 5.10. An important step in this analysis is to understand the strain difference of the configurations w^h and w^h_s; see Lemma 5.5.

Let us mention that a derivation of a nonlinear bending theory (see [23]) in the setting of viscoelasticity seems to be even more involved and remains an open problem: the fact that deformations are generically not near a single rigid motion does not comply with the model investigated in [22]. Even more severely, the characterization of the local slope in the limiting two-dimensional setting appears to be very difficult due to the nonlinear isometry constraint.

The plan of the paper is as follows: in Section 2, we introduce the nonlinear three-dimensional viscoelastic system and its two-dimensional vK limiting version in more detail. Then we state our main results. In particular, Proposition 2.1 shows existence of time-discrete solutions to the three-dimensional problem (1.2). In Theorem 2.2 we present an existence result for solutions to the vK plate equations (1.3) in the "time-continuous setting", which is based on identifying these solutions with so-called *curves of maximal slope* [19]. Finally, Theorem 2.3 shows the relationship between the two systems which relies on an abstract convergence result for curves of maximal slope and their approximation via the minimizing movement scheme.

Section 3 is devoted to definitions of generalized minimizing movements and curves of maximal slope. Here, we also recall results about sequences of curves of maximal slope and their approximation by minimizing movements. Section 4 is devoted to properties of the elastic energies and the dissipation distances. In particular, we show that the dissipation distances give rise to complete metric spaces, and derive some generalized convexity properties in the limiting two-dimensional setting, as well as a representation of the local slope. Section 5 discusses the relation between the three- and two-dimensional setting, compactness properties, and Γ-convergence results. Moreover, here we prove the fundamental lower semicontinuity properties for local slopes. Finally, proofs of our main results can be found in Section 6.

2. The Model and Main Results

2.1. Notation

In what follows, we use standard notation for Lebesgue spaces, $L^p(\Omega)$, which are measurable maps on $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, integrable with the p-th power (if $1 \leqq p < +\infty$) or essentially bounded (if $p = +\infty$). Sobolev spaces, that is, $W^{k,p}(\Omega)$ denote the linear spaces of maps which, together with their weak derivatives up to the order $k \in \mathbb{N}$, belong to $L^p(\Omega)$. Furthermore, $W^{k,p}_0(\Omega)$ contains maps from $W^{k,p}(\Omega)$ having zero boundary conditions (in the sense of traces). To emphasize the target space $\mathbb{R}^k$, $k = 1, 2, 3$, we write $L^p(\Omega; \mathbb{R}^k)$. If $k = 1$, we write $L^p(\Omega)$ as usual. We refer to [2] for more details on Sobolev spaces and their duals. We also denote the components of vector functions y by y_1, y_2, and y_3, and so on.

If $A \in \mathbb{R}^{d\times d\times d\times d}$ and $e \in \mathbb{R}^{d\times d}$ then $Ae \in \mathbb{R}^{d\times d}$ is such that for $i, j \in \{1, \dots, d\}$ we define $(Ae)_{ij} := A_{ijkl}e_{kl}$ where we use Einstein's summation convention. An analogous convention is used in similar occasions, in the sequel. By $\mathbf{Id} \subset \mathbb{R}^{3\times 3}$ we denote the identity matrix. We often drop dx at the end of integrals if the integration variable is clear from the context. Finally, at many spots, we closely follow notation introduced in [3] and [24] to ease readability of our work.

2.2. The setting

We first introduce a three-dimensional setting following the setup in [24]. We consider a right-handed orthonormal system $\{e_1, e_2, e_3\}$ and $S \subset \mathbb{R}^2$ open, bounded with Lipschitz boundary, in the span of e_1 and e_2. Let $h > 0$ small. We consider *deformations* $w : S \times (-\frac{h}{2}, \frac{h}{2}) \to \mathbb{R}^3$. It is convenient to work in a fixed domain $\Omega = S \times I$ with $I := (-\frac{1}{2}, \frac{1}{2})$ and to rescale deformations according to $y(x) = w(x', hx_3)$ such that $y : \Omega \to \mathbb{R}^3$, where we use the abbreviation $x' = (x_1, x_2)$. We also introduce the notation $\nabla' y = y_{,1} \otimes e_1 + y_{,2} \otimes e_2$ for the in-plane gradient, and the scaled gradient

$$\nabla_h y := \left(\nabla' y, \frac{1}{h} y_{,3}\right) = \nabla w. \tag{2.1}$$

Moreover, we define the scaled second gradient by

$$(\nabla_h^2 y)_{ijk} := h^{-\delta_{3j}-\delta_{3k}} (\nabla^2 y)_{ijk} = (\nabla^2 w)_{ijk} = \partial^2_{jk} w_i \ \text{ for } i, j, k \in \{1, 2, 3\}, \tag{2.2}$$

where δ_{3j}, δ_{3k} denotes the Kronecker delta.

Stored elastic energy density and body forces: We assume that $W : \mathbb{R}^{3\times 3} \to [0, \infty]$ is a single well, frame-indifferent stored energy density with the usual assumptions in nonlinear elasticity. We suppose that there exists $c > 0$ such that

$$\begin{aligned} &(i)\ \ W \text{ continuous and } C^3 \text{ in a neighborhood of } SO(3),\\ &(ii)\ \ \text{frame indifference: } W(QF) = W(F) \text{ for all } F \in \mathbb{R}^{3\times 3},\ Q \in SO(3),\\ &(iii)\ \ W(F) \geqq c \operatorname{dist}^2(F, SO(3)),\ \ W(F) = 0 \text{ iff } F \in SO(3), \end{aligned} \tag{2.3}$$

where $SO(3) = \{Q \in \mathbb{R}^{3\times 3} : Q^\top Q = \mathbf{Id},\ \det Q = 1\}$. Moreover, for $p > 3$, let $P : \mathbb{R}^{3\times 3\times 3} \to [0, \infty]$ be a higher order perturbation satisfying

$$\begin{aligned} &(i)\ \ \text{frame indifference: } P(QZ) = P(Z) \text{ for all } Z \in \mathbb{R}^{3\times 3\times 3},\ Q \in SO(3),\\ &(ii)\ \ P \text{ is convex and } C^1,\\ &(iii)\ \ \text{growth condition: for all } Z \in \mathbb{R}^{3\times 3\times 3} \text{ we have}\\ &\qquad c_1|Z|^p \leqq P(Z) \leqq c_2|Z|^p, \qquad |\partial_Z P(Z)| \leqq c_2|Z|^{p-1} \end{aligned} \tag{2.4}$$

for $0 < c_1 < c_2$. Finally, $f \in L^\infty(\Omega)$ denotes a volume normal force, that is, a force oriented in e_3 direction. Note that more general forces could in principle be included, for example, boundary forces (see [28]). This is neglected here for the sake of simplicity rather than generality.

Dissipation potential and viscous stress: We now introduce a dissipation potential. We follow here the discussion in [33, Section 2.2] and [22, Section 2]. Consider a time dependent deformation $y : [0, T] \times \Omega \to \mathbb{R}^3$. Viscosity is not only related to the strain rate $\partial_t \nabla_h y(t, x)$ but also to the strain $\nabla_h y(t, x)$. It can be expressed in terms of a dissipation potential $R(\nabla_h y, \partial_t \nabla_h y)$, where $R : \mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3} \to [0, \infty)$. An admissible potential has to satisfy frame indifference in the sense (see [4,33]) that

$$R(F, \dot{F}) = R(QF, Q(\dot{F} + AF)) \quad \forall Q \in SO(3),\ A \in \mathbb{R}^{3\times 3}_{\text{skew}} \tag{2.5}$$

for all $F \in GL_+(3)$ and $\dot{F} \in \mathbb{R}^{3\times 3}$, where $GL_+(3) = \{F \in \mathbb{R}^{3\times 3} : \det F > 0\}$ and $\mathbb{R}^{3\times 3}_{\text{skew}} = \{A \in \mathbb{R}^{3\times 3} : A = -A^\top\}$.

From the point of modeling, it is more convenient to assume the existence of a (smooth) global distance $D : GL_+(3) \times GL_+(3) \to [0, \infty)$ satisfying $D(F, F) = 0$ for all $F \in GL_+(3)$. From this, an associated dissipation potential R can be calculated by

$$R(F, \dot{F}) := \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon^2} D^2(F + \varepsilon \dot{F}, F) = \frac{1}{4} \partial^2_{F_1^2} D^2(F, F)[\dot{F}, \dot{F}] \tag{2.6}$$

for $F \in GL_+(3)$, $\dot{F} \in \mathbb{R}^{3\times 3}$. Here, $\partial^2_{F_1^2} D^2(F_1, F_2)$ denotes the Hessian of D^2 in direction of F_1 at (F_1, F_2), which is a fourth order tensor. For some $c > 0$ we suppose that D satisfies

(i) $D(F_1, F_2) > 0$ if $F_1^\top F_1 \neq F_2^\top F_2$,

$$
\begin{aligned}
&(ii)\ \ D(F_1, F_2) = D(F_2, F_1), \\
&(iii)\ \ D(F_1, F_3) \leqq D(F_1, F_2) + D(F_2, F_3), \\
&(iv)\ \ D(\cdot,\cdot) \text{ is } C^3 \text{ in a neighborhood of } SO(3)\times SO(3), \\
&(v)\ \ \text{Separate frame indifference: } D(Q_1F_1, Q_2F_2) = D(F_1, F_2) \\
&\qquad \forall Q_1, Q_2 \in SO(3),\ \forall F_1, F_2 \in GL_+(3), \\
&(vi)\ \ D(F, \mathbf{Id}) \geqq c\,\mathrm{dist}(F, SO(3)) \forall F \in \mathbb{R}^{3\times 3} \text{ in a neighborhood of } SO(3).
\end{aligned} \tag{2.7}
$$

Note that conditions (i),(iii) state that D is a true distance when restricted to symmetric matrices with nonnegative determinants. We cannot expect more due to the separate frame indifference (v). We also point out that (v) implies (2.5) as shown in [33, Lemma 2.1]. Note that in our model we do not require any conditions of polyconvexity [6] neither for W nor for D. One possible example of D satisfying (2.7) might be $D(F_1, F_2) = |F_1^\top F_1 - F_2^\top F_2|$. This leads to $R(F, \dot F) = |F^\top \dot F + \dot F^\top F|^2/2$ which is a standard choice. For further examples we refer to [33, Section 2.3].

Equations of viscoelasticity in three dimensions: Following the study in [28], we define clamped boundary conditions as follows. We consider functions $\hat u \in W^{2,\infty}(S;\mathbb{R}^2)$ and $\hat v \in W^{3,\infty}(S)$ which represent in-plane and out-of-plane boundary conditions, respectively. We introduce the set of admissible configurations by

$$
\mathscr{S}_h = \left\{ y \in W^{2,p}(\Omega;\mathbb{R}^3):\ y(x', x_3) = \begin{pmatrix} x' \\ hx_3 \end{pmatrix} + \begin{pmatrix} h^2\hat u(x') \\ h\hat v(x') \end{pmatrix} - x_3 \begin{pmatrix} h^2(\nabla'\hat v(x'))^\top \\ 0 \end{pmatrix} \right. \\ \left. \text{for } x' \in \partial S,\ x_3 \in I \right\}, \tag{2.8}
$$

where $I = (-\frac12, \frac12)$. Following [22] we formulate the equations of viscoelasticity for a nonsimple material involving the perturbation P (cf. (2.4)). We introduce a differential operator associated to P. To this end, we recall the notation of the scaled gradients in (2.1)–(2.2). For $i, j \in \{1, \dots, 3\}$, we denote by $(\partial_Z P(\nabla_h^2 y))_{ij*}$ the vector-valued function $((\partial_Z P(\nabla_h^2 y))_{ijk})_{k=1,2,3}$. We also introduce the scaled (distributional) divergence $\mathrm{div}_h g$ for a function $g \in L^1(\Omega;\mathbb{R}^3)$ by $\mathrm{div}_h g = \partial_1 g_1 + \partial_2 g_2 + \frac1h \partial_3 g_3$. We define

$$
\big(\mathcal{L}_P^h(\nabla_h^2 y)\big)_{ij} = -\mathrm{div}_h(\partial_Z P(\nabla_h^2 y))_{ij*}, \quad i, j \in \{1, \dots, 3\} \tag{2.9}
$$

for $y \in \mathscr{S}_h$. Let $\beta_1, \beta_2 > 0$. The equations of nonlinear viscoelasticity can be written as

$$
\begin{cases}
-\mathrm{div}_h\Big(\partial_F W(\nabla_h y) + h^{\beta_1}\mathcal{L}_P^h(\nabla_h^2 y) + \partial_{\dot F} R(\nabla_h y, \partial_t \nabla_h y)\Big) = h^{\beta_2} f e_3 & \text{in } [0,\infty)\times\Omega \\
y(0,\cdot) = y_0^h & \text{in } \Omega \\
y(t,\cdot) \in \mathscr{S}_h & \text{for } t \in [0,\infty)
\end{cases} \tag{2.10}
$$

for some $y_0^h \in \mathscr{S}_h$, where $\partial_F W(\nabla_h y) + h^{\beta_1}\mathcal{L}_P^h(\nabla_h^2 y)$ denotes the *first Piola-Kirchhoff stress tensor* and $\partial_{\dot F} R(\nabla_h y, \partial_t \nabla_h y)$ the *viscous stress* with R as introduced in (2.6). As no surface forces are applied, we implicitly assume zero Neumann boundary conditions for the stress and the hyperstress on $S \times \{-1/2, 1/2\}$, that is, on the top and the bottom of the cylinder Ω, see for example [26] for a specific form of such conditions. As (2.8) prescribes only the values of the function but not of the derivative, on the lateral boundary there arise additional Neumann conditions from the second deformation gradient. (We again refer to [26] for details.) As we will see below, however, they do not affect the effective two-dimensional plate model. Suitable scalings h^{β_1} and h^{β_2} related to $\mathcal{L}_P^h$ and the normal force f, respectively, will be discussed below. Note that $\varepsilon = h^{\beta_1}$ in (1.2).

Time-discrete solutions to (2.10): The first auxiliary goal of this paper is to show existence of time-discrete solutions to (2.10) for small $h > 0$. For this, we introduce a functional $I_h^{\beta_1,\beta_2} : W^{2,p}(\Omega;\mathbb{R}^3) \to \mathbb{R}$ describing the elastic energy of the body by

$$I_h^{\beta_1,\beta_2}(y) = \int_\Omega W(\nabla_h y(x))\,\mathrm{d}x + h^{\beta_1}\int_\Omega P(\nabla_h^2 y(x))\,\mathrm{d}x - h^{\beta_2}\int_\Omega f(x) y_3(x)\,\mathrm{d}x \tag{2.11}$$

for a deformation $y : W^{2,p}(\Omega;\mathbb{R}^3) \to \mathbb{R}^3$. We note that the energy takes into account the different scalings of the terms in (2.10).

We use an approximation scheme solving suitable time-incremental minimization problems: consider a fixed time step $\tau > 0$ and suppose that an initial datum $y_0^h \in \mathscr{S}_h$ is given. Set $Y_{h,\tau}^0 = y_0^h$. Whenever $Y_{h,\tau}^0, \ldots, Y_{h,\tau}^{n-1}$ are known, $Y_{h,\tau}^n$ is defined as (if existent)

$$Y_{h,\tau}^n = \mathrm{argmin}_{y\in\mathscr{S}_h}\Phi_h(\tau, Y_{h,\tau}^{n-1}; y), \quad \Phi_h(\tau, y_0; y_1) := I_h^{\beta_1,\beta_2}(y_1) + \frac{1}{2\tau}\mathcal{D}^2(y_0, y_1), \tag{2.12}$$

where $\mathcal{D}$ denotes the *global dissipation distance* between two deformations, defined by

$$\mathcal{D}(y_0, y_1) := \Big(\int_\Omega D^2(\nabla_h y_0, \nabla_h y_1)\Big)^{1/2}.$$

Suppose that, for a choice of τ, a sequence $(Y_{h,\tau}^n)_{n\in\mathbb{N}}$ solving (2.12) exists. We define the piecewise constant interpolation by

$$\tilde{Y}_{h,\tau}(0,\cdot) = Y_{h,\tau}^0, \quad \tilde{Y}_{h,\tau}(t,\cdot) = Y_{h,\tau}^n \text{ for } t \in ((n-1)\tau, n\tau],\ n \geqq 1. \tag{2.13}$$

In what follows, $\tilde{Y}_{h,\tau}$ will be called a *time-discrete solution*. We often drop the x-dependence and write $\tilde{Y}_{h,\tau}(t)$ for a time-discrete solution at time t. Note that the existence of such solutions is usually guaranteed by the direct method of the calculus of variations under suitable compactness, coercivity, and lower semicontinuity assumptions, see Proposition 2.1 and its proof. We point out that, in the setting of general (but not thin) bodies with suitable boundary conditions imposed on the entire $\partial\Omega$, it has been shown in [22, Theorem 2.1] that time-discrete solutions indeed converge to weak solutions to the system (2.10) as $\tau \to 0$. In the present context, time-discrete solutions will be the starting point to pass to a two-dimensional, time-continuous framework.

Scaling and displacement fields: Due to the scaling of the boundary conditions in (2.8), the energy of time-discrete solutions $\tilde{Y}_{h,\tau}$ is expected to be small in terms of h. More specifically, in our setting it will turn out that the energy is of order h^4 which corresponds to the so-called *von Kármán regime*. (For an exhaustive treatment of different scaling regimes we refer the reader to [24].) As y_3 scales like h, see (2.8), a suitable choice for the scaling of the forces in (2.11) is therefore h^3, that is, we set $\beta_2 = 3$. Moreover, we choose $\beta_1 = 4 - p\alpha$ for some $0 < \alpha < 1$. On the one hand, this will imply that $\|\nabla_h^2 \tilde{Y}_{h,\tau}\|_{L^p(\Omega)}$ is small, more precisely, of order h^α, cf. (2.4)(iii). On the other hand, $\alpha < 1$ will ensure that the higher order perturbation will vanish in the effective two-dimensional limiting model.

Based on this discussion, we introduce the rescaled nonlinear energy $\phi_h : W^{2,p}(\Omega;\mathbb{R}^3) \to [0,\infty]$ by

$$\phi_h(y) = h^{-4} I_h^{4-p\alpha,3}(y) = \frac{1}{h^4}\int_\Omega W(\nabla_h y(x))\,\mathrm{d}x + \frac{1}{h^{\alpha p}}\int_\Omega P(\nabla_h^2 y(x))\,\mathrm{d}x - \frac{1}{h}\int_\Omega f(x) y_3(x)\,\mathrm{d}x \tag{2.14}$$

for $y \in \mathscr{S}_h$. Similarly, for $y_0, y_1 \in \mathscr{S}_h$, the rescaled global dissipation distance is given by

$$\mathcal{D}_h(y_0, y_1) = h^{-2}\mathcal{D}(y_0, y_1) = h^{-2}\Big(\int_\Omega D^2(\nabla_h y_0, \nabla_h y_1)\Big)^{1/2}. \tag{2.15}$$

Following the discussion in [24], for $y \in \mathscr{S}_h$ we introduce the corresponding averaged, scaled in-plane and out-of-plane displacements which measure the deviation from the mapping $(x', 0)$:

$$u(x') := \frac{1}{h^2}\int_I \left(\begin{pmatrix} y_1^h \\ y_2^h \end{pmatrix}(x', x_3) - \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) \mathrm{d}x_3, \qquad v(x') := \frac{1}{h}\int_I y_3^h(x', x_3)\,\mathrm{d}x_3, \tag{2.16}$$

where again $I = (-\frac{1}{2}, \frac{1}{2})$. In a similar fashion, given a time-discrete solution $\tilde{Y}_{h,\tau}$, we introduce averaged functions $\tilde{U}_{h,\tau}$ and $\tilde{V}_{h,\tau}$ dependent on (t, x'). Via the minimizing movement scheme, we will later see that along a sequence $(\tilde{Y}_{h,\tau}(t))_{h,\tau}$ we get $\tilde{U}_{h,\tau}(t) \rightharpoonup u(t)$ weakly in $W^{1,2}(S;\mathbb{R}^2)$ and $\tilde{V}_{h,\tau}(t) \to v(t)$ strongly in $W^{1,2}(S)$ with $v(t) \in W^{2,2}(S)$ for each $t \geqq 0$, when $h, \tau \to 0$. The main goal of our work is to understand which equations are solved by $(u(t), v(t))$.

Quadratic forms: To formulate the effective two-dimensional problem for the scaled in-plane and out-of-plane displacements, we need to consider various quadratic forms. First, we define $Q_W^3 : \mathbb{R}^{3\times3} \to \mathbb{R}$ by $Q_W^3(F) = \partial^2_{F^2} W(\mathbf{Id})[F, F]$. One can show that it depends only on the symmetric part $\frac{1}{2}(F^\top + F)$ and that it is positive definite on $\mathbb{R}^{3\times3}_{\mathrm{sym}} = \{A \in \mathbb{R}^{3\times3} : A = A^\top\}$, cf. Lemma 4.1. We also introduce $Q_W^2 : \mathbb{R}^{2\times2} \to \mathbb{R}$ by

$$Q_W^2(G) = \min_{a\in\mathbb{R}^3} Q_W^3(G^* + a \otimes e_3 + e_3 \otimes a) \tag{2.17}$$

for $G \in \mathbb{R}^{2\times 2}$, where the entries of $G^* \in \mathbb{R}^{3\times 3}$ are given by $G^*_{ij} = G_{ij}$ for $i, j \in \{1, 2\}$ and zero otherwise. Note that (2.17) corresponds to a minimization over stretches in the e_3 direction. We will assume that the minimum in (2.17) is attained for $a = 0$. Similarly, we define

$$Q^3_D(F)=\frac{1}{2}\partial^2_{F_1^2} D^2(\mathbf{Id}, \mathbf{Id})[F, F], \qquad Q^2_D(G)= \min_{a\in\mathbb{R}^3} Q^3_D(G^*+a \otimes e_3+e_3 \otimes a). \tag{2.18}$$

(Notice that $Q^3_D(F) = 2R(\mathbf{Id}, F)$ with R from (2.6).) We again assume that the minimum is attained for $a = 0$.

The assumption that $a = 0$ is a minimum in (2.17)–(2.18) corresponds to a model with zero Poisson's ratio in the e_3 direction. This assumption is not needed in the purely static analysis [24,28]. In our setting, it is only needed in the proof of lower semicontinuity of slopes, see Theorem 5.10. Dropping this assumption would lead to a considerably more involved limiting description which we do not want to pursue here. We refer to Remark 5.11 for some details in that direction.

We also introduce corresponding symmetric fourth order tensors $\mathbb{C}^d_W$ and $\mathbb{C}^d_D$, $d = 2, 3$, satisfying

$$Q^3_W(F) = \mathbb{C}^3_W[F, F] \ \ \forall F \in \mathbb{R}^{3\times 3}, \quad Q^2_W(G) = \mathbb{C}^2_W[G, G] \ \ \forall G \in \mathbb{R}^{2\times 2}, \tag{2.19}$$

and likewise for $\mathbb{C}^d_D$, $d = 2, 3$.

Equations of viscoelasticity in two dimensions: We now present the effective two-dimensional equations. By recalling definition (2.16) and the boundary conditions in the three-dimensional setting (2.8), we first introduce the relevant space by

$$\mathscr{S}_0 = \{(u, v) \in W^{1,2}(S; \mathbb{R}^2) \times W^{2,2}(S) : \ u = \hat{u}, \ v = \hat{v}, \ \nabla' v = \nabla' \hat{v} \ \text{ on } \partial S\}. \tag{2.20}$$

The clamped boundary conditions correspond to the ones considered in [28]. We emphasize that Neumann conditions in (2.10) on the lateral boundary arising from the second deformation gradient do not affect the plate equations (2.21) below since our scaling makes the second gradient vanish as the thickness of the domain tends to zero.

