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Lindström theorems in graded model theory

Guillermo Badia [a], Carles Noguera [b,*]

[a] School of Historical and Philosophical Inquiry, University of Queensland, Brisbane, St Lucia, QLD 4072, Australia
[b] Institute of Information Theory and Automation, Czech Academy of Sciences, Pod vodárenskou věží 4, 182 00 Prague, Czech Republic

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ABSTRACT

Stemming from the works of Petr Hájek on mathematical fuzzy logic, graded model theory has been developed by several authors in the last two decades as an extension of classical model theory that studies the semantics of many-valued predicate logics. In this paper we take the first steps towards an abstract formulation of this model theory. We give a general notion of abstract logic based on many-valued models and prove six Lindström-style characterizations of maximality of first-order logics in terms of metalogical properties such as compactness, abstract completeness, the Löwenheim–Skolem property, the Tarski union property, and the Robinson property, among others. As necessary technical restrictions, we assume that the models are valued on finite MTL-chains and the language has a constant for each truth-value.

1. Introduction

First-order predicate logic is the best known example of a formal language whose model theory had a great impact on 20th century mathematics, from non-standard analysis to abstract algebra. The celebrated characterization of classical first-order logic obtained by Per Lindström in the 60s (published as [23]; a nice accessible exposition can be found in [19]) is a landmark in contemporary logic. The introduction of a notion of "extended first-order logic", that encompassed a great number of expressive extensions of first-order logic, allowed Lindström to establish, roughly, that there are no extensions of classical first-order logic that would also satisfy the compactness and Löwenheim–Skolem theorems, so this logic is *maximal* in terms of expressive power with respect to these properties (other similar characterizations soon followed). Expressive extensions of first-order logic are commonly called "abstract logics",[1] giving rise to the field of *abstract* or *soft* model

* Corresponding author.
E-mail addresses: guillebadia89@gmail.com (G. Badia), noguera@utia.cas.cz (C. Noguera).

[1] In fact, probably the term "model-theoretic language" (see [18]) is more accurate, depending on one's views of what a "logic" is.

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theory (cf. [5,6]). In this field, one uses "only very general properties of the logic, properties that carry over to a large number of other logics" ([5], p. 225). Common examples of such properties are compactness or the Craig interpolation theorem. Abstract model theory is concerned with the study of such properties and their mutual interaction.

In the context of mathematical fuzzy logic (MFL) the possibility of abstract model-theoretic results was briefly considered by Petr Hájek in a technical report from 2002 [20]. Later, two Lindström-style results for the important cases of $[0,1]$-valued Łukasiewicz and rational Pavelka logics appeared in the literature [8,9]. However, this work was not meant to present a general framework for other fuzzy logics, but rather ad hoc non-trivial results for particular systems.

Interestingly, Hájek had shown that the analogues of Lindström's first theorem fail for some of the main first-order fuzzy logics (BL∀, Π∀, and G∀) with their *standard semantics* (i.e. truth-values in the interval $[0,1]$). Furthermore, Hájek also established that the result cannot be obtained for any of BL∀, Ł∀, Π∀, or G∀ with their *general semantics* (the algebra of truth-values being allowed to vary among the elements of the variety corresponding to the logic in question). In fact, Hájek's argument holds for any fuzzy logic w.r.t. its general semantics if

- it satisfies the compactness and Löwenheim–Skolem theorems,
- it has the usual propositional connectives $\{\vee, \wedge, \&, \to, \overline{1}, \overline{0}\}$ and the quantifiers $\exists$ and $\forall$, and
- the compactness and Löwenheim–Skolem theorems remain true when adding the Baaz–Monteiro $\triangle$ connective.

Perhaps discouraged by these initial negative results, the MFL community has not attempted again, to the best of our knowledge, to build a corresponding abstract model theory.

In this paper we would like to show that such a theory is actually a viable one, at least under certain technical conditions. In particular, we will give a general framework (as general as we can see) in which the Lindström theorems hold. This will lead to two main restrictions:

(1) our algebras of truth-values will be *finite*, and
(2) we will have a *truth-constant* for each element of the algebra (allowing for the possibility that such constants be definable or, more generally, that the language has the same expressive power, in the sense described below, as the version with constants).

Both conditions are necessary. We are led to (1) by a result from [20] and, given (1), also (2) can be quickly seen to be necessary. The framework ends up being the same recently used in [2] for some pure model-theoretic results and in [15] for results connected to fuzzy constraint satisfaction.

We follow closely the classical arguments from [5], adapting the ideas and techniques to the many-valued context. In [12], a translation into a two-sorted first-order language provides a way to interpret formulas in the languages studied in this paper as making statements not about many-valued structures but classical two-sorted structures. Even though we use this trick in Proposition 7, we should stress here that the applicability of this translation is limited. Indeed, the Lindström-style theorems in this paper are not immediate consequences of the classical ones. For one thing, since the translation of atomic formulas into the two-sorted languages are identity formulas, these languages have only one relation symbol, namely equality. They may vary on the function symbols they possess depending on the signature of the original language but they only require one relation symbol. The definition of an abstract logic, however, allows for arbitrary signatures. Moreover, observe that the classical Lindström theorems for two-sorted first-order logic would consider *arbitrary* two-sorted structures (no restrictions on the finiteness of the domain of one of the sorts, no algebraic structure imposed on said sort, etc.). Hence, the framework of the classical Lindström theorem is not ours and one cannot just apply it gratuitously; results have to be established independently.

A similar reason was mentioned in [1] as to why the results there do not just follow from the classical Fraïssé theorem. In fact, the limitations of the translation are also pointed out in [12].

The paper is organized as follows: In §2 we review the basic notions of graded model theory, introduce the definition of an abstract graded logic, give some examples and present the key model-theoretic properties that will be involved in our characterizations (abstract completeness, the Löwenheim-Skolem property, compactness, the Karp property, and the Tarski union property among others). In §3 we show that the first-order logic based on an arbitrary (but fixed) finite MTL-chain has all the desired model-theoretic properties introduced in the previous section. In §4 we establish our six Lindström maximality results. Finally, in §5 we end with some concluding remarks.

2. Theoretical framework

2.1. Graded model theory

In this section we introduce the basic notions of the standard graded model theory framework which lies at the base of the abstract hierarchy that we will later propose. Let us start with the syntax and semantics of graded predicate logics, and recall the basic notions we will use in the paper. We (mostly) use the notation and definitions of the Handbook of Mathematical Fuzzy Logic [11].

Syntax The syntactical aspects of our logical setting are (almost) completely classical. We start from a basic propositional language that contains the binary connectives $\wedge$ (lattice conjunction), $\vee$ (lattice disjunction), & (residuated conjunction), and $\to$ (implication), and two truth-constants: $\overline{0}$ (falsum) and $\overline{1}$ (verum). Two other connectives are defined: $\neg\varphi = \varphi \to \overline{0}$ and $\varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$. A *signature* (or *predicate language*) τ is a triple $\langle Pred_\tau, Func_\tau, Ar_\tau\rangle$, where $Pred_\tau$ is a non-empty set of *predicate symbols*, $Func_\tau$ is a set of *function symbols* (disjoint from $Pred_\tau$), and Ar_τ represents the *arity function*, which assigns a natural number to each predicate symbol or function symbol. We call this natural number the *arity* of the symbol. The function symbols with arity zero are named *object constants* (*constants* for short). Object variables, τ-terms, τ-formulas, and the notions of free occurrence of a variable, open formula, substitutability, and sentence are defined as in classical predicate logic. A theory is a set of sentences. When τ is clear from the context, we will refer to τ-terms and τ-formulas simply as *terms* and *formulas*. Also, when no confusion can arise, we will identify τ with the set of its symbols (i.e. $Pred_\tau \cup Func_\tau$) and write expressions such as $\tau \subseteq \tau'$, meaning that $Pred_\tau \subseteq Pred_{\tau'}$ and $Func_\tau \subseteq Func_{\tau'}$ and $Ar_{\tau'}$ agrees with Ar_τ in the symbols of τ.

Semantics The non-classicality appears on the semantical side. In graded predicate logics we work with models based on an algebra of (possibly more than two) truth-values. Propositional connectives are semantically interpreted by the notion of an MTL-*algebra* [16], that is, a structure of the form $\boldsymbol{A} = \langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \&^{\boldsymbol{A}}, \to^{\boldsymbol{A}}, \overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}}\rangle$ such that

- $\langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}}\rangle$ is a bounded lattice,
- $\langle A, \&^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}}\rangle$ is a commutative monoid,
- for each $a, b, c \in A$, we have:

$$a \,\&^{\boldsymbol{A}}\, b \leq c \quad \text{iff} \quad b \leq a \to^{\boldsymbol{A}} c, \qquad \text{(residuation)}$$

$$(a \to^{\boldsymbol{A}} b) \vee^{\boldsymbol{A}} (b \to^{\boldsymbol{A}} a) = \overline{1}^{\boldsymbol{A}} \qquad \text{(prelinearity)}$$

$\boldsymbol{A}$ is called an MTL-*chain* if its underlying lattice is linearly ordered. Observe that the two-element Boolean algebra, $\boldsymbol{B}_2$, can be seen, in particular, as an MTL-algebra (identifying the operations & and

$\wedge$, and defining the complement as $\neg x = x \to \overline{0}$). Typical examples of non-Boolean MTL-chains are the algebras $[0,1]_{\mathrm{G}}$, $[0,1]_{\text{Ł}}$, and $[0,1]_{\Pi}$, respectively used in the semantics of Gödel–Dummett, Łukasiewicz, and Product logics (three prominent examples of fuzzy logics; see e.g. [11]). In all cases, $\wedge$, $\vee$, $\overline{0}$, $\overline{1}$ are interpreted respectively as the minimum, the maximum, the number 0, and the number 1, while the interpretations of the other operations differ:

$$\begin{aligned}
a \,\&^{[0,1]_{\mathrm{G}}}\, b &= \min\{a,b\},\\
a \,\&^{[0,1]_{\text{Ł}}}\, b &= \max\{a+b-1,0\},\\
a \,\&^{[0,1]_{\Pi}}\, b &= ab \text{ (standard product of reals)},\\
a \to^{[0,1]_{\mathrm{G}}} b &= \begin{cases} 1, & \text{if } a \leq b,\\ b, & \text{otherwise},\end{cases}\\
a \to^{[0,1]_{\text{Ł}}} b &= \begin{cases} 1, & \text{if } a \leq b,\\ 1-a+b, & \text{otherwise},\end{cases}\\
a \to^{[0,1]_{\Pi}} b &= \begin{cases} 1, & \text{if } a \leq b,\\ b/a, & \text{otherwise}.\end{cases}
\end{aligned}$$

The restriction to linearly-ordered algebras is common in the setting of mathematical fuzzy logic, particularly for the predicate case. In this article, linearity will be used without special mention. For example, in the proof of Theorem 13, we will assume that there is a co-atom a in the algebra $\boldsymbol{A}$ such that every element of $\boldsymbol{A}$ is either $\overline{1}^{\boldsymbol{A}}$ or $\leq_{\boldsymbol{A}} a$ (ruling out the possibility of having some element that is simply incomparable with a). It is possible, however, that our arguments can be reframed in a general setting that does not impose linearity but instead some weaker conditions. We leave the task of finding those conditions for future study.

An MTL-chain $\boldsymbol{A}$ may sometimes be expanded, for greater expressive power of the logic, with an operation for the Baaz–Monteiro unary connective $\triangle$ in the following the way:

$$\triangle^{\boldsymbol{A}}(a) = \begin{cases} \overline{1}^{\boldsymbol{A}}, & \text{if } a = \overline{1}^{\boldsymbol{A}},\\ \overline{0}^{\boldsymbol{A}}, & \text{otherwise}.\end{cases}$$

Also, it may be expanded with truth-constants (i.e. 0-ary connectives) $\overline{a}$, for each $a \in A$, demanding that they denote their corresponding element, that is, $\overline{a}^{\boldsymbol{A}} = a$ (see e.g. [17]).

Based on MTL-chains (and their expansions) as algebraic interpretations of the propositional language, now we can give the semantics of first-order predicate formulas. Given a signature $\tau = \langle Pred_\tau, Func_\tau, Ar_\tau\rangle$, we define a *$\tau$-structure* as a pair $\mathfrak{M} = \langle \boldsymbol{A}, \mathbf{M}\rangle$ where $\boldsymbol{A}$ is an MTL-chain and

$$\mathbf{M} = \langle M, (P_{\mathbf{M}})_{P\in Pred_\tau}, (F_{\mathbf{M}})_{F\in Func_\tau}\rangle,$$

where M is a non-empty set (the *domain*), $P_{\mathbf{M}}$ is an n-ary $\boldsymbol{A}$-valued relation for each n-ary predicate symbol P, i.e., a function from M^n to A, identified with an element of A if $n = 0$; and $F_{\mathbf{M}}$ is a function from M^n to M, identified with an element of M if $n = 0$. We will call $\langle \boldsymbol{A}, \mathbf{M}\rangle$ an *$\boldsymbol{A}$-structure* whenever we need to stress its algebraic part. An $\mathbf{M}$-evaluation of the object variables is a mapping v assigning to each object variable an element of M. If v is an $\mathbf{M}$-evaluation, x is an object variable and $d \in M$, we denote by $v[x \mapsto d]$ the $\mathbf{M}$-evaluation defined as $v[x \mapsto d](x) = d$ and $v[x \mapsto d](y) = v(y)$ for $y \neq x$. We define the *values* of terms and the *truth values* of formulas in M for an $\mathbf{M}$-evaluation v recursively as follows:

$\|x\|^{\boldsymbol{A}}_{\mathbf{M},v} = v(x)$;
$\|F(t_1,\dots,t_n)\|^{\boldsymbol{A}}_{\mathbf{M},v} = F_{\mathbf{M}}(\|t_1\|^{\boldsymbol{A}}_{\mathbf{M},v},\dots,\|t_n\|^{\boldsymbol{A}}_{\mathbf{M},v})$, for each $F \in Func_\tau$;
$\|P(t_1,\dots,t_n)\|^{\boldsymbol{A}}_{\mathbf{M},v} = P_{\mathbf{M}}(\|t_1\|^{\boldsymbol{A}}_{\mathbf{M},v},\dots,\|t_n\|^{\boldsymbol{A}}_{\mathbf{M},v})$, for each $P \in Pred_\tau$;

$\|\varphi \circ \psi\|^{\boldsymbol{A}}_{\mathbf{M},v} = \|\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v} \circ^{\boldsymbol{A}} \|\psi_n\|^{\boldsymbol{A}}_{\mathbf{M},v}$, for each binary connective $\circ$;
$\|\overline{0}\|^{\boldsymbol{A}}_{\mathbf{M},v} = \overline{0}^{\boldsymbol{A}}$;
$\|\overline{1}\|^{\boldsymbol{A}}_{\mathbf{M},v} = \overline{1}^{\boldsymbol{A}}$;
$\|(\forall x)\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v} = \inf\{\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v[x\mapsto d]} \mid d \in M\}$;
$\|(\exists x)\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v} = \sup\{\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v[x\mapsto d]} \mid d \in M\}$.

If the infimum or the supremum does not exist, we take the truth value of the formula to be undefined. A τ-structure $\langle \boldsymbol{A}, \mathbf{M}\rangle$ is said to be *safe* if the value $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v}$ is defined for each formula φ and each $\mathbf{M}$-evaluation v. Certainly, the semantics could be restricted to models based on completely ordered chains, that is, chains with suprema and infima of all their subsets, which would ensure that all models would be safe. However, in general, this gives rise to serious drawbacks regarding the axiomatizability of the corresponding first-order logics (see [11, Chapter XI]), which justifies the design choice of safe models instead.

For a set of formulas Φ, we write $\|\Phi\|^{\boldsymbol{A}}_{\mathbf{M},v} = \overline{1}^{\boldsymbol{A}}$, if $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v} = \overline{1}^{\boldsymbol{A}}$ for every $\varphi \in \Phi$. We denote by $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$ the fact that $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v} = \overline{1}^{\boldsymbol{A}}$ for all $\mathbf{M}$-evaluations v; analogously for sets Φ. We say that $\langle \boldsymbol{A}, \mathbf{M}\rangle$ is a *model of a set of formulas* Φ, if $\|\Phi\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$ (in symbols, $\langle \boldsymbol{A}, \mathbf{M}\rangle \models \Phi$).

Observe that in this general presentation we have not required yet the presence of an equality symbol in the language, but it can be added in the form of a binary relational symbol $\approx$ which can be forced to be interpreted as (crisp) equality in the models, i.e. $\|t_1 \approx t_2\|^{\boldsymbol{A}}_{\mathbf{M},v} = \overline{1}^{\boldsymbol{A}}$ if $\|t_1\|^{\boldsymbol{A}}_{\mathbf{M},v} = \|t_2\|^{\boldsymbol{A}}_{\mathbf{M},v}$, and $\|t_1 \approx t_2\|^{\boldsymbol{A}}_{\mathbf{M},v} = \overline{0}^{\boldsymbol{A}}$ otherwise.

