ATTRACTORS FOR STOCHASTIC REACTION-DIFFUSION EQUATION WITH ADDITIVE HOMOGENEOUS NOISE

Jakub Slavík, Praha

Received March 28, 2019. Published online August 18, 2020.

Abstract. We study the asymptotic behaviour of solutions of a reaction-diffusion equation in the whole space $\mathbb{R}^d$ driven by a spatially homogeneous Wiener process with finite spectral measure. The existence of a random attractor is established for initial data in suitable weighted L^2-space in any dimension, which complements the result from P. W. Bates, K. Lu, and B. Wang (2013). Asymptotic compactness is obtained using elements of the method of short trajectories.

Keywords: reaction-diffusion equation; random attractor; spatially homogeneous noise

MSC 2020: 35B41, 60H15, 37L55, 35K57

1. Introduction

We study asymptotic behaviour of the reaction-diffusion equation (RDE)

$$\partial_t u(t,x) - \Delta u(t,x) + f(u(t,x)) = g(t) + \dot{W}(t,x), \quad (t,x) \in (0,T) \times \mathbb{R}^d, \tag{1.1}$$

driven by a spatially homogeneous Wiener process W with nonlinear reaction term f, deterministic forcing g and initial data in weighted L^2-space.

The study of SPDEs driven by spatially homogeneous Wiener processes originates in Dawson and Salehi [15], where a linear equation driven by multiplicative noise has been considered as a model of growth of a population in random environment. Existence and uniqueness of semilinear parabolic SPDE in weighted L^2 spaces has been established by Peszat and Zabczyk in [23] and [24]. The theory has been extended to weighted L^p spaces by Brzeźniak and Peszat, see [8], for second order operators and to more general operators in Brzeźniak and van Neerven, see [9]. Concerning asymptotic properties, Dawson and Salehi in [15] have provided conditions for exis-

DOI: 10.21136/CMJ.2020.0144-19

tence of a stationary random field. Existence of invariant measures has been shown by Tessitore and Zabczyk, see [25].

In this paper we show that a random dynamical system generated by equation (1.1) in weighted Lebesgue spaces possesses a random attractor. A similar result has been obtained by Bates, Lu and Wang, see [4], for parabolic SPDEs in weighted spaces driven by noise of the form $\dot{w}(t)h(x)$, where w is a one-dimensional two-sided Wiener process and $h \in C^2(\mathbb{R}^d)$ is compactly supported.

From the mathematical point of view, the problem of the existence of a random attractor in this setting is interesting for several reasons. First, since (1.1) is posed in the whole space, we cannot rely on standard compact embeddings and need to employ localization arguments. Secondly, the L^2-regularity of the initial data does not allow us to obtain dissipative estimates in $W^{1,2}$. This is solved by using elements of the method of short trajectories, see Málek and Pražák [21].

In deterministic setting, the attractors of evolution equations in the whole space are usually studied in the context of locally uniform spaces, see e.g. Feireisl [16] and Zelik [28], since these spaces are considered to be better suited as the spaces of initial data, see Cholewa and Dlotko [11]. However, the nonseparability of locally uniform space make their use in SPDEs more difficult. In particular, one needs the separability of the phase space to obtain measurability of the cocycle mapping. Moreover, even in the deterministic setting the solutions of evolution equations in locally uniform spaces are not even strongly measurable in general. We postpone the study of asymptotic behaviour of solutions of (1.1) in locally uniform spaces to a subsequent paper.

The approach presented here combines the technique for deterministic RDEs from Grasselli, Pražák and Schimperna, see [19], with results on spatially homogeneous Wiener proceses and stochastic integration by Peszat and Zabczyk, see [23], and Bŕezniak and van Neerven, see [9]. The existence of a suitable Ornstein-Uhlenbeck process is obtained following a method by Brzeźniak and Li, see [7] originally used to establish the existence of an invariant measure for stochastic 2D Navier-Stokes equations in unbounded domains.

Let us now state the main result of this paper. For precise definitions of the random attractor, function spaces and spatially homogeneous Wiener process see Sections 2, 3 and 4, respectively. Let μ be a finite tempered symmetric measure on $\mathbb{R}^d$ and let H_μ be the reproducing kernel Hilbert space of the spatially homogeneous Wiener process W_μ given by (4.1) on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$. Let $\mathcal{D}$ denote the tempered sets in $L^2(\varrho)$.

Assume that the nonlinear function $f \in C(\mathbb{R}, \mathbb{R})$ satisfies $f(0) = 0$ and

$$|f(r) - f(s)| \leqslant c_1(1 + |r| + |s|)^{p-2}|r - s| \quad \forall\, r, s \in \mathbb{R}, \tag{1.2}$$

$$(f(r) - f(s))(r - s) \geqslant -c_2|r - s|^2 \quad \forall\, r, s \in \mathbb{R}, \tag{1.3}$$

$$c_3|r|^p - c_4 \leqslant f(r)r \leqslant c_4(|r|^p + 1) \quad \forall\, r \in \mathbb{R}, \tag{1.4}$$

hold with $c_i > 0$ for some $p \in (2, \infty)$. A standard example of a function satisfying (1.2)–(1.4) is the function $f(r) = r|r|^{p-2}$.

Theorem 1.1. *Assume that $f \in C(\mathbb{R}, \mathbb{R})$ satisfies $f(0) = 0$ and (1.2)–(1.4) and let $g \in L^2_{\rm loc}(\mathbb{R}, L^2(\varrho))$ be such that*

$$\int_{-\infty}^{\tau} \mathrm{e}^{\beta s}\|g(s)\|^2_{L^2(\varrho)}\, \mathrm{d}s < \infty \tag{1.5}$$

for all $\tau \in \mathbb{R}$ and for $\beta > 0$ sufficiently small. Then the random dynamical system associated to (1.1) possesses a random $\mathcal{D}$-attractor.

The rest of the paper is organised as follows. In Section 2 we recall basic definitions from the theory of random dynamical systems and an abstract existence result. In Section 3 we define the weighted function spaces used in the rest of the paper and recall their properties. Section 4 contains the definition and basic properties of spatially homogeneous Wiener processes and their reproducing kernel Hilbert spaces. In Section 5 we establish the existence of an Ornstein-Uhlenbeck process driven by a spatially homogeneous Wiener process. In Section 6 we recall the existence and uniqueness result for a time-dependent RDE in weighted spaces and show that the random dynamical system associated to (1.1) has an absorbing set and is asymptotically compact, which leads to the existence of a random attractor.

2. Random dynamical systems

We briefly review basic concepts from the theory of random attractors and dynamical systems (DS). For details on the theory of random attractors and proofs see [3], [12], [13], [17]. In particular, for the notion of random attractor suited for a nonautonomous stochastic equations see e.g. [10] and [27]. For more detailed treatment of the theory of random dynamical systems we refer the reader to the monograph [1].

Throughout this section, let (X, d) be a Polish space, Ω_1 be a nonempty set and $(\Omega_2, \mathcal{F}_2, \mathbb{P})$ be a complete probability space.

Definition 2.1. We call a couple $(\Omega_1, \{\vartheta_{1,t}\}_{t\in\mathbb{R}})$ a *parametric dynamical system* if $\vartheta_{1,t}\colon \Omega_1 \to \Omega_1$ for all $t \in \mathbb{R}$, $\vartheta_{1,0}$ is the identity on Ω_1 and $\vartheta_{1,t+s} = \vartheta_{1,t} \circ \vartheta_{1,s}$ for all $t, s \in \mathbb{R}$.

Definition 2.2. A triple $(\Omega_2, \mathcal{F}_2, \{\vartheta_{2,t}\}_{t\in\mathbb{R}})$ is called a *measurable dynamical system* if $\vartheta_2\colon \mathbb{R}\times\Omega_2 \to \Omega_2$ is $(\mathcal{B}(\mathbb{R})\otimes\mathcal{F}_2, \mathcal{F}_2)$-measurable, $\vartheta_{2,0}$ is the identity on Ω_2 and $\vartheta_{2,t+s} = \vartheta_{2,t}\circ\vartheta_{2,s}$ holds for all $s, t \in \mathbb{R}$.

The quadruple $(\Omega_2, \mathcal{F}_2, \mathbb{P}, \{\vartheta_{2,t}\}_{t\in\mathbb{R}})$ is called a *metric dynamical system* if the triple $(\Omega_2, \mathcal{F}_2, \{\vartheta_{2,t}\}_{t\in\mathbb{R}})$ is a measurable DS and for all $t \in \mathbb{R}$ the map $\vartheta_{2,t}\colon \Omega_2 \to \Omega_2$ preserves the measure $\mathbb{P}$.