Given $(u_0, v_0) \in \mathscr{S}_0$, we consider the equations

$$\begin{cases} \mathrm{div}_2\Big(\mathbb{C}^2_W\big(e(u) + \frac{1}{2}\nabla' v \otimes \nabla' v\big) + \mathbb{C}^2_D\big(e(\partial_t u) + \nabla'\partial_t v \odot \nabla' v\big)\Big) = 0, & \\ -\mathrm{div}_2\Big(\Big(\mathbb{C}^2_W\big(e(u) + \frac{1}{2}\nabla' v \otimes \nabla' v\big) + \mathbb{C}^2_D\big(e(\partial_t u) + \nabla'\partial_t v \odot \nabla' v\big)\Big)\nabla' v\Big) & \\ \quad + \frac{1}{12}\mathrm{div}_2\,\mathrm{div}_2\Big(\mathbb{C}^2_W(\nabla')^2 v + \mathbb{C}^2_D(\nabla')^2\partial_t v\Big) = f & \text{in } [0, \infty) \times S \\ u(0, \cdot) = u_0, \ v(0, \cdot) = v_0 & \text{in } S \\ (u(t, \cdot), v(t, \cdot)) \in \mathscr{S}_0 & \text{for } t \in [0, \infty), \end{cases} \tag{2.21}$$

where $\mathbb{C}^2_W$ and $\mathbb{C}^2_D$ are defined in (2.19), and $\odot$ denotes the symmetrized tensor product. Note that the frame indifference of the energy and the dissipation (see

(2.3)(ii) and (2.7)(v), respectively) imply that the contributions only depend on the symmetric part of the strain $e(u) := \frac{1}{2}(\nabla' u + (\nabla' u)^\top)$ and the strain rate $e(\partial_t u) := \frac{1}{2}(\partial_t \nabla' u + \partial_t (\nabla' u)^\top)$. Here, div_2 denotes the distributional divergence in dimension two.

We also say that $(u, v) \in W^{1,2}([0,\infty); \mathscr{S}_0)$ is a *weak solution* of (2.21) if $u(0,\cdot) = u_0$, $v(0,\cdot) = v_0$ and for a.e. $t \geqq 0$ we have

$$\int_S \Big(\mathbb{C}_W^2\big(e(u) + \tfrac{1}{2}\nabla' v \otimes \nabla' v\big) + \mathbb{C}_D^2\big(e(\partial_t u) + \nabla' \partial_t v \odot \nabla' v\big)\Big) : \nabla' \varphi_u = 0, \tag{2.22a}$$

$$\int_S \Big(\mathbb{C}_W^2\big(e(u) + \tfrac{1}{2}\nabla' v \otimes \nabla' v\big) + \mathbb{C}_D^2\big(e(\partial_t u) + \nabla' \partial_t v \odot \nabla' v\big)\Big) : \big(\nabla' v \odot \nabla' \varphi_v\big)$$
$$+ \frac{1}{12}\int_S \Big(\mathbb{C}_W^2 (\nabla')^2 v + \mathbb{C}_D^2 (\nabla')^2 \partial_t v\Big) : (\nabla')^2 \varphi_v = \int_S f \varphi_v \tag{2.22b}$$

for all $\varphi_u \in W_0^{1,2}(S; \mathbb{R}^2)$ and $\varphi_v \in W_0^{2,2}(S)$. Note that (2.22a) corresponds to two scalar equations and (2.22b) corresponds to one scalar equation, respectively. Our goal will be to show that time-discrete solutions to (2.10), as introduced in (2.13), converge to weak solutions to (2.21) in a suitable sense.

We also mention that the equations are related to the *von Kármán* functional

$$\phi_0(u, v) := \int_S \frac{1}{2} Q_W^2\Big(e(u) + \frac{1}{2}\nabla' v \otimes \nabla' v\Big) + \frac{1}{24} Q_W^2((\nabla')^2 v) - \int_S f v \tag{2.23}$$

for $(u, v) \in \mathscr{S}_0$; actually, we will also see that ϕ_0 can be related to ϕ_h (see (2.14)) in the sense of Γ-convergence, cf. Section 5.2 below. A similar relation holds for $\mathcal{D}_h$, introduced in (2.15), and the *global dissipation distance* in the two-dimensional setting, defined by

$$\begin{aligned}
&\mathcal{D}_0((u_0, v_0), (u_1, v_1)) \\
&:= \Big(\int_S Q_D^2\Big(e(u_1) - e(u_0) + \frac{1}{2}\nabla' v_1 \otimes \nabla' v_1 - \frac{1}{2}\nabla' v_0 \otimes \nabla' v_0\Big) \\
&\quad + \frac{1}{12} Q_D^2\big((\nabla')^2 v_1 - (\nabla')^2 v_0\big)\Big)^{1/2}
\end{aligned} \tag{2.24}$$

for $(u_0, v_0), (u_1, v_1) \in \mathscr{S}_0$.

2.3. Main results

Define the sublevel sets $\mathscr{S}_h^M := \{y \in \mathscr{S}_h : \phi_h(y) \leqq M\}$. Our general strategy will be to show that the spaces $(\mathscr{S}_h^M, \mathcal{D}_h)$ and $(\mathscr{S}_0, \mathcal{D}_0)$ are complete metric spaces (see Lemma 4.5 and Lemma 4.6 below) and to follow the methodology developed in [3].

To show the existence of solutions to the equations, we will apply the theory of [3] about curves of maximal slope. By using the property that in Hilbert spaces curves of maximal slope can be related to gradient flows, we then find weak solutions to (2.21). To understand the relation of solutions in the three-dimensional and two-dimensional setting, we will employ an abstract convergence result for curves of maximal slope and their approximation via the minimizing movement scheme, see [36,41]. The relevant results about curves of maximal slope are recalled in Section 3.

Our first main result addresses the existence of time-discrete solutions to the three-dimensional problem.

Proposition 2.1. (Time-discrete solutions in the three-dimensional setting) *Let* $M > 0$ *and* $\mathscr{S}_h^M = \{y \in \mathscr{S}_h : \phi_h(y) \leqq M\}$. *Let* $\beta_1 = 4 - \alpha p$ *and* $\beta_2 = 3$. *Let* $y_0^h \in \mathscr{S}_h^M$. *Then, for* $h > 0$ *sufficiently small only depending on* M, *the sequence of minimization problems in* (2.12) *has a solution, and gives rise to a time-discrete solution* $\tilde{Y}_{h,\tau}$ *with* $\tilde{Y}_{h,\tau}(0) = y_0^h$.

Note that the existence of weak (time-continuous) solutions to (2.10) has been addressed in [22] for bulk materials. However, due to the thinness of the domain representing the body and the imposed boundary conditions, the results obtained there are not applicable. This is due to the fact that specific constants depend on the domain and blow up for vanishing thickness. Therefore, here we only prove the existence of time-discrete solutions.

For the main definitions and notation for curves of maximal slope and strong upper gradients we refer to Section 3.1. In particular, we write $|\partial\phi_0|_{\mathcal{D}_0}$ for the (local) slopes, see Definition 3.1. For the two-dimensional problem we obtain the following results:

Theorem 2.2. (Solutions in the two-dimensional setting) *The limiting two-dimensional problem has the following properties:*

(i) (Curves of maximal slope) For all $(u_0, v_0) \in \mathscr{S}_0$ *there exists a curve of maximal slope* $(u, v) : [0, \infty) \to \mathscr{S}_0$ *for* ϕ_0 *with respect to the strong upper gradient* $|\partial\phi_0|_{\mathcal{D}_0}$ *satisfying* $(u, v)(0) = (u_0, v_0)$.

(ii) (Relation to PDE) For all $(u_0, v_0) \in \mathscr{S}_0$, *each curve of maximal slope* $(u, v) : [0, \infty) \to \mathscr{S}_0$ *with* $(u, v)(0) = (u_0, v_0)$ *is a weak solution to the partial differential equations* (2.21) *in the sense of (2.22).*

We mention that in [23, Theorem 4.1] we provide a slightly different proof of Theorem 2.2(i), without using the fact that the two-dimensional model is the limit of the time-discrete three-dimensional model. However, the proof still heavily relies on the properties of ϕ_0, $\mathcal{D}_0$, and $|\partial\phi_0|_{\mathcal{D}_0}$ derived in the present paper.

Finally, we study the relation of time-discrete solutions (2.13) and weak solutions to the equations (2.21). To this end, we need to specify the topology of the convergence. Given $y^h \in \mathscr{S}_h$ and $(u, v) \in \mathscr{S}_0$, we say $y^h \overset{\pi\sigma}{\to} (u, v)$ as $h \to 0$ if the corresponding averaged and scaled displacements fields defined in (2.16), denoted by u^h and v^h, satisfy $u^h \rightharpoonup u$ weakly in $W^{1,2}(S; \mathbb{R}^2)$ and $v^h \to v$ strongly in $W^{1,2}(S)$. (The symbol $\pi\sigma$ is used because of the abstract convergence result, see Section 3.2.)

Theorem 2.3. (Relation between three-dimensional and two-dimensional systems) *Let* $\beta_1 = 4 - p\alpha$ *and* $\beta_2 = 3$. *Let* $(u_0, v_0) \in \mathscr{S}_0$ *be an initial datum.*

(i) Then, for the family of sequences of initial data we have

$$\mathcal{B}(u_0, v_0) = \left\{(y_0^h)_h : \ y_0^h \in \mathscr{S}_h, \ y_0^h \overset{\pi\sigma}{\to} (u_0, v_0), \ \phi_h(y_0^h) \to \phi_0((u_0, v_0))\right\} \neq \emptyset. \tag{2.25}$$

(ii) We consider a sequence $(y_0^h)_h \in \mathcal{B}(u_0, v_0)$*, a null sequence* $(\tau_h)_h$*, and a sequence of time-discrete solutions* $\tilde{Y}_{h,\tau_h}$ *as in* (2.13) *with* $\tilde{Y}_{h,\tau_h}(0) = y_0^h$.
Then, there exists a curve of maximal slope $(u, v) : [0, \infty) \to \mathscr{S}_0$ *for* ϕ_0 *with respect to* $|\partial\phi_0|_{\mathcal{D}_0}$ *satisfying* $u(0) = u_0$ *and* $v(0) = v_0$ *such that up to a subsequence (not relabeled) it holds that*

$$\begin{aligned} &\tilde{Y}_{h,\tau_h}(t) \overset{\pi\sigma}{\to} (u(t), v(t)), \quad \phi_h(\tilde{Y}_{h,\tau_h}(t)) \\ &\quad \to \phi_0(u(t), v(t)) \quad \textit{for all } t \in [0, \infty) \quad \textit{as } h \to 0. \end{aligned}$$

We point out that (2.25) corresponds to the existence of recovery sequences for the static problem. We note that the existence of the time-discrete solutions $\tilde{Y}_{h,\tau_h}$ in (ii) is guaranteed by Proposition 2.1. Item (ii) shows the convergence of time-discrete solutions to (2.10) to (time-continuous) weak solutions to (2.21). Moreover, we also have convergence of the energies. From now on we set $f \equiv 0$ for convenience. The general case indeed follows with minor modifications, which are standard.

3. Preliminaries: Curves of Maximal Slope

In this section we recall the relevant definitions about curves of maximal slope and present a convergence result of time-discrete solutions to curves of maximal slope.

3.1. Definitions

We consider a complete metric space $(\mathscr{S}, \mathcal{D})$. We say a curve $y : (a, b) \to \mathscr{S}$ is *absolutely continuous* with respect to $\mathcal{D}$ if there exists $m \in L^1(a, b)$ such that

$$\mathcal{D}(y(s), y(t)) \leqq \int_s^t m(r)\, \mathrm{d}r \quad \text{for all } a \leqq s \leqq t \leqq b.$$

The smallest function m with this property, denoted by $|y'|_{\mathcal{D}}$, is called *metric derivative* of y and satisfies for a.e. $t \in (a, b)$ (see [3, Theorem 1.1.2] for the existence proof)

$$|y'|_{\mathcal{D}}(t) := \lim_{s \to t} \frac{\mathcal{D}(y(s), y(t))}{|s - t|}.$$

We define the notion of a *curve of maximal slope*. We only give the basic definition here and refer to [3, Section 1.2, 1.3] for motivations and more details. By $h^+ := \max(h, 0)$ we denote the positive part of a function h.

Definition 3.1. *(Upper gradients, slopes, curves of maximal slope)* We consider a complete metric space $(\mathscr{S}, \mathcal{D})$ with a functional $\phi : \mathscr{S} \to (-\infty, +\infty]$.

(i) A function $g : \mathscr{S} \to [0, \infty]$ is called a strong upper gradient for ϕ if for every absolutely continuous curve $y : (a, b) \to \mathscr{S}$ the function $g \circ y$ is Borel and

$$|\phi(y(t)) - \phi(y(s))| \leqq \int_s^t g(y(r))|y'|_{\mathcal{D}}(r)\, \mathrm{d}r \quad \text{for all } a < s \leqq t < b.$$

(ii) For each $y \in \mathscr{S}$ the local slope of ϕ at y is defined by

$$|\partial\phi|_{\mathcal{D}}(y) := \limsup_{z \to y} \frac{(\phi(y) - \phi(z))^+}{\mathcal{D}(y, z)}.$$

(iii) An absolutely continuous curve $y : (a, b) \to \mathscr{S}$ is called a curve of maximal slope for ϕ with respect to the strong upper gradient g if for a.e. $t \in (a, b)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(y(t)) \leqq -\frac{1}{2}|y'|^2_{\mathcal{D}}(t) - \frac{1}{2}g^2(y(t)).$$

3.2. Curves of maximal slope as limits of time-discrete solutions

We consider a sequence of complete metric spaces $(\mathscr{S}_k, \mathcal{D}_k)_k$, as well as a limiting complete metric space $(\mathscr{S}, \mathcal{D})$. Moreover, let $(\phi_k)_k$ be a sequence of functionals with $\phi_k : \mathscr{S}_k \to [0, \infty]$ and $\phi : \mathscr{S} \to [0, \infty]$.

We introduce time-discrete solutions for the energy ϕ_k and the metric $\mathcal{D}_k$ by solving suitable time-incremental minimization problems: consider a fixed time step $\tau > 0$ and suppose that an initial datum $Y^0_{k,\tau}$ is given. Whenever $Y^0_{k,\tau}, \ldots, Y^{n-1}_{k,\tau}$ are known, $Y^n_{k,\tau}$ is defined as (if existent)

$$Y^n_{k,\tau} = \mathrm{argmin}_{v \in \mathscr{S}_k} \Phi_k(\tau, Y^{n-1}_{k,\tau}; v), \quad \Phi_k(\tau, u; v) := \frac{1}{2\tau}\mathcal{D}_k(v, u)^2 + \phi_k(v). \tag{3.1}$$

We suppose that for a choice of τ a sequence $(Y^n_{k,\tau})_{n \in \mathbb{N}}$ solving (3.1) exists. Then we define the piecewise constant interpolation by

$$\tilde{Y}_{k,\tau}(0) = Y^0_{k,\tau}, \quad \tilde{Y}_{k,\tau}(t) = Y^n_{k,\tau} \ \text{ for } t \in ((n-1)\tau, n\tau],\ n \geqq 1. \tag{3.2}$$

We call $\tilde{Y}_{k,\tau}$ a *time-discrete solution*. Note that the existence of such solutions is usually guaranteed by the direct method of the calculus of variations under suitable compactness, coercivity, and lower semicontinuity assumptions.

Our goal is to study the limit of time-discrete solutions as $k \to \infty$. To this end, we need to introduce a suitable topology for the convergence. First, although $\mathcal{D}$ naturally induces a topology on the limiting space $\mathscr{S}$, it is often convenient to consider a weaker Hausdorff topology σ on $\mathscr{S}$ to have more flexibility in the derivation of compactness properties (see [3, Remark 2.0.5]). We assume that for each $k \in \mathbb{N}$ there exists a map $\pi_k : \mathscr{S}_k \to \mathscr{S}$. Given a sequence $(z_k)_k$, $z_k \in \mathscr{S}_k$, and $z \in \mathscr{S}$, we say

$$z_k \overset{\pi\sigma}{\to} z \quad \text{if } \pi_k(z_k) \overset{\sigma}{\to} z. \tag{3.3}$$

We suppose that the topology σ satisfies

$$z_k \stackrel{\pi\sigma}{\to} z,\ \bar{z}_k \stackrel{\pi\sigma}{\to} \bar{z} \quad \Rightarrow \quad \liminf_{k\to\infty} \mathcal{D}_k(z_k, \bar{z}_k) \geqq \mathcal{D}(z, \bar{z}). \tag{3.4}$$

Moreover, assume that there exists a σ-sequentially compact set $K_N \subset \mathscr{S}$ such that for all $k \in \mathbb{N}$

$$\{\pi_k(z) :\ z \in \mathscr{S}_k,\ \phi_k(z) \leqq N\} \subset K_N. \tag{3.5}$$

Specifically, for a sequence $(z_k)_k$ with $\phi_k(z_k) \leqq N$, we find a subsequence (not relabeled) and $z \in \mathscr{S}$ such that $\pi_k(z_k) \stackrel{\sigma}{\to} z$. We suppose lower semicontinuity of the energies and the slopes in the following sense: for all $z \in \mathscr{S}$ and $(z_k)_k$, $z_k \in \mathscr{S}_k$, we have

$$z_k \stackrel{\pi\sigma}{\to} z \quad \Rightarrow \quad \liminf_{k\to\infty} |\partial\phi_k|_{\mathcal{D}_k}(z_k) \geqq |\partial\phi|_{\mathcal{D}}(z), \quad \liminf_{k\to\infty} \phi_k(z_k) \geqq \phi(z). \tag{3.6}$$

We now formulate the main convergence result of time-discrete solutions to curves of maximal slope, proved in [36, Section 2].

Theorem 3.2. *Suppose that (3.4)–(3.6) hold. Moreover, assume that $|\partial\phi|_{\mathcal{D}}$ is a strong upper gradient for ϕ. Consider a null sequence $(\tau_k)_k$. Let $(Y^0_{k,\tau_k})_k$ with $Y^0_{k,\tau_k} \in \mathscr{S}_k$ and $\bar{z}_0 \in \mathscr{S}$ be initial data satisfying*

$$\begin{aligned} &(i)\ \ \sup\nolimits_k \mathcal{D}\big(\pi_k\big(Y^0_{k,\tau_k}\big), \bar{z}_0\big) < +\infty, \\ &(ii)\ \ Y^0_{k,\tau_k} \stackrel{\pi\sigma}{\to} \bar{z}_0, \quad \phi_k(Y^0_{k,\tau_k}) \to \phi(\bar{z}_0). \end{aligned} \tag{3.7}$$

Then for each sequence of discrete solutions $(\tilde{Y}_{k,\tau_k})_k$ starting from $(Y^0_{k,\tau_k})_k$ there exists a limiting function $z : [0, +\infty) \to \mathscr{S}$ such that up to a subsequence (not relabeled)

$$\tilde{Y}_{k,\tau_k}(t) \stackrel{\pi\sigma}{\to} z(t), \quad \phi_k(\tilde{Y}_{\tau_k}(t)) \to \phi(z(t)) \qquad \forall t \geqq 0$$

as $k \to \infty$, and z is a curve of maximal slope for ϕ with respect to $|\partial\phi|_{\mathcal{D}}$. In particular, z satisfies the energy identity

$$\frac{1}{2}\int_0^T |z'|^2_{\mathcal{D}}(t)\,dt + \frac{1}{2}\int_0^T |\partial\phi|^2_{\mathcal{D}}(z(t))\,dt + \phi(z(T)) = \phi(\bar{z}_0) \quad \forall T > 0. \tag{3.8}$$

The statement is a combination of convergence results for curves of maximal slope [22,41] with their approximation by time-discrete solutions via the minimizing movement scheme. We refer to [22, Theorem 3.6] for an abstract convergence result for curves of maximal slope in a setting where conditions (3.4)–(3.6) hold. We also mention the version in the seminal work [41] where condition (3.4) is replaced by a lower bound condition on the metric derivatives along the sequence.

For the proof of Theorem 3.2 we refer to [36, Section 2]. We also mention [22, Theorem 3.7] for a formulation of the result which is a bit closer to the statement given here. Strictly speaking, in [36], only the case of a single metric space is considered. The generalization to a sequence of spaces, however, is straightforward, cf. [41, equation (2.7)]. Note that the nonnegativity of the energies ϕ_k can be generalized to a suitable *coerciveness* condition; see [3, (2.1.2b)] or [36, (2.5)]. This is not included here for the sake of simplicity.

Let us also mention the recently obtained variant [14] where it is not necessary to require that $|\partial\phi|_{\mathcal{D}}$ is a *strong* upper gradient, cf. [3, Definition 1.2.1 and Definition 1.2.2] for the definition of strong and weak upper gradients. This comes at the expense of the fact that the lower semicontinuity along the sequence $(\phi_k)_k$ (see (3.6)) has to be replaced by a continuity condition along $(\phi_k)_k$ for sequences $(\pi_k(z_k))_k$ converging with respect to the metric $\mathcal{D}$.

4. Properties of Energies and Dissipation Distances

In this section we prove basic properties of the energies and dissipation distances, and we establish properties for the local slope in the two-dimensional setting. Let $h > 0$ and $0 < \alpha < 1$. We recall the definition of the nonlinear energy and the dissipation distance in (2.14) and (2.15), respectively. We also recall (2.8) and the notation for the sublevel sets $\mathscr{S}_h^M = \{y \in \mathscr{S}_h : \phi_h(y) \leqq M\}$. In the whole section, $C \geqq 1$ and $0 < c \leqq 1$ indicate generic constants, which may vary from line to line and depend on M, S, the exponent $p > 3$ (see (2.4)), α, on the constants in (2.3), (2.4), (2.7), and on the boundary data $\hat{u}$ and $\hat{v}$. However, all constants are always independent of the small parameter h and the deformations y.

4.1. Basic properties

We start with some properties about the Hessian of W and D^2. By $\partial^2 D^2$ we denote the Hessian and by $\partial^2_{F_1^2} D^2$, $\partial^2_{F_2^2} D^2$ the Hessian in direction of the first or second entry of D^2, respectively. Moreover, we define $\mathrm{sym}(F) = \frac{1}{2}(F + F^\top)$ for $F \in \mathbb{R}^{d\times d}$, $d = 2, 3$. Recall the definitions of the quadratic forms in (2.17)–(2.18). By $\mathbf{Id} \subset \mathbb{R}^{3\times 3}$ we again denote the identity matrix.

Lemma 4.1. (Properties of Hessian) *(i)* $\partial^2_{F_1^2} D^2(Y, Y) = \partial^2_{F_2^2} D^2(Y, Y)$ *for all* $Y \in \mathbb{R}^{3\times 3}$ *in a neighborhood of* $SO(3)$ *such that* $\partial^2 D^2(Y, Y)$ *exists.*

(ii) There exists a constant $c > 0$ *such that* $Q^d_W(F) = Q^d_W(\mathrm{sym}(F)) \geqq c|\mathrm{sym}(F)|^2$ *and* $Q^d_D(F) = Q^d_D(\mathrm{sym}(F)) \geqq c|\mathrm{sym}(F)|^2$ *for all* $F \in \mathbb{R}^{d\times d}$, $d = 2, 3$.

(iii) There exists $C > 0$ *such that for all* $F_0, F_1 \in \mathbb{R}^{3\times 3}$ *in a neighborhood of* $SO(3)$ *there holds* $|W(F_1) - W(F_0) - \frac{1}{2}(Q^3_W(F_1 - \mathbf{Id}) - Q^3_W(F_0 - \mathbf{Id}))| \leqq \sum_{k=1}^3 C|F_0 - \mathbf{Id}|^{3-k}|F_1 - F_0|^k$.