We use the notation $\overrightarrow{x}$ for a finite sequence of variables $x_1, \dots, x_n$, and $\overrightarrow{d}$ for a finite sequence of elements of a domain M (by a slight abuse of language, we write $\overrightarrow{d} \subseteq M$). Generalizing the previous notation, given an $\mathbf{M}$-evaluation v, we define $v[\overrightarrow{x} \mapsto \overrightarrow{d}]$ as the $\mathbf{M}$-evaluation such that $v[\overrightarrow{x} \mapsto \overrightarrow{d}](x_i) = d_i$ for each $i \in \{1, \dots, n\}$ and $v[\overrightarrow{x} \mapsto \overrightarrow{d}](y) = v(y)$ for each $y \notin \overrightarrow{x}$. We write $\varphi(\overrightarrow{x})$ to indicate that the free variables of φ are among $\{x_1, \dots, x_n\}$. Given a τ-structure $\langle \boldsymbol{A}, \mathbf{M}\rangle$ and a formula $\varphi(\overrightarrow{x})$, we say that $\overrightarrow{d} \subseteq M$ *satisfies* $\varphi(\overrightarrow{x})$ (or that $\varphi(\overrightarrow{x})$ is *satisfied* by $\overrightarrow{d}$) if $\|\varphi(\overrightarrow{x})\|^{\boldsymbol{A}}_{\mathbf{M},v[\overrightarrow{x}\mapsto\overrightarrow{d}]} = \overline{1}^{\boldsymbol{A}}$ for any $\mathbf{M}$-evaluation v (also written $\left\|\varphi[\overrightarrow{d}]\right\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$ or $\langle \boldsymbol{A}, \mathbf{M}\rangle \models \varphi[\overrightarrow{d}]$). Finally, we say that a set of sentences Φ is *satisfiable* if there is a safe τ-structure $\langle \boldsymbol{A}, \mathbf{M}\rangle$ such that $\|\Phi\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$, and we say that it is *finitely satisfiable* if each finite subset of Φ is satisfiable.

Equivalent formulas, elementary equivalence, elementary diagrams, homomorphisms, embeddings, elementary chains, and (partial) isomorphisms The many-valued semantics brings forth an interesting increase of complexity of basic notions of classical model theory, starting from the very notions of equivalence of formulas and elementary equivalence of structures. Indeed, given two formulas $\varphi(\overrightarrow{x})$ and $\psi(\overrightarrow{x})$, we can define their equivalence in two different ways:

- $\varphi(\overrightarrow{x})$ and $\psi(\overrightarrow{x})$ are *1-equivalent* if for any safe τ-structure $\langle \boldsymbol{A}, \mathbf{M}\rangle$ and any sequence of elements $\overrightarrow{d} \in M$: $\langle \boldsymbol{A}, \mathbf{M}\rangle \models \varphi[\overrightarrow{d}]$ iff $\langle \boldsymbol{A}, \mathbf{M}\rangle \models \psi[\overrightarrow{d}]$,
- $\varphi(\overrightarrow{x})$ and $\psi(\overrightarrow{x})$ are *equivalent* if for any safe τ-structure $\langle \boldsymbol{A}, \mathbf{M}\rangle$ and any sequence of elements $\overrightarrow{d} \in M$: $\langle \boldsymbol{A}, \mathbf{M}\rangle \models \varphi \leftrightarrow \psi[\overrightarrow{d}]$ (that is, for each v, $\|\varphi(\overrightarrow{x})\|^{\boldsymbol{A}}_{\mathbf{M},v[\overrightarrow{x}\mapsto\overrightarrow{d}]} = \|\psi(\overrightarrow{x})\|^{\boldsymbol{A}}_{\mathbf{M},v[\overrightarrow{x}\mapsto\overrightarrow{d}]}$).

Similarly, equivalence between two structures can be meaningfully defined in two different ways. We say that two safe τ-structures $\langle \boldsymbol{A}, \mathbf{M}\rangle$ and $\langle \boldsymbol{B}, \mathbf{N}\rangle$ are *elementarily equivalent* (in symbols: $\langle \boldsymbol{A}, \mathbf{M}\rangle \equiv \langle \boldsymbol{B}, \mathbf{N}\rangle$) if they are models of the same sentences, i.e. for every τ-sentence σ, $\|\sigma\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$ if and only if $\|\sigma\|^{\boldsymbol{B}}_{\mathbf{N}} = \overline{1}^{\boldsymbol{B}}$.

In case the two structures are based on the same algebra we can define a stronger notion of equivalence by requiring sentences to take the exact same values. More precisely, given safe τ-structures $\langle \boldsymbol{A}, \mathbf{M}\rangle$ and

$\langle \boldsymbol{A}, \mathbf{N} \rangle$, we say that they are *strongly elementarily equivalent* (in symbols: $\langle \boldsymbol{A}, \mathbf{M} \rangle \equiv^s \langle \boldsymbol{A}, \mathbf{N} \rangle$) if for every τ-sentence σ, $\|\sigma\|^{\boldsymbol{A}}_{\mathbf{M}} = \|\sigma\|^{\boldsymbol{A}}_{\mathbf{N}}$.

For classical structures, i.e. when $\boldsymbol{A} \cong \boldsymbol{B}_2$, these two definitions give the classical notion of elementary equivalence. But, in general, they differ as shown with counterexamples in [13].

Assume now that the propositional language has a truth-constant $\overline{a}$ for each $a \in A$ and the signature has an object constant c_m for each $m \in M$. We define the *elementary diagram* of a τ-structure $\langle \boldsymbol{A}, \mathbf{M} \rangle$ as the following set of sentences:

$$\mathrm{ElDiag}(\boldsymbol{A}, \mathbf{M}) = \{\sigma \leftrightarrow \overline{a} \mid \sigma \text{ is a } \tau\text{-sentence}, a \in A, \text{ and } \|\sigma\|^{\boldsymbol{A}}_{\mathbf{M}} = a\}.$$

Now, let us introduce the necessary notions of morphisms between structures. Given τ-structures $\langle \boldsymbol{A}, \mathbf{M} \rangle$ and $\langle \boldsymbol{B}, \mathbf{N} \rangle$, a mapping $f : A \to B$ and a mapping $g : M \to N$, the pair $\langle f, g \rangle$ is said to be a *strong homomorphism* from $\langle \boldsymbol{A}, \mathbf{M} \rangle$ to $\langle \boldsymbol{B}, \mathbf{N} \rangle$ if f is an algebraic homomorphism and for every n-ary function symbol $F \in Func_\tau$ and $d_1, \ldots, d_n \in M$,

$$g(F_{\mathbf{M}}(d_1, \ldots, d_n)) = F_{\mathbf{N}}(g(d_1), \ldots, g(d_n))$$

and for every n-ary predicate symbol $P \in Pred_\tau$ and $d_1, \ldots, d_n \in M$,

$$f(\|P(d_1 \ldots, d_n)\|^{\boldsymbol{A}}_{\mathbf{M}}) = \|P(g(d_1), \ldots, g(d_n))\|^{\boldsymbol{B}}_{\mathbf{N}}.$$

A strong homomorphism $\langle f, g \rangle$ is said to be *elementary* if we have, for every τ-formula $\varphi(x_1, \ldots, x_n)$ and $d_1, \ldots, d_n \in M$,

$$f(\|\varphi(d_1 \ldots, d_n)\|^{\boldsymbol{A}}_{\mathbf{M}}) = \|\varphi(g(d_1), \ldots, g(d_n)\|^{\boldsymbol{B}}_{\mathbf{N}}.$$

Let $\langle f, g \rangle$ be a strong homomorphism from $\langle \boldsymbol{A}, \mathbf{M} \rangle$ to $\langle \boldsymbol{B}, \mathbf{N} \rangle$, we say that $\langle f, g \rangle$ is an *embedding* from $\langle \boldsymbol{A}, \mathbf{M} \rangle$ to $\langle \boldsymbol{B}, \mathbf{N} \rangle$ if both functions f and g are injective, and we say that $\langle f, g \rangle$ is an *isomorphism* from $\langle \boldsymbol{A}, \mathbf{M} \rangle$ to $\langle \boldsymbol{B}, \mathbf{N} \rangle$ if $\langle f, g \rangle$ is an embedding and both functions f and g are surjective. For a general study of different kinds of homomorphisms and the formulas they preserve we refer to [12].

An indexed family $\{\langle \boldsymbol{A}_i, \mathbf{M}_i \rangle \mid i < \gamma\}$ of safe τ-structures is a called a *chain* when for all $i < j < \gamma$ we have that $\langle \boldsymbol{A}_i, \mathbf{M}_i \rangle$ is a substructure of $\langle \boldsymbol{A}_j, \mathbf{M}_j \rangle$. If, moreover, these substructures are elementary, we speak of an *elementary chain*. The *union* of the chain is the τ-structure $\langle \boldsymbol{A}, \mathbf{M} \rangle$ where $\boldsymbol{A}$ is the classical union of the chain of algebras $\{\boldsymbol{A}_i \mid i < \gamma\}$, while $\mathbf{M}$ is defined by taking as its domain $\bigcup_{i<\gamma} M_i$, interpreting the symbols of the signature as they were interpreted in each $\mathbf{M}_i$.

Finally, we need to recall the notions of partial isomorphism and finitely isomorphic structures studied in [13]. Let $\langle \boldsymbol{A}, \mathbf{M} \rangle$ and $\langle \boldsymbol{B}, \mathbf{N} \rangle$ be τ-structures, p be a partial mapping from A to B, and r be a partial mapping from M to N. We say that $\langle p, r \rangle$ is a *partial isomorphism* from $\langle \boldsymbol{A}, \mathbf{M} \rangle$ to $\langle \boldsymbol{B}, \mathbf{N} \rangle$ if

1. p and r are injective,
2. for every binary connective $\circ$, and every $a_1, a_2 \in \mathrm{dom}(p)$ such that $a_1 \circ^{\mathbf{A}} a_2 \in \mathrm{dom}(p)$,
$$p(a_1 \circ^{\boldsymbol{A}} a_2) = p(a_1) \circ^{\boldsymbol{B}} p(a_2),$$
3. if $\overline{0}^{\boldsymbol{A}} \in \mathrm{dom}(p)$, then $p(\overline{0}^{\boldsymbol{A}}) = \overline{0}^{\boldsymbol{B}}$,
4. if $\overline{1}^{\boldsymbol{A}} \in \mathrm{dom}(p)$, then $p(\overline{1}^{\boldsymbol{A}}) = \overline{1}^{\boldsymbol{B}}$,
5. for every n-ary functional symbol $F \in Func_\tau$ and every $d_1, \ldots, d_n \in M$ such that $d_1, \ldots, d_n$, $F_{\mathbf{M}}(d_1, \ldots, d_n) \in \mathrm{dom}(r)$,
$$r(F_{\mathbf{M}}(d_1, \ldots d_n)) = F_{\mathbf{N}}(r(d_1), \ldots, r(d_n)),$$

6. for every n-ary predicate symbol $P \in Pred_\tau$ and $d_1, \ldots, d_n \in M$ such that $d_1, \ldots, d_n \in \mathrm{dom}(r)$,

$$p(P_{\mathbf{M}}(d_1, \ldots, d_n)) = P_{\mathbf{N}}(r(d_1), \ldots, r(d_n)).$$

Two τ-structures $\langle \boldsymbol{A}, \mathbf{M}\rangle$ and $\langle \boldsymbol{B}, \mathbf{N}\rangle$ are said to be *finitely isomorphic*, written $\langle \boldsymbol{A}, \mathbf{M}\rangle \cong_f \langle \boldsymbol{B}, \mathbf{N}\rangle$, if there is a sequence $\langle I_n \mid n \in \mathsf{N}\rangle$ with the following properties:

1. Every I_n is a non-empty set of partial isomorphisms from $\langle \boldsymbol{A}, \mathbf{M}\rangle$ to $\langle \boldsymbol{B}, \mathbf{N}\rangle$.
2. For each $n \in \mathsf{N}$, $I_{n+1} \subseteq I_n$.
3. (Forth-property I) For every $\langle p, r\rangle \in I_{n+1}$ and $m \in M$, there is $\langle p, r'\rangle \in I_n$ such that $r \subseteq r'$ and $m \in \mathrm{dom}(r')$.
4. (Back-property I) For every $\langle p, r\rangle \in I_{n+1}$ and $n \in N$, there is $\langle p, r'\rangle \in I_n$ such that $r \subseteq r'$ and $n \in \mathrm{rg}(r')$.
5. (Forth-property II) For every $\langle p, r\rangle \in I_{n+1}$ and $a \in A$, there is $\langle p', r\rangle \in I_n$ such that $p \subseteq p'$ and $a \in \mathrm{dom}(p')$.
6. (Back-property II) For every $\langle p, r\rangle \in I_{n+1}$ and $b \in B$, there is $\langle p', r\rangle \in I_n$ such that $p \subseteq p'$ and $b \in \mathrm{rg}(p')$.

If the propositional language is expanded with the unary connective $\triangle$ and/or with truth-constants, all the mentioned notions are extended in the obvious way.

Observe that if the chosen MTL-algebras are (isomorphic to) the two-element Boolean algebra $\boldsymbol{B}_2$, all the defined semantical notions turn out to be exactly the usual classical definitions. Therefore, the presented graded model theory contains classical model theory.

2.2. Model-theoretic logics

In this subsection we introduce the general framework for an abstract graded model theory. As in the classical theory, we need a basic notion of "abstract logic" that builds on the basic language seen above and, moreover, abstracts away from any particular signature (i.e. includes all possible signatures), allows for any additional syntactical devices one might want, and has a corresponding semantical counterpart with enough corresponding interpretive devices. These models will certainly need to include (and possibly expand) the kind of structures seen above (that is, safe τ-structures valued on an MTL-chain $\boldsymbol{A}$), in order to account for the basic propositional language, the quantifiers $\exists, \forall$ and the crisp equality $\approx$. We will call these expanded structures $(\boldsymbol{A}, \tau)$-*models*. However, we cannot be completely general and allow for arbitrary MTL-chains. Instead, henceforth, we will assume that we have fixed a *finite* MTL-chain $\boldsymbol{A}$. The reason for this restriction is that we have a compactness property for satisfaction for first-order languages with semantics given over a fixed finite MTL-chain [14, Theorem 4.4]). In fact, it is not too difficult to find examples of logics with an infinite algebra of truth-values where compactness fails (for instance, with the addition of the Baaz–Monteiro $\triangle$ connective—but this could be dispensed with by having enough constants for the values of the algebra in the language—see [13]). However, a more important case is [20, p. 2], where it is shown that $\Pi\forall$ does not satisfy the compactness property for satisfaction over the standard semantics $[0, 1]_\Pi$. Note that in this example there is no extra expressive power obtained by either the connective $\triangle$ or truth-constants. So we have that no Lindström characterization involving compactness (or a stronger property) can be obtained, in general, when $\boldsymbol{A}$ is an arbitrary infinite MTL-chain.

Let us proceed with the fixed finite MTL-chain $\boldsymbol{A}$. We define an *abstract logic* (or *model-theoretic language*) as a pair of the form $\mathscr{L}^{\boldsymbol{A}} = \langle \mathcal{L}_{\mathscr{L}}, \|\cdot\|_{\mathscr{L}}\rangle$ such that:

- $\mathcal{L}_{\mathscr{L}}$ maps every signature τ to a set of $\mathscr{L}(\tau)$-formulas $\mathcal{L}_{\mathscr{L}}(\tau)$ in such a way that:
  - (Monotonicity). If $\tau \subseteq \tau'$, then $\mathcal{L}_{\mathscr{L}}(\tau) \subseteq \mathcal{L}_{\mathscr{L}}(\tau')$.