Definition 2.3. A map $\varphi\colon [0,\infty)\times\Omega_1\times\Omega_2\times X \to X$ is called a *measurable random dynamical system* (RDS) over a parametric DS $(\Omega_1, \{\vartheta_{1,t}\}_{t\in\mathbb{R}})$ and a metric DS $(\Omega_2, \mathcal{F}_2, \mathbb{P}, \{\vartheta_{2,t}\}_{t\in\mathbb{R}})$ if

(1) $\varphi(\cdot, \omega_1, \cdot, \cdot)$ is $(\mathcal{B}([0,\infty))\otimes\mathcal{F}_2\otimes\mathcal{B}(X), \mathcal{B}(X))$-measurable for all $\omega_1 \in \Omega_1$,
(2) $\varphi(0, \omega_1, \omega_2, \cdot)$ is the identity on X for all $\omega_1 \in \Omega_1$, $\omega_2 \in \Omega_2$,
(3) for all $s, t \geqslant 0$, $\omega_1 \in \Omega$ and $\omega_2 \in \Omega_2$, the *cocycle property* holds:

$$\varphi(t+s, \omega_1, \omega_2, \cdot) = \varphi(t, \vartheta_{1,s}\omega_1, \vartheta_{2,s}\omega_2, \cdot)\circ\varphi(s, \omega_1, \omega_2, \cdot). \tag{2.1}$$

Moreover, if the mapping $\varphi(t, \omega_1, \omega_2, \cdot)\colon X \to X$ is continuous for all $t \in [0,\infty)$, $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$, we call φ a *continuous RDS*.

In the rest of this subsection let φ be a continuous RDS over the parametric DS $(\Omega_1, \{\vartheta_{1,t}\}_{t\in\mathbb{R}})$ and the metric DS $(\Omega_2, \mathcal{F}_2, \mathbb{P}, \{\vartheta_{2,t}\}_{t\in\mathbb{R}})$. By $\mathcal{D}$ we will denote a collection of set-valued mappings $D\colon \Omega_1\times\Omega_2 \to 2^X\setminus\{\emptyset\}$ indexed by $\iota \in I$ if the need arises, i.e.

$$\mathcal{D} = \{D_\iota\}_{\iota\in I}, \quad D_\iota = \{\emptyset \neq D_\iota(\omega_1, \omega_2) \subseteq X\}_{\omega_1\in\Omega_1, \omega_2\in\Omega_2} \tag{2.2}$$

for all $\iota \in I$, $\omega_1 \in \Omega_1$, $\omega_2 \in \Omega_2$. For $A, B \subseteq X$ we define

$$\operatorname{dist}(A, B) = \sup_{a\in A}\inf_{b\in B} d(a, b).$$

Definition 2.4. Let $D\colon \Omega_1\times\Omega_2 \to 2^X$ be a set-valued mapping. We call D an *$\mathcal{F}_2$-measurable random set* if $D(\omega_1, \omega_2)$ is nonempty and closed for all $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$ and the mapping $\Omega_2 \ni \omega_2 \to \operatorname{dist}(x, D(\omega_1, \omega_2))$ is $(\mathcal{F}_2, \mathcal{B}(\mathbb{R}))$-measurable for all $\omega_1 \in \Omega_1$ and $x \in X$.

Definition 2.5. We say that a collection $D = \{D(\omega_1, \omega_2)\}_{\omega_1\in\Omega_1, \omega_2\in\Omega_2}$ of nonempty subsets of X is *tempered* w.r.t. $(\Omega_1, \{\vartheta_{1,t}\}_{t\in\mathbb{R}})$ and $(\Omega_2, \mathcal{F}_2, \mathbb{P}, \{\vartheta_{2,t}\}_{t\in\mathbb{R}})$ if there exists $x \in X$ such that for all $\beta > 0$, $\omega_1 \in \Omega_1$, $\omega_2 \in \Omega_2$

$$\lim_{t\to\infty} \mathrm{e}^{-\beta t}\operatorname{dist}(D(\vartheta_{1,-t}\omega_1, \vartheta_{2,-t}\omega_2), x) = 0.$$

We denote the set of all tempered random sets in X by $\mathcal{D}$.

Definition 2.6. A family $\mathcal{A} = \{\mathcal{A}(\omega_1, \omega_2)\}_{\omega_1 \in \Omega_1, \omega_2 \in \Omega_2} \in \mathcal{D}$ is called a *$\mathcal{D}$-random attractor* of the RDS φ if

(1) $\mathcal{A}$ is an $\mathcal{F}_2$-measurable random set and $\mathcal{A}(\omega_1, \omega_2)$ is compact for all $\omega_1 \in \Omega_1$, $\omega_2 \in \Omega_2$,

(2) $\mathcal{A}$ is forward φ-invariant, i.e. for all $t \geqslant 0$, $\omega_1 \in \Omega_2$ and $\omega_2 \in \Omega_2$

$$\varphi(t, \omega_1, \omega_2, \mathcal{A}(\omega_1, \omega_2)) = \mathcal{A}(\vartheta_{1,t}\omega_1, \vartheta_{2,t}\omega_2),$$

(3) $\mathcal{A}$ is pullback $\mathcal{D}$-attracting, i.e. for all $D \in \mathcal{D}$

$$\lim_{t \to \infty} \operatorname{dist}(\varphi(t, \vartheta_{1,-t}\omega_1, \vartheta_{2,-t}\omega_2, D(\vartheta_{1,-t}\omega_1, \vartheta_{2,-t}\omega_2)), A(\omega_1, \omega_2)) = 0.$$

Definition 2.7. A family $K \in \mathcal{D}$ of bounded sets is called *$\mathcal{D}$-absorbing* if for all $\omega_1 \in \Omega_1$, $\omega_2 \in \Omega_2$ and $D \in \mathcal{D}$ there exists $t_0 = t_0(\omega_1, \omega_2, D)$ such that for all $t \geqslant t_0$ it holds

$$\varphi(t, \vartheta_{1,-t}\omega_1, \vartheta_{2,-t}\omega_2, D(\vartheta_{1,-t}\omega_1, \vartheta_{2,-t}\omega_2)) \subseteq K(\omega_1, \omega_2).$$

If K is also an $\mathcal{F}_2$-measurable random set, we call K a *closed measurable $\mathcal{D}$-absorbing set.*

Definition 2.8. We call a RDS φ *$\mathcal{D}$-asymptotically compact* if the set

$$\{\varphi(t_n, \vartheta_{1,-t_n}\omega_1, \vartheta_{2,-t_n}\omega_2, x_n);\ n \in \mathbb{N}\}$$

is relatively compact in X for every positive sequence $t_n \to \infty$ and all $x_n \in D(\vartheta_{1,-t_n}\omega_1, \vartheta_{2,-t_n}\omega_2)$ for all $D \in \mathcal{D}$ and $\omega_1 \in \Omega_1$, $\omega_2 \in \Omega_2$.

We recall the following standard existence criterion. For details on the proof see [3], [27].

Theorem 2.1. *Assume that a continuous RDS φ is $\mathcal{D}$-asymptotically compact and possesses a closed measurable $\mathcal{D}$-absorbing set. Then φ has a $\mathcal{D}$-random attractor unique in the class of tempered sets.*

3. Function spaces

Unless stated otherwise, all the function spaces are considered on the spatial domain $\mathbb{R}^d$.

Definition 3.1. A function $\varrho\colon \mathbb{R}^d \to (0, \infty)$ is called a *weight function* if $\varrho \in C^2$ and there exist $\varrho_1, \varrho_2 > 0$ such that

$$\Big|\frac{\partial \varrho}{\partial x_j}(x)\Big| \leqslant \varrho_1 \varrho(x), \quad \Big|\frac{\partial^2 \varrho}{\partial x_j \partial x_k}(x)\Big| \leqslant \varrho_2 \varrho(x) \tag{3.1}$$

for all $x \in \mathbb{R}^d$ and $1 \leqslant j, k \leqslant d$.

If ϱ is a weight function in the sense of the previous definition, then also $\varrho \in L^1$. Let $\gamma, \varepsilon > 0$. Define

$$\langle x \rangle^{\gamma} = (1 + |x|^2)^{-\gamma/2}, \quad \varrho_{\gamma,\varepsilon}(x) = \langle \sqrt{\varepsilon} x \rangle^{\gamma}. \tag{3.2}$$

In the following sections we will be working mostly with the weight functions $\varrho = \varrho_{\gamma,\varepsilon}$ with $\gamma > d$, since then the weighted Lebesgue space $L^p(\varrho)$ defined below contains all bounded measurable functions, in particular constants and travelling wave solutions. Moreover, obviously

$$|\nabla \varrho_{\gamma,\varepsilon}(x)| \leqslant C\varepsilon \varrho_{\gamma,\varepsilon}(x) \tag{3.3}$$

for all $x \in \mathbb{R}^d$.

Let $p \in [1, \infty)$. The *weighted Lebesgue space* is defined by

$$L^p(\varrho) = \left\{ u \in L^p_{\rm loc};\ \|u\|_{L^p(\varrho)} = \left(\int_{\mathbb{R}^d} |u(x)|^p \varrho(x) \, {\rm d}x \right)^{1/p} < \infty \right\}.$$

The weighted space $L^p(\varrho)$ is an M-type 2 Banach space for $p \in [2, \infty)$ on account of being a UMD space of type 2, see e.g. [6], Proposition 2.11.

Let $T\colon L^p(\varrho) \to L^p$ be the map defined by $T(f) = f\varrho^{1/p}$. By [2], Section 5.2, the map T is an isomorphism between L^p and $L^p(\varrho)$ and therefore the spaces $L^p(\varrho)$ are separable for $p \in [1, \infty)$ and reflexive for $p \in (1, \infty)$. We note that for $p = 2$ the space $L^2(\varrho)$ is a Hilbert space. The inner product on L^2 and $L^2(\varrho)$ will be denoted by $(\cdot, \cdot)$ and $(\cdot, \cdot)_{\varrho}$, respectively.