Proof. For the proof of (i) and (ii) we refer to [22, Lemma 4.1]. To see (iii), we perform a Taylor expansion. First, we find

$$W(F_1) = W(F_0) + DW(F_0) : (F_1 - F_0)$$

$$+\tfrac{1}{2}D^2W(F_0)[F_1 - F_0, F_1 - F_0] + \mathrm{O}(|F_1 - F_0|^3).$$

We observe that $DW(F_0) = DW(\mathbf{Id}) + D^2W(\mathbf{Id})(F_0 - \mathbf{Id}) + \mathrm{O}(|F_0 - \mathbf{Id}|^2)$ and $DW(\mathbf{Id}) = 0$ by (2.3)(iii). Moreover, $|D^2W(F_0) - D^2W(\mathbf{Id})| \leqq C|F_0 - \mathbf{Id}|$ by the regularity of W. Thus, we get

$$\begin{aligned} W(F_1) &= W(F_0) + D^2W(\mathbf{Id})[F_0 - \mathbf{Id}, F_1 - F_0] + \tfrac{1}{2}D^2W(\mathbf{Id})[F_1 - F_0, F_1 - F_0] \\ &\quad + \mathrm{O}(|F_0 - \mathbf{Id}||F_1 - F_0|^2) + \mathrm{O}(|F_0 - \mathbf{Id}|^2|F_1 - F_0|) + \mathrm{O}(|F_1 - F_0|^3). \quad (4.1) \end{aligned}$$

By recalling that $Q_W^3(F) = D^2W(\mathbf{Id})[F, F]$ for $F \in \mathbb{R}^{3\times 3}$ and the fact that $D^2W(\mathbf{Id})[\cdot, \cdot]$ is symmetric in the two entries, an elementary computation yields

$$\begin{aligned} Q_W^3(F_1 - \mathbf{Id}) - Q_W^3(F_0 - \mathbf{Id}) &= 2D^2W(\mathbf{Id}) \\ [F_0 - \mathbf{Id}, F_1 - F_0] &+ D^2W(\mathbf{Id})[F_1 - F_0, F_1 - F_0]. \quad (4.2) \end{aligned}$$

The result follows by combination of (4.1) and (4.2). □

The following geometric rigidity result will be a key ingredient for our analysis:

Lemma 4.2. (Rigidity in thin domains) *For h sufficiently small, for all $y \in \mathscr{S}_h^M$ there exists a mapping $R(y) \in W^{1,2}(S; SO(3))$ satisfying*

$$\begin{aligned} (i)&\quad \|\nabla_h y - R(y)\|^2_{L^2(\Omega)} \leqq Ch^4, \\ (ii)&\quad \|\nabla_h y - \mathbf{Id}\|^2_{L^2(\Omega)} \leqq Ch^2, \\ (iii)&\quad \|\nabla' R(y)\|^2_{L^2(S)} \leqq Ch^2 \\ (iv)&\quad \|R(y) - \mathbf{Id}\|_{L^q(S)} \leqq C_q h, \\ (v)&\quad \|\nabla_h y - \mathbf{Id}\|_{L^\infty(\Omega)} \leqq Ch^\alpha, \\ (vi)&\quad \|R(y) - \mathbf{Id}\|_{L^\infty(S)} \leqq Ch^\alpha, \end{aligned} \quad (4.3)$$

where C_q depends also on $q \in [1, \infty)$. (In (i), $R(y)$ is extended to Ω by $R(y)(x', x_3) = R(y)(x')$.)

Proof. Property (i) is based on geometric rigidity [23] and is proved in [24, Theorem 6, Remark 5], where we use that $\|\operatorname{dist}(\nabla_h y, SO(3))\|^2_{L^2(\Omega)} \leqq CMh^4$ by (2.3)(iii), (2.14), and the fact that $y \in \mathscr{S}_h^M$. Also (ii)-(iv) are proved there with a rotation $\bar{Q}$ in place of $\mathbf{Id}$. The fact that we may choose $\bar{Q} = \mathbf{Id}$ is due to the boundary conditions, which break the rotational invariance, see [28, Lemma 13].

We now show (v). By the definition of ϕ_h and (2.4)(iii) we get $\|\nabla_h^2 y\|_{L^p(\Omega)} \leqq Ch^\alpha$ for all $y \in \mathscr{S}_h^M$, where the constant depends on M. In particular, by (2.1)–(2.2) this implies

$$\|\nabla y_{,3}\|_{L^p(\Omega)} \leqq \|\nabla_h y_{,3}\|_{L^p(\Omega)} = h\|\nabla_h h^{-1} y_{,3}\|_{L^p(\Omega)} \leqq h\|\nabla_h^2 y\|_{L^p(\Omega)} \leqq Ch^{1+\alpha}.$$

As $p > 3$, Poincaré's inequality yields some $F \in \mathbb{R}^{3\times 3}$ such that

$$\|\nabla' y - (Fe_1, Fe_2)\|_{L^\infty(\Omega)} \leqq C\|\nabla^2 y\|_{L^p(\Omega)} \leqq C\|\nabla_h^2 y\|_{L^p(\Omega)} \leqq Ch^\alpha,$$

$$\|y_{,3} - hFe_3\|_{L^\infty(\Omega)} \leqq C\|\nabla y_{,3}\|_{L^p(\Omega)} \leqq Ch^{1+\alpha}$$

for a constant additionally depending on Ω and p. This implies $\|\nabla_h y - F\|_{L^\infty(\Omega)} \leqq Ch^\alpha$. Along with (ii), the triangle inequality, and $\alpha < 1$ we obtain $|F - \mathbf{Id}| \leqq Ch^\alpha$. This concludes the proof of (v).

Finally, we show (vi). A careful inspection of the proof of [24, Theorem 6] shows that $R(y)(x')$ may be defined as the nearest-point projection onto $SO(3)$ of

$$\int_I \int_{x'+h(-1,1)^2} \frac{1}{h^2}\psi\Big(\frac{x'-z'}{h}\Big)\, \nabla_h y(z', z_3)\, \mathrm{d}z'\, \mathrm{d}z_3,$$

where $I = (-1/2, 1/2)$ and $\psi \in C_c^\infty((-1,1)^2)$ denotes a standard mollifier, that is, $\psi \geqq 0$ and $\int_{(-1,1)^2} \psi = 1$. Then (vi) follows from (v). □

By Lemma 4.2 we get that $\nabla_h y$ is approximated by the $SO(3)$-valued function $R(y)$. As the energy is invariant under rotations, the energy of y is essentially controlled by the distance of $R(y)^\top \nabla_h y$ from $\mathbf{Id}$. To this end, we introduce the quantity

$$G^h(y) := \frac{R(y)^\top \nabla_h y - \mathbf{Id}}{h^2}. \tag{4.4}$$

In what follows we set for shorthand $H_Y := \frac{1}{2}\partial^2_{F_1^2} D^2(Y, Y) = \frac{1}{2}\partial^2_{F_2^2} D^2(Y, Y)$ for $Y \in \mathbb{R}^{3\times 3}$ in a neighborhood of $SO(3)$. Given a deformation $y \in \mathscr{S}_h^M$, we also introduce the mapping $H_{\nabla_h y} : \Omega \to \mathbb{R}^{3\times3\times3\times3}$ by $H_{\nabla_h y}(x) = H_{\nabla_h y(x)}$ for $x \in \Omega$. Note that this is well defined for h sufficiently small by (4.3)(v). Recall the definition of $\mathcal{D}_h$ and $\mathcal{D}_0$ in (2.15) and (2.24), respectively, and the definition of Q_W^3, Q_D^3 in (2.17)–(2.18).

Lemma 4.3. *(Dissipation and energy) Let h sufficiently small. Then, for all $y, y_0, y_1 \in \mathscr{S}_h^M$ and all open subsets $U \subset \Omega$ we have*

(i) $\Big|\int_U D^2(\nabla_h y_0, \nabla_h y_1) - \int_U Q_D^3(\nabla_h y_1 - \nabla_h y_0)\Big| \leqq Ch^\alpha \|\nabla_h y_1 - \nabla_h y_0\|^2_{L^2(U)}$,

(ii) $\Big|\mathcal{D}_h(y_0, y_1)^2 - \int_\Omega Q_D^3\big(G^h(y_0) - G^h(y_1)\big)\Big| \leqq Ch^\alpha \|G^h(y_0) - G^h(y_1)\|^2_{L^2(\Omega)} \leqq Ch^\alpha$,

(iii) $|\Delta(y)| \leqq Ch^\alpha$, *where* $\Delta(y) := \frac{1}{h^4}\int_\Omega W(\nabla_h y) - \int_\Omega \frac{1}{2} Q_W^3(G^h(y))$,

(iv) $|\Delta(y_0) - \Delta(y_1)| \leqq Ch^\alpha \|G^h(y_0) - G^h(y_1)\|_{L^2(\Omega)} \leqq Ch^\alpha$.

Proof. As a preparation, we observe that by the uniform bound on $\nabla_h y_0$, $\nabla_h y_1$ (see (4.3)(v)) and a Taylor expansion at $(\nabla_h y_0, \nabla_h y_0)$ we obtain for all open subsets $U \subset \Omega$

$$\Big|\int_U D^2(\nabla_h y_0, \nabla_h y_1) - \int_U H_{\nabla_h y_0}[\nabla_h(y_1 - y_0), \nabla_h(y_1 - y_0)]\Big| \leqq C\|\nabla_h(y_1 - y_0)\|^3_{L^3(U)}. \tag{4.5}$$

We recall (4.4) and define $G(y_i) := h^2 G^h(y_i) = R(y_i)^\top \nabla_h y_i - \mathbf{Id}$, $i = 0, 1$, for convenience. Using the separate frame indifference (2.7)(v) we have

$$\int_\Omega D^2(\nabla_h y_0, \nabla_h y_1) = \int_\Omega D^2\big(R(y_0)^\top \nabla_h y_0, R(y_1)^\top \nabla_h y_1\big).$$

Thus, by $h^4 \mathcal{D}_h(y_0, y_1)^2 = \int_\Omega D^2(\nabla_h y_0, \nabla_h y_1)$ and again by Taylor expansion we also get

$$\Big| h^4 \mathcal{D}_h(y_0, y_1)^2 - \int_\Omega H_{R(y_0)^\top \nabla_h y_0} [G(y_1) - G(y_0), G(y_1) - G(y_0)] \Big| \leqq C \|G(y_1) - G(y_0)\|^3_{L^3(\Omega)}. \tag{4.6}$$

We now show (i). By the regularity of D and (4.3)(v) we get $\|H_{\nabla_h y_0} - \mathbb{C}^3_D\|_\infty \leqq Ch^\alpha$, where the fourth order tensor $\mathbb{C}^3_D$ associated to Q^3_D is defined in (2.19). Therefore, we obtain

$$\Big| \int_U H_{\nabla_h y_0}[\nabla_h (y_1 - y_0), \nabla_h (y_1 - y_0)] - \int_U Q^3_D(\nabla_h y_1 - \nabla_h y_0) \Big| \leqq Ch^\alpha \|\nabla_h y_1 - \nabla_h y_0\|^2_{L^2(U)}$$

for all open $U \subset \Omega$. By using (4.5) and again (4.3)(v) we get (i).

To see (ii), we observe $\|H_{R(y_0)^\top \nabla_h y_0} - \mathbb{C}^3_D\|_\infty \leqq C\|\nabla_h y_0 - R(y_0)\|_\infty \leqq Ch^\alpha$ by the regularity of D and (4.3)(v),(vi). Thus, we get

$$\Big| \int_\Omega H_{R(y_0)^\top \nabla_h y_0}[G(y_1) - G(y_0), G(y_1) - G(y_0)] - \int_\Omega Q^3_D\big(G(y_1) - G(y_0)\big) \Big| \leqq Ch^\alpha \|G(y_1) - G(y_0)\|^2_{L^2(\Omega)}. \tag{4.7}$$

In a similar fashion, (4.3)(v),(vi) also imply $\|G(y_i)\|_\infty \leqq Ch^\alpha$ for $i = 0, 1$ and thus

$$\|G(y_1) - G(y_0)\|^3_{L^3(\Omega)} \leqq Ch^\alpha \|G(y_1) - G(y_0)\|^2_{L^2(\Omega)}.$$

This together with (4.6)–(4.7) (divided by h^4), and $G^h(y_i) = h^{-2} G(y_i)$ for $i = 0, 1$ yields

$$\Big| \mathcal{D}_h(y_0, y_1)^2 - \int_\Omega Q^3_D\big(G^h(y_0) - G^h(y_1)\big) \Big| \leqq Ch^\alpha \|G^h(y_0) - G^h(y_1)\|^2_{L^2(\Omega)}.$$

This shows the first inequality of (ii). To see the second inequality, we use (4.3)(i) and (4.4).

We now show (iii) and (iv). We use the frame indifference of W and Lemma 4.1 (iii) (with $F_i = R(y_i)^\top \nabla_h y_i = \mathbf{Id} + G(y_i)$ for $i = 0, 1$) to obtain

$$|\Delta(y_1) - \Delta(y_0)| \leqq Ch^{-4} \sum\nolimits_{k=1}^{3} \int_\Omega |R(y_0)^\top \nabla_h y_0 - \mathbf{Id}|^{3-k} |R(y_1)^\top \nabla_h y_1 - R(y_0)^\top \nabla_h y_0|^k$$

$$= Ch^{-4} \sum_{k=1}^{3} \int_{\Omega} |G(y_0)|^{3-k} |G(y_1) - G(y_0)|^k$$
$$\leqq Ch^{-4} \int_{\Omega} (|G(y_1)| + |G(y_0)|)^2 |G(y_1) - G(y_0)|,$$

where $\Delta(y_0)$ and $\Delta(y_1)$ are defined in the statement of the lemma. The fact that $\|G(y_i)\|_\infty \leqq Ch^\alpha$ for $i = 0, 1$ (see (4.3)(v),(vi)) and Hölder's inequality yield

$$|\Delta(y_1)-\Delta(y_0)| \leqq Ch^{\alpha-4}\big(\|G(y_0)\|_{L^2(\Omega)}+\|G(y_1)\|_{L^2(\Omega)}\big)\, \|G(y_1)-G(y_0)\|_{L^2(\Omega)}.$$

Using $G^h(y_i) = h^{-2}G(y_i)$ for $i = 0, 1$ and (4.3)(i) we obtain the first inequality of (iv). The second inequality follows again by (4.3)(i). Finally, to see (iii), we apply (iv) for $y_0 = y$ and $y_1 = \mathbf{id}$, where we use $\Delta(y_1) = 0$. □

4.2. Metric spaces and their properties

In this section we prove that $(\mathscr{S}_h^M, \mathcal{D}_h)$ and $(\mathscr{S}_0, \mathcal{D}_0)$ are complete metric spaces. We start with the three-dimensional setting. Recall the definition of ϕ_h and $\mathcal{D}_h$ in (2.14)–(2.15). As a preparation, we address the positivity of $\mathcal{D}_h$.

Lemma 4.4. (Positivity of $\mathcal{D}_h$) *Let $M > 0$ and let h sufficiently small. Let $y_0, y_1 \in \mathscr{S}_h^M$ with $\mathcal{D}_h(y_0, y_1) = 0$. Then $y_0 = y_1$.*

Proof. It is convenient to formulate the problem for the original (not rescaled) functions w_0 and w_1 defined on $\Omega_h = S \times (-\frac{h}{2}, \frac{h}{2})$. To explain the main idea, we first assume that Ω_h is the union of pairwise disjoint cubes of sidelength h up to a set of negligible measure. Denote the family of cubes by $\mathcal{Q}$. By $\mathcal{Q}_1 \subset \mathcal{Q}$ we denote the cubes whose boundaries share at least one face with $\partial S \times (-\frac{h}{2}, \frac{h}{2})$. Let $\mathcal{Q}_2 \subset \mathcal{Q} \setminus \mathcal{Q}_1$ be the cubes whose boundaries share at least one face with a cube in $\mathcal{Q}_1$. In a similar fashion, we define $\mathcal{Q}_i$, $i \geqq 2$, and find $\mathcal{Q} = \bigcup_{i=1}^I \mathcal{Q}_i$ for some $I \in \mathbb{N}$.

We now first show that $w_0 = w_1$ on each $Q \in \mathcal{Q}_1$. To this end, fix $Q \in \mathcal{Q}_1$. From Lemma 4.3(i) (in terms of w_0, w_1 instead of y_0, y_1) we get

$$\Big| \int_Q D^2(\nabla w_0, \nabla w_1) - \int_Q Q_D^3(\nabla w_1 - \nabla w_0)\Big| \leqq Ch^\alpha \|\nabla w_1 - \nabla w_0\|^2_{L^2(Q)}. \tag{4.8}$$

Since $w_1 = w_0$ on $\partial S \times (-\frac{h}{2}, \frac{h}{2})$, we get that $w_1 = w_0$ on at least one face of ∂Q. Then Korn's inequality (see, for example, [24, Proposition 1]) implies

$$\|\nabla w_1 - \nabla w_0\|^2_{L^2(Q)} \leqq C\|\mathrm{sym}(\nabla w_1 - \nabla w_0)\|^2_{L^2(Q)}.$$

Together with Lemma 4.1(ii) this shows

$$\|\nabla w_1 - \nabla w_0\|^2_{L^2(Q)} \leqq C \int_Q Q_D^3(\nabla w_1 - \nabla w_0). \tag{4.9}$$

For h sufficiently small, (4.8)–(4.9) along with $\int_Q D^2(\nabla w_0, \nabla w_1) = 0$ show $\nabla w_1 = \nabla w_0$ almost everywhere on Q. Since $w_1 = w_0$ on at least one face of ∂Q, this also gives $w_1 = w_0$ almost everywhere on Q, as desired.

We now proceed iteratively to show that $w_1 = w_0$ on each $Q \in \mathcal{Q}$: suppose that the property has already been shown for all $Q \in \bigcup_{i=1}^{j} \mathcal{Q}_i$. Then $w_1 = w_0$ on each $Q \in \mathcal{Q}_{j+1}$ follows from the above arguments noting that $w_1 = w_0$ on at least one face of ∂Q since $w_1 = w_0$ on all squares $Q \in \bigcup_{i=1}^{j} \mathcal{Q}_i$.

This shows $w_1 = w_0$ on Ω_h in the case that Ω_h is the union of pairwise disjoint cubes of sidelength h up to a set of negligible measure. In the general case, we may cover Ω_h by cubes of sidelength h, again denoted by $\mathcal{Q}$, such that for each $Q \in \mathcal{Q}$ the set $Q \cap \Omega_h$ is bilipschitzly equivalent to a cube of sidelength h with a controlled Lipschitz constant. We note that the constant in Korn's inequality can be chosen uniformly for these sets. We may again decompose $\mathcal{Q}$ into pairwise disjoint families $(\mathcal{Q}_i)_i$ and show iteratively that w_1 coincides with w_0 on all cubes. □

Lemma 4.5. (Properties of $(\mathscr{S}_h^M, \mathcal{D}_h)$ and ϕ_h) *Let $M > 0$. For $h > 0$ sufficiently small we have*

(i) $(\mathscr{S}_h^M, \mathcal{D}_h)$ is a complete metric space.
(ii) Compactness: If $(y_n)_n \subset \mathscr{S}_h^M$, then $(y_n)_n$ admits a subsequence converging weakly in $W^{2,p}(\Omega; \mathbb{R}^3)$ and strongly in $W^{1,\infty}(\Omega; \mathbb{R}^3)$.
(iii) Topologies: The topology induced by $\mathcal{D}_h$ coincides with the weak $W^{2,p}(\Omega; \mathbb{R}^3)$ topology.
(iv) Lower semicontinuity: $\mathcal{D}_h(y_n, y) \to 0 \;\Rightarrow\; \liminf_{n\to\infty} \phi_h(y_n) \geqq \phi_h(y)$.

Proof. We start with (ii). We recall (2.4)(iii) and (2.14), and find $\|\nabla_h^2 y\|^p_{L^p(\Omega)} \leqq CMh^{p\alpha}$ for all $y \in \mathscr{S}_h^M$. This together with (4.3)(v) and the boundary conditions (2.8) shows $\sup_{y \in \mathscr{S}_h^M} \|y\|_{W^{2,p}(\Omega)} < \infty$. Since $p > 3$, (ii) follows from a standard compactness argument.

We now show (iii). (a) We first suppose that $y_n \rightharpoonup y$ weakly in $W^{2,p}(\Omega; \mathbb{R}^3)$. As $p > 3$, this implies $y_n \to y$ strongly in $W^{1,\infty}(\Omega; \mathbb{R}^3)$ and thus $\mathcal{D}_h(y_n, y) \to 0$ by dominated convergence. (b) On the other hand, assume that $\mathcal{D}_h(y_n, y) \to 0$. Item (ii) yields that a subsequence (not relabeled) of $(y_n)_n$ converges weakly in $W^{2,p}(\Omega; \mathbb{R}^3)$ to some $\tilde{y}$. By (a) we find $\mathcal{D}_h(y_n, \tilde{y}) \to 0$. The triangle inequality, see (2.7)(iii), then shows $\mathcal{D}_h(y, \tilde{y}) \leqq \lim_{n\to\infty}(\mathcal{D}_h(y, y_n) + \mathcal{D}_h(y_n, \tilde{y})) = 0$. This yields $y = \tilde{y}$ by Lemma 4.4, and therefore $y_n \rightharpoonup y$ weakly in $W^{2,p}(\Omega; \mathbb{R}^3)$. As the limit is independent of the subsequence, the convergence actually holds for the whole sequence.

The proof of (iv) follows again by the dominated convergence theorem and the convexity of P, see (2.4)(ii).

Finally, we show (i). Apart from the positivity, all properties of a metric follow directly from (2.7) and (2.15). The positivity has has been addressed in Lemma 4.4. It therefore remains to show that $(\mathscr{S}_h^M, \mathcal{D}_h)$ is complete. Let $(y_k)_k \subset \mathscr{S}_h^M$ be a Cauchy sequence with respect to $\mathcal{D}_h$. By (ii) and (iii) we find $y \in W^{2,p}(\Omega; \mathbb{R}^3)$ and a subsequence (not relabeled) such that $\lim_{k\to\infty} \mathcal{D}_h(y_k, y) = 0$. By (iv) and the trace theorem we get $y \in \mathscr{S}_h^M$. The fact that $(y_k)_k$ is a Cauchy sequence now implies that the whole sequence y_k converges to y with respect to $\mathcal{D}_h$. This concludes the proof. □

Similar properties can be derived in the two-dimensional setting. Recall the definition of ϕ_0 and $\mathcal{D}_0$ in (2.23) and (2.24), respectively. For convenience, we will use the notations

$$e(u) = \mathrm{sym}(\nabla' u), \qquad B(v_1, v_2) = \frac{1}{2}\mathrm{sym}(\nabla' v_1 \otimes \nabla' v_2). \tag{4.10}$$

Lemma 4.6. (Properties of $(\mathscr{S}_0, \mathcal{D}_0)$ and ϕ_0) *We have:*

(i) $(\mathscr{S}_0, \mathcal{D}_0)$ *is a complete metric space.*
(ii) Compactness: If $(u_n, v_n)_n \subset \mathscr{S}_0$ *is a sequence with* $\sup_n \phi_0(u_n, v_n) < +\infty$, *then* $(u_n, v_n)_n$ *is bounded in* $W^{1,2}(S; \mathbb{R}^2) \times W^{2,2}(S)$.
(iii) Topologies: The metrics $\mathcal{D}_0$ *and* $d((u, v), (u', v')) := \|u - u'\|_{W^{1,2}(S)} + \|v - v'\|_{W^{2,2}(S)}$ *are equivalent. In particular,* $\|v - v'\|_{W^{2,2}(S)} \leqq C\mathcal{D}_0((u, v), (u', v'))$ *for* $C = C(S) > 0$.
(iv) Continuity: $\mathcal{D}_0((u_n, v_n), (u, v)) \to 0 \;\Rightarrow\; \lim_{n\to\infty} \phi_0(u_n, v_n) = \phi_0(u, v)$.