  – (Occurrence). If $\varphi \in \mathcal{L}_{\mathscr{L}}(\tau)$, then there is a finite signature $\tau_\varphi \subseteq \tau$ such that for every signature τ', $\varphi \in \mathcal{L}_{\mathscr{L}}(\tau')$ iff $\tau_\varphi \subseteq \tau'$.
- $\|\cdot\|_{\mathscr{L}}$ is a function that, given a signature τ, a formula $\varphi \in \mathcal{L}_{\mathscr{L}}(\tau)$, and an $(\boldsymbol{A}, \tau)$-model $\mathfrak{M}$, maps every pair $\langle \varphi, \mathfrak{M} \rangle$ to an element of A in such a way that respects the interpretation of the basic connectives, the equality $\approx$, and the quantifiers $\forall, \exists$ (as defined in the previous subsection), and, in addition, the following conditions:
  – (Isomorphism). If $\mathfrak{M}$ and $\mathfrak{N}$ are isomorphic safe $(\boldsymbol{A}, \tau)$-models, then for every $\varphi \in \mathcal{L}_{\mathscr{L}}(\tau)$

$$\|\varphi\|_{\mathscr{L}}^{\mathfrak{M}} = \|\varphi\|_{\mathscr{L}}^{\mathfrak{N}}.$$

  – (Expansion). If $\tau \subseteq \tau'$ and $\mathfrak{M}$ is an $(\boldsymbol{A}, \tau')$-model, then

$$\|\varphi\|_{\mathscr{L}}^{\mathfrak{M}} = \|\varphi\|_{\mathscr{L}}^{\mathfrak{M}\restriction\tau}.$$

  – (Renaming property). Let $\pi : \tau \longrightarrow \sigma$ be a renaming of signatures. Then, for each $\varphi \in \mathscr{L}[\tau]$, there is a sentence $\varphi^\pi \in \mathscr{L}[\sigma]$ such that for each $(\boldsymbol{A}, \tau)$-model $\mathfrak{M}$ and corresponding $(\boldsymbol{A}, \sigma)$-model $(\mathfrak{M})^\pi$,

$$\|\varphi\|_{\mathscr{L}}^{\mathfrak{M}} = \|\varphi^\pi\|_{\mathscr{L}}^{(\mathfrak{M})^\pi}.$$

We will write $\mathscr{L}_1^{\boldsymbol{A}} \leq \mathscr{L}_2^{\boldsymbol{A}}$ to mean that for every formula of the first abstract logic there is a 1-equivalent formula in the second; in this case we say that $\mathscr{L}_2^{\boldsymbol{A}}$ is an *extension* of $\mathscr{L}_1^{\boldsymbol{A}}$. Clearly, $\leq$ is a partial order. If both $\mathscr{L}_1^{\boldsymbol{A}} \leq \mathscr{L}_2^{\boldsymbol{A}}$ and $\mathscr{L}_2^{\boldsymbol{A}} \leq \mathscr{L}_1^{\boldsymbol{A}}$ hold, then we say that $\mathscr{L}_1^{\boldsymbol{A}}$ and $\mathscr{L}_2^{\boldsymbol{A}}$ are *expressively equivalent* and write $\mathscr{L}_1^{\boldsymbol{A}} \simeq \mathscr{L}_2^{\boldsymbol{A}}$.

This notion of expressive equivalence between abstract logics in terms of 1-equivalence was first proposed in the fuzzy and many-valued setting in [20]. It seems appropriate given that it connects to satisfaction and expressivity in a natural way. Moreover, for the case of fuzzy predicate logics, Lindström theorems showing equivalence in this sense are the only kind known in the literature (for the particular example of Łukasiewicz logic on the standard algebra in $[0, 1]$; see [9]).

In [9] this comparison between logics using the relation $\leq$ is said to refer to their *axiomatic strength*. It is, of course, weaker than if we required that for any formula of the weaker logic there should be a formula of the stronger logic that would coincide on *all values*, not just on $\overline{1}^{\boldsymbol{A}}$. As we shall see, if we required this stronger condition, the corresponding version of the first Lindström theorem in this paper would be plainly false.

We denote by $\mathscr{L}_{\omega\omega}^{\boldsymbol{A}}$ the abstract logic, lying at the bottom of the hierarchy, obtained from considering the usual setting (with truth-constants and equality) presented in the previous subsection. The subindexes represent the finitary character of the quantifiers and the connectives $\vee, \wedge, \&$.

By a crisp predicate we mean one that takes values only in the set $\{\overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}}\}$. In the presence of crisp equality, a function symbol f can be represented as a crisp binary predicate which is functional.

Let us list the metalogical properties that we will use, in different combinations, to characterize $\mathscr{L}_{\omega\omega}^{\boldsymbol{A}}$. For any abstract logic $\mathscr{L}^{\boldsymbol{A}} = \langle \mathcal{L}_{\mathscr{L}}, \|\cdot\|_{\mathscr{L}} \rangle$, we define the following (extending notions introduced above in the obvious way):

- *Closure property*: $\mathcal{L}_{\mathscr{L}}(\tau)$ is closed under the MTL connectives.
- *Finite occurrence property*: whenever $\varphi \in \mathscr{L}_{\mathscr{L}}(\tau)$, there is a finite $\tau_\varphi \subseteq \tau$ such that for every signature τ', $\varphi \in \mathscr{L}_{\mathscr{L}}(\tau')$ iff $\tau_\varphi \subseteq \tau'$.
- *Effective abstract logic property*: the set of all formulas of $\mathscr{L}^{\boldsymbol{A}}$ is recursive.
- *Compactness property*: every set of sentences of $\mathscr{L}^{\boldsymbol{A}}$ which is finitely satisfiable is also satisfiable.
- *Löwenheim–Skolem property*: any countable set of sentences of $\mathscr{L}^{\boldsymbol{A}}$ which is satisfiable in a model with an infinite domain, is satisfiable in a model with a countable domain.

- *Karp property*: for every pair of models of $\mathscr{L}^{\boldsymbol{A}}$, if they are finitely isomorphic, then they are strongly elementarily equivalent.
- *Abstract completeness property*: in any countable signature, the collection of the validities of $\mathscr{L}^{\boldsymbol{A}}$ is recursively enumerable.
- *Tarski union property*: the Tarski–Vaught theorem on unions of chains, i.e. if $\langle \boldsymbol{A}, \mathbf{M}\rangle$ is the union of an elementary chain $\{\langle \boldsymbol{A}_i, \mathbf{M}_i\rangle \mid i < \gamma\}$, then, for each sequence $\overrightarrow{a}$ of elements of $\mathbf{M}_i$ and each formula $\varphi(\overrightarrow{x})$, $\|\varphi(\overrightarrow{a})\|^{\boldsymbol{A}}_{\mathbf{M}} = \|\varphi\overrightarrow{a})\|^{\boldsymbol{A}_i}_{\mathbf{M}_i}$ and, moreover, the union $\boldsymbol{A} = \langle \boldsymbol{A}, \mathbf{M}\rangle$ is a safe τ-structure.
- *Robinson property*: if τ_1, τ_2, τ_3 are such that $\tau_3 = \tau_1 \cap \tau_2$, φ_{τ_3} is a set of $\mathscr{L}^{\boldsymbol{A}}[\tau_3]$ sentences, φ_{τ_i} $(i = 1, 2)$ a set of $\mathscr{L}^{\boldsymbol{A}}[\tau_i]$ sentences, if φ_{τ_3} contains a formula of the form $\varphi \leftrightarrow \overline{b}$ (for some $b^{\boldsymbol{A}} \in A$) for each formula φ of $\mathscr{L}^{\boldsymbol{A}}[\tau_3]$ and $\varphi_{\tau_3} \cup \varphi_{\tau_i}$ $(i = 1, 2)$ is satisfiable, then $\varphi_{\tau_3} \cup \varphi_{\tau_1} \cup \varphi_{\tau_2}$ is also satisfiable.
- *κ-Omitting types property*: if Δ is a theory and $\Sigma(x)$ a collection of formulas, we say that $\Sigma(x)$ is an *unsupported κ-type* of Δ if $|\Delta \cup \Sigma(x)| \leq \kappa$, Δ has a model and for every set of formulas $\Theta(x)$ with cardinality $< \kappa$ for which a model of $\Delta \cup \Theta(x)$ exists, we can find some $\sigma \in \Sigma(x)$ such that $\Delta \cup \Theta(x) \cup \{\sigma \to \overline{a}\}$ (where a is the predecessor of $\overline{1}^{\boldsymbol{A}}$) has a model. The *$\kappa$-omitting types property* is just that for every Δ with a model and every unsupported κ-type of Δ, there is model of Δ omitting the type in question.

In the next section, we will justify that these properties are good candidates for characterizations of $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$, since they are all satisfied at the bottom level of the abstract hierarchy. Now we want to illustrate the hierarchy with some other examples and describe how they stand with respect to some of these properties.

Example 1. We can close $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$ under infinitary lattice disjunctions and conjunctions, e.g. by allowing formulas $\bigwedge_{i\in I} \varphi_i$ and $\bigvee_{i\in I} \varphi_i$ (where $|I| < \omega_1$) with the following semantics:

$$\begin{aligned}
&\left\|\textstyle\bigwedge_{i\in I} \varphi_i\right\|^{\boldsymbol{A}}_{\mathbf{M},v} = \inf\{\|\varphi_i\|^{\boldsymbol{A}}_{\mathbf{M},v} \mid i \in I\};\\
&\left\|\textstyle\bigvee_{i\in I} \varphi_i\right\|^{\boldsymbol{A}}_{\mathbf{M},v} = \sup\{\|\varphi_i\|^{\boldsymbol{A}}_{\mathbf{M},v} \mid i \in I\}.
\end{aligned}$$

We call the resulting abstract logic $\mathscr{L}^{\boldsymbol{A}}_{\omega_1\omega}$. In this setting, compactness and the finite occurrence properties are clearly lost but we preserve others such as the Tarski union property (easy exercise). The Löwenheim–Skolem property is preserved at least for the case $\boldsymbol{A} \cong \boldsymbol{B}_2$ [22, Theorem 17].

Example 2. We can close $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$ under monadic second-order quantifiers $(\exists X)\varphi(X)$ and $(\forall X)\varphi(X)$ with the following semantics:

$$\begin{aligned}
&\|(\forall X)\varphi(X)\|^{\boldsymbol{A}}_{\mathbf{M},v} = \inf\{\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v[X\mapsto f]} \mid f : M \longrightarrow \boldsymbol{A}\};\\
&\|(\exists X)\varphi(X)\|^{\boldsymbol{A}}_{\mathbf{M},v} = \sup\{\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v[X\mapsto f]} \mid f : M \longrightarrow \boldsymbol{A}\}.
\end{aligned}$$

This way we obtain the abstract logic $MSO(\mathscr{L}^{\boldsymbol{A}}_{\omega_1\omega})$, which does not have the abstract completeness property. To see this, consider the finite signature of Peano arithmetic. Recall that equality is crisp, i.e. equality formulas only take values $\overline{1}^{\boldsymbol{A}}$ or $\overline{0}^{\boldsymbol{A}}$. Now take the sentence $\sigma := (\forall X)(\forall y)(Xy \vee (Xy \to \overline{0}))$, which expresses that every predicate has to be crisp. But then $(\sigma \to \overline{a}) \vee \psi$ is a validity of $MSO(\mathscr{L}^{\boldsymbol{A}}_{\omega\omega})$ only if ψ is a validity of $MSO(\mathscr{L}^{\boldsymbol{B}_2}_{\omega\omega})$ (any countermodel based on the algebra $\boldsymbol{B}_2$ has an $\boldsymbol{A}$ counterpart satisfying the same formulas), and the latter is just classical monadic second-order logic. Since the validities of $MSO(\mathscr{L}^{\boldsymbol{B}_2}_{\omega\omega})$ in the signature of Peano arithmetic are not recursively enumerable (by Gödel's incompleteness), neither are the validities of $MSO(\mathscr{L}^{\boldsymbol{A}}_{\omega\omega})$. Furthermore, $MSO(\mathscr{L}^{\boldsymbol{A}}_{\omega\omega})$ does not satisfy compactness either in a signature containing at least one binary relation R: one can adapt the typical example using a second-order formula which expresses that R is a well-founded relation in terms of an induction property, since well-foundedness is equivalent the absence of an infinitely descending chain in R, compactness would give a contradiction.

Example 3. We can take the monadic existential fragment of $MSO(\mathscr{L}_{\omega\omega}^{\boldsymbol{A}})$ which consists of the formulas $(\exists X)\varphi(X)$ (where φ is an $\mathscr{L}_{\omega\omega}^{\boldsymbol{A}}$-formula) with the following semantics:

$$\|(\exists X)\varphi(X)\|_{\mathbf{M},v}^{\boldsymbol{A}} = \sup\{\|\varphi\|_{\mathbf{M},v[X\mapsto f]}^{\boldsymbol{A}} \mid f : M \longrightarrow \boldsymbol{A}\}.$$

This way we obtain the abstract logic $\Sigma_1^1(\mathscr{L}_{\omega\omega}^{\boldsymbol{A}})$. As in [9], the reason this fits our notion of abstract logic is that we have not included Closure as one of the defining properties, since $\Sigma_1^1(\mathscr{L}_{\omega\omega}^{\boldsymbol{A}})$ is not closed under the MTL-connectives. $\Sigma_1^1(\mathscr{L}_{\omega\omega}^{\boldsymbol{A}})$ is compact: adapt the proof of [10, Corollary 4.1.14] to show that formulas of $\Sigma_1^1(\mathscr{L}_{\omega_1\omega}^{\boldsymbol{A}})$ are preserved under the ultraproduct construction and reproduce the compactness argument from [14, Theorem 4.4]. Furthermore, $\Sigma_1^1(\mathscr{L}_{\omega\omega}^{\boldsymbol{A}})$ has the Löwenheim–Skolem property: take a countable set Φ of sentences of $\Sigma_1^1(\mathscr{L}_{\omega\omega}^{\boldsymbol{A}})$ which have an infinite model, and consider the set Φ^* of sentences $\mathscr{L}_{\omega\omega}^{\boldsymbol{A}}$ which come from Φ by instantiating the second-order existential quantifiers to new predicates in a uniform manner, so by the Löwenheim–Skolem property of $\mathscr{L}_{\omega\omega}^{\boldsymbol{A}}$ (see next section), Φ^* has a countable model, which is also a model of Φ.

Example 4. We can close $\mathscr{L}_{\omega\omega}^{\boldsymbol{A}}$ under the quantifier I with the following semantics:

$$\|(Ixy)(\varphi(x),\psi(y))\|_{\mathbf{M},v}^{\boldsymbol{A}} = \sup\{\overline{a} \in \boldsymbol{A} \mid |\{d \in M \mid \|\varphi\|_{\mathbf{M},v[x\mapsto d]}^{\boldsymbol{A}} = \overline{a}\}| = |\{d \in M \mid \|\psi\|_{\mathbf{M},v[x\mapsto d]}^{\boldsymbol{A}} = \overline{a}\}|\}.$$

We call this abstract logic $I(\mathscr{L}_{\omega\omega}^{\boldsymbol{A}})$. We can observe that $\mathfrak{M} \models (Ixy)(\varphi(x),\psi(y))$ iff $|\{d \in M \mid \mathfrak{M} \models \varphi[d]\}| = |\{d \in M \mid \mathfrak{M} \models \psi[d]\}|$, which means that I is the so called Härtig quantifier. For the case $\boldsymbol{A} \cong \boldsymbol{B}_2$, as observed in [21], $I(\mathscr{L}_{\omega\omega}^{\boldsymbol{A}})$ has neither the Karp nor the Tarski union properties.

Example 5. We can close $\mathscr{L}_{\omega\omega}^{\boldsymbol{A}}$ under the quantifier Q_1 with the following semantics:

$$\|(Q_1x)\varphi(x)\|_{\mathbf{M},v}^{\boldsymbol{A}} = \sup\{\overline{a} \in \boldsymbol{A} \mid |\{d \in M \mid \|\varphi\|_{\mathbf{M},v[x\mapsto d]}^{\boldsymbol{A}} = \overline{a}\}| \geq \aleph_1\}.$$

We call this logic $Q_1(\mathscr{L}_{\omega\omega}^{\boldsymbol{A}})$. Observe that $\mathfrak{M} \models (Q_1x)\varphi(x)$ iff $|\{d \in M \mid \mathfrak{M} \models \varphi[d]\}| \geq \aleph_1$. It is obvious that $Q_1(\mathscr{L}_{\omega\omega}^{\boldsymbol{A}})$ does not have the Löwenheim–Skolem property. When $\boldsymbol{A} \cong \boldsymbol{B}_2$, $Q_1(\mathscr{L}_{\omega\omega}^{\boldsymbol{A}})$ has compactness, abstract completeness and the omitting types property [6, Chap. II, Example 1].

3. "Hard" graded model theory: properties of $\mathscr{L}_{\omega\omega}^{\boldsymbol{A}}$

In this section we collect several results of graded model theory, in the standard framework developed recently in the literature. We add the necessary restrictions corresponding to the abstract logic $\mathscr{L}_{\omega\omega}^{\boldsymbol{A}}$, as defined in the previous section. Because they are constrained to a particular level of the abstract hierarchy (namely, the bottom level), these results may be seen as a "hard" graded model theory, as opposed to the "soft" theory that in the next section will give us results referring to unconstrained abstract logics. More precisely, we will show (by calling the corresponding sources in the literature or by offering new proofs when needed) that $\mathscr{L}_{\omega\omega}^{\boldsymbol{A}}$ has all the metalogical properties introduced above.

The closure, finite occurrence, and effectiveness properties hold simply by the definition of the syntax in $\mathscr{L}_{\omega\omega}^{\boldsymbol{A}}$. In contrast, the compactness and the Löwenheim–Skolem properties are non-trivial and have been studied in the literature. They hold in our setting of $\mathscr{L}_{\omega\omega}^{\boldsymbol{A}}$ thanks to [14, Theorem 4.4] and [12, Theorem 30].