Let $\gamma > d$, $0 < \varepsilon_1 < \varepsilon_2$ and $u \in L^p(\varrho_{\gamma,\varepsilon_2})$. Then clearly

$$\begin{aligned} \int_{\mathbb{R}^d} |u(x)|^p \varrho_{\gamma,\varepsilon_2}(x) \, {\rm d}x &\leqslant \int_{\mathbb{R}^d} |u(x)|^p \varrho_{\gamma,\varepsilon_1}(x) \, {\rm d}x \leqslant \int_{\mathbb{R}^d} |u(x)|^p \frac{\varrho_{\gamma,\varepsilon_1}(x)}{\varrho_{\gamma,\varepsilon_2}(x)} \varrho_{\gamma,\varepsilon_2}(x) \, {\rm d}x \\ &\leqslant C_{\varepsilon_1,\varepsilon_2} \int_{\mathbb{R}^d} |u(x)|^p \varrho_{\gamma,\varepsilon_2}(x) \, {\rm d}x, \end{aligned} \tag{3.4}$$

in other words, with $\gamma > d$ fixed, the $L^p(\varrho_{\gamma,\varepsilon_1})$-norm is equivalent to the $L^p(\varrho_{\gamma,\varepsilon_2})$-norm.

For $k \in \mathbb{N}$ and $p \in [1, \infty)$ we may define the *weighted Sobolev space* $W^{k,p}(\varrho)$ as the space of all $L^p(\varrho)$ functions with respective weak derivatives in $L^p(\varrho)$ with the obvious norm. It is straightforward to show that smooth compactly supported functions are dense in $W^{k,p}(\varrho)$.

The Fourier transform of $\phi \in \mathscr{S}$ and its inverse is defined by

$$(\mathcal{F}\phi)(\xi) = (2\pi)^{-d} \int_{\mathbb{R}^d} {\rm e}^{-{\rm i}\langle x,\xi \rangle} \phi(x) \, {\rm d}x, \quad (\mathcal{F}^{-1}\phi)(\xi) = (2\pi)^{-d} \int_{\mathbb{R}^d} {\rm e}^{{\rm i}\langle x,\xi \rangle} \phi(x) \, {\rm d}x,$$

for $\xi \in \mathbb{R}^d$. The Fourier transform on the space of distributions $\mathscr{S}'$ and its inverse is defined through duality

$$\langle \mathcal{F}\Phi, \phi \rangle = \langle \Phi, \mathcal{F}\phi \rangle, \quad \langle \mathcal{F}^{-1}\Phi, \phi \rangle = \langle \Phi, \mathcal{F}^{-1}\phi \rangle, \quad \phi \in \mathscr{S}, \Phi \in \mathscr{S}'.$$

For $\theta \in \mathbb{R}$ and $p \in [1, \infty)$ we define the *weighted Bessel potential spaces* $H_p^\theta(\varrho)$ by

$$H_p^\theta(\varrho) = \{u \in \mathscr{S}';\ \|u\|_{H_p^\theta(\varrho)} = \|\mathcal{F}^{-1}[(1 + |\cdot|^2)^{\theta/2}\mathcal{F}u\|_{L^p(\varrho)} < \infty\}.$$

Concerning embeddings of weighted spaces, the embedding $W^{k,p}(\varrho) \hookrightarrow L^q(\varrho)$ holds only for $1 \leqslant q \leqslant p$, which can be seen by considering the function $\varrho^{-a/p}$ for $a \in [0, 1]$, and cannot be improved. Embeddings similar to the ones in $\mathbb{R}^d$ can be obtained if we allow different rates of decrease (i.e. different exponents γ in (3.2)), for more information see e.g. [20].

We conclude the section by recalling the properties of the operator $A_\alpha = \Delta + \alpha I$ for $\alpha > 0$ in weighted spaces. For the proof see [2], Theorem 5.1.

Theorem 3.1. *Let $p \in (1, \infty)$. Then the operator A_α is a sectorial operator in $L^p(\varrho)$ with domain $W^{2,p}(\varrho)$. The operator $-A_\alpha$ generates a C^0 analytic semigroup and the fractional spaces of the operator coincide with the spaces $H_p^\theta(\varrho)$.*

4. Spatially homogeneous Wiener process

We follow the presentation from [9]. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be a complete filtered probability space. Recall that a measure μ on $\mathbb{R}^d$ is *tempered* if μ is a Radon measure on $\mathbb{R}^d$ that is also a tempered distribution and *symmetric* if $\mu(B) = \mu(-B)$ for all μ-measurable sets $B \subseteq \mathbb{R}^d$.

From now on let μ be a finite tempered symmetric measure on $\mathbb{R}^d$. Let $f_{(s)}(x) = \overline{f(-x)}$. We define

$$L^2_{(s)}(\mu) = \{f \in L^2_{\mathbb{C}}(\mu); f(x) = f_{(s)}(x)\ \forall\, x \in \mathbb{R}^d\}.$$

The space $L^2_{(s)}(\mu)$ is a closed subspace of $L^2_{\mathbb{C}}(\mu)$, the space of square μ-integrable functions with complex values, and forms a separable real Hilbert space with the inner product

$$[f, g]_{L^2_{(s)}(\mu)} = \int_{\mathbb{R}^d} f(\xi)\overline{g(\xi)}\,\mathrm{d}\mu(\xi), \quad f, g \in L^2_{(s)}(\mu).$$

Definition 4.1. The space $\mathscr{H}_\mu$ is the real separable Hilbert space obtained as the closure of $\mathscr{S}$ w.r.t. the norm generated by the inner product

$$[\phi,\psi]_{\mathscr{H}_\mu} = [\mathcal{F}\phi,\mathcal{F}\psi]_{L^2_{(s)}(\mu)}, \quad \phi,\psi\in\mathscr{S}.$$

Given $f\in L^2_{\mathbb{C}}(\mu)$ it is easily checked that the map $f\mu\colon \mathscr{S}\to\mathbb{R}$ given by

$$f\mu\colon \phi\to\int_{\mathbb{R}^d}\phi(\xi)f(\xi)\,\mathrm{d}\mu(\xi), \quad \phi\in\mathscr{S}_{\mathbb{C}},$$

is a tempered distribution $f\mu\in\mathscr{S}'_{\mathbb{C}}$.

Definition 4.2. The space H_μ is defined by

$$H_\mu = \{\mathcal{F}^{-1}(f\mu);\, f\in L^2_{(s)}(\mu)\}. \tag{4.1}$$

The space H_μ is a separable Hilbert space w.r.t. the inner product

$$[\mathcal{F}^{-1}(f\mu),\mathcal{F}^{-1}(g\mu)]_{H_\mu} = [f,g]_{L^2_{(s)}(\mu)}, \quad f,g\in L^2_{(s)}(\mu).$$

Definition 4.3. A *spatially homogeneous Wiener process* is a continuous $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted $\mathscr{S}'$-valued process $W=\{W(t)\}_{t\geqslant 0}$ with the following properties:

(i) For all $\phi\in\mathscr{S}$, $\{\langle\phi,W(t)\rangle\}_{t\geqslant 0}$ is a $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted real-valued Wiener process.

(ii) For all $t\geqslant 0$ the distribution of $W(t)$ is invariant with respect to transformations $\tau'_h\colon \mathscr{S}'\to\mathscr{S}'$ for all $h\in\mathbb{R}^d$, where τ_h is the translation defined by

$$(\tau_h\phi)(x)=\phi(x-h), \quad \phi\in\mathscr{S},\ x,h\in\mathbb{R}^d.$$

By [18], pages 169 and 264 the condition (ii) is equivalent to the following:

(ii′) There exists a symmetric tempered measure μ on $\mathbb{R}^d$ such that for all $\varphi,\psi\in\mathscr{S}$, $s,t\geqslant 0$ and with $\Gamma=\mathcal{F}\mu$ we have

$$\mathbb{E}[\langle\varphi,W(t)\rangle\cdot\langle\psi,W(s)\rangle] = (t\wedge s)\langle\Gamma,\varphi*\psi_{(s)}\rangle = (t\wedge s)[\varphi,\psi]_{\mathscr{H}_\mu}. \tag{4.2}$$

If (ii′) holds, the measure μ is uniquely determined by the spatially homogeneous Wiener process W and is called the *spectral measure* of the process W. In what follows, we denote the spatially homogeneous Wiener process with spectral measure μ by W_μ.

The space H_μ is the reproducing kernel Hilbert space of the spatially homogeneous Wiener process W_μ. By [22], Lemma 1 we have the following result: If there exists $\delta \in \mathbb{N} \cup \{0\}$ and $C_\delta > 0$ such that

$$\int_{\mathbb{R}^d} (C_\delta + |\xi|^2)^\delta \, \mathrm{d}\mu(\xi) < \infty, \tag{4.3}$$

then $H_\mu \hookrightarrow BUC^\delta(\mathbb{R}^d)$, which suffices to observe that $H_\mu \hookrightarrow L^p(\varrho)$. The following simple lemma provides additional information on the embeddings of the reproducing kernel Hilbert space H_μ into weighted Bessel spaces.

Lemma 4.1. *Let* (4.3) *hold for some* $\delta \in [0, \infty)$. *Let* $p \in [1, \infty)$ *and let* ϱ *be a weight function satisfying* (3.1). *Then* $H_\mu \hookrightarrow H_p^\delta(\varrho)$.