Proof. We first prove (ii). By Lemma 4.1(ii) and the triangle inequality we find

$$\|e(u)\|^2_{L^2(S)} \leqq C \int_S Q^2_W\big(e(u) + B(v, v)\big) + C\|B(v, v)\|^2_{L^2(S)}$$

for a universal constant $C > 0$. This, together with Korn's and Poincaré's inequality, and the fact that $u = \hat{u}$ on ∂S, gives

$$\begin{aligned}\|u\|^2_{W^{1,2}(S)} &\leqq C\|u - \hat{u}\|^2_{W^{1,2}(S)} + C\|\hat{u}\|^2_{W^{1,2}(S)} \leqq C\|e(u - \hat{u})\|^2_{L^2(S)} + C\|\hat{u}\|^2_{W^{1,2}(S)} \\ &\leqq C\|\hat{u}\|^2_{W^{1,2}(S)} + C\|B(v, v)\|^2_{L^2(S)} + C\int_S Q^2_W\big(e(u) + B(v, v)\big). \end{aligned}\tag{4.11}$$

Now consider $(u, v) \in \mathscr{S}_0$ with $\phi_0(u, v) \leqq M$ for $M > 0$. Then by Lemma 4.1(ii) and (2.23) we find $\|(\nabla')^2 v\|_{L^2(S)} \leqq C$, where the constant depends on M. Since $v = \hat{v}$ and $\nabla' v = \nabla' \hat{v}$ on ∂S, by an argumentation similar to (4.11) we also get

$$\|v\|_{W^{2,2}(S)} \leqq \|v - \hat{v}\|_{W^{2,2}(S)} + \|\hat{v}\|_{W^{2,2}(S)} \leqq C\|(\nabla')^2(v - \hat{v})\|_{L^2(S)} + \|\hat{v}\|_{W^{2,2}(S)} \leqq C, \tag{4.12}$$

where the constant additionally depends on $\|\hat{v}\|_{W^{2,2}(S)}$. Then $\phi_0(u, v) \leqq M$ together with (4.11) and the embedding $W^{2,2} \hookrightarrow W^{1,4}$ (in dimension two) yields $\|u\|_{W^{1,2}(S)} \leqq C$, where the constant also depends on $\|\hat{u}\|_{W^{1,2}(S)}$. This shows property (ii).

We now address (iii). As a preparation, consider $(u_0, v_0), (u_1, v_1) \in \mathscr{S}_0$. Arguing similarly as before, using Lemma 4.1(ii) and Poincaré's inequality, we find that

$$\|v_0 - v_1\|^2_{W^{2,2}(S)} \leqq C \int_S Q^2_D\big((\nabla')^2 v_0 - (\nabla')^2 v_1\big). \tag{4.13}$$

Here, we used that v_0 and v_1 as well as their first derivatives coincide on ∂S. Repeating the argumentation leading to (4.11) we find that

$$\begin{aligned}\|u_0 - u_1\|^2_{W^{1,2}(S)} &\leqq C\|e(u_0) - e(u_1)\|^2_{L^2(S)} \\ &\leqq C \int_S \big(|B(v_0, v_0) - B(v_1, v_1)|^2 + Q_D^2\big(e(u_0 - u_1) \\ &\quad + B(v_0, v_0) - B(v_1, v_1)\big)\big). \end{aligned} \tag{4.14}$$

Here, in the first step, we used Korn's and Poincaré's inequality. In the second step, we applied Lemma 4.1(ii) and the triangle inequality.

We now suppose that $\mathcal{D}_0((u_n, v_n), (u, v)) \to 0$ (see (2.24)). Then $v_n \to v$ strongly in $W^{2,2}(S)$ by (4.13). This also implies $v_n \to v$ in $W^{1,4}(S)$ and thus, in view of (4.14), we find that $u_n \to u$ strongly in $W^{1,2}(S; \mathbb{R}^2)$. Therefore, the sequence converges with respect to the metric d defined in the statement of (iii). Conversely, given a sequence $(u_n, v_n)_n$ with $u_n \to u$ strongly in $W^{1,2}(S; \mathbb{R}^2)$ and $v_n \to v$ in $W^{2,2}(S)$, we also get $v_n \to v$ in $W^{1,4}(S)$. We observe that $\mathcal{D}_0$ is continuous with respect to this convergence, that is, $\mathcal{D}_0((u_n, v_n), (u, v)) \to 0$. This yields the equivalence of the metrics, and (4.13) shows the second part of (iii).

The proof of (iv) is similar, noting also that ϕ_0 is continuous with respect to the topology induced by $\mathcal{D}_0$, see (2.23).

We finally prove (i). The positivity and the completeness follow from (iii). Thus, the only thing left to show is the triangle inequality. This can be derived by an elementary computation, using that $\mathcal{D}_0^2$ is the sum of two quadratic forms. We omit the details. □

Remark 4.7. For later purposes, we remark that the proof of (ii) can be generalized: a similar argument shows that for given $(\tilde{u}, \tilde{v}) \in \mathscr{S}_0$ and $v \in W^{2,2}(S)$ we get

$$\begin{aligned}&\int_S |e(\tilde{u}) + \mathrm{sym}(\nabla'\tilde{v} \otimes \nabla' v)|^2 + \int_S |(\nabla')^2\tilde{v}|^2 + \|v\|_{W^{2,2}(S)} \leqq M \\ &\quad \Rightarrow \quad \|\tilde{u}\|_{W^{1,2}(S)} + \|\tilde{v}\|_{W^{2,2}(S)} \leqq C,\end{aligned}$$

where C depends on M, $\hat{u}$ and $\hat{v}$. This follows by repeating the argument in (4.11)–(4.12) and using $\|B(\tilde{v}, v)\|_{L^2(S)} \leqq C\|\tilde{v}\|_{W^{1,4}(S)}\|v\|_{W^{1,4}(S)} \leqq C\|\tilde{v}\|_{W^{2,2}(S)}\|v\|_{W^{2,2}(S)}$.

4.3. Generalized geodesics and properties of slopes in the two-dimensional setting

In this section, we first derive convexity properties for the energy and the dissipation distance in the two-dimensional setting along generalized geodesics. (We refer to [33, Section 3.2, Section 3.4] for a discussion about generalized geodesics in a related problem.) Afterwards, we derive fundamental properties for the local slope which will be instrumental to use the theory in [3]. For $M > 0$, we define the sublevel sets $\mathscr{S}_0^M = \{(u, v) \in \mathscr{S}_0 : \ \phi_0(u, v) \leqq M\}$.

Lemma 4.8. (Convexity and generalized geodesics in the two-dimensional setting) *Let* $M > 0$. *Then there exist smooth functions* $\Phi^1, \Phi^2_M : [0,\infty) \to [0,\infty)$ *satisfying* $\lim_{t\to 0}\Phi^1(t)/t = 1$ *and* $\lim_{t\to 0}\Phi^2_M(t)/t = 0$ *such that for all* $(u_0, v_0) \in \mathscr{S}^M_0$ *and all* $(u_1, v_1) \in \mathscr{S}_0$ *we have*

(*i*) $\mathcal{D}_0\big((u_0, v_0), (u_s, v_s)\big) \leqq s\Phi^1\big(\mathcal{D}_0\big((u_0, v_0), (u_1, v_1)\big)\big)$,

(*ii*) $\phi_0(u_s, v_s) \leqq (1-s)\phi_0(u_0, v_0) + s\phi_0(u_1, v_1) + s\Phi^2_M\big(\mathcal{D}_0\big((u_0, v_0), (u_1, v_1)\big)\big)$,

where $u_s := (1-s)u_0 + su_1$ *and* $v_s := (1-s)v_0 + sv_1$, $s \in [0, 1]$.

Proof. For convenience, we first introduce some abbreviations and provide some preliminary estimates. We let $D = \mathcal{D}_0((u_0, v_0), (u_1, v_1))$. We use the notation defined in (4.10) and also set $B_{\mathrm{diff}} = B(v_0 - v_1, v_0 - v_1)$. Particularly, by Lemma 4.6(iii) and a Sobolev embedding we observe that

$$\|B_{\mathrm{diff}}\|_{L^2(S)} \leqq C\|\nabla' v_0 - \nabla' v_1\|^2_{L^4(S)} \leqq C\|v_0 - v_1\|^2_{W^{2,2}(S)} \leqq CD^2. \tag{4.15}$$

For brevity, we also introduce

$$G^s_0 = e(u_s) + B(v_s, v_s) \tag{4.16}$$

for $s \in [0, 1]$. (The notation G_0 is borrowed from [24], see also Lemma 5.3 below.) With the definition of $\mathcal{D}_0$ in (2.24) and Lemma 4.1(ii) we get

$$\|G^1_0 - G^0_0\|^2_{L^2(S)} \leqq C\int_S Q^2_D(G^1_0 - G^0_0) \leqq CD^2. \tag{4.17}$$

Similarly, we observe by (2.23) (with $f \equiv 0$), Lemma 4.1(ii), and the fact that $(u_0, v_0) \in \mathscr{S}^M_0$, that

$$\|G^0_0\|^2_{L^2(S)} \leqq C\int_S Q^2_W(G^0_0) \leqq C\phi_0(u_0, v_0) \leqq CM. \tag{4.18}$$

We now start with the proof of (i). First, we observe

$$\frac{1}{12}\int_S Q^2_D((\nabla')^2 v_s - (\nabla')^2 v_0) = s^2\frac{1}{12}\int_S Q^2_D((\nabla')^2 v_1 - (\nabla')^2 v_0).$$

We will show that there exists $C > 0$ independent of s such that

$$\int_S Q^2_D(G^s_0 - G^0_0) \leqq s^2\int_S Q^2_D(G^1_0 - G^0_0) + Cs^2D^3 + Cs^2D^4 \tag{4.19}$$

for $s \in [0, 1]$. Then recalling the definition of $\mathcal{D}_0$ in (2.24), (i) follows for the function $\Phi^1(t) = \sqrt{t^2 + Ct^3 + Ct^4}$. To show (4.19), recalling (4.10), we obtain by an elementary expansion

$$\begin{aligned} B(v_s, v_s) - B(v_0, v_0) &= s\big(B(v_1, v_1) - B(v_0, v_0)\big) - s(1-s) \\ &\quad \big(B(v_0, v_0) + B(v_1, v_1) - 2B(v_0, v_1)\big) \\ &= s(B(v_1, v_1) - B(v_0, v_0)) - s(1-s)B(v_0 - v_1, v_0 - v_1) \\ &= s(B(v_1, v_1) - B(v_0, v_0)) - s(1-s)B_{\mathrm{diff}}, \end{aligned}$$

where the last equality follows from the definition of B_{diff}. By recalling (4.16) this implies

$$G_0^s - G_0^0 = s\big(G_0^1 - G_0^0 - (1-s)B_{\mathrm{diff}}\big). \tag{4.20}$$

Recalling also $\mathbb{C}_D^2$ in (2.19), an expansion and the Cauchy-Schwartz inequality then yield

$$\begin{aligned}\int_S Q_D^2(G_0^s - G_0^0) &= s^2\int_S Q_D^2(G_0^1 - G_0^0) + s^2(1-s)^2\int_S Q_D^2(B_{\mathrm{diff}})\\ &\quad - 2s^2(1-s)\int_S \mathbb{C}_D^2[G_0^1 - G_0^0, B_{\mathrm{diff}}]\\ &\leqq s^2\int_S Q_D^2(G_0^1 - G_0^0) + Cs^2\|B_{\mathrm{diff}}\|^2_{L^2(S)}\\ &\quad + Cs^2\|G_0^1 - G_0^0\|_{L^2(S)}\|B_{\mathrm{diff}}\|_{L^2(S)}.\end{aligned}$$

By using (4.15) and (4.17) we now see that (4.19) holds. This concludes the proof of (i).

Recall the definition of ϕ_0 in (2.23). We show (ii) for the function $\Phi_M^2(t) = C\sqrt{M}t^2 + Ct^3 + Ct^4$ for some $C > 0$. Due to convexity of $s \mapsto \int_S \frac{1}{24}Q_W^3((\nabla')^2 v_s)$, it suffices to show

$$\begin{aligned}\frac{1}{2}\int_S Q_W^2(G_0^s)\,\mathrm{d}x' &\leqq \frac{1-s}{2}\int_S Q_W^2(G_0^0)\\ &\quad + \frac{s}{2}\int_S Q_W^2(G_0^1) + C\sqrt{M}sD^2 + CsD^3 + Cs^2D^4.\end{aligned} \tag{4.21}$$

In view of (2.19) and (4.20), an elementary expansion yields

$$\begin{aligned}&Q_W^2(G_0^s)\\ &= Q_W^2(G_0^0 + G_0^s - G_0^0) = Q_W^2\big((1-s)G_0^0 + sG_0^1 - s(1-s)B_{\mathrm{diff}}\big)\\ &= (1-s)Q_W^2(G_0^0) + sQ_W^2(G_0^1) - (1-s)sQ_W^2(G_0^0 - G_0^1)\\ &\quad - 2s(1-s)^2\mathbb{C}_W^2[G_0^0, B_{\mathrm{diff}}] - 2s^2(1-s)\mathbb{C}_W^2[G_0^1, B_{\mathrm{diff}}]\\ &+ s^2(1-s)^2Q_W^2(B_{\mathrm{diff}}).\end{aligned}$$

Then taking the integral we obtain

$$\begin{aligned}\int_S Q_W^2(G_0^s) &\leqq (1-s)\int_S Q_W^2(G_0^0) + s\int_S Q_W^2(G_0^1)\\ &\quad + Cs\big(\|G_0^0\|_{L^2(S)} + \|G_0^1\|_{L^2(S)}\big)\|B_{\mathrm{diff}}\|_{L^2(S)} + Cs^2\|B_{\mathrm{diff}}\|^2_{L^2(S)}.\end{aligned}$$

By using $\|G_0^1\|_{L^2(S)} \leqq \|G_0^0\|_{L^2(S)} + \|G_0^1 - G_0^0\|_{L^2(S)}$, (4.15), and (4.17)–(4.18) we get (4.21). This concludes the proof of (ii). □

Now we derive representations and properties of the local slope corresponding to ϕ_0. Recall the notation $\mathscr{S}_0^M = \{(u, v) \in \mathscr{S}_0 : \phi_0(u, v) \leqq M\}$.

Lemma 4.9. (Local slope in the two-dimensional setting) *Let $M > 0$. The local slope for the energy ϕ_0 admits the representation*

$$|\partial\phi_0|_{\mathcal{D}_0}(u, v) = \sup_{\substack{(u',v')\in\mathscr{S}_0 \\ (u',v')\neq(u,v)}} \frac{\Big(\phi_0(u, v) - \phi_0(u', v') - \Phi^2_M\big(\mathcal{D}_0((u, v), (u', v'))\big)\Big)^+}{\Phi^1\big(\mathcal{D}_0((u, v), (u', v'))\big)}$$

for all $(u, v) \in \mathscr{S}^M_0$, where Φ^1 and Φ^2_M are the functions from Lemma 4.8. The slope is a strong upper gradient for ϕ_0.

Proof. We follow the lines of the proofs of Theorem 2.4.9 and Corollary 2.4.10 in [3]. Let $M > 0$ and $(u, v) \in \mathscr{S}^M_0$. Let Φ^1, Φ^2_M be the functions from Lemma 4.8, and recall that $\lim_{t\to 0} \Phi^1(t)/t = 1$ and $\lim_{t\to 0} \Phi^2_M(t)/t = 0$. We recall the definition of the local slope in Definition 3.1 and obtain

$$\begin{aligned}|\partial\phi_0|_{\mathcal{D}_0}(u, v) &= \limsup_{(u',v')\to(u,v)} \frac{(\phi_0(u, v) - \phi_0(u', v'))^+}{\mathcal{D}_0((u, v), (u', v'))} \\ &= \limsup_{(u',v')\to(u,v)} \frac{\Big(\phi_0(u, v) - \phi_0(u', v') - \Phi^2_M\big(\mathcal{D}_0((u, v), (u', v'))\big)\Big)^+}{\Phi^1\big(\mathcal{D}_0((u, v), (u', v'))\big)} \\ &\leqq \sup_{(u',v')\neq(u,v)} \frac{\Big(\phi_0(u, v) - \phi_0(u', v') - \Phi^2_M\big(\mathcal{D}_0((u, v), (u', v'))\big)\Big)^+}{\Phi^1\big(\mathcal{D}_0((u, v), (u', v'))\big)},\end{aligned}$$

where the supremum is taken over functions in $\mathscr{S}_0$. In the second equality we used that $(u', v') \to (u, v)$ means $\mathcal{D}_0((u, v), (u', v')) \to 0$, and the fact that $\lim_{t\to 0} \Phi^1(t)/t = 1$ and $\lim_{t\to 0} \Phi^2_M(t)/t = 0$.

To see the other inequality, we fix $(u', v') \neq (u, v)$. It is not restrictive to suppose that

$$\phi_0(u, v) - \phi_0(u', v') - \Phi^2_M\big(\mathcal{D}_0((u, v), (u', v'))\big) > 0.$$

Define $u_s = (1 - s)u + su'$ and $v_s = (1 - s)v + sv'$ for $s \in [0, 1]$. By Lemma 4.8 with $(u_0, v_0) = (u, v)$ and $(u_1, v_1) = (u', v')$ we get

$$\frac{\phi_0(u, v) - \phi_0(u_s, v_s)}{\mathcal{D}_0((u_0, v_0), (u_s, v_s))} \geqq \frac{s\phi_0(u, v) - s\phi_0(u', v') - s\Phi^2_M\big(\mathcal{D}_0((u, v), (u', v'))\big)}{s\Phi^1\big(\mathcal{D}_0((u, v), (u', v'))\big)}.$$

Note $(u_s, v_s) \to (u, s)$ as $s \to 0$ with respect to the topology induced by $\mathcal{D}_0$, see Lemma 4.6(iii). Therefore,

$$|\partial\phi|_{\mathcal{D}_0}(u, v) \geqq \frac{\phi_0(u, v) - \phi_0(u', v') - \Phi^2_M\big(\mathcal{D}_0((u, v), (u', v'))\big)}{\Phi^1\big(\mathcal{D}_0((u, v), (u', v'))\big)}.$$

The claim now follows by taking the supremum with respect to (u', v').

It remains to show that $|\partial\phi_0|_{\mathcal{D}_0}$ is a strong upper gradient. Let us first recall from [3, Lemma 1.2.5] that in the complete metric space $(\mathscr{S}_0, \mathcal{D}_0)$, the global slope

$$\mathcal{G}_{\phi_0}(u, v) := \sup_{\substack{(u',v')\in\mathscr{S}_0 \\ (u',v')\neq(u,v)}} \frac{(\phi_0(u, v) - \phi_0(u', v'))^+}{\mathcal{D}_0((u, v), (u', v'))} \tag{4.22}$$

is a strong upper gradient for ϕ_0 since ϕ_0 is $\mathcal{D}_0$-lower semicontinuous (see Lemma 4.6 (iv)). Moreover, [3, Lemma 1.2.5] also states that $|\partial\phi_0|_{\mathcal{D}_0}$ is a weak upper gradient for ϕ_0 in the sense of [3, Definition 1.2.2]. We do not repeat the definition of weak upper gradients, but only mention that weak upper gradients are also strong upper gradients if for each absolutely continuous curve $z : (a, b) \to \mathscr{S}_0$ with respect to $\mathcal{D}_0$ satisfying $|\partial\phi_0|_{\mathcal{D}_0}(z)|z'|_{\mathcal{D}_0} \in L^1(a, b)$, the function $\phi_0 \circ z$ is absolutely continuous.

To check that $\phi_0 \circ z$ is absolutely continuous, we first extend the curve z continuously to $[a, b]$ and introduce the compact metric space $\mathscr{S}' = z([a, b])$ with the metric induced by $\mathcal{D}_0$. Choose $M > 0$ such that $\mathscr{S}' \subset \mathscr{S}_0^M$, which is possible due to (2.23), (2.24), and the fact that $\mathrm{diam}(\mathscr{S}') := \sup_{s,t\in[a,b]} \mathcal{D}_0(z(s), z(t)) < +\infty$. Let $\mathcal{G}'_{\phi_0}$ be the global slope as introduced in (4.22) with respect to $\mathscr{S}'$ (instead of $\mathscr{S}_0$). The representation of the local slope implies

$$\mathcal{G}'_{\phi_0}(u, v) = \sup_{\substack{(u',v')\in\mathscr{S}' \\ (u',v')\neq(u,v)}} \frac{(\phi_0(u, v) - \phi_0(u', v'))^+}{\mathcal{D}_0((u, v), (u', v'))} \leqq C_1|\partial\phi_0|_{\mathcal{D}_0}(u, v) + C_2$$

for all $(u, v) \in \mathscr{S}' \subset \mathscr{S}_0^M$, where

$$C_1 := \sup_{t\in[0,\mathrm{diam}(\mathscr{S}')]} \frac{\Phi^1(t)}{t} < +\infty,$$

$$C_2 := \sup_{t\in[0,\mathrm{diam}(\mathscr{S}')]} \frac{\Phi^2_M(t)}{t} < +\infty.$$

In particular, since $|\partial\phi_0|_{\mathcal{D}_0}(z)|z'|_{\mathcal{D}_0} \in L^1(a, b)$, it follows $\mathcal{G}'_{\phi_0}(z)|z'|_{\mathcal{D}_0} \in L^1(a, b)$. As discussed above, $\mathcal{G}'_{\phi_0}$ is a strong upper gradient. Thus, we indeed get that $\phi_0 \circ z$ is absolutely continuous, see Definition 3.1. □

5. Relation Between Three-Dimensional and Two-Dimensional Setting

In this section we consider sequences $(y^h)_h$ with $y^h \in \mathscr{S}_h^M = \{y \in \mathscr{S}_h : \phi_h(y) \leqq M\}$. In Section 5.1 we derive compactness properties and identify suitable limiting objects in terms of scaled in-plane and out-of-plane displacement fields. Afterwards, in Section 5.2 we derive a Γ-convergence result and prove lower semicontinuity for the local slopes along the passage from the three-dimensional to the two-dimensional setting. As in Section 4, $C > 0$ indicates a generic constant which is independent of h, but may depend on M, S, $p > 3$, $\alpha \in (0, 1)$, on the constants in (2.3), (2.4), (2.7), and the boundary data $\hat{u}$ and $\hat{v}$.

5.1. Compactness and identification of limiting strain

Given $y^h \in \mathscr{S}_h^M$, we define the averaged, scaled in-plane and out-of-plane displacement fields by

$$u^h(x') := \frac{1}{h^2}\int_I \left(\begin{pmatrix} y_1^h \\ y_2^h \end{pmatrix}(x', x_3) - \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) \mathrm{d}x_3, \quad v^h(x') := \frac{1}{h}\int_I y_3^h(x', x_3)\,\mathrm{d}x_3, \tag{5.1}$$

where $I = (-\frac{1}{2}, \frac{1}{2})$. By recalling the boundary conditions (2.8) and (2.20) we observe $(u^h, v^h) \in \mathscr{S}_0$.

In view of the rigidity estimate in Lemma 4.2, a deformation $y^h \in \mathscr{S}_h^M$ is close to the affine map $(x', x_3) \mapsto (x', hx_3)$ and thus the displacements defined in (5.1) are suitably controlled. More specifically, we have the following compactness result.

Lemma 5.1. (Compactness for displacements) *Consider a sequence $(y^h)_h$ with $y^h \in \mathscr{S}_h^M$ for all h. Then up to passing to a subsequence (not relabeled) we find $(u, v) \in \mathscr{S}_0$ such that*

$$\begin{aligned} &u^h \rightharpoonup u \ \textit{ weakly in } W^{1,2}(S;\mathbb{R}^2), \\ &v^h \to v \ \textit{ strongly in } W^{1,2}(S). \end{aligned} \tag{5.2}$$

For the proof we refer to [24, Lemma 1] and [28, Lemma 13].