The Karp property is obtained in the next proposition, which relies heavily on the finiteness of the fixed MTL-chain $\boldsymbol{A}$ (cf. [13]).

Proposition 6. *Consider the setting of $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$ with a finite signature τ. Then, the following are equivalent:*

(i) $\mathfrak{M} \cong_f \mathfrak{N}$, *i.e., $\mathfrak{M}$ and $\mathfrak{N}$ are finitely isomorphic.*
(ii) $\mathfrak{M} \equiv^s \mathfrak{N}$, *i.e., $\mathfrak{M}$ and $\mathfrak{N}$ are strongly elementary equivalent.*

Furthermore, (i) $\implies$ (ii) does not depend on the finiteness of τ (so $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$ has the Karp property).

Proof. $(i) \implies (ii)$: This follows from Theorem 14 (b) from [13].

$(ii) \implies (i)$: We start by defining inductively, for any model $\mathfrak{M}$ and finite sequence $\overrightarrow{e}$ of elements of M, the formulas $\psi^m_{\langle \mathfrak{M}, \overrightarrow{e} \rangle}(\overrightarrow{x})$ $(m < \omega)$ as follows. As the initial one, we take:

$$\psi^0_{\langle \mathfrak{M}, \overrightarrow{e} \rangle} = \bigwedge \{\varphi(\overrightarrow{x}) \leftrightarrow \overline{d_{\varphi(\overrightarrow{e})}} \mid \varphi(\overrightarrow{x}) \text{ is atomic and } d_{\varphi(\overrightarrow{e})} = \|\varphi[\overrightarrow{e}]\|^{\mathfrak{M}}\},$$

and observe that this conjunction is finitary since the signature is finite. Moreover, since $\boldsymbol{A}$ is finite, we have only finitely many possibilities for $\psi^0_{\langle \mathfrak{M}, \overrightarrow{e} \rangle}$, that is, when we allow to $\overrightarrow{e}$ to vary over all finite sequences of elements of M, we obtain a finite number of formulas $\psi^0_{\langle \mathfrak{M}, \overrightarrow{e} \rangle}$ up to 1-equivalence. Then, assuming that $\psi^n_{\langle \mathfrak{M}, \overrightarrow{d} \rangle}$ is given for arbitrary finite sequences of elements $\overrightarrow{d}$ of $\mathfrak{M}$ and that there are only finitely many of them, we continue by defining

$$\psi^{n+1}_{\langle \mathfrak{M}, \overrightarrow{e} \rangle} = \bigwedge_{c \in M} (\exists y)\, \psi^n_{\langle \mathfrak{M}, \overrightarrow{e} c \rangle}(\overrightarrow{x}, y) \wedge (\forall y) \bigvee_{c \in M} \psi^n_{\langle \mathfrak{M}, \overrightarrow{e} c \rangle}(\overrightarrow{x}, y).$$

Then, this is a finitary formula and, moreover, there are only finitely many possibilities for it. We can easily observe that, for any $n < \omega$,

$$\mathfrak{M} \models \psi^n_{\langle \mathfrak{M}, \overrightarrow{e} \rangle}[\overrightarrow{e}],$$
$$\psi^n_{\langle \mathfrak{M}, \overrightarrow{e} c \rangle} \models \psi^n_{\langle \mathfrak{M}, \overrightarrow{e} \rangle},$$

and, from the latter fact, it also follows that

$$\psi^{n+1}_{\langle \mathfrak{M}, \overrightarrow{e} \rangle} \models \psi^n_{\langle \mathfrak{M}, \overrightarrow{e} \rangle}.$$

To show that $\mathfrak{M} \cong_f \mathfrak{N}$, we define the following system $\langle I_m \mid m < \omega \rangle$ of sets of partial isomorphisms:

$$I_m = \{\langle Id, p \rangle \mid \langle Id, p \rangle \text{ is a partial isomorphism from } \mathfrak{M} \text{ into } \mathfrak{N}, \mathrm{dom}(p) = \{e_0, \dots, e_k\}, \mathfrak{N} \models \psi^m_{\langle \mathfrak{M}, \overrightarrow{e} \rangle}[p(\overrightarrow{e})]\}.$$

Note that the I_ms are non-empty since at least they contain $\langle Id, \emptyset \rangle$.

Next, we check the forth property. Suppose that $\langle Id, p \rangle \in I_{m+1}$, $\mathrm{dom}(p) = \{e_0, \dots, e_k\}$ and let e_{k+1} be an arbitrary element of M. Then, $\mathfrak{N} \models \psi^{m+1}_{\langle \mathfrak{M}, \overrightarrow{e} \rangle}[p(\overrightarrow{e})]$, so $\mathfrak{N} \models \exists y \psi^m_{\langle \mathfrak{M}, \overrightarrow{e}, e_{k+1} \rangle}(\overrightarrow{x}, y)[p(\overrightarrow{e})]$. Hence, say that $\mathfrak{N} \models \psi^m_{\langle \mathfrak{M}, \overrightarrow{e}, e_{k+1} \rangle}[p(\overrightarrow{e}), e'_{k+1}]$, so also, $\mathfrak{N} \models \psi^0_{\langle \mathfrak{M}, \overrightarrow{e}, e_{k+1} \rangle}[p(\overrightarrow{e}), e'_{k+1}]$, and, consequently, $\langle Id, p \cup \{\langle e_{k+1}, e'_{k+1} \rangle\} \rangle$ is a partial isomorphism in I_m.

Finally, we check the back property. Suppose that $\langle Id, p \rangle \in I_{m+1}$, $\mathrm{dom}(p) = \{e_0, \dots, e_k\}$ and let e'_{k+1} be some new element of N. Then, $\mathfrak{N} \models \psi^{m+1}_{\langle \mathfrak{M}, \overrightarrow{e} \rangle}[p(\overrightarrow{e})]$, so $\mathfrak{N} \models \bigvee_{c \in M} \psi^m_{\langle \mathfrak{M}, \overrightarrow{e}, c \rangle}[p(\overrightarrow{e}), e'_{k+1}]$, and hence for some $e_{k+1} \in M$, $\mathfrak{N} \models \psi^0_{\langle \mathfrak{M}, \overrightarrow{e}, e_{k+1} \rangle}[p(\overrightarrow{e}), e'_{k+1}]$, and, consequently, $\langle Id, p \cup \{\langle e_{k+1}, e'_{k+1} \rangle\} \rangle$ is a partial isomorphism in I_m. $\square$

Recall that, for an abstract logic, the abstract completeness property simply says that the collection of its validities is recursively enumerable.

Proposition 7. *$\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$ has the abstract completeness property.*

Proof. The cheap way to establish this without any syntactic calculus makes a detour through a translation into a two-sorted first-order language. In [12], a translation is defined such that for each formula φ of $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$, we assign a formula E_φ in an appropriate two-sorted first-order language (one sort for the elements of $\boldsymbol{A}$, another sort for the individuals of the domain of an $\boldsymbol{A}$-valued model) such that for any model $\mathfrak{M}$ and $b \in \boldsymbol{A}$, $\|\varphi(d_1, \dots, d_n)\|^{\mathfrak{M}} = b$ iff $\mathfrak{M} \models E_\varphi(d_1, \dots, d_n, b)$ when seen as a classical two-sorted model. In general, this translation is not a very precise surgical tool and hence we make limited use of it (see [12] for some commentary), but it suffices for our current purposes. In classical two-sorted first-order logic we can write a sentence $\psi_{\boldsymbol{A}}$ axiomatizing the isomorphism type of $\boldsymbol{A}$ by standard methods, since only finitely many elements and relations are involved. Note that only variables of the sort used for the elements of the algebra will be involved. Then, it is clear that $\psi_{\boldsymbol{A}} \to E_\varphi(\overline{1})$ is a validity of classical two-sorted first-order logic iff φ is a validity of $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$. Consequently, by the recursive enumerability of the theorems of multi-sorted first-order logic, we obtain the abstract completeness of $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$. □

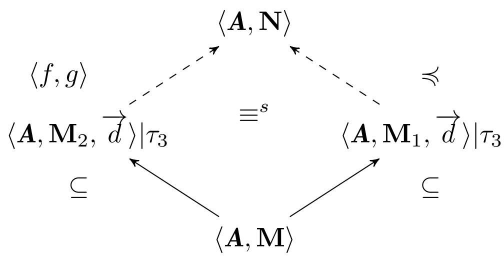
Finally, we focus on the three remaining metalogical properties of $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$.

Proposition 8. *([4, Theorem 2]) $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$ has the Tarski union property.*

Lemma 9. *Let $\langle \boldsymbol{A}, \mathbf{M}_1\rangle$ and $\langle \boldsymbol{A}, \mathbf{M}_2\rangle$ be two structures of signatures τ_1 and τ_2 with a common part $\langle \boldsymbol{A}, \mathbf{M}\rangle$ of signature $\tau_3 = \tau_1 \cap \tau_2$ with domain generated by a sequence of elements $\overrightarrow{d}$. Moreover, suppose that*

$$\langle \boldsymbol{A}, \mathbf{M}_2, \overrightarrow{d}\rangle|\tau_3 \equiv^s \langle \boldsymbol{A}, \mathbf{M}_1, \overrightarrow{d}\rangle|\tau_3.$$

Then, there is a structure $\langle \boldsymbol{A}, \mathbf{N}\rangle$ in the signature τ_1 into which the restriction of $\langle \boldsymbol{A}, \mathbf{M}_2\rangle$ to the signature τ_3, $\langle \boldsymbol{A}, \mathbf{M}_2\rangle|\tau_3$, can be elementarily embedded by $\langle f, g\rangle$, while $\langle \boldsymbol{A}, \mathbf{M}_1\rangle$ is (taking isomorphic copies) an elementary substructure in the signature τ_1. The situation is described by the following picture:



Proof. It is not difficult to show that $\mathrm{ElDiag}(\boldsymbol{A}, \mathbf{M}_1) \cup \mathrm{ElDiag}(\boldsymbol{A}, \mathbf{M}_2|\tau_3)$ (where we let the elements of the domain serve as constants to name themselves) has a model, which suffices for the purposes of the result. Suppose otherwise, that is, for some finite $\mathrm{ElDiag}_0(\boldsymbol{A}, \mathbf{M}_2|\tau_3) \subseteq \mathrm{ElDiag}(\boldsymbol{A}, \mathbf{M}_2|\tau_3)$, we have that

$$\mathrm{ElDiag}(\boldsymbol{A}, \mathbf{M}_1) \vDash (\bigwedge \mathrm{ElDiag}_0(\boldsymbol{A}, \mathbf{M}_2|\tau_3)) \to \overline{a},$$

where a is the immediate predecessor of $\overline{1}^{\boldsymbol{A}}$ in the lattice order of $\boldsymbol{A}$.

Quantifying away the new object constants, we have

$$\mathrm{ElDiag}(\boldsymbol{A}, \mathbf{M}_1) \vDash (\exists \overrightarrow{x})((\bigwedge \mathrm{ElDiag}^*_0(\boldsymbol{A}, \mathbf{M}_2|\tau_3))) \to \overline{a},$$

where $\mathrm{ElDiag}_0^*(\boldsymbol{A}, \mathbf{M}_2|\tau_3)$ is the result of substituting the constants by variables in $\mathrm{ElDiag}_0(\boldsymbol{A}, \mathbf{M}_2|\tau_3)$. Since $\langle \boldsymbol{A}, \mathbf{M}_2, \overrightarrow{d}\rangle|\tau_3 \equiv^s \langle \boldsymbol{A}, \mathbf{M}_1, \overrightarrow{d}\rangle|\tau_3$, then

$$\langle \boldsymbol{A}, \mathbf{M}_2\rangle \not\models (\exists \overrightarrow{x})(\bigwedge \mathrm{ElDiag}_0^*(\boldsymbol{A}, \mathbf{M}_2|\tau_3)),$$

which is a contradiction. □

Corollary 10. *Let $\langle \boldsymbol{A}, \mathbf{M}_1\rangle$ and $\langle \boldsymbol{A}, \mathbf{M}_2\rangle$ be two structures of signatures τ_1, τ_2 with a common part $\langle \boldsymbol{A}, \mathbf{M}\rangle$ of signature $\tau_3 = \tau_1 \cap \tau_2$ with domain generated by a sequence of elements $\overrightarrow{d}$. Moreover, suppose that*

$$\langle \boldsymbol{A}, \mathbf{M}_2, \overrightarrow{d}\rangle|\tau_3 \equiv^s \langle \boldsymbol{A}, \mathbf{M}_1, \overrightarrow{d}\rangle|\tau_3.$$

Then, there is a structure $\langle \boldsymbol{A}, \mathbf{N}\rangle$ in the signature $\tau_1 \cup \tau_2$ into which $\langle \boldsymbol{A}, \mathbf{M}_2\rangle$ can be elementarily embedded by $\langle f, g\rangle$ in the signature τ_2 while $\langle \boldsymbol{A}, \mathbf{M}_1\rangle$ is (taking isomorphic copies) an elementary substructure in the signature τ_1.

Proof. We will build simultaneously two elementary chains:

$$\langle \boldsymbol{A}, \mathbf{M}_i\rangle_{i<\omega} \qquad \langle \boldsymbol{A}, \mathbf{S}_i\rangle_{i<\omega}.$$

First, let $\langle \boldsymbol{A}, \mathbf{M}_0\rangle = \langle \boldsymbol{A}, \mathbf{M}\rangle$ and $\langle \boldsymbol{A}, \mathbf{S}_0\rangle = \langle \mathbf{A}, \mathbf{S}\rangle$.

Now, given $\langle \boldsymbol{A}, \mathbf{M}_{i+1}\rangle$ and $\langle \boldsymbol{A}, \mathbf{S}_i\rangle$ with the property that $\langle \boldsymbol{A}, \mathbf{M}_{i+1}\rangle|\tau_3 \equiv^s \langle \boldsymbol{A}, \mathbf{S}_i\rangle|\tau_3$, we use Lemma 9 to obtain a τ_2-structure $\langle \boldsymbol{A}, \mathbf{S}_{i+1}\rangle$ into which $\langle \boldsymbol{A}, \mathbf{S}_i\rangle$ is elementarily embedded while $\langle \boldsymbol{A}, \mathbf{M}_{i+1}\rangle|\tau_3$ is elementarily embedded into $\langle \boldsymbol{A}, \mathbf{S}_{i+1}\rangle|\tau_3$.

On the other hand, given $\langle \boldsymbol{A}, \mathbf{M}_i\rangle$ and $\langle \boldsymbol{A}, \mathbf{S}_i\rangle$ with the property that $\langle \boldsymbol{A}, \mathbf{M}_i\rangle)\tau_3 \equiv \langle \boldsymbol{A}, \mathbf{S}_i\rangle|\tau_3$, we use Lemma 9 to obtain a τ_1-structure $\langle \boldsymbol{A}, \mathbf{M}_{i+1}\rangle$ into which $\langle \boldsymbol{A}, \mathbf{M}_i\rangle$ is elementarily embedded while $\langle \boldsymbol{A}, \mathbf{S}_i\rangle|\tau_3$ is elementarily embedded into $\langle \boldsymbol{A}, \mathbf{M}_{i+1}\rangle|\tau_3$.

By the elementary chain construction, we have that indeed

$$\langle \bigcup_{i<\omega} \boldsymbol{A}, \bigcup_{i<\omega} \mathbf{M}_i\rangle|\tau_3 = \langle \bigcup_{i<\omega} \boldsymbol{A}, \bigcup_{i<\omega} \mathbf{S}_i\rangle|\tau_3.$$

Then, we can use the τ_2-structure $\langle \bigcup_{i<\omega} \boldsymbol{A}, \bigcup_{i<\omega} \mathbf{S}_i\rangle$ as a template to expand the τ_1-structure $\langle \bigcup_{i<\omega} \boldsymbol{A}, \bigcup_{i<\omega} \mathbf{M}_i\rangle$ to a $(\tau_1 \cup \tau_2)$-structure. By the Tarski union property, we have our result. □

Proposition 11. *$\mathscr{L}_{\omega\omega}^{\boldsymbol{A}}$ has the Robinson property.*

Proof. Immediate from Corollary 10. □

Proposition 12. *$\mathscr{L}_{\omega\omega}^{\boldsymbol{A}}$ has the κ-omitting types property.*

Proof. Here once more we will apply the cheap trick of the translation. It will work because we have constants for the values of the algebra and the algebra is finite. The possibility to express the classical Boolean negation of our metatheory in $\mathscr{L}_{\omega\omega}^{\boldsymbol{A}}$ (since we have a constant for the co-atom of the algebra) is what makes the difference and why this translation trick does not work in [3]. It suffices to establish that if $\Sigma(x)$ is an unsupported κ-type of Δ, then $\Sigma^*(x)$ is an unsupported κ-type of $\Delta^* \cup \{\psi_{\boldsymbol{A}}\}$, where $\psi_{\boldsymbol{A}}$ is as in Proposition 7, $\Sigma^*(x)$ is $\{E_\varphi(x, \overline{1}) \mid \varphi(x) \in \Sigma(x)\}$ and similarly for Δ^*. It is enough to observe that for every formula $\theta(x)$ of the two-sorted language such that x is a variable of the element sort (note that we have no way of translating formulas with free variables of the algebra sort!), there is a formula $\theta^*(x)$ of $\mathscr{L}_{\omega\omega}^{\boldsymbol{A}}$ with the same models of $\psi_{\boldsymbol{A}}$. □

4. "Soft" graded model theory: Lindström theorems for $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$

After the preliminary presentation of "hard" graded model theory and the introduction of the our setting for a "soft" or abstract graded model theory, we can finally start substantiating it with results. As promised, we will offer several Lindström-style theorems that characterize $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$ as the maximal abstract logic based on a finite MTL-chain with certain (combinations of) metalogical properties. It is important to observe that our theorems could not be obtained without the expressive power of having truth-constants in our language. By this we mean that the considered combinations of properties can only characterize languages that are expressively identical to their own expansions with truth-constants for all the elements of the algebra. This is because said language expansions also satisfy the model-theoretic properties that we are discussing and, hence, if they increase the expressive power of the original language, the original language cannot be maximal with respect to the properties in question.