Proof. Let $h \in H_\mu$ and $f \in L^2_{(s)}(\mu)$ be such that $h = \mathcal{F}^{-1}(f\mu)$. By Hölder's inequality we have

$$\begin{aligned}\|\mathcal{F}^{-1}(f\mu)\|^p_{H^\delta_p} &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\langle x,\xi\rangle} (1 + |\xi|^2)^{\delta/2} f(\xi) \, \mathrm{d}\mu(\xi) \right|^p \varrho(x) \, \mathrm{d}x \\ &\leqslant C\|f\|^p_{L^2_{(s)}(\mu)} \|(1 + |\cdot|^2)^{\delta/2}\|^p_{L^2_{(s)}(\mu)} \int_{\mathbb{R}^d} \varrho(x) \, \mathrm{d}x \leqslant C' \|\mathcal{F}^{-1}(f\mu)\|^p_{H_\mu}.\end{aligned}$$

□

Lemma 4.2. *The embedding* $H_\mu \to L^p(\varrho)$ *is* γ*-radonifying if and only if* $\varrho \in L^1$. *Moreover, if the embedding is* γ*-radonifying, the spatially homogeneous Wiener process* W_μ *is* $L^p(\varrho)$*-valued.*

Proof. Let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis of $L^2_{(s)}(\mu)$ and let $h_n = \mathcal{F}^{-1}(e_n\mu)$. It is easy to check that $\{h_n\}_{n=1}^\infty$ is an orthonormal basis of H_μ. Using Parseval's identity we have

$$\sum_{n=1}^\infty |h_n(x)|^2 = (2\pi)^{-d} \|\mathrm{e}^{\mathrm{i}\langle x,\cdot\rangle}\|^2_{L^2_{(s)}(\mu)} = (2\pi)^{-d} \mu(\mathbb{R}^d) \tag{4.4}$$

for all $x \in \mathbb{R}^d$. By [26], Lemma 2.1 the embedding $H_\mu \to L^p(\varrho)$ is γ-radonifying if and only if $\left(\sum_n |h_n(x)|^2\right)^{1/2} \in L^p(\varrho)$, which by (4.4) is true if and only if $\varrho \in L^1$.

Let E be a separable Banach space. By a variant of [23], Proposition 1.1, if the embedding $H_\mu \to E$ is γ-radonifying, the spatially homogeneous Wiener process W_μ is E-valued. The final assertion immediately follows. □

The definition of stochastic integral w.r.t. a spatially homogeneous Wiener process can be found e.g. in [14], Section 4.2.

5. Ornstein-Uhlenbeck process

In this section we show that given $\alpha > 0$ and a spatially homogeneous Wiener process $\{W_\mu(t)\}_{t\in\mathbb{R}}$ on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$, the equation

$$\mathrm{d}z(t) + (-\Delta - \alpha I)z(t)\,\mathrm{d}t = \mathrm{d}W_\mu(t), \quad t \in \mathbb{R}, \tag{5.1}$$

has an $L^p(\varrho)$-valued solution. We adapt the definition of weak solution from [9].

Definition 5.1. A predictable $\mathscr{S}'$-valued process $\{z(t)\}_{t\in\mathbb{R}}$ is called a *weak solution* of (5.1) if for all $\psi \in \mathscr{S}$ we have $r \to \langle z(r), \Delta\psi\rangle \in L^1_{\mathrm{loc}}(\mathbb{R})$ a.s. and for all $t, s \in \mathbb{R}$, $t > s$, we have

$$\langle z(t), \psi\rangle = \langle z(s), \psi\rangle + \int_s^t \langle z(r), \Delta\psi\rangle + \alpha\langle z(r), \psi\rangle\,\mathrm{d}r + \langle W_\mu(t) - W_\mu(s), \psi\rangle \quad \text{a.s.}$$

As before, let $A_\alpha = \Delta + \alpha I$. Let $q_\alpha(\xi) = |\xi|^2 + \alpha$ and define

$$D(\tilde{A}_\alpha) = \{\Psi \in \mathscr{S}';\ q_\alpha(\cdot)\mathcal{F}\Psi \in \mathscr{S}'\}, \quad \tilde{A}_\alpha\Psi = \mathcal{F}^{-1}(q(\cdot)\mathcal{F}\Psi), \quad \Psi \in D(\tilde{A}_\alpha).$$

The operator $-\tilde{A}_\alpha$ is the infinitesimal generator of the semigroup

$$S(t)\Psi = \mathcal{F}^{-1}(\mathrm{e}^{-tq_\alpha(\cdot)}\mathcal{F}\Psi), \quad \Psi \in \mathscr{S}'. \tag{5.2}$$

Lemma 5.1. *Let $\varrho \in L^1$. The integral*

$$z_\alpha(t) = \int_{-\infty}^t S(t-s)\,\mathrm{d}W_\mu(s) \tag{5.3}$$

converges in $L^p(\varrho)$ for all $t \in \mathbb{R}$.

P r o o f. Since the semigroup $S(t)$ and the semigroup e^{-tA_α} defined in Theorem 3.1 coincide on H_μ, by [9], Proposition 6.3 and a limiting argument, it suffices to show the convergence of the integral

$$z_\alpha(t) = \int_{-\infty}^t S(t-s)\,\mathrm{d}W_{H_\mu}, \tag{5.4}$$

where $\{W_{H_\mu}(t)\}_{t\in\mathbb{R}}$ is the two-sided H_μ-cylindrical Wiener process from [9], Proposition 6.2 and [23], Proposition 1.1 associated to the spatially homogenous Wiener process $\{W_\mu(t)\}_{t\in\mathbb{R}}$ and the integral is the Itô integral on the M-type 2 Banach space $L^p(\varrho)$ in the sense of [6]. It is clear that it suffices to show that

$$\int_0^\infty \|S(t)\|^2_{\gamma(H_\mu,L^p(\varrho))}\,\mathrm{d}t<\infty. \tag{5.5}$$

Since

$$(S(t)h)(x)=(2\pi)^{-d}(\mathcal{F}^{-1}[\mathrm{e}^{-q_\alpha(\cdot)}\mathrm{e}^{\mathrm{i}\langle x,\cdot\rangle}\mu],h)_{H_\mu}$$

for all $x\in\mathbb{R}^d$ and $h\in H_\mu$, by [8], Proposition 2.1 we have

$$\|S(t)\|^2_{\gamma(H_\mu,L^p(\varrho))}\leqslant C\biggl(\int_{\mathbb{R}^d}\biggl|\int_{\mathbb{R}^d}\mathrm{e}^{-2tq_\alpha(\xi)}\,\mathrm{d}\mu(\xi)\biggr|^{p/2}\varrho(x)\,\mathrm{d}x\biggr)^{2/p}\leqslant C\mathrm{e}^{-2t\alpha}\|\varrho\|^{2/p}_{L^1}\mu(\mathbb{R}^d).$$

The integral (5.4) converges since (5.5) holds by the estimate above. □

In the rest of this section we recall a technique from [7] that imposes additional structure on the probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in\mathbb{R}},\mathbb{P})$ required later. Let ϱ be a weight function satisfying (3.1). Let $\gamma\in(0,1/2)$ and define

$$E=\mathrm{cl}_{\|\cdot\|_E}A_\alpha^{-\gamma}L^p(\varrho),\quad \text{where}\quad \|x\|_E=\|A_\alpha^{-\gamma}x\|_X,\ x\in L^p(\varrho). \tag{5.6}$$

Note that then $X\subseteq E$ and the operator $A_\alpha^{-\gamma}\colon E\to X$ is bounded. For $\xi\in(0,1/2)$ we set

$$\|\omega\|_{C^\xi_{1/2}(\mathbb{R},E)}=\sup_{t,s\in\mathbb{R},t\neq s}\frac{\|\omega(t)-\omega(s)\|_E}{|t-s|^\xi(1+|t|+|s|)^{1/2}}.$$

We also define

$$C^\xi_{1/2}(\mathbb{R},E)=\{\omega\in C(\mathbb{R},E);\ \omega(0)=0,\|\omega\|_{C^\xi_{1/2}(\mathbb{R},E)}<\infty\},$$
$$\Omega(\xi,E)=\mathrm{cl}_{C^\xi_{1/2}(\mathbb{R},E)}\ \{\omega\in C_0^\infty(\mathbb{R},E);\ \omega(0)=0\}.$$

It can be shown that the space $\Omega(\xi,E)$ is a separable Banach space. We also define

$$C_{1/2}(\mathbb{R},E)=\Bigl\{\omega\in C(\mathbb{R},E);\ \|\omega\|_{C_{1/2}(\mathbb{R},E)}=\sup_{t\in\mathbb{R}}\frac{\|\omega(t)\|_E}{1+|t|^{1/2}}<\infty\Bigr\}.$$

Let $\mathcal{F}$ denote the Borel σ-algebra on $\Omega(\xi,E)$. For $t\in\mathbb{R}$ let $\mathcal{F}_t=\sigma(\{w_s;s\leqslant t\})$. Using the technique from [5] one can show that there exists a probability measure $\mathbb{P}$ on $(\Omega(\xi,E),\mathcal{F})$ such that the canonical process

$$w(t)(\omega)=i_t(\omega),\quad \omega\in\Omega(\xi,E),$$

where $i_t(\omega) = \omega(t)$, is an E-valued two-sided Wiener process. The map

$$\mathcal{W}\colon\ E^* \to L^2(\Omega(\xi, E)), \quad \mathcal{W}(z) = \langle i_t(\cdot), z\rangle_{E,E^*}$$

can be uniquely extended to a bounded linear map

$$W_{H_\mu}(t)\colon\ H_\mu \to L^2(\Omega(\xi, E), \{\mathcal{F}_t\}_{t\in\mathbb{R}}).$$

The process $\{W_{H_\mu}(t)\}_{t\in\mathbb{R}}$ is a two-sided H_μ-cylindrical Wiener process on the filtered probability space $(\Omega(\xi, E), \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$.