Remark 5.2. (Ansatz for recovery sequences) Note that the convergence results in Lemma 5.1 are compatible with the ansatz

$$y^h(x', x_3) = \begin{pmatrix} x' \\ hx_3 \end{pmatrix} + \begin{pmatrix} h^2 u(x') \\ hv(x') \end{pmatrix} - h^2 x_3 \begin{pmatrix} (\nabla' v(x'))^\top \\ 0 \end{pmatrix} + h^3 x_3 d^h(x'), \tag{5.3}$$

for $(u, v) \in \mathscr{S}_0$ and $d^h \in W_0^{2,\infty}(S;\mathbb{R}^3)$, which additionally satisfy the regularity

$$\|u\|_{W^{2,\infty}(S)} + \|v\|_{W^{3,\infty}(S)} + \sqrt{h}\|d^h\|_{W^{2,\infty}(S)} \leqq C'. \tag{5.4}$$

We observe that the boundary conditions (2.8) are satisfied, that is, $y^h \in \mathscr{S}_h$. For later purposes, we note that the scaled gradient is given by

$$\begin{aligned} \nabla_h y^h = \mathbf{Id} &+ \left(\begin{array}{c|c} h^2 \nabla' u & -h(\nabla' v)^\top \\ \hline h\nabla' v & 0 \end{array} \right) - h^2 x_3 \left(\begin{array}{c|c} (\nabla')^2 v & 0 \\ \hline 0 & 0 \end{array} \right) \\ &+ h^2 d^h \otimes e_3 + h^3 x_3 (\nabla' d^h \,|\, 0). \end{aligned} \tag{5.5}$$

We also note that the (scaled) second gradient has the form

$$\begin{aligned} (\nabla')^2 y^h &= \begin{pmatrix} h^2(\nabla')^2 u - h^2 x_3 (\nabla')^3 v \\ h(\nabla')^2 v \end{pmatrix} + h^3 x_3 (\nabla')^2 d^h, \quad \frac{1}{h}\nabla' y^h_{,3} \\ &= \begin{pmatrix} -h(\nabla')^2 v \\ 0 \end{pmatrix} + h^2 \nabla' d^h \end{aligned}$$

and $\frac{1}{h^2} y^h_{,33} = 0$. By (5.4) this implies $\|(\nabla')^2 y^h\|_\infty \leqq Ch$, $\|\nabla'(\frac{1}{h} y^h_{,3})\|_\infty \leqq Ch$ and $y^h_{,33} = 0$ for a constant C depending on C'. Consequently, by using $\alpha < 1$ and (2.4)(iii) we compute

$$\begin{aligned}
&\limsup_{h\to\infty} \frac{1}{h^{\alpha p}} \int_\Omega P(\nabla_h^2 y^h) \\
&\leqq \limsup_{h\to\infty} \frac{C}{h^{\alpha p}} \int_\Omega \Big(\big|(\nabla')^2 y^h\big|^p + \Big|\nabla'\Big(\frac{1}{h} y^h_{,3}\Big)\Big|^p + \Big|\frac{1}{h^2} y^h_{,33}\Big|^p \Big) \\
&\leqq \lim_{h\to 0} C h^{p(1-\alpha)} = 0.
\end{aligned} \tag{5.6}$$

Later in Theorem 5.6 and Theorem 5.10 we will use this ansatz to construct recovery sequences for the energy and the dissipations. We remark that, on the one hand, our ansatz is slightly simpler than the one in [24, (119)] since we consider a model with zero Poisson's ratio in e_3 direction, see Remark 5.7 and Remark 5.11 for more details. On the other hand, some additional effort arises from the fact that our three-dimensional model contains second gradient terms.

In (4.4) we have already introduced the quantity $G^h(y^h) = h^{-2}(R(y^h)^\top \nabla_h y^h - \mathbf{Id})$ which essentially controls the energy of y^h. By (4.3)(i) we get that $G^h(y^h)$ converges weakly in L^2 (up to a subsequence) to some G. The next result shows that the symmetric part of the in-plane components of G can be identified in terms of the in-plane displacement u and the out-of-plane displacement v.

Lemma 5.3. (Identification of scaled limiting strain) *Consider the setting of Lemma 5.1. Let $R(y^h) : S \to SO(3)$ be the corresponding mappings from Lemma 4.2. Then*

$$\frac{1}{h}(R(y^h) - \mathbf{Id}) \to A(v) := e_3 \otimes \nabla' v - \nabla' v \otimes e_3 \ \ in \ \ L^q(S; \mathbb{R}^{3\times 3}),\ 1 \leqq q < \infty. \tag{5.7}$$

Moreover, we find $G \in L^2(\Omega; \mathbb{R}^{3\times 3})$ such that

$$G^h(y^h) = \frac{R(y^h)^\top \nabla_h y^h - \mathbf{Id}}{h^2} \rightharpoonup G \ \ weakly\ in \ \ L^2(\Omega; \mathbb{R}^{3\times 3}).$$

The 2×2 submatrix G'' given by $G''_{ij} = G_{ij}$ for $i, j \in \{1, 2\}$ satisfies

$$G''(x', x_3) = G_0(x') + x_3 G_1(x')$$

with

$$\mathrm{sym}(G_0) = e(u) + \frac{1}{2}\nabla' v \otimes \nabla' v \ \ and\ \ G_1 = -(\nabla')^2 v.$$

Here, we again use the notation $e(u) = \mathrm{sym}(\nabla' u)$, see (4.10). For the proof we refer to [24, Lemma 1, Lemma 2] (see also [28, Lemma 16]) from which we also adopted the notation for ease of readability. Note that, in fact, an inspection of the proof shows that the mappings $R(y^h)$ introduced there can be chosen as the ones from Lemma 4.2. In particular, the result shows that the relevant components of G are affine in the thickness variable x_3. In the following, we will also use the notation

$$G(u,v)(x',x_3) = \mathrm{sym}(G_0)(x') + x_3 G_1(x') = e(u) + \frac{1}{2}\nabla' v \otimes \nabla' v - x_3(\nabla')^2 v. \tag{5.8}$$

Note that $G(u,v) \in L^2(\Omega;\mathbb{R}^{2\times 2}_{\mathrm{sym}})$.

Remark 5.4. Using (5.8) and the fact that Q^2_W only depends on the symmetric part of matrices (see Lemma 4.1(ii)), the energy $\phi_0(u,v)$ defined in (2.23) (with $f \equiv 0$) can be written as

$$\begin{aligned}\phi_0(u,v) &= \int_S \Big(\frac{1}{2} Q^2_W(G_0) + \frac{1}{24} Q^2_W(G_1)\Big)\\ &= \int_S \int_{-1/2}^{1/2} \frac{1}{2} Q^2_W(G_0(x') + x_3 G_1(x'))\,\mathrm{d}x_3\,\mathrm{d}x' = \int_\Omega \frac{1}{2} Q^2_W(G(u,v)).\end{aligned} \tag{5.9}$$

In view of the definition of $\mathcal{D}_0$ in (2.24), a similar computation yields

$$\int_\Omega Q^2_D\big(G(u_1,v_1) - G(u_2,v_2)\big) = \mathcal{D}_0\big((u_1,v_1),(u_2,v_2)\big)^2. \tag{5.10}$$

In what follows we will mainly use the representation in terms of an integral over Ω as it is convenient for many proofs. Only at the very end we will pass to an integral over S as indicated in (5.9)–(5.10).

The next lemma, pertaining to strong convergence of strain differences, will be instrumental in the proof of the lower semicontinuity of the local slopes. We introduce the notation $W^{2,p}_{0,\partial S}(\Omega;\mathbb{R}^3) = \{y \in W^{2,p}(\Omega;\mathbb{R}^3) : y(x',x_3) = 0 \text{ for } x' \in \partial S,\ x_3 \in I\}$, where $I = (-\frac{1}{2},\frac{1}{2})$. We also define $\mathrm{skew}(F) = \frac{1}{2}(F - F^\top)$ for $F \in \mathbb{R}^{3\times 3}$.

Lemma 5.5. (Strong convergence of strain differences) *Let $M > 0$. Let $(y^h)_h$ be a sequence with $y^h \in \mathscr{S}^M_h$ and let $(z^h_s)_{s,h} \subset W^{2,p}_{0,\partial S}(\Omega;\mathbb{R}^3)$, $h > 0$, $s \in (0,1)$, be functions such that*

(i) $\|\nabla_h z^h_s\|_{L^\infty(\Omega)} + \|\nabla^2_h z^h_s\|_{L^\infty(\Omega)} \leqq Msh.$

(ii) $\|\mathrm{sym}(\nabla_h z^h_s)\|_{L^2(\Omega)} \leqq Msh^2$

(iii) $\Big|\mathrm{skew}(\nabla_h z^h_s)(x',x_3) - \int_I \mathrm{skew}(\nabla_h z^h_s)(x',x_3)\,dx_3\Big|$
$\leqq Msh^{5/2}$ *for a.e. $x \in \Omega$,*

(iv) *there exist $E^s, F^s \in L^2(\Omega;\mathbb{R}^{3\times 3})$ for $s \in (0,1)$, and $\eta(h) \to 0$ as $h \to 0$ such that*

$$\|h^{-2}\mathrm{sym}(\nabla_h z_s^h) - E^s\|_{L^2(\Omega)} + \|h^{-1}\mathrm{skew}(\nabla_h z_s^h) - F^s\|_{L^2(\Omega)} \leqq s\eta(h). \tag{5.11}$$

Then the following holds for a subsequence of $(y^h)_h$ *(not relabeled):*

(a) For all h *sufficiently small,* $w_s^h := y^h + z_s^h$ *lies in* $\mathscr{S}_h^{M'}$ *for some* $M' = M'(M) > 0$.

(b) Let $(G^h(y^h))_h$, $(G^h(w_s^h))_h$ *be the sequences in Lemma 5.3 and let* G_y, G_w^s *be their limits. Then there exists* $C = C(M) > 0$ *and* $\rho(h)$ *with* $\rho(h) \to 0$ *as* $h \to 0$ *such that*

$$(i)\ \ \big\|\big(G^h(y^h) - G^h(w_s^h)\big) - \big(G_y - G_w^s\big)\big\|_{L^2(\Omega)} \leqq s\rho(h),$$
$$(ii)\ \ \|G^h(y^h) - G^h(w_s^h)\|_{L^2(\Omega)} \leqq Cs.$$

(c) Let (u, v) *and* $(\bar{u}_s, \bar{v}_s)$ *be limits corresponding to* y^h *and* w_s^h*, respectively, as given in Lemma 5.1. Then* $\mathrm{sym}(G_w^s - G_y)\, e_3 = E^s\, e_3 - \frac{1}{2}(|\nabla' v|^2 - |\nabla' \bar{v}_s|^2) e_3$ *almost everywhere in* Ω.

We note that weak convergence of the strain differences is already guaranteed by Lemma 5.3. The important point here is that we actually obtain strong convergence with linear control in terms of s, see (b). Moreover, (c) provides a characterization of the out-of-plane components of the limiting strain difference. Later in the proof of Theorem 5.10 we will use this lemma to construct competitor sequences for the local slope in the three-dimensional setting.

Proof. Let $R(y^h)$ be the $SO(3)$-valued mappings given by Lemma 4.2. For brevity, we introduce notations for the symmetric and skew-symmetric part of $\nabla_h z_s^h$ by

$$E(z_s^h) = \mathrm{sym}(\nabla_h z_s^h), \qquad F(z_s^h) = \mathrm{skew}(\nabla_h z_s^h), \qquad \bar{F}(z_s^h) = \int_I F(z_s^h)\,\mathrm{d}x_3.$$

The crucial point is to find a suitable $SO(3)$-valued mapping $R(w_s^h)$ associated to $w_s^h = y^h + z_s^h$ satisfying the properties stated in Lemma 4.2 (Step 1). Once $R(w_s^h)$ has been defined, we can prove properties (a)-(c) (Step 2).

Step 1: Definition of $R(w_s^h)$. We first define

$$\tilde{R} = R(y^h)\big(\mathbf{Id} + \bar{F}(z_s^h) - \tfrac{1}{2}\bar{F}(z_s^h)^\top \bar{F}(z_s^h)\big)$$

on S. By (4.3)(iii) and (5.11)(i) we can check that $\tilde{R}$ is in a small tubular neighborhood of $SO(3)$ and satisfies $\|\nabla' \tilde{R}\|^2_{L^2(S)} \leqq Ch^2$. We let $R(w_s^h) \in W^{1,2}(S; SO(3))$ be the map obtained from $\tilde{R}$ by nearest-point projection on $SO(3)$, see [24, Remark 5] for a similar argument. By (5.11)(i) and $\bar{F}(z_s^h)(x') \in \mathbb{R}^{3\times 3}_{\mathrm{skew}}$ for all $x' \in S$, it is elementary to check that

$$\big\|R(w_s^h) - R(y^h)\big(\mathbf{Id} + \bar{F}(z_s^h) - \tfrac{1}{2}\bar{F}(z_s^h)^\top \bar{F}(z_s^h)\big)\big\|_{L^\infty(S)} \leqq C\|\bar{F}(z_s^h)\|^3_{L^\infty(S)} \leqq Csh^3.$$

Indeed, this follows from the fact that $|((\mathbf{Id}+A-\frac{1}{2}A^\top A)^\top(\mathbf{Id}+A-\frac{1}{2}A^\top A))^{1/2}-\mathbf{Id}| \leqq C|A|^3$ for all $A \in \mathbb{R}^{3\times 3}_{\text{skew}}$. Along with (5.11)(iii) we thus get

$$\big\|R(w_s^h) - R(y^h)\big(\mathbf{Id} + F(z_s^h) - \tfrac{1}{2}F(z_s^h)^\top F(z_s^h)\big)\big\|_{L^\infty(\Omega)} \leqq Csh^{5/2}. \tag{5.12}$$

We now check that $R(w_s^h)$ satisfies the properties stated in Lemma 4.2, see (4.3). First, $\|\nabla'\tilde{R}\|^2_{L^2(S)} \leqq Ch^2$ implies $\|\nabla' R(w_s^h)\|^2_{L^2(S)} \leqq Ch^2$. Moreover, (5.11)(i), (5.12), and the fact that $R(y^h)$ satisfies (4.3)(iv) shows $\|R(w_s^h)-\mathbf{Id}\|_{L^q(S)} \leqq C_q h$ for $q \in [1,\infty)$. In a similar fashion, (4.3)(vi) and $\alpha < 1$ yield $\|R(w_s^h)-\mathbf{Id}\|_{L^\infty(S)} \leqq Ch^\alpha$. It thus remains to check (4.3)(i), that is, that

$$\|R(w_s^h)^\top \nabla_h w_s^h - \mathbf{Id}\|^2_{L^2(\Omega)} \leqq Ch^4 \tag{5.13}$$

holds. For notational convenience, we denote by $\omega_i^h \in L^2(\Omega;\mathbb{R}^{3\times 3})$, $i \in \mathbb{N}$, (generic) matrix valued functions whose L^2-norm is controlled in terms of a constant independent of h and s. By (4.3)(v) (applied for y^h), (5.11)(i), and (5.12) we find

$$R(w_s^h)^\top \nabla_h w_s^h = \big(\mathbf{Id} + F(z_s^h) - \tfrac{1}{2}F(z_s^h)^\top F(z_s^h)\big)^\top R(y^h)^\top(\nabla_h y^h + \nabla_h z_s^h) + sh^{5/2}\omega_1^h. \tag{5.14}$$

We now consider the asymptotic expansion of $R(y^h)^\top(\nabla_h y^h + \nabla_h z_s^h)$ in terms of h: in view of (5.11)(ii), $\nabla_h z_s^h = E(z_s^h) + F(z_s^h)$, and $\|R(y^h) - \mathbf{Id}\|_{L^\infty(\Omega)} \leqq Ch^\alpha$ (see (4.3)(vi)) we find

$$R(y^h)^\top(\nabla_h y^h + \nabla_h z_s^h) = R(y^h)^\top(\nabla_h y^h + F(z_s^h)) + E(z_s^h) + h^{2+\alpha}s\omega_2^h.$$

In a similar fashion, by using (4.3)(i),(iv) (for y^h) and (5.11)(i),(ii), we compute

$$R(y^h)^\top(\nabla_h y^h + \nabla_h z_s^h) = \mathbf{Id} + R(y^h)^\top F(z_s^h) + h^2\omega_3^h = \mathbf{Id} + F(z_s^h) + h^2\omega_4^h$$

as well as

$$R(y^h)^\top(\nabla_h y^h + \nabla_h z_s^h) = \mathbf{Id} + h\omega_5^h.$$

By inserting these three estimates in (5.14) and using (5.11)(i) we find

$$\begin{aligned} R(w_s^h)^\top \nabla_h w_s^h &= R(y^h)^\top(\nabla_h y^h + F(z_s^h)) + E(z_s^h) + F(z_s^h)^\top + F(z_s^h)^\top F(z_s^h) \\ &\quad - \tfrac{1}{2}F(z_s^h)^\top F(z_s^h) + s(h^{5/2} + h^{2+\alpha})\,\omega_6^h \\ &= R(y^h)^\top \nabla_h y^h + (R(y^h) - \mathbf{Id})^\top F(z_s^h) \\ &\quad + E(z_s^h) + \tfrac{1}{2}F(z_s^h)^\top F(z_s^h) + s\omega_6^h \mathrm{o}(h^2), \end{aligned} \tag{5.15}$$

where in the last step we used $F(z_s^h)^\top + F(z_s^h) = 0$. We now check that (5.13) holds. Indeed, it suffices to use (5.15), (5.11)(i),(ii), $\omega_6^h \in L^2(\Omega;\mathbb{R}^{3\times 3})$, and the fact that (4.3)(i),(iv) holds for y^h. In conclusion, this implies that the mapping $R(w_s^h)$ satisfies the properties stated in Lemma 4.2.

Step 2: Proof of the statement. We are now in a position to prove the statement.

(a) By (4.3)(v) and (5.11)(i), $\nabla_h w_s^h$ is in a neighborhood of $\mathbf{Id}$. Thus, by (2.3)(iii) there holds $\int_\Omega W(\nabla_h w_s^h) \leqq C\int_\Omega \mathrm{dist}^2(\nabla_h w_s^h, SO(3))$, and then by (5.13) we get $\int_\Omega W(\nabla_h w_s^h) \leqq M' h^4$ for some M' sufficiently large (depending on M). Moreover, from (2.4)(iii) and the triangle inequality we get $P(\nabla_h^2 w_s^h) \leqq CP(\nabla_h^2 y^h) + CP(\nabla_h^2 z_s^h)$. Therefore, by possibly passing to a larger M', we obtain $h^{-\alpha p}\int_\Omega P(\nabla_h^2 w_s^h) \leqq M'$ by (2.4)(iii), (5.11)(i), $\alpha < 1$, and the fact that $y^h \in \mathscr{S}_h^M$. Summarizing, since $z_s^h \in W^{2,p}_{0,\partial S}(\Omega;\mathbb{R}^3)$ and thus w_s^h also satisfies the boundary conditions (2.8), we have shown that $w_s^h \in \mathscr{S}_h^{M'}$ for some $M' = M'(M) > 0$ independent of h. In particular, this implies that the statement of Lemma 4.2 holds for w_s^h with $R(w_s^h)$ as defined in Step 1.

(b) By (u, v) we denote the limit corresponding to y^h as given in Lemma 5.1. Recalling the definition $G^h(y^h) = h^{-2}(R(y^h)^\top \nabla_h y^h - \mathbf{Id})$ we find by (5.7), (5.11)(iv), and (5.15) that

$$\big\|\big(G^h(w_s^h) - G^h(y^h)\big) - \big(A(v)^\top F^s + E^s + \tfrac{1}{2}(F^s)^\top F^s\big)\big\|_{L^2(\Omega)} \leqq s\rho(h), \quad (5.16)$$

where $\rho(h) \to 0$ as $\rho \to 0$. By G_w^s and G_y we denote the weak L^2-limits of $G^h(w_s^h)$ and $G^h(y^h)$, respectively, which exist by Lemma 5.3. Then (5.16) implies

$$G_w^s - G_y = A(v)^\top F^s + E^s + \tfrac{1}{2}(F^s)^\top F^s, \quad (5.17)$$

and the first part of (b) holds. The second part of (b) is a consequence of (4.3)(iv) (for y^h), (5.11)(i),(ii), (5.15), and the fact that $\omega_6^h \in L^2(\Omega;\mathbb{R}^{3\times 3})$.

(c) By $(\bar u_s, \bar v_s)$ we denote the limit corresponding to w_s^h as given in Lemma 5.1. By (5.7) (for w_s^h and y^h, respectively), (5.11)(i),(iv), and (5.12) we observe that pointwise almost everywhere in Ω it holds that

$$A(\bar v_s) = \lim_{h\to 0}\frac{1}{h}(R(w_s^h) - \mathbf{Id}) = \lim_{h\to 0} \Big(\frac{1}{h}(R(y^h) - \mathbf{Id})(\mathbf{Id} + F(z_s^h)) + \frac{1}{h}F(z_s^h)\Big) = A(v) + F^s.$$

Then by (5.17) and an expansion we get

$$\begin{aligned}\mathrm{sym}(G_w^s - G_y) &= E^s + \mathrm{sym}(A(v)^\top F^s) + \tfrac{1}{2}(F^s)^\top F^s\\ &= E^s + \tfrac{1}{2}(A(\bar v_s))^\top A(\bar v_s) - \tfrac{1}{2}(A(v))^\top A(v)\\ &= E^s - \tfrac{1}{2}(A(\bar v_s))^2 + \tfrac{1}{2}(A(v))^2,\end{aligned}$$

where in the last step we used that $A(v) \in \mathbb{R}^{3\times 3}_{\mathrm{skew}}$ pointwise almost everywhere in Ω and thus $A(v)^\top A(v) = -(A(v))^2$. Therefore, recalling the definition of $A(v)$ in (5.7) we obtain

$$\begin{aligned}\mathrm{sym}(G_w^s - G_y)\,e_3 &= E^s\,e_3 + \tfrac{1}{2}(A(v))^2\,e_3 - \tfrac{1}{2}(A(\bar v_s))^2\,e_3\\ &= E^s\,e_3 - \tfrac{1}{2}(|\nabla' v|^2 - |\nabla' \bar v_s|^2)e_3.\end{aligned}$$

This concludes the proof. □

5.2. Γ-convergence and lower semicontinuity of slopes

In this section we establish a Γ-convergence result for the energies which is essentially proved in [24,28]. However, some adaptions are necessary due to the second order perturbation P in the energy. For an exhaustive treatment of Γ-convergence we refer the reader to [18]. Afterwards, we prove lower semicontinuity for the dissipation distances and the local slopes which is fundamental to use the theory in [36,41] (see also Section 3.2, in particular (3.4) and (3.6)).

We first fix a topology for the convergence of the scaled in-plane and out-of-plane displacements induced by the compactness result in Section 5.1, see (5.2). We define mappings $\pi_h : \mathscr{S}_h \to \mathscr{S}_0$ by $\pi_h(y^h) = (u^h, v^h)$ for each $y^h \in \mathscr{S}_h$, where u^h and v^h are the scaled in-plane and out-of-plane displacements corresponding to y^h (see (5.1)). We say that

$$\pi_h(y^h) = (u^h, v^h) \stackrel{\sigma}{\to} (u, v) \text{ if } u^h \rightharpoonup u \text{ in } W^{1,2}(S;\mathbb{R}^2) \text{ and } v^h \to v \text{ in } W^{1,2}(S). \tag{5.18}$$

We also say that $y^h \stackrel{\pi\sigma}{\to} (u, v)$ if $\pi_h(y^h) \stackrel{\sigma}{\to} (u, v)$, cf. (3.3). Recall the definitions (2.14) and (2.23).