4.1. Löwenheim–Skolem property + compactness property

Our first result is a generalization of the classical Lindström theorem that characterizes the standard first-order language as the maximal abstract logic with the compactness and Löwenheim–Skolem properties.

Theorem 13. (First Lindström theorem) *Let $\mathscr{L}^{\boldsymbol{A}}$ be an abstract logic such that $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega} \leq \mathscr{L}^{\boldsymbol{A}}$. If $\mathscr{L}^{\boldsymbol{A}}$ is closed under the connectives of $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$, it has the Löwenheim–Skolem property, and the compactness property for countable sets of formulas, then $\mathscr{L}^{\boldsymbol{A}} \leq \mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$.*

Proof. We start by showing that any formula φ of a signature τ in $\mathscr{L}^{\boldsymbol{A}}$ depends only on a finite subset $\tau_0 \subseteq \tau$; in other words, for any two $(\boldsymbol{A}, \tau)$-models $\mathfrak{M}$ and $\mathfrak{N}$, we have that $\mathfrak{M} \upharpoonright \tau_0 \cong \mathfrak{N} \upharpoonright \tau_0$ implies $\|\varphi\|^{\mathfrak{M}} = \|\varphi\|^{\mathfrak{N}}$. In order to see this, consider the following collection Γ of formulas in the signature $\tau \cup \tau'$ (where τ' is a disjoint copy of τ, that is, it has a symbol λ' of the corresponding arity for any symbol λ of τ):

$$\begin{aligned}\{(\forall \overrightarrow{x})(R(\overrightarrow{x}) \leftrightarrow R'(\overrightarrow{x})) \mid R \in Pred_\tau\}&\\ &\cup \{(\forall \overrightarrow{x})(f(\overrightarrow{x}) \approx f'(\overrightarrow{x})) \mid f \in Func_\tau\}\\ &\cup \{c \approx c' \mid c \in Func_\tau, Ar_\tau(c) = 0\}.\end{aligned}$$

It is clear that $\Gamma \vDash \varphi \leftrightarrow \varphi'$ (where φ' is the result of replacing the non-logical symbols from τ in φ by their corresponding copies from τ'). This follows by standard substitution properties. Observe the essential use of crisp equality in this step.

For the remainder of the proof let a be the immediate predecessor of $\overline{1}^{\boldsymbol{A}}$ in the lattice order of $\boldsymbol{A}$. Now, there must be some finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vDash \varphi \leftrightarrow \varphi'$. This follows by compactness, for suppose otherwise, that is, for every finite $\Gamma_0 \subseteq \Gamma$, $\Gamma_0 \nvDash \varphi \leftrightarrow \varphi'$, i.e., $\Gamma_0 \cup \{(\varphi \leftrightarrow \varphi') \to \overline{a}\}$ has a model. Hence, $\Gamma \cup \{(\varphi \leftrightarrow \varphi') \to \overline{a}\}$ would also have a model, which is a contradiction.

Suppose now, by the way of contradiction, that the conclusion of our theorem does not hold. This means that there is a formula φ of $\mathscr{L}^{\boldsymbol{A}}$ which is not 1-equivalent to any formula from $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$. Let τ_0 be the finite signature φ depends on.

We enumerate the formulas from $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$ in the signature τ_0 as $\psi_1, \psi_2, \ldots$ and then we define a list of $\mathscr{L}^{\boldsymbol{A}}$-formulas $(\varphi_i)_{i\in\omega}$ such that for any n,

$$\bigwedge_{i=0}^{n} \varphi_i$$

is not 1-equivalent to any $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$-formula as follows. First, $\varphi_0 = \varphi$. Now, suppose that

$$\bigwedge_{i=0}^{k} \varphi_i$$

is not 1-equivalent to any $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$-formula. Then, either

$$(\bigwedge_{i=0}^{k} \varphi_i) \wedge \psi_{k+1}$$

or

$$(\bigwedge_{i=0}^{k} \varphi_i) \wedge (\psi_{k+1} \to \overline{a})$$

is not 1-equivalent to any $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$-formula. For otherwise,

$$(\bigwedge_{i=0}^{k} \varphi_i) \wedge \psi_{k+1}$$

is 1-equivalent to an $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$-formula θ_0, and

$$(\bigwedge_{i=0}^{k} \varphi_i) \wedge (\psi_{k+1} \to \overline{a})$$

is 1-equivalent to an $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$-formula θ_1. But $\models \psi_{k+1} \vee (\psi_{k+1} \to \overline{a})$. Hence,

$$\bigwedge_{i=0}^{k} \varphi_i \models \theta_0 \vee \theta_1.$$

Moreover, it is easy to see that

$$\theta_0 \vee \theta_1 \models \bigwedge_{i=0}^{k} \varphi_i.$$

Then, we would have that

$$\bigwedge_{i=0}^{k} \varphi_i$$

is 1-equivalent to an $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$-formula, which is a contradiction. Finally, let $\varphi_{k+1} = \psi_{k+1}$ or $\varphi_{k+1} = (\psi_{k+1} \to \overline{a})$ according to which alternative holds (if both, then make an arbitrary choice).

We observe that

$$(\varphi \to \overline{a}) \wedge (\bigwedge_{i=1}^{n} \varphi_i)$$

is satisfiable. Otherwise, every model $\mathfrak{M}$ of $(\bigwedge_{i=1}^{n}\varphi_i)$ would be one in which $\|\varphi\|^{\mathfrak{M}}_{\mathscr{L}^{\boldsymbol{A}}} \not\leq_{\boldsymbol{A}} a$, and, by linearity, $a <_{\boldsymbol{A}} \|\varphi\|^{\mathfrak{M}}_{\mathscr{L}^{\boldsymbol{A}}}$ so that $\overline{1}^{\boldsymbol{A}} = \|\varphi\|^{\mathfrak{M}}_{\mathscr{L}^{\boldsymbol{A}}}$, that is, $\mathfrak{M} \models_{\mathscr{L}^{\boldsymbol{A}}} \varphi$. And, hence, $\varphi \wedge (\bigwedge_{i=1}^{n}\varphi_i)$ is 1-equivalent to $(\bigwedge_{i=1}^{n}\varphi_i)$, which is a contradiction.

Then, for any n,

$$\varphi \wedge (\bigwedge_{i=1}^{n}\varphi_i) \text{ and } (\varphi \to \overline{a}) \wedge (\bigwedge_{i=1}^{n}\varphi_i)$$

both have models. For if the first conjunct does not have a model, it would be 1-equivalent to $\overline{0}$. Hence, by compactness, we can obtain models $\mathfrak{M}$ and $\mathfrak{N}$ of the sets $\{\varphi_i \mid 1 \leq i\} \cup \{\varphi\}$ and $\{\varphi_i \mid 1 \leq i\} \cup \{\varphi \to \overline{a}\}$ respectively. Moreover, by the Löwenheim–Skolem property for countable sets of formulas, we can assume that the domains of $\mathfrak{M}$ and $\mathfrak{N}$ are countable.

Furthermore, $\mathfrak{M} \upharpoonright \tau_0 \equiv^s \mathfrak{N} \upharpoonright \tau_0$ since there are formulas of $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$ in the signature τ_0 expressing that a given formula ψ takes a value b in the algebra $\boldsymbol{A}$, namely $\psi \leftrightarrow \overline{b}$. Now, also we must have that $\mathfrak{M} \upharpoonright \tau_0 \ncong \mathfrak{N} \upharpoonright \tau_0$ since otherwise, the models could not differ on the value for φ. Moreover, these models cannot be finite (since given the presence of crisp equality we could axiomatize the isomorphism type of the model with one formula of $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$; we leave this as a simple exercise for the reader). Hence, we might assume that both have the same infinite domain.

Consider now the signature $\tau^* = \tau \cup \tau' \cup \{f_n, g_n \mid n \in \omega\}$, where $f_n(g_n)$ are new $2n+1$-ary function symbols. Take enumerations, for every $n \in \omega$, $\{\chi_i(x_1, \ldots, x_n, x) \mid i \in \omega\}$ of all $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$-formulas in τ_0 with variables among $x_1, \ldots, x_n, x$.

Now we consider the collection $\Sigma = \Gamma \cup \Delta$ where Γ is

$\varphi, \varphi' \to \overline{a}$
$\psi \leftrightarrow \psi'$ for every $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$-formula in the signature τ_0,

whereas Δ is the set having for every $n, k \in \omega$ (letting $\overrightarrow{x}, \overrightarrow{y}$ be n-tuples of variables) the formulas:

$$(\forall \overrightarrow{x}, \overrightarrow{y}, x)(((\exists y)(\bigwedge_{i=0}^{k}(\chi_i(\overrightarrow{x}, x) \leftrightarrow \chi_i'(\overrightarrow{y}, y))) \to \overline{a}) \vee (\bigwedge_{i=0}^{k}(\chi_i(\overrightarrow{x}, x) \leftrightarrow \chi_i(\overrightarrow{y}, f_n(\overrightarrow{x}, \overrightarrow{y}, x))))),$$
$$(\forall \overrightarrow{x}, \overrightarrow{y}, y)(((\exists x)(\bigwedge_{i=0}^{k}(\chi_i(\overrightarrow{x}, x) \leftrightarrow \chi_i'(\overrightarrow{y}, y))) \to \overline{a}) \vee (\bigwedge_{i=0}^{k}(\chi_i(\overrightarrow{x}, g_n(\overrightarrow{x}, \overrightarrow{y}, y)) \leftrightarrow \chi_i(\overrightarrow{y}, y)))).$$

Take any finite subset $\Delta_0 \subseteq \Delta$, we can observe that we can expand any $\tau \cup \tau'$ model to a model of Δ_0. Take any f_n (the case for g_n is analogous) appearing in Δ_0 and a $\tau \cup \tau'$ model $\mathfrak{M}$. We define $f_n(\overrightarrow{e}, \overrightarrow{d}, c)$ for arbitrary n-tuples $\overrightarrow{e}, \overrightarrow{d}$ and element c of the domain of $\mathfrak{M}$ as follows. Let k be the biggest index of conjunctions appearing in the formulas of Δ_0. If,

$$\mathfrak{M} \models (\exists y)(\bigwedge_{i=0}^{k}(\chi_i(\overrightarrow{e}, c) \leftrightarrow \chi_i'(\overrightarrow{d}, y))),$$

then for some element c' of $\mathfrak{M}$,

$$\left\| \bigwedge_{i=0}^{k}(\chi_i(\overrightarrow{e}, c) \leftrightarrow \chi_i'(\overrightarrow{d}, c')) \right\|^{\mathfrak{M}} = \overline{1}^{\boldsymbol{A}}.$$

And we can set $f_n(\overrightarrow{e}, \overrightarrow{d}, c) = c'$. Alternatively, if

$$\mathfrak{M} \not\models (\exists y)(\bigwedge_{i=0}^{k}(\chi_i(\overrightarrow{e}, c) \leftrightarrow \chi'_i(\overrightarrow{d}, y))),$$

we let $f_n(\overrightarrow{e}, \overrightarrow{d}, c)$ be arbitrary.

Hence, with the models $\mathfrak{M}$ and $\mathfrak{N}$ of the sets $\{\varphi_i \mid 1 \leq i\} \cup \{\varphi\}$ and $\{\varphi_i \mid 1 \leq i\} \cup \{\varphi \to \overline{a}\}$ that we obtained above, we can observe that, in fact, every finite subset $\Sigma_0 \subseteq \Sigma$ has a model. By compactness, we might take a model $\mathfrak{M}'$ of Σ. Now we may consider $\mathfrak{M}' \upharpoonright \tau$ and $(\mathfrak{M}' \upharpoonright \tau')^{h^{-1}}$ (where $h : \tau \longrightarrow \tau'$ is the renaming sending every symbol λ of τ to the corresponding symbol λ' of τ'). Then, $\mathfrak{M}'$, $\mathfrak{M}' \upharpoonright \tau$ and $(\mathfrak{M}' \upharpoonright \tau')^{h^{-1}}$ all have the same domain, $\mathfrak{M}' \upharpoonright \tau \models \varphi$, and $(\mathfrak{M}' \upharpoonright \tau')^{h^{-1}} \models \varphi \to \overline{a}$, i.e., $(\mathfrak{M}' \upharpoonright \tau')^{h^{-1}} \not\models \varphi$. Moreover,

$$(\mathfrak{M}' \upharpoonright \tau) \upharpoonright \tau_0 \equiv^s (\mathfrak{M}' \upharpoonright \tau')^{h^{-1}} \upharpoonright \tau_0.$$

The goal is to show that

$$(\mathfrak{M}' \upharpoonright \tau) \upharpoonright \tau_0 \cong (\mathfrak{M}' \upharpoonright \tau')^{h^{-1}} \upharpoonright \tau_0,$$

contradicting the fact that φ depends only on the signature τ_0.

Enumerate the elements of the common domain of these three models as $e_1, e_2, \ldots$ From the fact that

$$(\mathfrak{M}' \upharpoonright \tau) \upharpoonright \tau_0 \equiv^s (\mathfrak{M}' \upharpoonright \tau')^{h^{-1}} \upharpoonright \tau_0,$$

taking e_1 and arbitrary r, letting for each $\chi_i(x)$ $(0 \leq i \leq r)$, $d_{\chi_i[e_1]}$ be $\|\chi_i[e_1]\|^{(\mathfrak{M}' \upharpoonright \tau) \upharpoonright \tau_0}$, we have that

$$(\mathfrak{M}' \upharpoonright \tau')^{h^{-1}} \upharpoonright \tau_0 \models \exists y(\chi_i(y) \leftrightarrow d_{\chi_i[e_1]}).$$

Then, we must also have that

$$\mathfrak{M}' \models \exists y(\bigwedge_{i=0}^{k}(\chi_i(e_1) \leftrightarrow \chi'_i(y))),$$

which means that

$$\mathfrak{M}' \models \bigwedge_{i=0}^{k}(\chi_i(e_1) \leftrightarrow \chi_i(f_0(e_1))).$$

Then,

$$\langle(\mathfrak{M}' \upharpoonright \tau) \upharpoonright \tau_0, e_1\rangle \equiv^s \langle(\mathfrak{M}' \upharpoonright \tau')^{h^{-1}} \upharpoonright \tau_0, f_0(e_1)\rangle.$$

Proceeding in a similar way we can establish that

$$\langle(\mathfrak{M}' \upharpoonright \tau) \upharpoonright \tau_0, e_1, g_1(e_1, f_0(e_1), e_1)\rangle \equiv^s \langle(\mathfrak{M}' \upharpoonright \tau')^{h^{-1}} \upharpoonright \tau_0, f_0(e_1), e_1\rangle$$
$$\langle(\mathfrak{M}' \upharpoonright \tau) \upharpoonright \tau_0, e_1, g_1(e_1, f_0(e_1), e_1), e_2\rangle \equiv^s \langle(\mathfrak{M}' \upharpoonright \tau')^{h^{-1}} \upharpoonright \tau_0, f_0(e_1), e_1, f_1(\ldots)\rangle \ldots$$

This way we have enumerated, possibly with repetitions, the elements of $(\mathfrak{M}' \upharpoonright \tau) \upharpoonright \tau_0$ and $(\mathfrak{M}' \upharpoonright \tau')^{h^{-1}} \upharpoonright \tau_0$ respectively as $c_1, c_2, \ldots$ and $d_1, d_2, \ldots$ in such a way that

$$\langle(\mathfrak{M}' \upharpoonright \tau) \upharpoonright \tau_0, c_1, c_2, \ldots\rangle \equiv^s \langle(\mathfrak{M}' \upharpoonright \tau')^{h^{-1}} \upharpoonright \tau_0, d_1, d_2, \ldots\rangle.$$

We can define an isomorphism $\langle Id, i\rangle$ where

$$i : (\mathfrak{M}' \upharpoonright \tau) \upharpoonright \tau_0 \longrightarrow (\mathfrak{M}' \upharpoonright \tau')^{h^{-1}} \upharpoonright \tau_0$$

as $i(c_k) = d_k$. This function is well defined given that we have crisp equality in the language. For if $c_k = c_m$, then by strong elementarity, also $d_k = d_m$, so $i(c_k)$ is indeed unique. A similar argument shows injectivity. □

By examining the proof above, let us see where the argument breaks downs if we wanted to obtain the result using a stronger notion $\leq'$ of inclusion between logics requiring that, for any φ in the weaker logic, we find a φ' in the stronger one such that $\varphi \leftrightarrow \varphi'$ holds in every model.