Let $\{\vartheta_t\}_{t\in\mathbb{R}}$ be the group on $C_{1/2}(\mathbb{R}, X)$ defined by

$$(\vartheta_t\omega)(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in C_{1/2}(\mathbb{R}, X), \quad t \in \mathbb{R}. \tag{5.7}$$

It is easily seen that the spaces $C^{\xi}_{1/2}(\mathbb{R}, X)$ and $\Omega(\xi, X)$ are invariant under ϑ_t. For simplicity we will denote the restriction of ϑ_t to these spaces by ϑ_t as well.

Before we proceed to the main theorem, we recall an analytical existence result, see [7], Proposition 6.2.

Proposition 5.1. *Let X be a separable Banach space and let A be a generator of an analytic semigroup e^{-tA} such that for some $C > 0$, $\gamma \in (0, 1/2)$ and $\nu > 0$ we have*

$$\|A^{1+\gamma}\mathrm{e}^{-tA}\|_{L(X)} \leqslant Ct^{-1-\gamma}\mathrm{e}^{-\nu t} \quad \forall\, t > 0.$$

For $t \in \mathbb{R}$, $\xi \in (\gamma, 1/2)$ and $\widetilde{\omega} \in C^{\xi}_{1/2}(\mathbb{R}, X)$ let

$$\tilde{z}(t) = \tilde{z}(\widetilde{\omega})(t) = \int_{-\infty}^{t} A^{1+\gamma}\mathrm{e}^{-(t-r)A}(\widetilde{\omega}(t) - \widetilde{\omega}(r))\,\mathrm{d}r.$$

Then $\tilde{z}(t)$ is a well-defined element of X and the mapping

$$C^{\xi}_{1/2}(\mathbb{R}, X) \ni \widetilde{\omega} \to \tilde{z}(t) \in X$$

is continuous. Moreover, the map $\tilde{z}\colon\ C^{\xi}_{1/2}(\mathbb{R}, X) \to C_{1/2}(\mathbb{R}, X)$ is linear and bounded. In particular, there exists $C > 0$ such that for all $\widetilde{\omega} \in C^{\xi}_{1,2}(\mathbb{R}, X)$ we have

$$\|\tilde{z}(\widetilde{\omega})(t)\|_X \leqslant C(1 + |t|^{1/2})\|\widetilde{\omega}\|_{C^{\xi}_{1/2}(\mathbb{R},X)}. \tag{5.8}$$

Theorem 5.1. *Let $\{W_\mu(t)\}_{t\in\mathbb{R}}$ be a two-sided spatially homogeneous Wiener process on a complete filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in\mathbb{R}},\mathbb{P})$. Then the process $\{z_\alpha(t)\}_{t\in\mathbb{R}}$ defined by (5.3) is an $L^p(\varrho)$-valued solution of (5.1). Moreover, if ϱ satisfies (3.1), then $z_\alpha\colon \mathbb{R}\to L^p(\varrho)$ is continuous and for all $\omega\in\Omega$ there exists $C(\omega)>0$ such that*

$$\|z_\alpha(\omega)(t)\|_{L^p(\varrho)} \leqslant C(\omega)(1+|t|^{1/2}),\quad t\in\mathbb{R}. \tag{5.9}$$

In particular, the growth estimate (5.9) implies

$$\int_{-\infty}^{\tau} \mathrm{e}^{\beta s}\|z(\omega)(s)\|^p_{L^p(\varrho)}\,\mathrm{d}s<\infty \tag{5.10}$$

for all $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $\beta>0$.

Proof. The existence proof follows immediately from Lemma 5.1, see also [9], Theorem 9.1. The rest of the proof follows the technique from [7], Section 6. By Lemma 4.2 the embedding $H_\mu\to L^p(\varrho)$ is γ-radonifying. Let $\gamma\in(0,1/2)$ and let $\xi\in(\gamma,1/2)$. Define the space E by (5.6). By the argument from [5] there exists a canonical representation of $\{W_\mu(t)\}_{t\in\mathbb{R}}$ on the space $(\Omega(\xi,E),\mathcal{F},\{\mathcal{F}_t\}_{t\in\mathbb{R}},\mathbb{P})$.

Let e^{-tA_α} be the C_0 analytic semigroup generated by operator $-A_\alpha$ in the space $L^p(\varrho)$ from Theorem 3.1. Using standard estimates for C_0 analytic semigroups and sectorial operators we obtain

$$\|A_\alpha^{1+\gamma}\mathrm{e}^{-tA_\alpha}\|_{L(L^p(\varrho))}=\|A_\alpha\mathrm{e}^{-A_\alpha t/2}A_\alpha^{\gamma}\mathrm{e}^{-A_\alpha t/2}\|_{L(L^p(\varrho))}\leqslant C_3t^{-1-\gamma}\mathrm{e}^{-\beta t}$$

for some $\beta>0$ and all $t>0$. Let $\omega\in C^\xi_{1/2}(\mathbb{R},E)$ and define

$$\tilde z_\alpha(\omega)(t)=\int_{-\infty}^{t}A_\alpha^{1+\gamma}\mathrm{e}^{-(t-r)A_\alpha}(A_\alpha^{-\gamma}\omega(t)-A_\alpha^{-\gamma}\omega(r))\,\mathrm{d}r.$$

By Proposition 5.1, $\tilde z_\alpha(t)\in L^p(\varrho)$ is well defined for all $t\in\mathbb{R}$. Similarly as in [7], Proposition 6.10 we can show that the process $\{\tilde z_\alpha(t)\}_{t\in\mathbb{R}}$ is stationary and is a solution of the equation

$$\mathrm{d}\tilde z_\alpha(t)+(-\Delta-\alpha I)\tilde z_\alpha\,\mathrm{d}t=\mathrm{d}W_{H_\mu}(t),\quad t\in\mathbb{R},$$

in particular that $\tilde z_\alpha$ is given by the right-hand side of (5.4). Using the integral equivalence from [9], Proposition 6.3 we get $z_\alpha=\tilde z_\alpha$. Estimate (5.9) follows directly from (5.8) by setting $C(\omega)=C\|A_\alpha^{-\gamma}\omega\|_{C^\xi_{1/2}(\mathbb{R},L^p(\varrho))}$. □

6. Random dynamical system and random attractor

In this section we show that the equation

$$\mathrm{d}u(t) + (-\Delta u(t) + f(u(t)))\,\mathrm{d}t = g(t)\,\mathrm{d}t + \mathrm{d}W_\mu(t), \quad u(t_0) = x, \tag{6.1}$$

where $\{W_\mu\}_{t\in\mathbb{R}}$ is a two-sided spatially homogeneous Wiener process on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$, has a unique solution and the random dynamical system associated to (6.1) possesses a random attractor.

In the first part of this section we recall an existence result for nonautonomous RDEs from [19] and establish a dissipative estimate. Then we show the existence of a RDS associated to (6.1). Finally, we prove the main result, Theorem 1.1.

From now on we assume that $\varrho = \varrho_{\gamma,1}$ with $\gamma > d$ in the notation of (3.2) and $f \in C(\mathbb{R},\mathbb{R})$ satisfies $f(0) = 0$ and (1.2)–(1.4).

6.1. Well-posedness of time dependent RDE. To establish the well-posedness of the stochastic RDE (6.1) driven by the spatially homogeneous Wiener process $\{W_\mu(t)\}_{t\in\mathbb{R}}$, we employ the classical argument and split the solution $u = v + z$, where z is the Ornstein-Uhlenbeck process from Theorem 5.1 associated to the operator $A_\alpha = \Delta + \alpha I$ and v is the solution of the random PDE

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} v(t) - \Delta v(t) + f(v(t) + z(t)) + \alpha z(t) = g(t), \quad t > t_0, \\ v(t_0) = v_{t_0}. \end{cases} \tag{6.2}$$

We assume that the deterministic and random forcing g and z have the regularity

$$g \in L^2_{\mathrm{loc}}(\mathbb{R}; L^2(\varrho)), \quad z \in L^p_{\mathrm{loc}}(\mathbb{R}; L^p(\varrho)). \tag{6.3}$$

Definition 6.1. A function v is a solution to (6.2) if for all $T > t_0$ we have

$$v \in L^\infty(t_0, T; L^2(\varrho)) \cap L^2(t_0, T; W^{1,2}(\varrho)) \cap L^p(t_0, T; L^p(\varrho))$$

and the equality

$$\begin{aligned} (v(T), \psi(T))_\varrho &+ \int_{t_0}^T (\nabla v(t), \nabla\psi(t))_\varrho + (\nabla v(t), \psi(t)\nabla\varrho(t))\,\mathrm{d}t \\ &+ \int_{t_0}^T \alpha(z(t), \psi(t))_\varrho + (f(v(t) + z(t), \psi(t))_\varrho\,\mathrm{d}t \\ &= (v_{t_0}, \psi)_\varrho + \int_{t_0}^T (g(t), \psi(t))_\varrho\,\mathrm{d}t \end{aligned} \tag{6.4}$$

holds for all

$$\psi \in L^{\infty}(t_0, T; L^2(\varrho)) \cap L^2(t_0, T; W^{1,2}(\varrho)) \cap L^{p'}(t_0, T; L^{p'}(\varrho)). \tag{6.5}$$

Note that even though the last integral on the first line of (6.4) contains standard L^2 inner product, it is well defined thanks to (3.1). We recall the existence result from [19].