Theorem 5.6. *(Γ-convergence) Suppose that W and P satisfy the assumptions* (2.3) *and* (2.4)*. Then ϕ_h converges to ϕ_0 in the sense of Γ-convergence. More precisely,*
(i) (Lower bound) For all $(u, v) \in \mathscr{S}_0$ and all sequences $(y^h)_h$ such that $y^h \stackrel{\pi\sigma}{\to} (u, v)$ we find

$$\liminf_{h\to 0} \phi_h(y^h) \geqq \phi_0(u, v).$$

(ii) (Optimality of lower bound) For all $(u, v) \in \mathscr{S}_0$ there exists a sequence $(y^h)_h$, $y^h \in \mathscr{S}_h$ for all h, such that $y^h \stackrel{\pi\sigma}{\to} (u, v)$ and

$$\lim_{h\to 0} \phi_h(y^h) = \phi_0(u, v).$$

Proof. (i) The result is essentially proved in [28]. We give here the main steps for convenience of the reader. By the representation (5.9) and the fact that P is nonnegative, for the lower bound it suffices to prove

$$\liminf_{h\to 0} h^{-4} \int_\Omega W(\nabla_h y^h) \geqq \int_\Omega \frac{1}{2} Q_W^2(G(u, v)) \tag{5.19}$$

for all sequences $(y^h)_h$ with $y^h \stackrel{\pi\sigma}{\to} (u, v)$. We may suppose that $\liminf_{h\to 0} \phi_h(y^h)$ is finite as otherwise there is nothing to prove. Thus, we can assume that $y^h \in \mathscr{S}_h^M$ for $M > 0$ large enough. By Lemma 4.3(iii) we find

$$\liminf_{h\to 0} h^{-4} \int_\Omega W(\nabla_h y^h) = \liminf_{h\to 0} \int_\Omega \frac{1}{2} Q_W^3\big(G^h(y^h)\big).$$

Moreover, Lemma 5.1 and Lemma 5.3 imply that, possibly passing to a subsequence (not relabeled), $G^h(y^h) \rightharpoonup G$ in $L^2(\Omega;\mathbb{R}^{3\times 3})$, where the 2×2 submatrix G'' satisfies $\mathrm{sym}(G'') = G(u, v)$, see (5.8). This along with the lower semicontinuity in L^2 (note that Q_W^3 is a positive semidefinite quadratic form) yields

$\liminf_{h\to 0} h^{-4}\int_\Omega W(\nabla_h y^h) \geqq \int_\Omega \frac{1}{2}Q_W^3(G)$. The fact that $\mathrm{sym}(G'') = G(u,v)$ and (2.17) give the desired lower bound (5.19).

(ii) By a general approximation argument in the theory of Γ-convergence it suffices to establish the optimality of the lower bound only for sufficiently smooth mappings, precisely for $u \in W^{2,\infty}(S;\mathbb{R}^2)$ and $v \in W^{3,\infty}(S)$ with

$$u = \hat{u}, \quad v = \hat{v}, \quad \nabla' v = \nabla'\hat{v} \quad \text{on } \partial S. \tag{5.20}$$

Indeed, a general (u,v) can be approximated strongly in $W^{1,2}(S;\mathbb{R}^2)\times W^{2,2}(S)$ by such functions, see Lemma 5.8 below. Note that the limiting energy ϕ_0 is continuous with respect to this topology, see Lemma 4.6(iii),(iv).

Let us now construct recovery sequences for $u \in W^{2,\infty}(S;\mathbb{R}^2)$ and $v \in W^{3,\infty}(S)$ satisfying (5.20). Define $d = -\frac{1}{2}|\nabla' v|^2 e_3 \in W^{2,\infty}(S;\mathbb{R}^3)$ and let $(d^h)_h \subset W_0^{2,\infty}(S;\mathbb{R}^3)$ with $d^h \to d = -\frac{1}{2}|\nabla' v|^2 e_3$ in $L^2(S;\mathbb{R}^3)$ and $\sup_h \sqrt{h}\|d^h\|_{W^{2,\infty}(S)} < \infty$. We take the ansatz for y^h as given in Remark 5.2. In Remark 5.2 we have already discussed that this ansatz is compatible with the convergence $y^h \overset{\pi\sigma}{\to} (u,v)$. By the representation of the scaled gradient in (5.5), an elementary computation yields for the nonlinear strain

$$\begin{aligned}(\nabla_h y^h)^\top \nabla_h y^h = \mathbf{Id} &+ 2h^2\big(e(u) - x_3(\nabla')^2 v\big) + h^2\big(\nabla' v\otimes\nabla' v + |\nabla' v|^2 e_3\otimes e_3\big)\\ &+ 2h^2\mathrm{sym}(d^h\otimes e_3) + \mathrm{O}(h^{5/2}),\end{aligned}$$

where we used $\sup_h \sqrt{h}\|d^h\|_{W^{2,\infty}(S)}0 < \infty$. Since it holds that $d^h \to d = -\frac{1}{2}|\nabla' v|^2 e_3$ in $L^2(S;\mathbb{R}^3)$ and $\sup_h \sqrt{h}\|d^h\|_{L^\infty(\Omega)} < \infty$, we get

$$(\nabla_h y^h)^\top \nabla_h y^h = \mathbf{Id} + 2h^2\big(e(u) + \tfrac{1}{2}\nabla' v\otimes\nabla' v - x_3(\nabla')^2 v\big) + h^2\omega^h$$

for functions $\omega^h : \Omega \to \mathbb{R}^{3\times 3}$ with $\|\omega^h\|_{L^2(\Omega)} \to 0$ and $\sup_h \sqrt{h}\|\omega^h\|_{L^\infty(\Omega)} < \infty$. Therefore,

$$(\nabla_h y^h)^\top \nabla_h y^h = \mathbf{Id} + 2h^2 G(u,v)^* + h^2\omega^h,$$

where $G(u,v)^* \in L^2(\Omega;\mathbb{R}^{3\times 3})$ denotes the mapping with $(G(u,v)^*)_{ij}=(G(u,v))_{ij}$ for $1 \leqq i,j \leqq 2$ and zero otherwise, see (5.8). Taking the square root, using the frame indifference of W, (2.3)(i), and a Taylor expansion, we derive (cf. also [28, Proposition 19])

$$\frac{1}{h^4}\int_\Omega W(\nabla_h y^h) = \frac{1}{h^4}\int_\Omega W\Big(\big((\nabla_h y^h)^\top\nabla_h y^h\big)^{1/2}\Big) \to \int_\Omega \frac{1}{2}Q_W^3(G(u,v)^*)$$

as $h \to 0$. Definition (2.17) (and the assumption that the minimum is attained for $a = 0$) yield $Q_W^3(G(u,v)^*) = Q_W^2(G(u,v))$. Then (5.9) implies $\frac{1}{h^4}\int_\Omega W(\nabla_h y^h) \to \phi_0(u,v)$. The proof is now concluded by observing $\lim_{h\to 0} h^{-\alpha p}\int_\Omega P(\nabla_h^2 y^h) = 0$; see (5.6). □

Remark 5.7. We remark that the assumption on Q_W^2 is actually not needed at the expense of a more involved recovery sequence, see [24, equation (119)]. However, the assumption will be instrumental for the lower semicontinuity of slopes, see Theorem 5.10 and Remark 5.11.

In the previous proof we have used the following density result:

Lemma 5.8. (Density of smooth functions with same boundary conditions) *For each* $(u, v) \in \mathscr{S}_0$ *we find sequences* $(u^h)_h \subset W^{2,\infty}(S; \mathbb{R}^2)$ *and* $(v^h)_h \subset W^{3,\infty}(S)$ *such that*

$$(i)\;\; u^h = \hat{u}, \quad v^h = \hat{v}, \quad \nabla' v^h = \nabla' \hat{v} \quad on\ \partial S,$$
$$(ii)\;\; u^h \to u \;\; in \;\; W^{1,2}(S; \mathbb{R}^2), \quad v^h \to v \;\; in \;\; W^{2,2}(S).$$

Proof. The proof is standard: we approximate $u - \hat{u}$ and $v - \hat{v}$ by smooth functions with compact support in S and add $\hat{u} \in W^{2,\infty}(S; \mathbb{R}^2)$, $\hat{v} \in W^{3,\infty}(S)$, respectively. □

We now proceed with the lower semicontinuity of the dissipation distances. Recall the definitions in (2.15) and (2.24).

Theorem 5.9. (Lower semicontinuity of dissipation distances) *Suppose that D satisfies the assumptions* (2.7)*. Let* $M > 0$*. Then for sequences* $(y_1^h)_h$ *and* $(y_2^h)_h$, $y_1^h, y_2^h \in \mathscr{S}_h^M$, *with* $y_1^h \overset{\pi\sigma}{\to} (u_1, v_1)$ *and* $y_2^h \overset{\pi\sigma}{\to} (u_2, v_2)$ *we have*

$$\liminf_{h \to 0} \mathcal{D}_h(y_1^h, y_2^h) \geqq \mathcal{D}_0\big((u_1, v_1), (u_2, v_2)\big).$$

Proof. The argument is similar to the one in (5.19), see the proof of Theorem 5.6(i), with the difference that we employ Lemma 4.3(ii) in place of Lemma 4.3 (iii) and (5.10) in place of (5.9). □

We close this section with the fundamental property that the local slopes are lower semicontinuous along the passage from the three-dimensional to the two-dimensional setting. As emphasized before, this is crucial for the application of the theory in [36,41], see (3.6). Recall the definition of Q_W^2, Q_D^2 in (2.17)–(2.18). The fact that the minimum is attained for $a = 0$ implies

$$Q_W^3(F) = Q_W^2(F'') + Q_W^3(F - F^*),$$
$$Q_D^3(F) = Q_D^2(F'') + Q_D^3(F - F^*) \tag{5.21}$$

for all $F \in \mathbb{R}^{3\times3}$, where F'' denotes the 2×2 matrix with entries $F''_{ij} = F_{ij}$ for $1 \leqq i, j \leqq 2$, and F^* denotes the 3×3 matrix with entries $F^*_{ij} = F_{ij}$ for $1 \leqq i, j \leqq 2$, and zero otherwise.

Theorem 5.10. (Lower semicontinuity of slopes) *For each sequence* $(y^h)_h$ *with* $y^h \in \mathscr{S}_h^M$ *such that* $y^h \overset{\pi\sigma}{\to} (u, v)$ *we have*

$$\liminf_{n \to \infty} |\partial \phi_h|_{\mathcal{D}_h}(y^h) \geqq |\partial \phi_0|_{\mathcal{D}_0}(u, v).$$

Proof. We divide the proof into several steps. We first define approximations of (u, v) which allow us to work with more regular functions (Step 1). We then construct *competitor sequences* $(w^h_s)_{h,s}$ for the local slope in the three-dimensional setting satisfying $w^h_s \to y^h$ as $s \to 0$ (Step 2). Afterwards, we identify the limiting strain of the sequences $(w^h_s)_h$ (Step 3), and we prove the lower semicontinuity (Step 4). Some technical estimates are contained in Steps 5–7.

Step 1: Approximation. By Lemma 5.8, for $\varepsilon > 0$ we fix $u_\varepsilon \in W^{2,\infty}(S; \mathbb{R}^2)$ and $v_\varepsilon \in W^{3,\infty}(S)$ with $(u_\varepsilon, v_\varepsilon) \in \mathscr{S}_0$ and

$$\|u_\varepsilon - u\|_{W^{1,2}(S)} + \|v_\varepsilon - v\|_{W^{2,2}(S)} \leqq \varepsilon. \tag{5.22}$$

This approximation will be necessary to construct sufficiently regular competitor sequences for the local slope of the three-dimensional setting.

We further fix $\tilde{u} \in W^{2,\infty}(S; \mathbb{R}^2)$ and $\tilde{v} \in W^{3,\infty}(S)$ with $(\tilde{u}, \tilde{v}) \in \mathscr{S}_0$, and satisfying $\tilde{u} \neq u, u_\varepsilon$, $\tilde{v} \neq v, v_\varepsilon$. The pair $(\tilde{u}, \tilde{v})$ will represent the competitor in the local slope of the two-dimensional setting, see Lemma 4.9. Below in (5.36), we will see that by approximation it is enough to work with functions of this regularity. The convex combinations

$$(\tilde{u}_s, \tilde{v}_s) := (1 - s)(u_\varepsilon, v_\varepsilon) + s(\tilde{u}, \tilde{v}), \quad s \in [0, 1] \tag{5.23}$$

will be the starting point for the construction of competitor sequences $(w^h_s)_{h,s}$ for the three-dimensional setting. In the following, $\tilde{C}$, C_ε denote generic constant which may vary from line to line, where $\tilde{C}$ may depend on $\tilde{u}$, u, $\tilde{v}$, v, and C_ε additionally on ε.

Step 2: Construction of competitor sequences $(w^h_s)_{h,s}$. We choose recovery sequences y^h_ε, $\tilde{y}^h_s$ related to $(u_\varepsilon, v_\varepsilon)$ and $(\tilde{u}_s, \tilde{v}_s)$, exactly as in the proof of Theorem 5.6(ii): define $d_\varepsilon = -\frac{1}{2}|\nabla' v_\varepsilon|^2 e_3$, $\tilde{d}_s = -\frac{1}{2}|\nabla' \tilde{v}_s|^2 e_3$ and let $(d^h_\varepsilon)_h, (\tilde{d}^h_s)_h \subset W^{2,\infty}_0(S; \mathbb{R}^3)$ be sequences with $d^h_\varepsilon \to d_\varepsilon$ and $\tilde{d}^h_s \to \tilde{d}_s$ in $L^2(S; \mathbb{R}^3)$. In view of (5.23), this can be done in such a way that it holds that

$$\|\tilde{d}^h_s - d^h_\varepsilon\|_{L^2(S)} \leqq s\rho(h), \quad \sqrt{h}\|\tilde{d}^h_s - d^h_\varepsilon\|_{W^{2,\infty}(S)} \leqq C_\varepsilon s, \tag{5.24}$$

where $\rho(h)$ depends on v_ε, $\tilde{v}$, and satisfies $\rho(h) \to 0$ as $h \to 0$.

We take the ansatz for y^h_ε, $\tilde{y}^h_s$ as given in Remark 5.2 and observe that y^h_ε, $\tilde{y}^h_s$ satisfy the boundary conditions, that is, $z^h_s := \tilde{y}^h_s - y^h_\varepsilon \in W^{2,p}_{0,\partial S}(\Omega; \mathbb{R}^3)$. For $h > 0$ small and $s \in [0, 1]$, we define

$$w^h_s := y^h + z^h_s = y^h - y^h_\varepsilon + \tilde{y}^h_s. \tag{5.25}$$

By (5.24), (5.5)–(5.6) (with $\tilde{u}_s$, u_ε and $\tilde{v}_s$, v_ε in place of u and v, respectively) and the fact that $(\tilde{u}_s - u_\varepsilon, \tilde{v}_s - v_\varepsilon) = s(\tilde{u} - u_\varepsilon, \tilde{v} - v_\varepsilon)$ we see that

$$\begin{aligned}
&(i)\, \|\nabla_h z^h_s\|_{L^\infty(\Omega)} + \|\nabla^2_h z^h_s\|_{L^\infty(\Omega)} \leqq C_\varepsilon s h, \quad \|\mathrm{sym}(\nabla_h z^h_s)\|_{L^2(\Omega)} \leqq C_\varepsilon s h^2,\\
&(ii)\, \Big|\mathrm{skew}(\nabla_h z^h_s)(x', x_3) - \int_I \mathrm{skew}(\nabla_h z^h_s)(x', x_3)\, \mathrm{d}x_3\Big|\\
&\qquad \leqq C_\varepsilon s h^{5/2} \text{ for a.e. } x \in \Omega.
\end{aligned} \tag{5.26}$$

This shows that the assumptions (5.11)(i)–(iii) are satisfied for $(z_s^h)_{s,h}$ (for a constant $M = M(C_\varepsilon)$). From (5.5) and (5.24) we also get that (5.11)(iv) holds for suitable E^s and F^s. In particular, by the definition of $\tilde{d}_s^h$ and d_ε^h we observe that

$$
\begin{aligned}
E^s\, e_3 &= \lim_{h\to 0} \frac{1}{h^2} \mathrm{sym}(\nabla_h z_s^h)\, e_3 \\
&= \lim_{h\to 0} \mathrm{sym}\big((\tilde{d}_s^h - d_\varepsilon^h) \otimes e_3\big)\, e_3 = \frac{1}{2}(|\nabla' v_\varepsilon|^2 - |\nabla' \tilde{v}_s|^2)\, e_3. \qquad (5.27)
\end{aligned}
$$

Then Lemma 5.5(a) implies $w_s^h \in \mathscr{S}_h^{M'}$ for a constant $M' > 0$ sufficiently large depending on ε, but independent of s, h. Thus, by (5.3), (5.25), $\tilde{d}_s^h \to d_\varepsilon^h$ in $L^2(S; \mathbb{R}^3)$ as $s \to 0$ (see (5.24)), and a compactness argument (see Lemma 4.5(ii)), one can check that

$$
w_s^h \rightharpoonup y^h \text{ in } W^{2,p}(\Omega; \mathbb{R}^3) \ \text{ as } s \to 0. \qquad (5.28)
$$

Step 3: Identification of limiting strains. Since the ansatz for $(y_\varepsilon^h)_h$ and $(\tilde{y}_s^h)_h$ is compatible with the convergence results in Lemma 5.1, the convergence in (5.18) holds, that is, the scaled displacement fields corresponding to $(y_\varepsilon^h)_h$ and $(\tilde{y}_s^h)_h$ converge to $(u_\varepsilon, v_\varepsilon)$ and $(\tilde{u}_s, \tilde{v}_s)$, respectively. Thus, in view of (5.25), the scaled displacement fields corresponding to w_s^h converge to $(u - u_\varepsilon + \tilde{u}_s, v - v_\varepsilon + \tilde{v}_s)$. In what follows, it will be convenient to work with the convex combination defined by

$$
(\hat{u}_s^\varepsilon, \hat{v}_s^\varepsilon) := (u, v) + s(\tilde{u} - u_\varepsilon, \tilde{v} - v_\varepsilon), \quad s \in [0, 1]. \qquad (5.29)
$$

In fact, by (5.23), we see that the scaled displacement fields corresponding to w_s^h converge to $(\hat{u}_s^\varepsilon, \hat{v}_s^\varepsilon)$.

The limits of the mappings $G^h(y^h)$ and $G^h(w_s^h)$ given by Lemma 5.3 are denoted by G_y and G_w^s, i.e, we have (up to a subsequence)

$$
G^h(y^h) \rightharpoonup G_y, \quad G^h(w_s^h) \rightharpoonup G_w^s \text{ weakly in } L^2(\Omega; \mathbb{R}^{3\times 3}), \qquad (5.30)
$$

where the 2×2 submatrices G_y'' and $(G_w^s)''$ satisfy

$$
\mathrm{sym}(G_y'') = G(u, v) \qquad \mathrm{sym}((G_w^s)'') = G(\hat{u}_s^\varepsilon, \hat{v}_s^\varepsilon), \qquad (5.31)
$$

respectively. (Recall notation (5.8).) Above we have checked that the assumptions (5.11) hold for $(z_s^h)_{s,h}$. We can therefore apply Lemma 5.5(b) and obtain

$$
\begin{aligned}
&(i) \ \ \|(G^h(y^h) - G^h(w_s^h)) - \big(G_y - G_w^s\big)\|_{L^2(\Omega)} \leqq s\rho_\varepsilon(h), \\
&(ii) \ \ \|G^h(y^h) - G^h(w_s^h)\|_{L^2(\Omega)} \leqq C_\varepsilon s, \qquad (5.32)
\end{aligned}
$$

where $\rho_\varepsilon(h)$ depends on ε and satisfies $\rho_\varepsilon(h) \to 0$ as $h \to 0$. Moreover, in view of (5.23) and (5.29), an elementary but tedious computation leads to

$$
|\nabla' v_\varepsilon|^2 - |\nabla' \tilde{v}_s|^2 - |\nabla' v|^2 + |\nabla' \hat{v}_s^\varepsilon|^2 = 2s\langle \nabla' v_\varepsilon - \nabla' v, \nabla' v_\varepsilon - \nabla' \tilde{v}\rangle.
$$

Then ([5.27]) and Lemma 5.5(c) yield $\mathrm{sym}(G_y - G^s_w)\, e_3 = s\langle \nabla' v_\varepsilon - \nabla' v, \nabla' v_\varepsilon - \nabla' \tilde v\rangle\, e_3$. Thus, by (5.22) and Hölder's inequality we get

$$\|\mathrm{sym}(G_y - G^s_w)\, e_3\|_{L^2(\Omega)} \leqq s\|\nabla'\tilde v - \nabla' v_\varepsilon\|_{L^4(\Omega)}\|\nabla' v_\varepsilon - \nabla' v\|_{L^4(\Omega)} \leqq \tilde C s\varepsilon. \tag{5.33}$$

Likewise, by (5.8), (5.29), and (5.31) one can check by an elementary expansion that

$$\mathrm{sym}(G_y - G^s_w) = s g_1 + s^2 g_2 \text{ for } g_1, g_2 \in L^2(S; \mathbb{R}^{3\times 3}_{\mathrm{sym}}), \tag{5.34}$$

where g_1, g_2 depend on $u, \tilde u, u_\varepsilon, v, \tilde v, v_\varepsilon$.