We begin by supposing, by the way of contradiction, that the conclusion of the new version of the theorem does not hold, i.e., there is a formula φ of $\mathscr{L}^{\boldsymbol{A}}$ such that there is no φ' from $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$ which is equivalent to it. Let τ_0 be the finite signature φ depends on. We enumerate the formulas from $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$ in the signature τ_0 as $\psi_1, \psi_2, \dots$ and then we can define a list of $\mathscr{L}^{\boldsymbol{A}}$-formulas $(\varphi_i)_{i\in\omega}$ with $\varphi = \varphi_0$ such that for any n,

$$\bigwedge_{i=0}^{n} \varphi_i$$

is not 1-equivalent to any $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$-formula.

However, this is as far as we can go, for now the fact that $\bigwedge_{i=0}^{n} \varphi_i$ is not equivalent to any formula of $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$, does not guarantee that there is a model where $\bigwedge_{i=0}^{n} \varphi_i$. It only tells us that there is a model where the value of $\bigwedge_{i=0}^{n} \varphi_i$ is certainly not $\overline{0}^{\boldsymbol{A}}$, but then it could be anything between the successor of $\overline{0}^{\boldsymbol{A}}$ and $\overline{1}^{\boldsymbol{A}}$. Hence, the proof breaks apart.

Indeed, it seems that the strategy of this kind of result where from the fact that a formula is not equivalent (in some sense) to a formula in another logic we construct a couple of models that are used to derive a contradiction is not going to work here.

In fact, we can show that this stronger version of the theorem cannot be obtained. So, in a sense, our result is the best possible. To see this we can adapt an argument suggested to the first author by Xavier Caicedo in the context of many-valued modal logics[2] showing that, in general, constants do not suffice to make the Baaz–Monteiro $\triangle$ connective definable.

Consider a three-element Gödel chain $\boldsymbol{G}_3^c$ with constants and say the domain is $\{0, a, 1\}$ with the obvious order. In an arbitrary $\mathscr{L}^{\boldsymbol{G}_3^c}_{\omega\omega}$-structure $\mathfrak{M} = \langle \boldsymbol{G}_3^c, \mathbf{M}\rangle$ we can check that $\mathfrak{M} \models (\overline{1} \leftrightarrow \overline{a}) \to (\varphi(\overline{1}) \leftrightarrow \varphi(\overline{a}))$ for any $\varphi(\cdot)$ of $\mathscr{L}^{\boldsymbol{G}_3^c}_{\omega\omega}$ and where $\varphi(\overline{a})$ comes from replacing $\overline{1}$ wherever it appears in $\varphi(\overline{1})$ by $\overline{a}$. This can be established by induction on the complexity of φ. We check some cases and leave the rest to the reader. Suppose that $\varphi(\overline{1})$ does not contain any appearances of $\overline{1}$. Then, $\left\|\overline{1} \leftrightarrow \overline{a}\right\|^{\mathfrak{M}} = \min\{1, a\} = a$, and $\|\varphi \leftrightarrow \varphi\|^{\mathfrak{M}} = \min\{1, 1\} = 1$. Suppose that $\varphi(\overline{1}) = \psi(\overline{1}) \mathbin{\&} \chi(\overline{1})$. Then, $\left\|\psi(\overline{1}) \mathbin{\&} \chi(\overline{1}) \leftrightarrow \psi(\overline{a}) \mathbin{\&} \chi(\overline{a})\right\|^{\mathfrak{M}} = \min\{\left\|\psi(\overline{1}) \mathbin{\&} \chi(\overline{1}) \to \psi(\overline{a}) \mathbin{\&} \chi(\overline{a})\right\|^{\mathfrak{M}}, \left\|\psi(\overline{a}) \mathbin{\&} \chi(\overline{a}) \to \psi(\overline{1}) \mathbin{\&} \chi(\overline{1})\right\|^{\mathfrak{M}}\}$. By inductive hypothesis

$$a \leq \min\{\left\|\psi(\overline{1}) \to \psi(\overline{a})\right\|^{\mathfrak{M}}, \left\|\psi(\overline{a}) \to \psi(\overline{1})\right\|^{\mathfrak{M}}\}$$
$$a \leq \min\{\left\|\chi(\overline{1}) \to \chi(\overline{a})\right\|^{\mathfrak{M}}, \left\|\chi(\overline{a}) \to \chi(\overline{1})\right\|^{\mathfrak{M}}\}.$$

But then

$$a \leq \min\{\left\|\psi(\overline{1}) \to \psi(\overline{a})\right\|^{\mathfrak{M}}, \left\|\chi(\overline{1}) \to \chi(\overline{a})\right\|^{\mathfrak{M}}\}$$

[2] The original example is for the context of the many-valued modal logic S5 on crisp frames evaluated on a finite Gödel chain $\boldsymbol{G}_3^c$. Then, the formula $\Box(p \leftrightarrow q) \to (\varphi(p) \leftrightarrow \varphi(q))$ is valid but $\Box(\overline{1} \leftrightarrow \overline{a}) \to (\triangle(\overline{1}) \leftrightarrow \triangle(\overline{a}))$ is not.

and, similarly,

$$a \leq \min\{\left\|\psi(\overline{a}) \to \psi(\overline{1})\right\|^{\mathfrak{M}}, \left\|\chi(\overline{a}) \to \chi(\overline{1})\right\|^{\mathfrak{M}}\}.$$

Since $((\varphi_1 \to \psi_1) \mathbin{\&} (\varphi_2 \to \psi_2)) \to ((\varphi_1 \mathbin{\&} \varphi_2) \to (\psi_1 \mathbin{\&} \psi_2))$ is a logical truth, we see that

$$a \leq \min\{\left\|\psi(\overline{1}) \mathbin{\&} \chi(\overline{1}) \to \psi(\overline{a}) \mathbin{\&} \chi(\overline{a})\right\|^{\mathfrak{M}}, \left\|\psi(\overline{a}) \mathbin{\&} \chi(\overline{a}) \to \psi(\overline{1}) \mathbin{\&} \chi(\overline{1})\right\|^{\mathfrak{M}}\}.$$

Suppose that $\varphi(\overline{1}) = \psi(\overline{1}) \to \chi(\overline{1})$. Then,

$$\left\|(\psi(\overline{1}) \to \chi(\overline{1})) \leftrightarrow (\psi(\overline{a}) \to \chi(\overline{a}))\right\|^{\mathfrak{M}} =$$
$$\min\{\left\|(\psi(\overline{1}) \to \chi(\overline{1})) \to (\psi(\overline{a}) \to \chi(\overline{a}))\right\|^{\mathfrak{M}}, \left\|(\psi(\overline{a}) \to \chi(\overline{a})) \to (\psi(\overline{1}) \to \chi(\overline{1}))\right\|^{\mathfrak{M}}\}.$$

But observe that $(\psi_1 \to \varphi_1) \mathbin{\&} (\varphi_2 \to \psi_2) \to ((\varphi_1 \to \varphi_2) \to (\psi_1 \to \psi_2))$ is a logical truth and, reasoning as in the case of &, we get what we desire.

Suppose that $\varphi(\overline{1}) = (\forall x)\psi(\overline{1})$. Then, $\left\|(\forall x)\psi(\overline{1}) \leftrightarrow (\forall x)\psi(\overline{a})\right\|^{\mathfrak{M}} = \min\{\left\|(\forall x)\psi(\overline{1}) \to (\forall x)\psi(\overline{a})\right\|^{\mathfrak{M}}, \left\|(\forall x)\psi(\overline{a}) \to (\forall x)\psi(\overline{1})\right\|^{\mathfrak{M}}\}$. Furthermore, $(\forall x)(\psi(\overline{1}) \to \psi(\overline{a})) \to ((\forall x)\psi(\overline{1}) \to (\forall x)\psi(\overline{a}))$ and $(\forall x)(\psi(\overline{a}) \to \psi(\overline{1})) \to ((\forall x)\psi(\overline{a}) \to (\forall x)\psi(\overline{1}))$ are logical truths. Using the inductive hypothesis for ψ, we can conclude that

$$a \leq \left\|(\forall x)(\psi(\overline{1}) \to \psi(\overline{a}))\right\|^{\mathfrak{M}}$$

and

$$a \leq \left\|(\forall x)(\psi(\overline{a}) \to \psi(\overline{1}))\right\|^{\mathfrak{M}},$$

and this suffices to give us what we want.

On the other hand,

$$\mathfrak{M} \not\models (\overline{1} \leftrightarrow \overline{a}) \to (\triangle(\overline{1}) \leftrightarrow \triangle(\overline{a}))$$

because $\left\|\overline{1} \leftrightarrow \overline{a}\right\|^{\mathfrak{M}} = \min\{1, a\} = a$. However, $\left\|\triangle(\overline{1})\right\|^{\mathfrak{M}} = 1$ and $\|\triangle(\overline{a})\|^{\mathfrak{M}} = 0$, so $\left\|\triangle(\overline{1}) \leftrightarrow \triangle(\overline{a})\right\|^{\mathfrak{M}} = \min\{1, 0\} = 0$. And, clearly, $a \nleq 0$.

Hence, the Baaz–Monteiro $\triangle$ is not equivalent in the sense of $\leftrightarrow$ to any combination of connectives from $\mathscr{L}_{\omega\omega}^{\boldsymbol{G}_3^c}$, and, therefore, the logic $\mathscr{L}_{\omega\omega}^{\boldsymbol{G}_3^c} + \triangle$ would give a proper expressive extension of $\mathscr{L}_{\omega\omega}^{\boldsymbol{G}_3^c}$ (in terms of $\leftrightarrow$) which would also satisfy the compactness and Löwenheim–Skolem properties. So our Lindström theorem in this section would be false. However, in its current form this causes no problem since, clearly, a formula $\triangle(\varphi)$ is 1-equivalent to φ.

4.2. Löwenheim–Skolem property + abstract completeness property

In this subsection, we will obtain a second Lindström-style theorem by using abstract completeness in place of the compactness property (in the presence of the necessary syntactic requirements). To this end, we will need two interesting auxiliary facts. The first one follows from the results in [7]:

Theorem 14. (Trakhtenbrot's theorem) *There is a finite signature δ such that the collection* $\mathrm{Fmla}_\delta^{<\omega}$ *of all formulas of $\mathscr{L}_{\omega\omega}^{\boldsymbol{A}}$ in the signature δ true in all finite models is not recursively enumerable.*

Given a crisp predicate P in a given structure $\mathfrak{M}$ (i.e., a predicate P such that $(\forall\overrightarrow{x})((P(\overrightarrow{x}) \to \overline{0}) \vee P(\overrightarrow{x}))$ holds in $\mathfrak{M}$) and a first-order formula φ we define the relativization of φ to P, in symbols φ^P, inductively as follows:

- φ^P is just φ if φ is atomic.
- $\varphi^P = \circ(\psi_0^P, \ldots, \psi_n^P)$ when $\varphi = \circ(\psi_0, \ldots, \psi_n)$ for an n-ary connective $\circ$.
- $\varphi^P = (\exists\overrightarrow{x})(P(\overrightarrow{x}) \wedge \psi^P(\overrightarrow{x}))$ if $\varphi = (\exists\overrightarrow{x})\psi(\overrightarrow{x})$.
- $\varphi^P = (\forall\overrightarrow{x})((P(\overrightarrow{x}) \to \overline{0}) \vee \psi^P(\overrightarrow{x}))$ if $\varphi = (\forall\overrightarrow{x})\psi(\overrightarrow{x})$.

Lemma 15. (Separation Lemma) *Let $\mathscr{L}^{\boldsymbol{A}}$ be an abstract logic with the finite occurrence property such that $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega} \leq \mathscr{L}^{\boldsymbol{A}}$. If $\mathscr{L}^{\boldsymbol{A}}$ is closed under the connectives of $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$, it has the Löwenheim–Skolem property, and for some τ_0 there are disjoint classes $Mod(\varphi)$, $Mod(\chi)$ (for φ, χ formulas in τ_0 of $\mathscr{L}^{\boldsymbol{A}}$) such that there is no $Mod(\psi)$ (ψ a formula in τ_0 of $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$) separating $Mod(\varphi)$ and $Mod(\chi)$, i.e.,*

$$Mod(\varphi) \subseteq Mod(\psi) \quad \textit{and} \quad Mod(\chi) \cap Mod(\psi) = \emptyset.$$

Then, for some signature δ containing at least a unary predicate U, there is a formula θ in δ of $\mathscr{L}^{\boldsymbol{A}}$ such that:

(i) *for any $\mathfrak{M} \models \theta$, we have that $U^{\mathfrak{M}}$ is crisp and finite with cardinality $\geqslant 1$.*
(ii) *for every $n \geqslant 1$, we can find $\mathfrak{M} \models \theta$ and $|\{a \in M \mid \mathfrak{M} \models U[a]\}| = n$.*

Proof. Suppose, by the way of contradiction, that the conclusion of the lemma does not hold having assumed the hypothesis. So there is no θ as described above.

We can show that for any χ which is a formula of $\mathscr{L}^{\boldsymbol{A}}$ and not 1-equivalent to a formula of $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$, there is a countable (infinite) model $\mathfrak{N} \models \chi$. For either (1) χ has an infinite model or (2) it has no infinite models. If (1), then add a new function symbol f and note that the infinite model $\mathfrak{N}' \models \chi$ can be expanded to a model of the lattice conjunction of the following formulas (recall that $\approx$ is crisp)

$$\begin{gathered} \chi, \\ (\forall x, y)((f(x) = f(y) \to \overline{0}) \vee x = y), \\ (\exists y)((\exists x)(y = f(x)) \to \overline{0}). \end{gathered}$$

All that we need is to pick some bijection of the infinite domain of $\mathfrak{N}'$ with one of its proper subsets (Dedekind's definition of infinity). Now, by the Löwenheim–Skolem property, we can take a countable infinite model $\mathfrak{N}$ of said lattice conjunction. On the other hand, if (2) holds, take a finite τ containing all the non-logical symbols appearing in χ. One can observe that since both the algebra $\boldsymbol{A}$ and τ are finite, there will be only finitely many structures for any given finite cardinality k that we can construct (up to isomorphism). If χ only has models up to some finite cardinality m, then we can write a lattice disjunction (a formula of $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$) which will be 1-equivalent to χ (just list the formulas describing the isomorphism types of each model of χ), which would contradict our assumption. Then, χ must have models of arbitrary finite cardinality. Then, introducing a new unary predicate U, the formula $\chi \wedge (\exists x)Ux \wedge (\forall x)((Ux \to \overline{0}) \vee Ux)$ satisfies the properties (i) and (ii), which we have assumed do not hold. This is because we can interpret U as the crisp set $\{1, \ldots, n\}$ in a sufficiently large model (by taking isomorphic copies if need be).

Now, φ and χ as in the statement of the theorem defined disjoint classes of models that cannot be separated by a class definable by a formula of $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$. Let τ_0 be the finite signature φ and χ depend on.

We enumerate the formulas from $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$ in the signature τ_0 as $\psi_1, \psi_2, \ldots$ and then we define inductively a list of $\mathscr{L}^{\boldsymbol{A}}$-formulas $(\varphi_i)_{1 \leq i < \omega}$ such that for any n,

$$\varphi \wedge \bigwedge_{i=1}^{n} \varphi_i \text{ and } \chi \wedge \bigwedge_{i=1}^{n} \varphi_i$$

are not separable by an $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$-formula.