Theorem 6.1. *Let* (1.2)–(1.4) *and* (6.3) *hold. Then given* $u_0 \in L^2(\varrho)$, *equation* (6.2) *has a unique solution* $v = v(\cdot, t_0, z, v_{t_0})$. *Moreover, if* $z_n \in L^p(t_0, T; L^p(\varrho))$ *and* $v_n \in L^2(\varrho)$ *satisfy*

$$z_n \to z \text{ in } L^p(t_0, T; L^p(\varrho)), \quad v_n \to v_0 \text{ in } L^2(\varrho) \tag{6.6}$$

for some $T > t_0$, *then*

$$v(t, t_0, z_n, v_n) \to v(t, t_0, z, v_0) \text{ in } L^2(\varrho) \quad \forall\, t_0 \leqslant t \leqslant T. \tag{6.7}$$

P r o o f. The existence and uniqueness is shown in [19], Theorem 3.2. Choosing $\varepsilon > 0$ sufficiently small, we test equation (6.2) by v, integrate and use (1.4), (1.2) and (3.3) to get

$$\begin{aligned}
&\|v(t)\|^2_{L^2(\varrho_{\gamma,\varepsilon})} + \int_{t_0}^t \|v(s)\|^2_{W^{1,2}(\varrho_{\gamma,\varepsilon})} + \|v(s) + z(s)\|^p_{L^p(\varrho_{\gamma,\varepsilon})} \,\mathrm{d}s \\
&\quad \leqslant C\biggl(\|v_{t_0}\|^2_{L^2(\varrho_{\gamma,\varepsilon})} + \int_{t_0}^t \|g(s)\|^2_{L^2(\varrho_{\gamma,\varepsilon})} + \|z(s)\|^p_{L^p(\varrho_{\gamma,\varepsilon})} + \|z(s)\|^2_{L^2(\varrho_{\gamma,\varepsilon})} \,\mathrm{d}s\biggr).
\end{aligned} \tag{6.8}$$

Let g_n, z_n and v_n be as in (6.6). Let $v^n = v(t, t_0, z_n, v_n)$ and $v = v(t, t_0, z, v_0)$. We subtract the equations for v and v^n, test by $w = v^n - v$ and integrate to obtain

$$\begin{aligned}
\|w(t)\|^2_{L^2(\varrho)} &+ \int_{t_0}^t \|\nabla w(s)\|^2_{L^2(\varrho)} + (\nabla w(s), w(s)\nabla\varrho) \,\mathrm{d}s \\
&+ \int_{t_0}^t (f(v^n(s) + z_n(s)) - f(v(s) + z(s)), w(s))_{L^2(\varrho)} \,\mathrm{d}s \\
&+ \alpha(z_n(s) - z(s), w(s))_{L^2(\varrho)} \,\mathrm{d}s = \|v_0 - v_n\|^2_{L^2(\varrho)}.
\end{aligned}$$

Since v, v^n are bounded in $L^p(t_0, T; L^p(\varrho))$, by (6.8) and the equivalence of norms (3.4), we may use (1.3) and (1.2) to obtain

$$\begin{aligned}
\|w(t)\|^2_{L^2(\varrho)} &\leqslant \|v_n - v_0\|^2_{L^2(\varrho)} + C_T \int_{t_0}^t \|w(s)\|^2_{L^2(\varrho)} \,\mathrm{d}s \\
&\quad + C_T \biggl(\int_{t_0}^t \|z_n - z\|^p_{L^p(\varrho)} \,\mathrm{d}s\biggr)^{1/p} + C_T \int_{t_0}^t \|z_n - z\|^2_{L^2(\varrho)} \,\mathrm{d}s.
\end{aligned}$$

Applying Gronwall's lemma we get

$$\|v^n(t)-v(t)\|^2_{L^2(\varrho)} \leqslant C_T\|v_n-v_0\|^2_{L^2(\varrho)} + C_T\biggl(\int_{t_0}^t \|z_n-z\|^p_{L^p(\varrho)}\,\mathrm{d}s\biggr)^{1/p} + C_T\int_{t_0}^t \|z_n-z\|^2_{L^2(\varrho)}\,\mathrm{d}s,$$

which gives the desired continuity properties. □

The following estimate will be useful later for establishing the existence of absorbing set.

Proposition 6.1. *Let g satisfy* (1.5) *and let z be such that*

$$\int_{-\infty}^{\tau} \mathrm{e}^{\beta s}\|z(s)\|^p_{L^p(\varrho)}\,\mathrm{d}s < \infty \tag{6.9}$$

for all $\tau\in\mathbb{R}$ and some $\beta>0$ sufficiently small. Let $\tau\in\mathbb{R}$ and $v_{\tau-t}\in L^2(\varrho)$ for all $t\geqslant 0$. Then there exist $\varepsilon, M, c>0$ independent of z, $\tau\in\mathbb{R}$ and $v_{\tau-t}$ such that for $t\geqslant 1$ we have

$$\begin{aligned}\|v(\tau,\tau-t,z,v_{\tau-t})\|^2_{L^2(\varrho_{\gamma,\varepsilon})} &+ c\int_{\tau-1}^{\tau}\|v(s,\tau-t,z,v_{\tau-t})\|^2_{W^{1,2}(\varrho_{\gamma,\varepsilon})}\,\mathrm{d}s\\ &+ c\int_{\tau-1}^{\tau}\|v(s,\tau-t,z,v_{\tau-t}+z(s)\|^p_{L^p(\varrho_{\gamma,\varepsilon})} + \|\partial_t v(s,\tau-t,z,v_{\tau-t})\|^2_{W^{-1,2}(\varrho_{\gamma,\varepsilon})}\,\mathrm{d}s\\ &\leqslant \mathrm{e}^{-\beta t}\|v_{\tau-t}\|^2_{L^2(\varrho_{\gamma,\varepsilon})} + M\biggl(1+\int_{-\infty}^{\tau}\mathrm{e}^{\beta(s-\tau)}(\|g(s)\|^2_{L^2(\varrho_{\gamma,\varepsilon})} + \|z(s)\|^p_{L^p(\varrho_{\gamma,\varepsilon})})\,\mathrm{d}s\biggr).\end{aligned}$$

Proof. Let $\varepsilon>0$ be sufficiently small and let $v=v(\cdot,\tau-t,z,v_{\tau-t})$. Testing equation (6.2) by v and using standard estimates as in (6.8) we get

$$\frac{\mathrm{d}}{\mathrm{d}s}\|v(s)\|^2_{L^2(\varrho_{\gamma,\varepsilon})} + \sigma(\|v(t)\|^2_{W^{1,2}(\varrho_{\gamma,\varepsilon})} + \|v(s)+z(s)\|^p_{L^p(\varrho_{\gamma,\varepsilon})}) \leqslant C(1+\|g(s)\|^2_{L^2(\varrho_{\gamma,\varepsilon})} + \|z(s)\|^p_{L^p(\varrho_{\gamma,\varepsilon})})$$

for some $\sigma>0$. Multiplying by $\mathrm{e}^{\beta s}$ for $\beta<\sigma/2$, integrating over $s\in(\tau-t,\tau)$ with $t>0$ and using integration by parts formula we obtain

$$\begin{aligned}\|v(\tau,\tau-t,z,v_{\tau-t})\|^2_{L^2(\varrho_{\gamma,\varepsilon})} &+ \frac{\sigma}{2}\int_{\tau-t}^{\tau}\mathrm{e}^{\beta(s-\tau)}\|v(s,\tau-t,z,v_{\tau-t})\|^2_{W^{1,2}(\varrho_{\gamma,\varepsilon})}\,\mathrm{d}s\\ &+ \frac{\sigma}{2}\int_{\tau-t}^{\tau}\mathrm{e}^{\beta(s-\tau)}\|v(s,\tau-t,z,v_{\tau-t})+z(s)\|^p_{L^p(\varrho_{\gamma,\varepsilon})}\,\mathrm{d}s\\ &\leqslant \mathrm{e}^{-\beta t}\|v_{\tau-t}\|^2_{L^2(\varrho_{\gamma,\varepsilon})} + C\int_{\tau-t}^{\tau}\mathrm{e}^{\beta(s-\tau)}(1+\|g(s)\|^2_{L^2(\varrho_{\gamma,\varepsilon})} + \|z(s)\|^p_{L^p(\varrho_{\gamma,\varepsilon})})\,\mathrm{d}s.\end{aligned}$$

Clearly the right-hand side of the previous inequality can be estimated by the desired expression. Concerning the left-hand side, we consider the time integrals over $(\tau-1,\tau)$. Then the exponentials can be removed by appropriately changing the constant σ.