Step 4: Lower semicontinuity of slopes. We will show that there exist a continuous function $\eta_\varepsilon : [0,\infty) \to [0,\infty)$ with $\eta_\varepsilon(0) = 0$ and a constant $\tilde C$ depending on $u, v, \tilde u, \tilde v$ such that for all $s \in [0,1]$ it holds that

$$\begin{aligned}
&(i)\ \ \mathcal{D}_h(y^h, w^h_s) \leqq \mathcal{D}_0\big((u,v), (\hat u^\varepsilon_s, \hat v^\varepsilon_s)\big) + s\eta_\varepsilon(h) + \tilde C s\varepsilon,\\
&(ii)\ \ h^{-4}\int_\Omega \big(W(\nabla_h y^h) - W(\nabla_h w^h_s)\big) \geqq \phi_0(u,v) - \phi_0(\hat u^\varepsilon_s, \hat v^\varepsilon_s) - s\eta_\varepsilon(h) - \tilde C s\varepsilon,\\
&(iii)\ \ h^{-p\alpha}\int_\Omega \big(P(\nabla^2_h y^h) - P(\nabla^2_h w^h_s)\big) \geqq -s\eta_\varepsilon(h).
\end{aligned} \tag{5.35}$$

We defer the proof of (5.35) to Steps 5–7 below and now prove the lower semicontinuity. Recall the definition of ϕ_h in (2.14). By combining the three estimates in (5.35), we obtain for all $s \in [0,1]$ that

$$\frac{(\phi_h(y^h) - \phi_h(w^h_s))^+}{\mathcal{D}_h(y^h, w^h_s)} \geqq \frac{\big(\phi_0(u,v) - \phi_0(\hat u^\varepsilon_s, \hat v^\varepsilon_s) - 2s\eta_\varepsilon(h) - s\tilde C\varepsilon\big)^+}{\mathcal{D}_0((u,v),(\hat u^\varepsilon_s, \hat v^\varepsilon_s)) + s\eta_\varepsilon(h) + s\tilde C\varepsilon}.$$

Recall that $y^h \in \mathscr{S}^M_h$ and Theorem 5.6(i) imply $\phi_0(u,v) \leqq M$. By applying Lemma 4.8 with $(u_0, v_0) = (u,v)$ and $(u_1, v_1) = (\hat u^\varepsilon_1, \hat v^\varepsilon_1)$ we get

$$\begin{aligned}
&\frac{(\phi_h(y^h) - \phi_h(w^h_s))^+}{\mathcal{D}_h(y^h, w^h_s)}\\
&\quad\geqq \frac{s\big(\phi_0(u,v) - \phi_0(\hat u^\varepsilon_1, \hat v^\varepsilon_1) - \Phi^2_M\big(\mathcal{D}_0((u,v),(\hat u^\varepsilon_1, \hat v^\varepsilon_1))\big) - 2\eta_\varepsilon(h) - \tilde C\varepsilon\big)^+}{s\Phi^1\big(\mathcal{D}_0((u,v),(\hat u^\varepsilon_1, \hat v^\varepsilon_1))\big) + s\eta_\varepsilon(h) + s\tilde C\varepsilon},
\end{aligned}$$

where Φ^1 and Φ^2_M are the functions introduced in Lemma 4.8. Thus, in view of (5.28), Lemma 4.5(iii), and Definition 3.1, we find, by letting $s \to 0$, that

$$|\partial\phi_h|_{\mathcal{D}_h}(y^h) \geqq \frac{\big(\phi_0(u,v) - \phi_0(\hat u^\varepsilon_1, \hat v^\varepsilon_1) - \Phi^2_M\big(\mathcal{D}_0((u,v),(\hat u^\varepsilon_1, \hat v^\varepsilon_1))\big) - 2\eta_\varepsilon(h) - \tilde C\varepsilon\big)^+}{\Phi^1\big(\mathcal{D}_0((u,v),(\hat u^\varepsilon_1, \hat v^\varepsilon_1))\big) + \eta_\varepsilon(h) + \tilde C\varepsilon}.$$

Letting $h \to 0$ we then derive

$$\liminf_{h\to 0} |\partial\phi_h|_{\mathcal{D}_h}(y^h) \geqq \frac{\big(\phi_0(u,v) - \phi_0((\hat u^\varepsilon_1, \hat v^\varepsilon_1)) - \Phi^2_M\big(\mathcal{D}_0((u,v),(\hat u^\varepsilon_1, \hat v^\varepsilon_1))\big) - \tilde C\varepsilon\big)^+}{\Phi^1\big(\mathcal{D}_0((u,v),(\hat u^\varepsilon_1, \hat v^\varepsilon_1))\big) + \tilde C\varepsilon}.$$

We observe that $\hat{u}_1^\varepsilon \to \tilde{u}$ in $W^{1,2}(S;\mathbb{R}^2)$ and $\hat{v}_1^\varepsilon \to \tilde{v}$ in $W^{2,2}(S;\mathbb{R}^2)$ as $\varepsilon \to 0$, see (5.22) and (5.29). Thus, letting $\varepsilon \to 0$, using Lemma 4.6(iii),(iv), and then taking the supremum with respect to $(\tilde{u}, \tilde{v})$ we get

$$\liminf_{h\to 0} |\partial\phi_h|_{\mathcal{D}_h}(y^h) \geqq \sup\Bigg\{ \frac{\Big(\phi_0(u,v) - \phi_0(\tilde{u},\tilde{v}) - \Phi_M^2\big(\mathcal{D}_0((u,v),(\tilde{u},\tilde{v}))\big)\Big)^+}{\Phi^1\big(\mathcal{D}_0((u,v),(\tilde{u},\tilde{v}))\big)} : (\tilde{u},\tilde{v}) \in \bar{\mathscr{S}}_0^{\mathrm{reg}} \setminus \{(u,v)\}\Bigg\}, \tag{5.36}$$

where $\bar{\mathscr{S}}_0^{\mathrm{reg}} \subset \mathscr{S}_0$ denotes the subset consisting of functions u, v with regularity $W^{2,\infty}$ and $W^{3,\infty}$, respectively. Since each $(\tilde{u},\tilde{v}) \in \mathscr{S}_0$ can be approximated in $W^{1,2}(S;\mathbb{R}^2) \times W^{2,2}(S)$ by a sequence of functions in $\bar{\mathscr{S}}_0^{\mathrm{reg}}$ (see Lemma 5.8) and the right hand side is continuous with respect to that convergence (see Lemma 4.6), the previous inequality also holds for $\mathscr{S}_0$ instead of $\bar{\mathscr{S}}_0^{\mathrm{reg}}$. The representation given in Lemma 4.9 then implies

$$\liminf_{h\to 0} |\partial\phi_h|_{\mathcal{D}_h}(y^h) \geqq |\partial\phi_0|_{\mathcal{D}_0}(u,v).$$

To conclude the proof, it therefore remains to show (5.35).

Step 5: Proof of (5.35)*(i).* By using Lemma 4.3(ii) and (5.32) we get

$$\begin{aligned}\mathcal{D}_h(y^h, w_s^h)^2 &\leqq \int_\Omega Q_D^3\big(G^h(y^h) - G^h(w_s^h)\big) + Ch^\alpha \|G^h(y^h) - G^h(w_s^h)\|^2_{L^2(\Omega)} \\ &\leqq \int_\Omega Q_D^3(G_y - G_w^s) + s^2\big(C_\varepsilon h^\alpha + C(\rho_\varepsilon(h))^2\big) \\ &= \int_\Omega Q_D^3(\mathrm{sym}(G_y - G_w^s)) + s^2\big(C_\varepsilon h^\alpha + C(\rho_\varepsilon(h))^2\big).\end{aligned}$$

Here, the last step follows from the fact that $Q_D^3(F) = Q_D^3(\mathrm{sym}(F))$ for $F \in \mathbb{R}^{3\times 3}$, see Lemma 4.1(ii). Then, using (5.21) and (5.31) we find

$$\begin{aligned}\mathcal{D}_h(y^h, w_s^h)^2 \leqq &\int_\Omega Q_D^2\big(G(u,v) - G(\hat{u}_s^\varepsilon, \hat{v}_s^\varepsilon)\big) \\ &+ C\int_\Omega |\mathrm{sym}(G_y - G_w^s)\, e_3|^2 + s^2\big(C_\varepsilon h^\alpha + C(\rho_\varepsilon(h))^2\big).\end{aligned}$$

By (5.10) and (5.33) we conclude

$$\mathcal{D}_h(y^h, w_s^h)^2 \leqq \mathcal{D}_0\big((u,v),(\hat{u}_s^\varepsilon, \hat{v}_s^\varepsilon)\big)^2 + (\tilde{C}s\varepsilon)^2 + s^2\big(C_\varepsilon h^\alpha + C(\rho_\varepsilon(h))^2\big).$$

This yields (5.35)(i).

Step 6: Proof of (5.35)*(ii).* First, by Lemma 4.3(iv) and (5.32)(ii) we get

$$\begin{aligned}\frac{2}{h^4}\int_\Omega \Big(W(y^h) - W(w_s^h)\Big) \geqq &\int_\Omega \Big(Q_W^3(G^h(y^h)) - Q_W^3(G^h(w_s^h))\Big) \\ &- Ch^\alpha \|G^h(y^h) - G^h(w_s^h)\|_{L^2(\Omega)}\end{aligned}$$

$$\geqq \int_\Omega \Big(Q_W^3(G^h(y^h)) - Q_W^3(G^h(w_s^h)) \Big) - C_\varepsilon h^\alpha s. \tag{5.37}$$

Recall the definition of $\mathbb{C}_W$ in (2.19). An expansion and (5.32)(i) yield

$$\begin{aligned} &\int_\Omega \Big(Q_W^3(G^h(y^h)) - Q_W^3(G^h(w_s^h)) \Big) \\ &\quad = - \int_\Omega \Big(Q_W^3\big(G^h(w_s^h) - G^h(y^h)\big) + 2\mathbb{C}_W^3[G^h(y^h), G^h(w_s^h) - G^h(y^h)] \Big) \\ &\quad \geqq - \int_\Omega \Big(Q_W^3(G_w^s - G_y) + 2\mathbb{C}_W^3[G^h(y^h), G_w^s - G_y] \Big) - Cs\rho_\varepsilon(h). \end{aligned} \tag{5.38}$$

Inequalities (5.37)–(5.38), the weak convergence $G^h(y^h) \rightharpoonup G_y$ in $L^2(\Omega; \mathbb{R}^{3\times 3})$ (see (5.30)) and (5.34) yield

$$\begin{aligned} &\frac{2}{h^4} \int_\Omega \Big(W(y^h) - W(w_s^h) \Big) \\ &\geqq - \int_\Omega \Big(Q_W^3(G_w^s - G_y) + 2\mathbb{C}_W^3[G_y, G_w^s - G_y] \Big) - s\tilde{\rho}_\varepsilon(h) \end{aligned}$$

for some $\tilde{\rho}_\varepsilon(h)$, still satisfying $\tilde{\rho}_\varepsilon(h) \to 0$ as $h \to 0$. Using the fact that $Q_W^3(F) = Q_W^3(\mathrm{sym}(F))$ (see Lemma 4.1(ii)), (5.21), (5.31), and (5.33) we conclude

$$\begin{aligned} &\frac{1}{h^4} \int_\Omega \Big(W(y^h) - W(w_s^h) \Big) \\ &\geqq \int_\Omega \frac{1}{2} \Big(Q_W^3(G_y) - Q_W^3(G_w^s) \Big) - s\tilde{\rho}_\varepsilon(h) \\ &\geqq \int_\Omega \frac{1}{2} \Big(Q_W^2(G(u,v)) - Q_W^2(G(\hat{u}_s^\varepsilon, \hat{v}_s^\varepsilon)) \Big) - s\tilde{\rho}_\varepsilon(h) - \tilde{C}s\varepsilon \\ &= \phi_0(u,v) - \phi_0(\hat{u}_s^\varepsilon, \hat{v}_s^\varepsilon) - s\tilde{\rho}_\varepsilon(h) - \tilde{C}s\varepsilon, \end{aligned}$$

where the last step follows from (5.9).

Step 7: Proof of (5.35)*(iii).* By convexity of P and the definition $w_s^h = y^h - y_\varepsilon^h + \tilde{y}_s^h$, see (5.25), we find

$$h^{-p\alpha} \int_\Omega \big(P(\nabla_h^2 y^h) - P(\nabla_h^2 w_s^h) \big) \geqq h^{-p\alpha} \int_\Omega \partial_Z P(\nabla_h^2 w_s^h) : (\nabla_h^2 y_\varepsilon^h - \nabla_h^2 \tilde{y}_s^h). \tag{5.39}$$

By Hölder's inequality and (2.4)(iii) we get

$$\begin{aligned} &\int_\Omega |\partial_Z P(\nabla_h^2 w_s^h) : (\nabla_h^2 y_\varepsilon^h - \nabla_h^2 \tilde{y}_s^h)| \\ &\quad \leqq \|\partial_Z P(\nabla_h^2 w_s^h)\|_{L^{p/(p-1)}(\Omega)} \|\nabla_h^2 \tilde{y}_s^h - \nabla_h^2 y_\varepsilon^h\|_{L^p(\Omega)} \\ &\quad \leqq C \Big(\int_\Omega P(\nabla_h^2 w_s^h) \Big)^{\frac{p-1}{p}} \|\nabla_h^2 \tilde{y}_s^h - \nabla_h^2 y_\varepsilon^h\|_{L^p(\Omega)}. \end{aligned}$$

Using $\phi_h(w_s^h) \leqq M'$, since $w_s^h \in \mathscr{S}_h^{M'}$ (see Lemma 5.5(a)) and (5.26)(i), we then derive

$$\int_\Omega |\partial_Z P(\nabla_h^2 w_s^h) : (\nabla_h^2 y_\varepsilon^h - \nabla_h^2 \tilde{y}_s^h)| \leqq Csh\Big(\int_\Omega P(\nabla_h^2 w^h)\Big)^{\frac{p-1}{p}}$$
$$\leqq C_\varepsilon sh(h^{\alpha p})^{\frac{p-1}{p}} \leqq C_\varepsilon shh^{\alpha(p-1)},$$

where C_ε depends also on M'. By (5.39) and $1 + \alpha(p-1) - \alpha p > 0$, we finally get that (5.35)(iii) holds. □

Remark 5.11. The previous proof is the only point where we need the assumption that the minimum in (2.17)–(2.18) is attained for $a = 0$ which corresponds to a model with zero Poisson's ratio in e_3 direction. Although this is a restrictive assumption, it is a good approximation for cellular materials such as cork. We also note that similar assumptions already appeared in the literature, see [9]. In fact, the sequence $(w_s^h)_h$ has to be constructed in such a way that it is a recovery sequence (up to an ε-error) for *both* (5.35)(i) and (5.35)(ii). Without this assumption, a sound two-dimensional model would necessarily have to depend on Q_W^3 and Q_D^3 (instead of Q_W^2 and Q_D^2) and extra variables in addition to u and v would be required to capture the extension or contraction of the vertical fibers along the evolution. (Still, we are not sure whether our analysis can be adapted to this case or not.)

6. Proof of the Main Results

In this section we give the proofs of Proposition 2.1-Theorem 2.3.

6.1. Existence of time-discrete solutions in three dimensions and passage from three dimensions to two dimensions

In this short subsection we prove Proposition 2.1 and Theorem 2.3.

Proof of Proposition 2.1. Let $y_0^h \in \mathscr{S}_h^M = \{y \in \mathscr{S}_h : \phi_h(y) \leqq M\}$ for some $M > 0$. We recall that for the choices $\beta_1 = 4 - \alpha p$ and $\beta_2 = 3$ we have $I_h^{\beta_1,\beta_2} = h^4\phi_h$, see (2.14). Moreover, there holds $\mathcal{D}_h = h^{-2}\mathcal{D}$ by (2.15).

It is clear that the minimization problem (2.12) on $\mathscr{S}_h$ can be restricted to the set $\mathscr{S}_h^M$. Then the existence of solutions to the incremental problem (2.12) follows from the direct method of the calculus of variations: Lemma 4.5(ii),(iii) yield compactness with respect to the topology induced by $\mathcal{D}_h$ and Lemma 4.5(iv) implies lower semicontinuity. □

We now proceed with the proof of Theorem 2.3. We formulate our problem in the setting of Section 3.2. We consider the complete metric spaces $(\mathscr{S}_h^M, \mathcal{D}_h)$ and the limiting space $(\mathscr{S}_0, \mathcal{D}_0)$ together with the functionals ϕ_h and ϕ_0. Let σ be the topology on $\mathscr{S}_0$ introduced in (5.18). Recall the definition of the mappings $\pi_h : \mathscr{S}_h^M \to \mathscr{S}_0$ defined by $\pi_h(y^h) = (u^h, v^h)$ for each $y^h \in \mathscr{S}_h$ and the convergence $y^h \stackrel{\pi\sigma}{\to} (u, v)$, see below (5.18) and see also (3.3).

Proof of Theorem 2.3. We consider an initial datum $(u_0, v_0) \in \mathscr{S}_0$. We first see that the family of sequences of initial data $\mathcal{B}(u_0, v_0)$ defined in (2.25) is nonempty. This follows from Theorem 5.6(ii). We check that all assumptions of Theorem 3.2 are satisfied. First, (3.4) holds by Theorem 5.9 and (3.5) follows from Lemma 5.1. Also (3.6) is satisfied by the Γ-liminf inequality (Theorem 5.6(i)) and Theorem 5.10. Finally, the local slope $|\partial\phi_0|_{\mathcal{D}_0}$ is a strong upper gradient for ϕ_0 by Lemma 4.9.

Now we consider a sequence $(y_0^h)_h \in \mathcal{B}(u_0, v_0)$ and a null sequence $(\tau_h)_h$. The definition of $\mathcal{B}(u_0, v_0)$ (see (2.25)) implies (3.7)(ii) with $\bar{z}_0 = (u_0, v_0)$. In particular, as $\pi_h(y_0^h) \overset{\pi\sigma}{\to} (u_0, v_0)$, we get that the sequence $(\pi_h(y_0^h))_h$ is bounded in $W^{1,2}(S; \mathbb{R}^2) \times W^{2,2}(S)$. In view of Lemma 4.6(iii), this yields (3.7)(i).

Let $\tilde{Y}_{h,\tau_h}$ be a sequence of time-discrete solutions as in (2.13) with $\tilde{Y}_{h,\tau_h}(0) = y_0^h$. Then the scalings $I_h^{4-\alpha p,3} = h^4\phi_h$ and $\mathcal{D} = h^2\mathcal{D}_h$ (see (2.14) and (2.15)) imply that $\tilde{Y}_{h,\tau_h}$ is also a time-discrete solution as in (3.1)–(3.2). The statement of Theorem 2.3 now follows from the abstract convergence result formulated in Theorem 3.2. □

6.2. *Fine representation of the slope and solutions to the equations in two dimensions*

This subsection is devoted to the proof of Theorem 2.2. We first note that Theorem 2.2(i) follows directly from Theorem 2.3. Therefore, we only need to show Theorem 2.2(ii). To this end, we derive a fine representation for the local slope in the two-dimensional setting which will allow us to relate curves of maximal slope to solutions to the equations (2.21).

For the following proofs we introduce the abbreviation

$$H(u, v|\tilde{v}) = \mathrm{sym}(\nabla' u) + \mathrm{sym}(\nabla' v \otimes \nabla' \tilde{v}) - x_3(\nabla')^2 v \in L^2(\Omega; \mathbb{R}^{2\times 2}_{\mathrm{sym}}) \quad (6.1)$$

for $(u, v) \in \mathscr{S}_0$ and $\tilde{v} \in W^{2,2}(S)$. This definition captures the linear part of the difference of two strains. More precisely, with $G(u, v)$ and $G(\bar{u}, \bar{v})$ as defined in (5.8), we have by an elementary computation

$$G(u, v) - G(\bar{u}, \bar{v}) = H(u - \bar{u}, v - \bar{v}|v) - \frac{1}{2}(\nabla' v - \nabla' \bar{v}) \otimes (\nabla' v - \nabla' \bar{v}). \quad (6.2)$$

Recall that $\mathbb{C}_D^2$ defined in (2.19) is a fourth order symmetric tensor inducing the quadratic form $G \mapsto Q_D^2(G)$ which is positive definite on $\mathbb{R}^{2\times 2}_{\mathrm{sym}}$ (cf. Lemma 4.1(ii)). Moreover, it maps $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}_{\mathrm{sym}}$, denoted by $G \mapsto \mathbb{C}_D^2 G$ in the following. More precisely, the mapping $G \mapsto \mathbb{C}_D^2 G$ from $\mathbb{R}^{2\times 2}_{\mathrm{sym}}$ to $\mathbb{R}^{2\times 2}_{\mathrm{sym}}$ is bijective. By $\sqrt{\mathbb{C}_D^2}$ we denote its (unique) root and by $\sqrt{\mathbb{C}_D^2}^{-1}$ the inverse of $\sqrt{\mathbb{C}_D^2}$, both mappings defined on $\mathbb{R}^{2\times 2}_{\mathrm{sym}}$. The same properties also hold with $\mathbb{C}_W^2$ in place of $\mathbb{C}_D^2$.

We now prove the following fine representation for the local slope.

Lemma 6.1. (Slope in the two-dimensional setting) *There exists a differential operator $\mathcal{L} : \mathscr{S}_0 \to L^2(\Omega; \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ satisfying*

$$\int_\Omega \mathcal{L}(u, v) : H(\varphi_u, \varphi_v|v) = 0 \quad \textit{for all } (u, v) \in \mathscr{S}_0 \quad \textit{and}$$
$$(\varphi_u, \varphi_v) \in W^{1,2}_0(S; \mathbb{R}^2) \times W^{2,2}_0(S) \tag{6.3}$$

such that the local slope at $(u, v) \in \mathscr{S}_0$ can be represented by

$$|\partial\phi_0|_{\mathcal{D}_0}(u, v) = \left\| \sqrt{\mathbb{C}^2_D}^{-1} \big(\mathbb{C}^2_W G(u, v) + \mathcal{L}(u, v)\big) \right\|_{L^2(\Omega)}.$$

Proof. To simplify the notation, we will write $(\bar{u}, \bar{v}) \to (u, v)$ instead of $\mathcal{D}_0((\bar{u}, \bar{v}), (u, v)) \to 0$. Recall the definition of the energy ϕ_0 and the dissipation $\mathcal{D}_0$ in (2.23) and (2.24), as well as their representations in (5.9)–(5.10). By Definition 3.1(ii) we have

$$\begin{aligned}
|\partial\phi_0|_{\mathcal{D}_0}(u, v) &= \limsup_{(\bar{u},\bar{v})\to(u,v)} \frac{(\phi_0(u, v) - \phi_0(\bar{u}, \bar{v}))^+}{\mathcal{D}_0\big((u, v), (\bar{u}, \bar{v})\big)} \\
&= \limsup_{(\bar{u},\bar{v})\to(u,v)} \frac{\Big(\int_\Omega \frac{1}{2}\big(Q^2_W\big(G(u, v)\big) - Q^2_W\big(G(\bar{u}, \bar{v})\big)\big)\Big)^+}{\Big(\int_\Omega Q^2_D\big(G(u, v) - G(\bar{u}, \bar{v})\big)\Big)^{1/2}} \\
&= \limsup_{(\bar{u},\bar{v})\to(u,v)} \frac{\Big(\int_\Omega \mathbb{C}^2_W[G(u, v), G(u, v) - G(\bar{u}, \bar{v})] - \frac{1}{2} Q^2_W\big(G(u, v) - G(\bar{u}, \bar{v})\big)\Big)^+}{\Big(\int_\Omega Q^2_D\big(G(u, v) - G(\bar{u}, \bar{v})\big)\Big)^{1/2}}.
\end{aligned}$$

This leads to

$$|\partial\phi_0|_{\mathcal{D}_0}(u, v) = \limsup_{(\bar{u},\bar{v})\to(u,v)} \frac{\Big(\int_\Omega \mathbb{C}^2_W[G(u, v), G(u, v) - G(\bar{u}, \bar{v})]\Big)^+}{\Big(\int_\Omega Q^2_D\big(G(u, v) - G(\bar{u}, \bar{v})\big)\Big)^{1/2}}.$$

Indeed, to see this, we use that $(\bar{u}, \bar{v}) \to (u, v)$ implies $G(\bar{u}, \bar{v}) \to G(u, v)$ strongly in $L^2(S; \mathbb{R}^{2\times 2})$ by Lemma 4.6(iii) and (5.8). Thus, we get

$$\int_\Omega Q^2_W\big(G(u, v) - G(\bar{u}, \bar{v})\big)\Big(\int_\Omega Q^2_D\big(G(u, v) - G(\bar{u}, \bar{v})\big)\Big)^{-1/2} \to 0.$$

Lemma 4.6(iii) and a Sobolev embedding also give

$$\begin{aligned}
\|\nabla'(v_0 - v_1) \otimes \nabla'(v_0 - v_1)\|_{L^2(S)} &\leqq C\|v_0 - v_1\|^2_{W^{1,4}(S)} \\
&\leqq C\|v_0 - v_1\|^2_{W^{2,2}(S)} \leqq C\mathcal{D}_0\big((u, v), (\bar{u}, \bar{v})\big)^2.
\end{aligned}$$

This, together with (5.10), (6.1)–(6.2), and the Cauchy-Schwartz inequality, shows that

$$|\partial\phi_0|_{\mathcal{D}_0}(u, v) = \limsup_{(\bar{u},\bar{v})\to(u,v)} \frac{\Big(\int_\Omega \mathbb{C}^2_W[G(u, v), H(u - \bar{u}, v - \bar{v}|v)]\Big)^+}{\Big(\int_\Omega Q^2_D(H(u - \bar{u}, v - \bar{v}|v))\Big)^{1/2}}.$$

We introduce the space of test functions $\mathscr{T} = W^{1,2}_0(S;\mathbb{R}^2) \times W^{2,2}_0(S)$. Due to the linearity of $H(\cdot,\cdot|v)$ we find

$$|\partial\phi_0|_{\mathcal{D}_0}(u,v) = \sup_{(u',v')\in\mathscr{T}} \frac{\int_\Omega \mathbb{C}^2_W[G(u,v), H(u',v'|v)]}{\big(\int_\Omega Q^2_D(H(u',v'|v))\big)^{1/2}} = \sup_{(u',v')\in\mathscr{T}} \frac{\int_\Omega \mathbb{C}^2_W[G(u,v), H(u',v'|v)]}{\|\sqrt{\mathbb{C}^2_D}H(u',v'|v)\|_{L^2(\Omega)}}, \tag{6.4}$$

where in the second step we used the properties of $\mathbb{C}^2_D$. We now consider the minimization problem

$$\min_{(u',v')\in\mathscr{T}} \mathcal{F}(u',v'),$$

where

$$\mathcal{F}(u',v') := \frac{1}{2}\int_\Omega \Big|\sqrt{\mathbb{C}^2_D}H(u',v'|v)\Big|^2 - \int_\Omega \mathbb{C}_W[G(u,v), H(u',v'|v)].$$

We note that the existence of a solution can be guaranteed by the direct method of the calculus of variations; to show coercivity, suppose that $\mathcal{F}(u',v') \leqq C$. We note that $\|H(u',v'|v)\|^2_{L^2(\Omega)} \leqq C$ by Lemma 4.1(ii) with C depending on $G(u,v)$. A standard argument involving Poincaré's inequality and the boundary values yields $\|u'\|_{W^{1,2}(S)} + \|v'\|_{W^{2,2}(S)} \leqq C$, where C additionally depends on $\hat{u}$, $\hat{v}$, and $\|v\|_{W^{2,2}(S)}$. We refer to Remark 4.7 for details. Moreover, the functional is lower semicontinuous as it is convex in $H(u',v'|v)$ and $H(u',v'|v)$ is linear in (u',v').