Now, suppose that

$$\varphi \wedge \bigwedge_{i=1}^{k} \varphi_i \text{ and } \chi \wedge \bigwedge_{i=1}^{k} \varphi_i$$

are not separable by an $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$-formula. Then, either

$$\varphi \wedge (\bigwedge_{i=1}^{k} \varphi_i \wedge \psi_{k+1}) \text{ and } \chi \wedge (\bigwedge_{i=1}^{k} \varphi_i \wedge \psi_{k+1})$$

are not separable or

$$\varphi \wedge (\bigwedge_{i=1}^{k} \varphi_i \wedge (\psi_{k+1} \to \overline{a})) \text{ and } \chi \wedge (\bigwedge_{i=1}^{k} \varphi_i \wedge (\psi_{k+1} \to \overline{a}))$$

are not separable. For otherwise,

$$\varphi \wedge (\bigwedge_{i=1}^{k} \varphi_i \wedge \psi_{k+1}) \text{ and } \psi \wedge (\bigwedge_{i=1}^{k} \varphi_i \wedge \psi_{k+1})$$

are separable by an $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$-formula θ_0, and

$$\varphi \wedge (\bigwedge_{i=1}^{k} \varphi_i \wedge (\psi_{k+1} \to \overline{a})) \text{ and } \psi \wedge (\bigwedge_{i=1}^{k} \varphi_i \wedge (\psi_{k+1} \to \overline{a}))$$

are separable by an $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$-formula θ_1. But

$$\vDash \psi_{k+1} \vee (\psi_{k+1} \to \overline{a})$$

and

$$\vDash ((\psi_{k+1} \to \overline{a}) \to \overline{a}) \vee (\psi_{k+1} \to \overline{a}).$$

Hence,

$$\varphi \wedge \bigwedge_{i=1}^{k} \varphi_i \vDash \theta_0 \vee \theta_1.$$

Moreover, notice that

$$\theta_0 \vee \theta_1 \vDash ((\chi \wedge \bigwedge_{i=1}^{k} \varphi_i \wedge \psi_{k+1}) \to \overline{a}) \vee ((\chi \wedge \bigwedge_{i=1}^{k} \varphi_i \wedge (\psi_{k+1} \to \overline{a})) \to \overline{a})$$

$$\vDash ((\chi \wedge \bigwedge_{i=1}^{k} \varphi_i) \to \overline{a}) \vee (\psi_{k+1} \to \overline{a}) \vee ((\chi \wedge \bigwedge_{i=1}^{k} \varphi_i) \to \overline{a}) \vee ((\psi_{k+1} \to \overline{a}) \to \overline{a})$$

$$\models (\chi \wedge \bigwedge_{i=1}^{k} \varphi_i) \to \overline{a}.$$

Then, we would have that

$$\mathrm{Mod}(\chi \wedge \bigwedge_{i=1}^{k} \varphi_i) \cap \mathrm{Mod}(\theta_0 \vee \theta_1) = \emptyset$$

which implies that

$$\varphi \wedge \bigwedge_{i=1}^{k} \varphi_i \text{ and } \chi \wedge \bigwedge_{i=1}^{k} \varphi_i$$

are indeed separable by an $\mathscr{L}_{\omega\omega}^{\boldsymbol{A}}$-formula, contradicting our assumption.

Then, for any n,

$$\varphi \wedge \bigwedge_{i=1}^{n} \varphi_i \text{ and } \chi \wedge \bigwedge_{i=1}^{n} \varphi_i$$

both have infinite countable models $\mathfrak{M}$ and $\mathfrak{N}$ (so, $\mathfrak{M} \models \varphi_i$ iff $\mathfrak{N} \models \varphi_i$, $1 \leq i \leq n$) using the first part of the proof of the theorem. For otherwise, one of them would be 1-equivalent to $\overline{0}^{\boldsymbol{A}}$, and then they would be trivially separable by an $\mathscr{L}_{\omega\omega}^{\boldsymbol{A}}$-formula.

Then, looking at the proof of Proposition 6, for each $k < \omega$, we can pick n large enough in our enumeration $\psi_1, \psi_2, \ldots$ and countable models $\mathfrak{M}$ and $\mathfrak{N}$ with

$$M = N \qquad \mathfrak{M} \models \varphi_i \text{ iff } \mathfrak{N} \models \varphi_i \ (1 \leq i \leq n) \qquad \mathfrak{M} \models \varphi \qquad \mathfrak{N} \models \chi$$

such that there is a sequence $\langle I_i \mid 0 \leq i \leq k \rangle$ of sets of partial isomorphisms from $\mathfrak{M}$ into $\mathfrak{N}$ with the back-and-forth property defined as follows:

$$I_0 = \{\langle Id, \emptyset \rangle\}$$

$$I_i = \{\langle Id, p \rangle \mid \langle Id, p \rangle \text{ is a partial isomorphism from } \mathfrak{M} \text{ into } \mathfrak{N}, \mathrm{dom}(p) = \{e_1, \ldots, e_{k-i}\}, \mathfrak{N} \models \psi^i_{\langle \mathfrak{M}, \overrightarrow{e} \rangle}[p(\overrightarrow{e})]\}.$$

The trick is to pick n large enough so that all the finitely many formulas needed to define the sequence $\langle I_i \mid 0 \leq i \leq k \rangle$ appear in the first n formulas of the enumeration (in this case, we write $\mathfrak{M} \sim_k \mathfrak{N}$).

We may assume that $\bigcup_{0 \leq i \leq k} I_i$ is countable, and moreover, taking isomorphic copies, that $\{0, \ldots, k\}$, $\bigcup_{0 \leq i \leq k} I_i \subseteq M$ (the domain of the model $\mathfrak{M}$). Expand the signature $\tau_0 \cup \tau_0'$ (where τ_0' is renaming of τ_0) by adding the set of symbols $\{<, U, B, I, G\}$, where B and U are unary predicates, $<$ and I are binary predicates, while G is a ternary predicate. Call the new signature δ. Then, consider the model $\mathfrak{M}^*$ for δ where

$$\mathfrak{M}^* \upharpoonright \tau_0 = \mathfrak{M}, \quad \mathfrak{M}^* \upharpoonright \tau_0' = \mathfrak{N}' \text{ (where } \mathfrak{N}' \text{ is the renaming of } \mathfrak{N} \text{ for the signature } \tau_0'),$$

$$\|U[a]\|^{\mathfrak{M}^*} = \begin{cases} \overline{1}^{\boldsymbol{A}} & \text{if } a \in \{0, \ldots, k\} \\ \overline{0}^{\boldsymbol{A}} & \text{otherwise.} \end{cases}$$

$$\|a < b\|^{\mathfrak{M}^*} = \begin{cases} \overline{1}^{\boldsymbol{A}} & \text{if } a, b \in \{0, \ldots, k\}, a < b \\ \overline{0}^{\boldsymbol{A}} & \text{otherwise.} \end{cases}$$

$$\|B[a]\|^{\mathfrak{M}^*} = \begin{cases} \overline{1}^{\boldsymbol{A}} & \text{if } a \in \bigcup_{0\leq i\leq k} I_i \\ \overline{0}^{\boldsymbol{A}} & \text{otherwise.} \end{cases}$$

$$\|I[m,y]\|^{\mathfrak{M}^*} = \begin{cases} \overline{1}^{\boldsymbol{A}} & \text{if } m \leq k, y \in I_m \\ \overline{0}^{\boldsymbol{A}} & \text{otherwise.} \end{cases}$$

$$\|G[dab]\|^{\mathfrak{M}^*} = \begin{cases} \overline{1}^{\boldsymbol{A}} & \text{if } \ d = \langle Id, p\rangle \in \bigcup_{0\leq i\leq k} I_i,\ p(a) = b \\ \overline{0}^{\boldsymbol{A}} & \text{otherwise.} \end{cases}$$

Take the formula θ to be the lattice conjunction of the following (for simplicity we assume that all symbols in $Pred_{\tau_0}$ are binary):

φ, χ'
"$G, I, B, <, U$ are all crisp"
$(\forall x)(x < x \to \overline{0})$, $(\forall x, y, z)(((x < y \wedge y < z) \to \overline{0}) \vee x < z)$
$(\forall x)(\exists y)(x < y \wedge ((\exists z)(x < z < y) \to \overline{0}))$, $(\forall x, y)(x < y \vee y < x \vee x = y)$
$(\exists x)(\forall y)((x = y \to \overline{0}) \vee x < y)$
$(\exists x)(\forall y)((x = y \to \overline{0}) \vee y < x)$
$(\forall x, y)((x < y \to \overline{0}) \vee (Ux \wedge Uy))$
$(\forall x)((Bx \to \overline{0}) \vee (\forall y, z, v, w)(((Gxyz \wedge Gxvw) \to \overline{0}) \vee (y = v \leftrightarrow z = w)))$
$(\forall x)((Ux \to \overline{0}) \vee (\exists y)(By \wedge Ixy))$
$(\forall x)((Bx \to \overline{0}) \vee (\forall y, z, v, w)(((Gxyz \wedge Gxvw) \to \overline{0}) \vee (Ryv \leftrightarrow Rzw)))$ $\quad (R \in Pred_{\tau_0})$
$(\forall u, v)((v < u \to \overline{0}) \vee (\forall p)((Iup \to \overline{0}) \vee (\forall x)(\exists q, y)(Ivq \wedge Gqxy \wedge (\forall z, w)((Gpzw \to \overline{0}) \vee Gqzw))))$
(forth property)
$(\forall u, v)((v < u \to \overline{0}) \vee (\forall p)((Iup \to \overline{0}) \vee (\forall x)(\exists q, y)(Ivq \wedge Gqyx \wedge (\forall z, w)((Gpzw \to \overline{0}) \vee Gqzw))))$
(back property)

$\mathfrak{M}^*$ is a model of θ, so putting everything together, θ satisfies property (ii). Now, for the property (i), suppose that it does not hold. Then, there is a model $\mathfrak{M}^{**}$ of θ such that $|\{a \in M \mid \mathfrak{M} \models U[a]\}|$ is infinite. Add a new function symbol f to our signature and consider the formula θ' which is the lattice conjunction of the following:

θ,
$(\forall x, y)((f(x) = y \to \overline{0}) \vee (Ux \wedge Uy))$,
$(\forall x, y)((f(x) = f(y) \to \overline{0}) \vee x = y)$,
$(\exists y)(Uy \wedge ((\exists x)((Ux \wedge y = f(x)) \to \overline{0}))$.

We can expand $\mathfrak{M}^{**}$ to a model of θ' by interpreting f as any injection of $\{a \in M \mid \mathfrak{M} \models U[a]\}$ into a proper subset of itself. Hence, by the Löwenheim–Skolem theorem, we can assume that $\mathfrak{M}^{**}$ is countable and $|\{a \in M \mid \mathfrak{M} \models U[a]\}| = \omega$. Hence, $\{\langle a, b\rangle \in M^{**} \times M^{**} \mid \mathfrak{M}^{**} \models a < b\}$ contains an infinitely descending sequence (since it has a last element). Then, we observe that we can build a sequence of sets of partial isomorphisms from $\mathfrak{M}^{**} \restriction \tau_0$ into $(\mathfrak{M}^{**} \restriction \tau_0')^{-'}$ such that $\mathfrak{M}^{**} \restriction \tau_0 \cong_f (\mathfrak{M}^{**} \restriction \tau_0')^{-'}$. But since $\mathfrak{M}^{**} \restriction \tau_0$ and $(\mathfrak{M}^{**} \restriction \tau_0')^{-'}$ are countable, then they are, in fact, isomorphic. But $\mathfrak{M}^{**} \restriction \tau_0 \models \varphi$ and $(\mathfrak{M}^{**} \restriction \tau_0')^{-'} \models \psi$, contradicting the hypothesis of the theorem. □

Theorem 16. (Second Lindström theorem) *Let $\mathscr{L}^{\boldsymbol{A}}$ be an effective abstract logic with the finite occurrence property such that $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega} \leq \mathscr{L}^{\boldsymbol{A}}$. If $\mathscr{L}^{\boldsymbol{A}}$ is closed under the connectives of $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$, it has the Löwenheim–Skolem property, and the abstract completeness property, then $\mathscr{L}^{\boldsymbol{A}} \leq \mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$.*

Proof. Once more, we take $a \in A$ as the immediate predecessor of $\overline{1}^{\boldsymbol{A}}$ in the lattice order of $\boldsymbol{A}$. Suppose, by the way of contradiction, that there is some $\varphi \in \mathscr{L}^{\boldsymbol{A}}$ which is not 1-equivalent to any formula of $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$. This means that there is no $\mathrm{Mod}(\psi)$ (for any formula $\psi \in \mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$) separating $\mathrm{Mod}(\varphi)$ and $\mathrm{Mod}(\varphi \to \overline{a})$, i.e.,

$$\mathrm{Mod}(\varphi) \subseteq \mathrm{Mod}(\psi) \quad \text{and} \quad \mathrm{Mod}(\varphi \to \overline{a}) \cap \mathrm{Mod}(\psi) = \emptyset.$$

But then, by the Separation Lemma, there is a $\theta \in \mathscr{L}^{\boldsymbol{A}}$ as in the lemma. From Trakhtenbrot's theorem for this context, for some finite signature τ (that we may assume disjoint from the signature in our use of the Separation Lemma), the collection $\mathrm{Fmla}_{\tau}^{<\omega}$ of all formulas of $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$ in the signature τ true in all finite models is not recursively enumerable. Then,

$$\chi \in \mathrm{Fmla}_{\tau}^{<\omega} \quad \text{iff} \quad \models (\theta \to \overline{a}) \vee \chi^{U}$$

where χ^U is the relativization of χ to the crisp predicate U introduced in the Separation Lemma. The left-to-right direction is as follows: let $\chi \in \mathrm{Fmla}_{\tau}^{<\omega}$ and suppose that in an arbitrary model $\mathfrak{M}$, we have $\mathfrak{M} \models \theta$. Then, $\mathfrak{M}$ is finite with size $\geqslant 1$, and then, in particular, the restriction of $\mathfrak{M}$ to the extension of the crisp predicate U is finite, and, hence, χ takes value $\overline{1}^{\boldsymbol{A}}$ there, so $\mathfrak{M} \models \chi^U$, as desired. On the other hand, the right-to-left direction is as follows: suppose that $\models (\theta \to \overline{a}) \vee \chi^U$, by property (ii) of θ in the Separation Lemma and the fact that the relevant signatures are disjoint, every finite structure for τ can be seen as the restriction to the crisp extension of U of an expansion to the signature τ of some model of θ, and, hence, in every such finite structure, χ must hold.

Finally, together with our assumptions, this would imply that $\mathrm{Fmla}_{\tau}^{<\omega}$ is recursively enumerable, which is a contradiction. □

4.3. Compactness property + Karp property

It is natural to ask whether a characterization of maximality can be obtained without reference to the Löwenheim–Skolem property. The next theorem shows that it is indeed possible if we use instead the Karp property and recover compactness.

Theorem 17. (Third Lindström theorem) *Let $\mathscr{L}^{\boldsymbol{A}}$ be an abstract logic such that $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega} \leq \mathscr{L}^{\boldsymbol{A}}$. If $\mathscr{L}^{\boldsymbol{A}}$ is closed under the connectives of $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$, it has the compactness property, and the Karp property, then $\mathscr{L}^{\boldsymbol{A}} \leq \mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$.*

Proof. Suppose, by the way of contradiction, that the conclusion of the theorem does not hold. This means that there is a formula φ of $\mathscr{L}^{\boldsymbol{A}}$ such that there is no φ' from $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$ which is 1-equivalent to it. Let τ_0 be the finite signature φ depends on (by the compactness argument used in the proof of Theorem 13).

We enumerate the formulas from $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$ in the signature τ_0 as $\psi_0, \psi_1, \ldots$ and then we define a list of $\mathscr{L}^{\boldsymbol{A}}$-formulas $(\varphi_i)_{i\in\omega}$ with $\varphi_0 = \varphi$ such that for any n,

$$\bigwedge_{i=0}^{n} \varphi_i$$

is not 1-equivalent to any $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$-formula as in the First Lindström Theorem.

Then, for any n,

$$\varphi \wedge (\bigwedge_{i=1}^{n} \varphi_i) \text{ and } (\varphi \to \overline{a}) \wedge (\bigwedge_{i=1}^{n} \varphi_i)$$

both have models.