Let ψ be as in (6.5). Since

$$\begin{aligned}(\partial_t v,\psi)_{\varrho_{\gamma,\varepsilon}} = &-(\nabla v,\nabla\psi)_{\varrho_{\gamma,\varepsilon}} - (\nabla v,\psi\nabla\varrho_{\gamma,\varepsilon})\\ &-(f(v+z),\psi)_{\varrho_{\gamma,\varepsilon}} - \alpha(z,\psi)_{\varrho_{\gamma,\varepsilon}} + (g,\psi)_{\varrho_{\gamma,\varepsilon}},\end{aligned}$$

using the definition of the $W^{-1,2}(\varrho_{\gamma,\varepsilon})$-norm, (1.2) and the Cauchy-Schwartz inequality we obtain

$$\begin{aligned}&\int_{\tau-1}^{\tau}\|\partial_s v(s,\tau-t,z,v_{\tau-t})\|^2_{W^{-1,2}(\varrho_{\gamma,\varepsilon})}\,\mathrm{d}s\\ &\qquad\leqslant C + C\int_{\tau-1}^{\tau}\|v(s,\tau-t,z,v_{\tau-t})\|^2_{W^{1,2}(\varrho_{\gamma,\varepsilon})} + \|g(s)\|^2_{L^2(\varrho_{\gamma,\varepsilon})}\,\mathrm{d}s\\ &\qquad\quad + C\int_{\tau-1}^{\tau}\|v(s,\tau-t,z,v_{\tau-t}) + z(s)\|^p_{L^p(\varrho_{\gamma,\varepsilon})} + \|z(s)\|^2_{L^2(\varrho_{\gamma,\varepsilon})}\,\mathrm{d}s.\end{aligned}$$

All the elements on the right-hand side of the above estimate with the exception of the $L^2(\tau-1,\tau;L^2(\varrho_{\gamma,\varepsilon}))$-norm of z have been bounded above, the remaining element can be bounded using Hölder's and Young's inequalities. □

6.2. RDS associated to RDE. Let $\{\vartheta_t\}_{t\in\mathbb{R}}$ be the group defined in (5.7). Set $\Omega_1=\mathbb{R}$ and $\vartheta_{1,t}(s)=s+t$. Let g be such that (1.5) holds. Finally, let $\mathcal{D}$ be the set of $L^2(\varrho)$-tempered sets.

Definition 6.2. A stochastic process $\{u(t)\}_{t\geqslant t_0}$ is a solution to (6.1) if $u=v+z$, where $z=z(\omega)$ is the solution of (5.1) and $v=v(t,t_0,z(\omega),x-z(t_0))$ is the solution of (6.2) in $[t_0,\infty)$ with forcing $z=z(\omega)$ and initial condition $v(t_0)=x-z(t_0)$.

In particular, the solution u satisfies

$$\begin{aligned}&\langle u(t),\psi\rangle + \int_{t_0}^{t}\langle u(s),\Delta\psi\rangle + \langle f(u(s)),\psi\rangle\,\mathrm{d}s\\ &\qquad\qquad = \langle x,\psi\rangle + \int_{t_0}^{t}\langle g(s),\psi\rangle\,\mathrm{d}s + \langle W_\mu(t)-W_\mu(t_0),\psi\rangle\end{aligned}$$

for all $\psi\in\mathscr{S}$.

Theorem 6.2. *The map* $\varphi\colon [0,\infty)\times\mathbb{R}\times\Omega\times L^2(\varrho)\to L^2(\varrho)$ *defined by*

$$\varphi(t,t_0,\vartheta_{t_0}\omega,x) = u(t+t_0,t_0,\omega,x),\quad t_0\in\mathbb{R},\ t\geqslant 0,\ x\in L^2(\varrho),$$

where $u(s,t_0,\omega,x)$ *is the solution of* (6.1) *in the sense of the above definition, is a continuous cocycle over the parametric DS* $(\mathbb{R},\{\vartheta_{1,t}\}_{t\in\mathbb{R}})$ *and the metric DS* $(\Omega,\mathcal{F},\mathbb{P},\{\vartheta_t\}_{t\in\mathbb{R}})$.

Proof. First, we show the existence of the solution u. Let $\alpha>0$. By Theorem 5.1 there exists a stationary Ornstein-Uhlenbeck process $\{z_\alpha(t)\}_{t\in\mathbb{R}}$ solving (5.1). In fact, by Theorem 5.1 we have $z(\omega)\in L^p_{\rm loc}(\mathbb{R};L^p(\varrho))$ for all $t_0\in\mathbb{R}$ $\mathbb{P}$-a.s. and so (6.3) is satisfied. By Theorem 6.1 there exists a unique solution $v=v(\cdot,t_0,z(\omega),x-z(t_0))$ of equation (6.2) with initial time t_0, initial data $x-z(t_0)$ and forcing $z(\omega)$. For $t\geqslant t_0$ we define

$$u_\alpha(t,t_0,\omega,x)=z_\alpha(\omega)(t)+v_\alpha(t,t_0,z_\alpha(\omega),x-z_\alpha(\omega)(t_0)). \tag{6.10}$$

Let us show that the definition of φ does not depend on α. We proceed similarly as in [7], Proposition 6.16. Let $t_0\in\mathbb{R}$ and $x\in L^2(\varrho)$. Let $\alpha,\beta>0$ and let $\{z_\alpha(t)\}_{t\in\mathbb{R}}$, $\{z_\beta(t)\}_{t\in\mathbb{R}}$ be the respective processes solving (5.1). Let u_α and u_β be given by (6.10). Then $w=u_\alpha-u_\beta$ satisfies the equation

$$\partial_t w(t)-\Delta w+f(u_\alpha)-f(u_\beta)=0,\quad w(t_0)=0.$$

Multiply by $w\varrho$, integrate over $\mathbb{R}^d$ and use (1.3) to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2_{L^2(\varrho)}\leqslant C\|w(t)\|^2_{L^2(\varrho)}.$$

From Gronwall's lemma it follows that $w(t)=0$ for $t\geqslant t_0$.

The required measurability and continuity properties of the cocycle φ follow from Theorems 5.1 and 6.1. □

Proposition 6.2. *Under the assumptions of Theorem* 6.2, *the RDS* φ *possesses a closed* $\mathcal{D}$*-absorbing set.*

Proof. Let $\omega\in\Omega$ and $\tau\in\mathbb{R}$. Given a tempered set $D=\{D(\tau,\omega)\}_{\tau\in\mathbb{R},\omega\in\Omega}\in\mathcal{D}$ we may use (5.9) to see that the set $\tilde D=\{D(\tau,\omega)+z(\tau,\omega)\}_{\tau\in\mathbb{R},\omega\in\Omega}$ is also tempered, therefore we may find $t_0=t_0(D,\tau,\omega)$ such that

$$\mathrm{e}^{-\beta t}\|u_{\tau-t}-z(\omega)(\tau-t)\|^2_{L^2(\varrho)}\leqslant 1$$

for $u_{\tau-t}\in D(\tau-t,\vartheta_{-t}\omega)$ and for $t\geqslant t_0$. Then, assumption (6.9) being satisfied by (5.10), by Proposition 6.1 we have

$$\begin{aligned}\|u(\tau,\tau-t,\omega,u_{-t})\|^2_{L^2(\varrho_{\gamma,\varepsilon})}&\leqslant M(1+\|z(\omega)(\tau)\|^2_{L^2(\varrho_{\gamma,\varepsilon})})\\&\quad+M\Big(\int_{-\infty}^{\tau}\mathrm{e}^{\beta s}(\|g(s)\|^2_{L^2(\varrho_{\gamma,\varepsilon})}+\|z(\omega)(s)\|^p_{L^p(\varrho_{\gamma,\varepsilon})})\,\mathrm{d}s\Big)=R(\tau,\omega).\end{aligned}$$

Recalling the notation of (3.4), we define the set $B=\{B(\tau,\omega)\}_{\tau\in\mathbb{R},\omega\in\Omega}$ by

$$B(\tau,\omega)=\{u\in L^2(\varrho);\ \|u\|^2_{L^2(\varrho)}\leqslant 2C_{\varepsilon,1}R(\tau,\omega)\}.$$

Also from (1.5) and (5.9) it follows that B is tempered. Moreover, the function $R(\tau,\cdot)$ is $(\mathcal{F},\mathcal{B}(\mathbb{R}))$-measurable and so B is $\mathcal{F}$-measurable. □

Lemma 6.1. *Let $D\in\mathcal{D}$, $\tau\in\mathbb{R}$, $\omega\in\Omega$, $\eta>0$ and let $v_{\tau-t}\in D(\tau-t,\vartheta_{\tau-t})$. Then there exists $T=T(\eta,\tau,\omega,D)$, $K=K(\eta,\tau,\omega)>0$ such that for all $t\geqslant T$*

$$\int_{|x|\geqslant K}|v(\tau,\tau-t,z(\omega),v_{\tau-t}-z(\omega)(\tau-t))|^2\varrho(x)\,\mathrm{d}x\leqslant\eta. \tag{6.11}$$

P r o o f. The proof is standard, see e.g. [4], Lemma 4.4. Let $\zeta\in C^\infty(\mathbb{R}_0^+)$ be such that $0\leqslant\zeta\leqslant 1$, $|\zeta'|<C$ and

$$\zeta(r)=\begin{cases}0, & 0\leqslant s\leqslant 1,\\ 1, & s\geqslant 2.\end{cases}$$