We denote a solution by $(u_*,v_*)\in\mathscr{T}$ and we observe that (u_*,v_*) satisfies

$$\int_\Omega \sqrt{\mathbb{C}^2_D}H(u_*,v_*|v) : \sqrt{\mathbb{C}^2_D}H(\varphi_u,\varphi_v|v) - \int_\Omega \mathbb{C}^2_W[G(u,v), H(\varphi_u,\varphi_v|v)] = 0$$

for all $(\varphi_u,\varphi_v)\in\mathscr{T}$. This equation can also be formulated as

$$\int_\Omega \mathcal{L}(u,v) : H(\varphi_u,\varphi_v|v) = 0 \tag{6.5}$$

for all $(\varphi_u,\varphi_v)\in\mathscr{T}$, where we define the operator

$$\mathcal{L}(u,v) := \mathbb{C}^2_D H(u_*,v_*|v) - \mathbb{C}^2_W G(u,v).$$

By the definition (6.1) and the regularity of the functions, we find $\mathcal{L}(u,v) \in L^2(\Omega;\mathbb{R}^{2\times 2}_{\mathrm{sym}})$. By (6.4), (6.5), and the definition of $\mathcal{L}$ we then get

$$|\partial\phi_0|_{\mathcal{D}_0}(u,v) \geqq \frac{\int_\Omega \big(\mathbb{C}^2_W G(u,v) + \mathcal{L}(u,v)\big) : H(u_*,v_*|v)}{\|\sqrt{\mathbb{C}^2_D}H(u_*,v_*|v)\|_{L^2(\Omega)}},$$
$$= \frac{\int_\Omega \sqrt{\mathbb{C}^2_D}^{-1}\big(\mathbb{C}^2_W G(u,v) + \mathcal{L}(u,v)\big) : \sqrt{\mathbb{C}^2_D}H(u_*,v_*|v)}{\|\sqrt{\mathbb{C}^2_D}H(u_*,v_*|v)\|_{L^2(\Omega)}}$$

$$=\Big\|\sqrt{\mathbb{C}_D^2}H(u_*,v_*|v)\Big\|_{L^2(\Omega)}=\Big\|\sqrt{\mathbb{C}_D^2}^{-1}\big(\mathbb{C}_W^2 G(u,v)+\mathcal{L}(u,v)\big)\Big\|_{L^2(\Omega)}.$$

On the other hand, by a similar argument, in view of (6.4) and (6.5), we find

$$\begin{aligned}|\partial\phi_0|_{\mathcal{D}_0}(u,v) &= \sup_{(u',v')\in\mathscr{T}} \frac{\int_\Omega \big(\mathbb{C}_W^2 G(u,v)+\mathcal{L}(u,v)\big) : H(u',v'|v)}{\|\sqrt{\mathbb{C}_D^2}H(u',v'|v)\|_{L^2(\Omega)}} \\ &\leqq \Big\|\sqrt{\mathbb{C}_D^2}^{-1}\big(\mathbb{C}_W^2 G(u,v)+\mathcal{L}(u,v)\big)\Big\|_{L^2(\Omega)},\end{aligned}$$

where in the inequality we again distributed $\sqrt{\mathbb{C}_D^2}$ suitably to the two terms and used the Cauchy-Schwartz inequality. This concludes the proof. □

Following ideas in [3, Section 1.4], we now finally relate curves of maximal slope to solutions to the equations (2.21).

Proof of Theorem 2.2(ii). Since $(u(t), v(t))$ is a curve of maximal slope, we get that $\phi_0(u(t), v(t))$ is decreasing in time, see (3.8). This together with Lemma 4.6(ii) gives

$$(u,v)\in L^\infty([0,\infty);W^{1,2}(S;\mathbb{R}^2)\times W^{2,2}(S)).$$

Moreover, since $|(u,v)'|_{\mathcal{D}_0}\in L^2([0,\infty))$ by (3.8) and $\mathcal{D}_0$ is equivalent to the strong topology on $W^{1,2}(S;\mathbb{R}^2)\times W^{2,2}(S)$ (see Lemma 4.6(iii)), we observe that u and v are absolutely continuous curves in the Hilbert spaces $W^{1,2}(S;\mathbb{R}^2)$ and $W^{2,2}(S;\mathbb{R})$, respectively. By using [3, Remark 1.1.3] we observe that u and v are differentiable for a.e. t with $\partial_t u(t)\in W^{1,2}(S;\mathbb{R}^2)$ and $\partial_t v(t)\in W^{2,2}(S)$ for a.e. t. More precisely, we have

$$(u,v)\in W^{1,2}([0,\infty);W^{1,2}(S;\mathbb{R}^2)\times W^{2,2}(S)) \tag{6.6}$$

and for all $0\leqq s<t$, and almost everywhere in S it holds that

$$\begin{aligned}&\nabla' u(t)-\nabla' u(s)=\int_s^t \partial_t\nabla' u(r)\,\mathrm{d}r,\\ &\nabla' v(t)-\nabla' v(s)=\int_s^t \partial_t\nabla' v(r)\,\mathrm{d}r,\quad (\nabla')^2 v(t)-(\nabla')^2 v(s)=\int_s^t \partial_t(\nabla')^2 v(r)\,\mathrm{d}r.\end{aligned} \tag{6.7}$$

As a preparation for the representation of the metric derivative, we now consider the difference $G(u(s), v(s)) - G(u(t), v(t)$. For a.e. t and a.e. $x\in\Omega$, we obtain by (6.2) and the linearity of $H(\cdot,\cdot\,|\,v(t))$ that

$$\begin{aligned}&\lim_{s\to t}\frac{G(u(t),v(t))-G(u(s),v(s))}{t-s}\\ &\quad=\lim_{s\to t} H\Big(\frac{u(t)-u(s)}{t-s},\frac{v(t)-v(s)}{t-s}\,\Big|\,v(t)\Big)\\ &\quad-\lim_{s\to t}(\nabla' v(t)-\nabla' v(s))\otimes\frac{\nabla' v(t)-\nabla' v(s)}{2(t-s)}\end{aligned}$$

$$= H\big(\partial_t u(t), \partial_t v(t) \,|\, v(t)\big). \tag{6.8}$$

Similarly, by (6.2), taking the integral over Ω, using (6.7), Poincaré's inequality, and Hölder's inequality, we get, for all $0 \leqq s < t$,

$$\begin{aligned}
&\Big\| \big(G(u(t), v(t)) - G(u(s), v(s))\big) - \int_s^t H(\partial_t u(r), \partial_t v(r) \,|\, v(t))\, \mathrm{d}r \Big\|^2_{L^2(\Omega)} \\
&\quad \leqq \int_\Omega |\nabla' v(t) - \nabla' v(s)|^4 \leqq C\Big(\int_\Omega |(\nabla')^2 (v(t) - v(s))|^2 \Big)^2 \\
&\quad = C\Big(\int_\Omega \Big| \int_s^t (\nabla')^2 \partial_t v(r,x)\, \mathrm{d}r \Big|^2 \mathrm{d}x \Big)^2 \\
&\quad \leqq C\Big(|t-s| \int_\Omega \int_s^t |(\nabla')^2 \partial_t v(r,x)|^2\, \mathrm{d}r\, \mathrm{d}x \Big)^2 \\
&\quad = C|t-s|^2 \Big(\int_s^t \|(\nabla')^2 \partial_t v(r)\|^2_{L^2(\Omega)}\, \mathrm{d}r \Big)^2.
\end{aligned} \tag{6.9}$$

We now estimate the metric derivative $|(u,v)'|_{\mathcal{D}_0}$. By (5.10), (6.8), and Fatou's lemma, we get, for a.e. t,

$$\begin{aligned}
|(u,v)'|_{\mathcal{D}_0}(t) &= \lim_{s\to t} \left(\frac{\mathcal{D}_0\big((u(t), v(t)), (u(s), v(s))\big)^2}{|t-s|^2} \right)^{1/2} \\
&\geqq \Big(\int_\Omega \liminf_{s\to t} Q_D^2 \Big(\frac{G(u(t), v(t)) - G(u(s), v(s))}{|t-s|} \Big) \Big)^{1/2} \\
&= \Big\| \sqrt{\mathbb{C}_D^2}\, H\big(\partial_t u(t), \partial_t v(t) \,|\, v(t)\big) \Big\|_{L^2(\Omega)}.
\end{aligned} \tag{6.10}$$

We now analyze the derivative $\frac{\mathrm{d}}{\mathrm{d}t}\phi_0(u(t), v(t))$ of the absolutely continuous curve $\phi_0 \circ (u,v)$. Note that for a.e. t we have $\lim_{s\to t} \int_s^t \|(\nabla')^2 \partial_t v(r)\|^2_{L^2(\Omega)}\, \mathrm{d}r = 0$ by (6.6) and, in a similar fashion, $\lim_{s\to t} (s-t)^{-1} \| \int_s^t H(\partial_t u(r), \partial_t v(r) \,|\, v(t)) \|^2_{L^2(\Omega)}\, \mathrm{d}r = 0$ by (6.6) and Hölder's inequality. Thus, using (5.9) and (6.9), we get, for a.e. t, that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\phi_0(u(t), v(t)) &= \lim_{s\to t} \frac{\phi_0(u(t), v(t)) - \phi_0(u(s), v(s))}{t-s} \\
&\geqq \liminf_{s\to t} \frac{1}{(t-s)} \int_\Omega \mathbb{C}_W^2 [G(u(t), v(t)), G(u(t), v(t)) - G(u(s), v(s))] \\
&\quad - \limsup_{s\to t} \frac{1}{2(t-s)} \int_\Omega Q_W^2 \big(G(u(t), v(t)) - G(u(s), v(s))\big) \\
&\geqq \liminf_{s\to t} \frac{1}{(t-s)} \int_\Omega \mathbb{C}_W^2 [G(u(t), v(t)), G(u(t), v(t)) - G(u(s), v(s))] \\
&= \int_\Omega \mathbb{C}_W^2 [G(u(t), v(t)), H(\partial_t u(t), \partial_t v(t) \,|\, v(t))].
\end{aligned}$$

By the property of $\mathcal{L}$ stated in (6.3) we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\phi_0(u(t), v(t)) &\geqq \int_\Omega \big(\mathbb{C}_W^2 G(u(t), v(t)) + \mathcal{L}(u(t), v(t)) \big) : H\big(\partial_t u(t), \partial_t v(t) \,|\, v(t)\big) \\
&= \int_\Omega \Big(\sqrt{\mathbb{C}_D^2}^{-1} \big(\mathbb{C}_W^2 G(u(t), v(t)) + \mathcal{L}(u(t), v(t)) \big)
\end{aligned}$$

$$: \sqrt{\mathbb{C}_D^2} H\big(\partial_t u(t), \partial_t v(t) \,|\, v(t)\big)\Big).$$

We find by Lemma 6.1, (6.10), and Young's inequality,

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_0(u(t), v(t)) \geqq -\frac{1}{2}\Big(|\partial\phi_0|^2_{\mathcal{D}_0}(u(t), v(t)) + |(u,v)'|^2_{\mathcal{D}_0}(t)\Big) \geqq \frac{\mathrm{d}}{\mathrm{d}t}\phi_0(u(t), v(t)),$$

where the last step is a consequence of the fact that $(u(t), v(t))$ is a curve of maximal slope with respect to ϕ_0. Consequently, all inequalities employed in the proof are in fact equalities and we get

$$\sqrt{\mathbb{C}_D^2}^{-1}\big(\mathbb{C}_W^2 G(u(t), v(t)) + \mathcal{L}(u(t), v(t))\big) = -\sqrt{\mathbb{C}_D^2} H(\partial_t u(t), \partial_t v(t) \,|\, v(t))$$

pointwise a.e. in Ω for a.e. t. Multiplying the equation with $\sqrt{\mathbb{C}_D^2}$ from the left and testing with $H(\varphi_u, \varphi_v \,|\, v(t))$ from the right for $(\varphi_u, \varphi_v) \in W_0^{1,2}(S;\mathbb{R}^2) \times W_0^{2,2}(S)$, we obtain

$$\int_\Omega \Big(\mathbb{C}_W^2 G(u(t), v(t)) + \mathbb{C}_D^2 H\big(\partial_t u(t), \partial_t v(t) \,|\, v(t)\big)\Big) : H(\varphi_u, \varphi_v \,|\, v(t)) = 0, \tag{6.11}$$

where we again used property (6.3). In what follows we will again use the representation $G(u(t), v(t)) = \mathrm{sym}(G_0)(t) + x_3 G_1(t)$ with the abbreviations

$$\mathrm{sym}(G_0)(t) = e(u(t)) + \frac{1}{2}\nabla' v(t) \otimes \nabla' v(t), \qquad G_1(t) = -(\nabla')^2 v(t)$$

introduced in (5.8). Consider also the corresponding time derivatives

$$\partial_t G_0(t) = e(\partial_t u(t)) + \nabla'\partial_t v(t) \odot \nabla' v(t), \qquad \partial_t G_1(t) = -(\nabla')^2\partial_t v(t),$$

where for convenience we use the notation $\odot$ for the symmetrized vector product. Recall also (6.1) and observe that $H(\partial_t u(t), \partial_t v(t) \,|\, v(t)) = \partial_t G_0(t) + x_3\partial_t G_1(t)$. Evaluating (6.11) for $\varphi_v = 0$ leads to the equations

$$\int_S \big(\mathbb{C}_W^2 G_0(t) + \mathbb{C}_D^2 \partial_t G_0(t)\big) : \nabla'\varphi_u = 0. \tag{6.12}$$

We observe that (6.12) gives (2.22a). Evaluating (6.11) for $\varphi_u = 0$ yields

$$\int_S \big(\mathbb{C}_W^2 G_0(t) + \mathbb{C}_D^2 \partial_t G_0(t)\big) : \big(\nabla' v(t) \odot \nabla'\varphi_v\big)$$
$$- \frac{1}{12}\Big(\mathbb{C}_W^2 G_1(t) + \mathbb{C}_D^2 \partial_t G_1(t)\Big) : (\nabla')^2\varphi_v = 0.$$

This gives the second equation (2.22b) and confirms that (u, v) is a weak solution to (2.21). □

Acknowledgements. This work was funded by the DFG Project FR 4083/1-1 and by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy EXC 2044 -390685587, Mathematics Münster: Dynamics–Geometry–Structure. M.F. acknowledges support by the Alexander von Humboldt Stiftung and thanks for the warm hospitality at ÚTIA AVČR, where this project has been initiated. M.K. acknowledges support by the GAČR project 17-04301S and GAČR-FWF project 19-29646L.

References

1. Abels, H., Mora, M.G., Müller, S.: The time-dependent von Kármán plate equation as a limit of 3d nonlinear elasticity. *Calc. Var. PDE* **41**, 241–259, 2011
2. Adams, R.A., Fournier, J.J.F.: *Sobolev Spaces*, 2nd edn. Elsevier, Amsterdam 2003
3. Ambrosio, L., Gigli, N., Savaré, G.: *Gradient Flows in Metric Spaces and in the Space of Probability Measures. Lectures Math. ETH Zürich*. Birkhäuser, Basel 2005
4. Antman, S.S.: Physically unacceptable viscous stresses. *Z. Angew. Math. Phys*. **49**, 980–988, 1998
5. Antman, S.S.: *Nonlinear Problems of Elasticity*. Springer, New York 2004
6. Ball, J.M.: Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Ration. Mech. Anal*. **63**, 337–403, 1977
7. Ball, J.M., Currie, J.C., Olver, P.L.: Null Lagrangians, weak continuity, and variational problems of arbitrary order. *J. Funct. Anal*. **41**, 135–174, 1981
8. Batra, R.C.: Thermodynamics of non-simple elastic materials. *J. Elasticity* **6**, 451–456, 1976
9. Bella, P., Kohn, R.V.: Coarsening of folds in hanging drapes. *Commun. Pure Appl. Math*. **70**, 978–1021, 2017
10. Benešová, B., Kružík, M.: Weak lower semicontinuity of integral functionals and applications. *SIAM Rev*. **59**, 703–766, 2017
11. Bock, I.: On Von Kármán equations for viscoelastic plates. *J. Comput. Appl. Math*. **63**, 277–282, 1995
12. Bock, I., Jarušek, J.: Solvability of dynamic contact problems for elastic von Kármán plates. *SIAM J. Math. Anal*. **41**, 37–45, 2009
13. Bock, I., Jarušek, J., Šilhavý, M.: On the solutions of a dynamic contact problem for a thermoelastic von Kármán plate. *Nonlinear Anal. Real World Appl*. **32**, 111–135, 2016
14. Braides, A., Colombo, M., Gobbino, M., Solci, M.: Minimizing movements along a sequence of functionals and curves of maximal slope. *C. R. Math*. **354**, 685–689, 2016
15. Capriz, G.: Continua with latent microstructure. *Arch. Ration. Mech. Anal*. **90**, 43–56, 1985
16. Casarino, V., Percivale, D.: A variational model for nonlinear elastic plates. *J. Convex Anal*. **3**, 221–243, 1996
17. Ciarlet, P.G.: *Mathematical Elasticity, Vol. I: Three-Dimensional Elasticity*. North-Holland, Amsterdam 1988
18. Dal Maso, G.: *An Introduction to*Γ*-Convergence*. Birkhäuser, Boston 1993
19. De Giorgi, E., Marino, A., Tosques, M.: Problems of evolution in metric spaces and maximal decreasing curve. *Att. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur*. **68**, 180–187, 1980
20. Dunn, J.E., Serrin, J.: On the thermomechanics of interstitial working. *Arch. Ration. Mech. Anal*. **88**, 95–133, 1985
21. Föppl, A.: Vorlesungen über technische Mechanik. *Leipzig* **5**, 132–139, 1907

22. Friedrich, M., Kružík, M., Valdman, J.: *Numerical approximation of von Kármán viscoelastic plates*. Disc. Cont. Dynam. Syst.-S https://doi.org/10.3934/dcdss.2020322
23. Friesecke, G., James, R.D., Müller, S.: A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity. *Commun. Pure Appl. Math*. **55**, 1461–1506, 2002
24. Friesecke, G., James, R.D., Müller, S.: A hierarchy of plate models derived from nonlinear elasticity by Gamma-Convergence. *Arch. Ration. Mech. Anal*. **180**, 183–236, 2006
25. Healey, T.J., Krömer, S.: Injective weak solutions in second-gradient nonlinear elasticity. *ESAIM Control Optim. Calc. Var*. **15**, 863–871, 2009
26. Krömer, S., Roubícek, T.: Quasistatic viscoelasticity with self-contact at large strains. *J. Elasticity* (to appear).
27. Kružík, M., Roubícek, T.: *Mathematical Methods in Continuum Mechanics of Solids*. Springer, Cham 2019
28. Lecumberry, M., Müller, S.: Stability of slender bodies under compression and validity of the von Kármán theory. *Arch. Ration. Mech. Anal*. **193**, 255–310, 2009
29. Le Dret, H., Raoult, A.: The nonlinear membrane model as a variational limit of nonlinear three-dimensional elasticity. *J. Math. Pures Appl*. **73**, 549–578, 1995
30. Le Dret, H., Raoult, A.: The membrane shell model in nonlinear elasticity: a variational asymptotic derivation. *J. Nonlinear Sci*. **6**, 59–84, 1996
31. Lewicka, M., Mahadevan, L., Pakzad, M.R.: The Monge-Ampére constraint matching: of isometries, density and regularity and elastic theories of shallow shells. *Ann. IHP (C) Nonlinear Anal*. **34**, 45–67, 2017
32. Maddalena, F., Percivale, D., Tomarelli, F.: Variational problems for Föppl-von Kármán plates. *SIAM J. Math. Anal*. **50**, 251–282, 2018
33. Mielke, A., Ortner, C., Şengül, Y.: An approach to nonlinear viscoelasticity via metric gradient flows. *SIAM J. Math. Anal*. **46**, 1317–1347, 2014
34. Mielke, A., Roubícek, T.: Rate-independent elastoplasticity at finite strains and its numerical approximation. *Math. Models Methods Appl. Sci*. **26**, 2203–2236, 2016
35. Mielke, A., textscRoubíček, T.: *Thermoviscoelasticity in Kelvin-Voigt rheology at large strains*. Arch. Ration. Mech. Anal. https://doi.org/10.1007/s00205-020-01537-z. Preprint at arXiv:1903.11094
36. Ortner, C.: *Two Variational Techniques for the Approximation of Curves of Maximal Slope. Technical report NA05/10*. Oxford University Computing Laboratory, Oxford 2005
37. Pantz, O.: On the justification of the nonlinear inextensional plate model. *Arch. Ration. Mech. Anal*. **167**, 179–209, 2003
38. Park, J.Y., Kang, J.R.: Uniform decay of solutions for von Karman equations of dynamic viscoelasticity with memory. *Acta Appl. Math*. **110**, 1461–1474, 2010
39. Podio-Guidugli, P.: Contact interactions, stress, and material symmetry for nonsimple elastic materials. *Theor. Appl. Mech*. **28–29**, 261–276, 2002
40. Sandier, E., Serfaty, S.: Gamma-convergence of gradient flows with applications to Ginzburg-Landau. *Commun. Pure Appl. Math*. **57**, 1627–1672, 2004
41. Serfaty, S.: Gamma-convergence of gradient flows on Hilbert and metric spaces and applications. *Discrete Contin. Dyn. Syst. Ser. A* **31**, 1427–1451, 2011
42. Toupin, R.A.: Elastic materials with couple stresses. *Arch. Ration. Mech. Anal*. **11**, 385–414, 1962
43. Toupin, R.A.: Theory of elasticity with couple stress. *Arch. Ration. Mech. Anal*. **17**, 85–112, 1964
44. von Kármán, T.: *Festigkeitsprobleme im Maschinenbau in Encyclopädie der Mathematischen Wissenschaften*. vol. IV/4, Leipzig, 311–385, 1910.

M. FRIEDRICH
Applied Mathematics,
Universität Münster,
Einsteinstr. 62,
48149 Münster
Germany.
e-mail: manuel.friedrich@uni-muenster.de

and

M. KRUŽÍK
Czech Academy of Sciences, Institute of Information Theory and Automation,
Pod vodárenskou věží 4,
182 00 Praha 8
Czechia.

and

Faculty of Civil Engineering,
Czech Technical University,
Thákurova 7,
166 29 Praha 6
Czechia.
e-mail: kruzik@utia.cas.cz

(*Received February 27, 2019 / Accepted June 3, 2020*)
Published online June 16, 2020