For $k\in\mathbb{N}$ we denote $\zeta_k(x)=\zeta(|x|^2/k^2)$ and $v(\cdot)=v(\cdot,\tau-t,z(\omega),v_{\tau-t}-z(\omega)(\tau-t))$. Testing equation (6.2) by $v\zeta_k$, using (1.3) and (1.2) and choosing k sufficiently large and $\varepsilon>0$ sufficiently small we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\int_{\mathbb{R}^d}|v(s)|^2\varrho_{\gamma,\varepsilon}(x)\zeta_k(x)\,\mathrm{d}x+\sigma\int_{\mathbb{R}^d}|v(s)|^2\varrho_{\gamma,\varepsilon}(x)\zeta_k(x)\,\mathrm{d}x\\ \leqslant C\int_{\mathbb{R}^d}(1+|g(s,x)|^2+|z(\omega)(s)(x)|^p)\varrho_{\gamma,\varepsilon}(x)\zeta_k(x)\,\mathrm{d}x\end{aligned}$$

for some $\sigma>0$. Applying Gronwall's lemma and increasing the integration range on the right-hand side we obtain

$$\begin{aligned}\int_{\mathbb{R}^d}|v(\tau)|^2\varrho_{\gamma,\varepsilon}(x)\zeta_k(x)\,\mathrm{d}x\leqslant \mathrm{e}^{-\sigma t}\int_{\mathbb{R}^d}|v(\tau-t)|^2\varrho_{\gamma,\varepsilon}(x)\zeta_k(x)\\ +C\int_{-\infty}^{\tau}\int_{\mathbb{R}^d}(1+|g(s,x)|^2+|z(\omega)(s)(x)|^p)\varrho_{\gamma,\varepsilon}(x)\zeta_k(x)\,\mathrm{d}x.\end{aligned}$$

Since D is tempered, we may choose $T>0$ such that

$$\mathrm{e}^{-\sigma t}\|v(\tau-t)\|^2_{L^2(\varrho_{\gamma,\varepsilon})}<\frac{\eta}{2}$$

for all $t\geqslant T$. The final estimate follows from the bounds (1.5) and (5.10) and $\varrho_{\gamma,1}\leqslant\varrho_{\gamma,\varepsilon}$. □

Proposition 6.3. *Let the assumptions of Theorem* 6.2 *hold and let g satisfy* (1.5). *Then the RDS φ is $\mathcal{D}$-asymptotically compact.*

Proof. Let $\omega \in \Omega$, $t_n \to \infty$, $\tau \in \mathbb{R}$ and let $D \in \mathcal{D}$ be a tempered set. For all $n \in \mathbb{N}$ fix $x_n \in D(\tau - t_n, \vartheta_{-t_n}\omega)$. Let $v_n = x_n - z(\omega)(-t_n)$. From the definition of solution we have

$$\varphi(t_n, \tau - t_n, \vartheta_{-t_n}\omega, x_n) = u(\tau, \tau - t_n, \omega, x_n) = z(\omega)(\tau) + v(\tau, \tau - t_n, z(\omega), v_n).$$

We observe that it suffices to show that then the sequence $\{v(\tau, \tau - t_n, z(\omega), v_n)\}_{n=1}^{\infty}$ is relatively compact in $L^2(\varrho)$.

The proof relies on localization and uses ideas from [19], Sections 4 and 5. By Lemma 6.1 for all $\eta > 0$ there exist $T = T(\tau, \omega, D, \eta)$, $K = K(\tau, \omega, \nu) > 0$ such that (6.11) holds. Without loss of generality we may assume $T \geqslant 1$. It suffices to show that the sequence $\{v(\tau, \tau - t_n, z(\omega), v_n)\}_{n=1}^{\infty}$ has an $L^2(B(0,K))$-convergent subsequence. To that end we use elements of the method of short trajectories, see [21]. Without loss of generality assume that $t_n \geqslant 1$ for all $n \in \mathbb{N}$. Let $v^n \colon [\tau - 1, \tau] \to L^2(\varrho_{\gamma,\varepsilon})$ be the restrictions of the unique solutions

$$\tilde{v}^n = \tilde{v}^n(\cdot, \tau - t_n, z(\omega), v_n) \tag{6.12}$$

to the interval $t \in [\tau - 1, \tau]$. Define $\mathcal{B} = \{v^n; n \in \mathbb{N}\}$. Let $\varsigma \in C^\infty(\mathbb{R}_0^+)$ be such that $0 \leqslant \varsigma \leqslant 1$, $|\varsigma'| \leqslant C$ and

$$\varsigma(r) = \begin{cases} 1, & 0 \leqslant s \leqslant 1, \\ 0, & s \geqslant 2. \end{cases}$$

Let $\varsigma_K(x) = \varsigma(|x|^2/(K+1)^2)$ and let $\varepsilon > 0$ be as in Proposition 6.1. Then we have

$$\|v(\tau, \tau - t_n, z(\omega), v_n)\|^2_{L^2(\varsigma_K \varrho_{\gamma,\varepsilon})} + \int_{\tau-1}^{\tau} \|v(s, \tau - t_n, z(\omega), v_n)\|^2_{L^2(\varsigma_K \varrho_{\gamma,\varepsilon})} \, \mathrm{d}s + \int_{\tau-1}^{\tau} \|\partial_t v(s, \tau - t_n, z(\omega), v_n)\|^2_{W^{-1,2}(\varsigma_K \varrho_{\gamma,\varepsilon})} \, \mathrm{d}s \leqslant C$$

for $n \in \mathbb{N}$. By the Aubin-Lions lemma there exists a subsequence $v^n \in \mathcal{B}$ and $v \in L^2(\tau - 1, \tau; L^2(\varsigma_K \varrho_{\gamma,\varepsilon}))$ such that $v^n \to v$ in $L^2(\tau - 1, \tau; L^2(\varsigma_K \varrho_{\gamma,\varepsilon}))$. For $u \in \mathcal{B}$, let us define the end-point mapping

$$e_\tau \colon \mathcal{B} \to L^2(B(0,K)), \quad e(u) = u(\tau)|_{B(0,K)}.$$

The mapping e_τ is well defined since $\mathcal{B} \subseteq C([\tau - 1, \tau]; L^2(\varrho_{\gamma,\varepsilon}))$. If we show that e_τ is Lipschitz continuous, the proof is finished since then the sequence $\{v^n(\tau)\}_{n=1}^{\infty}$ is

Cauchy by the estimate

$$\begin{aligned}\|v^n(\tau) - v^m(\tau)\|_{L^2(B(0,K))} &= \|e_\tau(v^n) - e_\tau(v^m)\|_{L^2(B(0,K))} \\ &\leqslant C\|v^n - v^m\|_{L^2(\tau-1,\tau;L^2(\varsigma_K \varrho_{\gamma,\varepsilon}))}\end{aligned}$$

and therefore has a convergent subsequence.

To show that the mapping e_τ is Lipschitz continuous, let $m, n \in \mathbb{N}$ and let $\tilde{v}^n$ and $\tilde{v}^m$ be the respective solutions from (6.12). Let $w = \tilde{v}^n - \tilde{v}^m$. On the interval $I = (\tau - (t_n \wedge t_m), \tau)$ the function w satisfies the equation

$$\partial_t w - \Delta w + f(v^n + z) - f(v^m + z) = 0.$$

We test the equation by $w\varsigma_K \varrho_{\gamma,\varepsilon}$ and use (1.3) to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2_{L^2(\varrho_{\gamma,\varepsilon})} \leqslant C\|w(t)\|^2_{L^2(\varrho_{\gamma,\varepsilon})}.$$

Let $s_1 < s_2 \in I$. Integrate the previous inequality over $t \in (s_1, s_2)$ to get

$$\|w(s_2)\|^2_{L^2(\varsigma_K \varrho_{\gamma,\varepsilon})} \leqslant \|w(s_1)\|^2_{L^2(\varsigma_K \varrho_{\gamma,\varepsilon})} + C\int_{s_1}^{s_2} \|w(t)\|^2_{L^2(\varsigma_K \varrho_{\gamma,\varepsilon})}\,\mathrm{d}t.$$

Using Gronwall's lemma we arrive at

$$\|w(s_2)\|^2_{L^2(\varsigma_K \varrho_{\gamma,\varepsilon})} \leqslant C\|w(s_1)\|^2_{L^2(\varsigma_K \varrho_{\gamma,\varepsilon})},$$

where the constant C does not depend on n, m. Set $s_2 = \tau$. Integrating over $s_1 \in (\tau - 1, \tau)$, we obtain the desired Lipschitz continuity

$$\|e(v^n) - e(v^m)\|^2_{L^2(\varrho_{\gamma,\varepsilon})} \leqslant C\|v^n - v^m\|^2_{L^2(\tau-1,\tau;L^2(\varrho_{\gamma,\varepsilon}))}.$$

□

P r o o f of Theorem 1.1. The assertion follows from Theorem 2.1, Proposition 6.2 and Proposition 6.3. □

Acknowledgement. The author wishes to thank J. Seidler for discussions leading to the results of this paper and M. Ondreját for simplification of several important arguments.

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Author's address: Jakub Slavík, Institute of Information Theory and Automation, Czech Academy of Sciences, Pod Vodárenskou věží 4, 182 00 Praha 8, Czech Republic, e-mail: slavik@utia.cas.cz.