This can be rewritten as saying that, for each $n < \omega$, we can pick models $\mathfrak{M}_n$ and $\mathfrak{N}_n$ such that

$$\mathfrak{M}_n \models \varphi_i \text{ iff } \mathfrak{N}_n \models \varphi_i \ (1 \leq i \leq n) \qquad \mathfrak{M}_n \models \varphi \qquad \mathfrak{N}_n \not\models \varphi.$$

As before, this means that for each $n < \omega$, we can pick models $\mathfrak{M}$ and $\mathfrak{N}$ such that

$$(+) \qquad \mathfrak{M}_n \sim_n \mathfrak{N}_n \qquad \mathfrak{M}_n \models \varphi \qquad \mathfrak{N}_n \not\models \varphi.$$

Expand the signature $\tau_0 \cup \tau_0'$ (where τ_0' is renaming of τ_0) by adding the set of symbols $\{<, U, P, I, G, V, W\}$, where P and U are unary predicates, $<, V, W$, and I are binary predicates, while G is a ternary predicate. We now consider the sentence $\varphi(<, U, P, I, G, V, W)$ formed by the lattice conjunction of the following elements (for simplicity we assume that all symbols in $Pred_{\tau_0}$ are binary):

"$G, I, <, U, P$ are all crisp"
$(\forall x)(\varphi^{\{y|Vxy\}})$, $(\forall x)((\varphi \to \overline{a})'^{\{y|Wxy\}})$
"The crisp extensions of U, P, as well as $\{y \mid Vxy\}, \{y \mid Wxy\}$ (for every x) are all disjoint"
$(\forall x, y)(((Vxy \vee Wxy) \to \overline{a}) \vee Ux)$
$(\forall x, y)((Ux \to \overline{a}) \vee (\exists y, z)(Vxy \vee Wxz))$
$(\forall x)(x < x \to \overline{0})$, $(\forall x, y, z)(((x < y \wedge y < z) \to \overline{0}) \vee x < z)$
$(\forall x)(\exists y)(x < y \wedge ((\exists z)(x < z < y) \to \overline{0}))$, $(\forall x, y)(x < y \vee y < x \vee x = y)$
$(\exists x)(\forall y)((x = y \to \overline{0}) \vee x < y)$
$(\exists x)(\forall y)((x = y \to \overline{0}) \vee y < x)$
$(\forall x, y)((x < y \to \overline{0}) \vee (Ux \wedge Uy))$
$(\forall x, v, w)(((Px \wedge ((\forall y)((v = y \to \overline{0}) \vee v < y) \to \overline{a}) \wedge w < v \wedge Iwx) \to \overline{a}) \vee \forall y, z((Gxyz \to \overline{a}) \vee (Vvy \wedge Wvz)))$
$(\forall x, v)(((Px \wedge (\forall y)((v = y \to \overline{0}) \vee v < y) \wedge Ivx) \to \overline{a}) \vee (\forall y, z)((Gxyz \to \overline{a}) \vee (Vvy \wedge Wvz)))$
$(\forall x, v)((Px \wedge Ivx) \to (\forall y, z)((Gxyz \to \overline{0}) \vee (Vvy \wedge Wvz)))$
$(\forall x)((Px \to \overline{0}) \vee (\forall y, z, v, w)(((Gxyz \wedge Gxvw) \to \overline{0}) \vee (y = v \leftrightarrow z = w)))$
$(\forall x)((Ux \to \overline{0}) \vee (\exists y)(Py \wedge Ixy))$
$(\forall x)((Px \to \overline{0}) \vee (\forall y, z, v, w)(((Gxyz \wedge Gxvw) \to \overline{0}) \vee (Ryv \leftrightarrow Rzw))) \quad (R \in Pred_{\tau_0})$
$(\forall u, v)((v < u \to \overline{0}) \vee (\forall p)((Iup \to \overline{0}) \vee (\forall x)(\exists q, y)(Ivq \wedge Gqxy \wedge (\forall z, w)((Gpzw \to \overline{0}) \vee Gqzw))))$
(forth property)
$(\forall u, v)((v < u \to \overline{0}) \vee (\forall p)((Iup \to \overline{0}) \vee (\forall x)(\exists q, y)(Ivq \wedge Gqyx \wedge (\forall z, w)((Gpzw \to \overline{0}) \vee Gqzw))))$
(back property)

The above sentence $\varphi(<, U, P, I, G, V, W)$ can be seen to have a model, using (+). First, observe that from (+) we obtain a union $\bigcup_{i\in\omega} I_i$ of collections of partial isomorphisms. To see this, we consider the following structure $\mathfrak{M}^*$:

For every $n < \omega$, $\mathfrak{M}^* \upharpoonright \tau_0|\{y \mid V^{\mathfrak{M}^*}ny\} = \mathfrak{M}_n$, $\quad \mathfrak{M}^* \upharpoonright \tau_0'|\{y \mid W^{\mathfrak{M}^*}ny\} = \mathfrak{N}_n'$ (where $\mathfrak{N}_n'$ is the renaming of $\mathfrak{N}_n$ for the signature τ_0'),

$$\|U[a]\|^{\mathfrak{M}^*} = \begin{cases} \overline{1}^{\boldsymbol{A}} & \text{if } a \in \omega \\ \overline{0}^{\boldsymbol{A}} & \text{otherwise.} \end{cases}$$

$$\|a < b\|^{\mathfrak{M}^*} = \begin{cases} \overline{1}^{\boldsymbol{A}} & \text{if } a, b \in \omega, a < b \\ \overline{0}^{\boldsymbol{A}} & \text{otherwise.} \end{cases}$$

$$\|P[a]\|^{\mathfrak{M}^*} = \begin{cases} \overline{1}^{\boldsymbol{A}} & \text{if } a \in \bigcup_{i\in\omega} I_i \\ \overline{0}^{\boldsymbol{A}} & \text{otherwise.} \end{cases}$$

$$\|I[m,y]\|^{\mathfrak{M}^*} = \begin{cases} \overline{1}^{\boldsymbol{A}} & \text{if } m < \omega, y \in I_m \\ \overline{0}^{\boldsymbol{A}} & \text{otherwise.} \end{cases}$$
$$\|G[dab]\|^{\mathfrak{M}^*} = \begin{cases} \overline{1}^{\boldsymbol{A}} & \text{if } \ d = \langle Id, p\rangle \in \bigcup_{i\in\omega} I_i,\ p(a) = b \\ \overline{0}^{\boldsymbol{A}} & \text{otherwise.} \end{cases}$$

If a structure $\mathfrak{M}$ is a model of $\varphi(<, U, P, I, G, V, W)$, $\langle \omega, <\rangle$ (seen as a crisp $\boldsymbol{A}$-structure, namely, the $\boldsymbol{A}$-structure where ω is the domain and $<$ is interpreted as the obvious crisp relation) is isomorphic to $\mathfrak{M} \restriction \{<\}|U^{\mathfrak{M}}$, and then we get a contradiction with the compactness of $\mathscr{L}^{\boldsymbol{A}}$. To see this, suppose that $\langle \omega, <\rangle$ were not isomorphic to $\mathfrak{M} \restriction \{<\}|U^{\mathfrak{M}}$. This would mean that the latter would contain a non-standard element c (in the sense of an element having infinitely many predecessors in the order). So we can construct an infinite sequence

$$e_0 < e_1 < e_2 < \cdots < e_\omega = c.$$

In the model $\mathfrak{M}$, if $I^{\mathfrak{M}}xp$, we can define a relation $p^* = \{\langle a, b\rangle \mid G^{\mathfrak{M}}pab\}$. Then, $\mathfrak{M} \restriction \tau_0|\{y \mid V^{\mathfrak{M}}cy\}$ and $(\mathfrak{M} \restriction \tau_0'|\{y \mid W^{\mathfrak{M}}cy\})^{-\prime}$ are seen to be partially isomorphic by considering

$$I = \{p^* \mid I^{\mathfrak{M}}e_\alpha p \text{ for some } \alpha \leq \omega\}.$$

However,

$$\mathfrak{M} \restriction \tau_0|\{y \mid V^{\mathfrak{M}}cy\} \models \varphi$$

whereas

$$(\mathfrak{M} \restriction \tau_0'|\{y \mid W^{\mathfrak{M}}cy\})^{-\prime} \not\models \varphi.$$

But this is a contradiction with the Karp property. □

4.4. Löwenheim–Skolem property + Robinson property

In the proof of the last result, Theorem 17, if, instead of the Karp property, we had used the Löwenheim–Skolem property as a hypothesis, we could have produced an argument to the effect that the structure $\langle \omega, <\rangle$ (seen as a crisp $\boldsymbol{A}$-structure) can be axiomatized by a sentence $\varphi(<, P, I, G, V, W)$ instead of $\varphi(<, P, I, U, G, V, W)$. That is, we do not need a predicate to pick out the field of $<$, we can simply let $<$ crisply order the domain of our model. This is so because in (+) we may assume that the given structures are countable. And hence, in deriving the contradiction in the theorem with the Karp property, we can argue instead with the Löwenheim–Skolem theorem by recalling that the existence of a system of partial isomorphisms between two countable structures gives us an isomorphism in our setting. This gives us the following form of Lindström theorem:

Theorem 18. (Fourth Lindström theorem) *Let $\mathscr{L}^{\boldsymbol{A}}$ be an abstract logic with the finite occurrence property such that $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega} \leq \mathscr{L}^{\boldsymbol{A}}$. If $\mathscr{L}^{\boldsymbol{A}}$ is closed under the connectives of $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$, it has the Löwenheim–Skolem property, the Robinson property, and for every model $\mathfrak{M}$, $Th_{\mathscr{L}^{\boldsymbol{A}}}(\mathfrak{M})$ is a set, then $\mathscr{L}^{\boldsymbol{A}} \leq \mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$.*

Proof. Suppose that, in fact, $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega} < \mathscr{L}^{\boldsymbol{A}}$, so there is φ in $\mathscr{L}^{\boldsymbol{A}}$ which is not 1-equivalent to any formula ψ of $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$. But then, by the remark at the beginning of this section, $\langle \omega, <\rangle$ (again seen as a crisp $\boldsymbol{A}$-structure) can be axiomatized by a sentence $\varphi(<, P, I, G, V, W)$ in an expanded signature.

By the assumption that $\mathscr{L}^{\boldsymbol{A}}$ has the Löwenheim–Skolem property and the finite occurrence property, by a reasoning that should be familiar by now, we can observe that it must have the Karp property. The idea is to argue for a contradiction and find models which are isomorphic but differ on the value of some formula. The isomorphism is built from a system of partial isomorphisms between two countable models (which we obtain by the Löwenheim–Skolem property).

Now let $\mathfrak{M}_1$ be a model of $\varphi(<, P, I, G, V, W)$, $\mathfrak{M}_2$ have no relations (other than $\approx$) and only ω_1-many individual constants with its domain having cardinality ω_1. Next, let $\mathfrak{M}_3^1$ be just an $\boldsymbol{A}$-structure with domain $M_3^1 = \omega$ and no other relations, constants or functions other than $=$, and $\mathfrak{M}_3^2$ be defined similarly with $M_3^2 = \omega_1$. We can form the disjoint unions of the models $[\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3^i]$ $(i = 1, 2)$ by taking isomorphic copies in the, by now, usual way.

Clearly, one can build a system of partial isomorphisms between $[\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3^1]$ and $[\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3^2]$. Now, we let

$$\Phi = \mathrm{Th}_{\mathscr{L}^{\boldsymbol{A}}}([\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3^1]) = \mathrm{Th}_{\mathscr{L}^{\boldsymbol{A}}}([\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3^2]).$$

The equalities hold because of the Karp property. Using the techniques from the proof of the Separation Lemma, we can write down a sentence θ_i that expresses that a new function symbol f_i $(i = 1, 2)$ defines a bijection between $\mathfrak{M}_i$ and $\mathfrak{M}_3^i$. It is not difficult to see that both $\Phi \cup \{\theta_1\}$ and $\Phi \cup \{\theta_2\}$ have models (we can expand $[\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3^1]$ and $[\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3^2]$ to models of the former and the latter, respectively). However, $\Phi \cup \{\theta_1, \theta_2\}$ cannot have a model since by $\varphi(<, P, I, G, V, W)$, $\mathfrak{M}_1$ is countable whereas $\mathfrak{M}_2$ is, by definition of its signature, uncountable. Hence, the Robinson property fails. □

4.5. Compactness property + Tarski union property

In this subsection, we present yet another variation of the classical Lindström characterization in which, instead of the Löwenheim–Skolem property, we use the Tarski union property.

Theorem 19. (Fifth Lindström theorem) *Let $\mathscr{L}^{\boldsymbol{A}}$ be an abstract logic such that $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega} \leq \mathscr{L}^{\boldsymbol{A}}$. If $\mathscr{L}^{\boldsymbol{A}}$ is closed under the connectives of $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$, it has the compactness property, the Tarski union property, and for every model $\mathfrak{M}$, $ElDiag_{\mathscr{L}^{\boldsymbol{A}}}(\mathfrak{M})$ is a set, then $\mathscr{L}^{\boldsymbol{A}} \leq \mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$.*

Proof. Note that if the following property holds

$$(\#) \quad \text{For all } \mathfrak{M}, \mathfrak{N}, \qquad \mathfrak{M} \equiv^s_{\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}} \mathfrak{N} \qquad \Longrightarrow \qquad \mathfrak{M} \equiv^s_{\mathscr{L}^{\boldsymbol{A}}} \mathfrak{N},$$

then $\mathscr{L}^{\boldsymbol{A}} \leq \mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$ follows. This is because under the assumption that (#), any formula φ of $\mathscr{L}^{\boldsymbol{A}}[\tau]$ is 1-equivalent to a formula (of an abstract infinitary language)

$$\bigvee_{\mathfrak{M} \models \varphi} \bigwedge_{\substack{\psi \in \mathscr{L}^{\boldsymbol{A}}_{\omega\omega} \\ \mathfrak{M} \models \psi}} \psi.$$

With compactness, we can bring down this second formula to a finitary one.

Suppose now that $\mathscr{L}^{\boldsymbol{A}} \not\leq \mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$. Then, we have an $\mathscr{L}^{\boldsymbol{A}}$-formula φ, a pair of models $\mathfrak{M}$ and $\mathfrak{N}$, and $a \in \boldsymbol{A}$ such that

$$\mathfrak{M} \equiv^s_{\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}} \mathfrak{N} \qquad \mathfrak{M} \models \varphi \leftrightarrow \overline{a} \quad \mathfrak{N} \not\models \varphi \leftrightarrow \overline{a}.$$

By compactness, there is a strong $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$-elementary extension $\mathfrak{M}_1 \not\models \varphi \leftrightarrow \overline{a}$ of $\mathfrak{M} = \mathfrak{M}_0$. Assuming $\mathfrak{M}_n$ is defined, by another compactness argument, we obtain $\mathfrak{M}_{n+1}$ as a strong $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$-elementary extension of $\mathfrak{M}_n$ and a strong $\mathscr{L}^{\boldsymbol{A}}$-elementary extension of $\mathfrak{M}_{n-1}$. Simply show that the theory

$$\mathrm{ElDiag}_{\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}}(\mathfrak{M}_n)\bigcup \mathrm{ElDiag}_{\mathscr{L}^{\boldsymbol{A}}}(\mathfrak{M}_{n-1})$$

has a model, but if this were not the case we would obtain a contradiction with the fact that $\mathfrak{M}_n$ is a strong $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$-elementary extension of $\mathfrak{M}_{n-1}$. After countably many steps, we take the union $\mathfrak{M}' = \bigcup_{n\in\omega}\mathfrak{M}_{2n} = \bigcup_{n\in\omega}\mathfrak{M}_{2n+1}$. But with the Tarski union property this gives the contradiction that $\mathfrak{M}' \models \varphi \leftrightarrow \overline{a}$ and $\mathfrak{M}' \not\models \varphi \leftrightarrow \overline{a}$. □

4.6. κ-Omitting types property

Finally, we will prove a last characterization in terms of the omitting types property. A version of this result was already known for a particular setting of fuzzy logic: rational Pavelka logic (see [8]).

Theorem 20. (Sixth Lindström theorem) *Let $\mathscr{L}^{\boldsymbol{A}}$ be an abstract logic such that $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega} \leq \mathscr{L}^{\boldsymbol{A}}$ and is closed under the connectives of $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$. For every uncountable regular cardinal κ, if $\mathscr{L}^{\boldsymbol{A}}$ has the κ-omitting types property, then $\mathscr{L}^{\boldsymbol{A}} \leq \mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$.*

Proof. Suppose, by the way of contradiction, that $\mathscr{L}^{\boldsymbol{A}} < \mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$. By inspecting the proof of our Theorem 17, if we know there is a structure $\mathfrak{M}$ which is a model of $\varphi(<, P, I, U, G, V, W)$ but fails to be such that $\langle\omega, <\rangle$ is isomorphic to $\mathfrak{M} \restriction \{<\}|U^{\mathfrak{M}}$, the Karp property must fail for $\mathscr{L}^{\boldsymbol{A}}$.

Take a list of new constants $\{c_\alpha \mid \alpha < \kappa\}$. We can observe that $\Sigma(x)$, where

$$\Sigma(x) = \{Ux\} \cup \{c_\alpha \leq x \mid \alpha < \kappa\},$$

is an unsupported κ-type of the theory S where

$$S = \{\varphi(<, P, I, U, G, V, W)\} \cup \{c_\alpha \leq c_\beta \mid \alpha < \beta < \kappa\}.$$

To see this suppose that $|\Delta(x)| < \kappa$ and, moreover, that $\Delta(x)\cup S$ has a model. Either $\Delta(x)\cup S\cup\{Ux\}$ has a model, or not. If the second, then $\Delta(x)\cup S\cup\{Ux \to \overline{a}\}$ has a model. So suppose that $\Delta(x)\cup S\cup\{Ux\}$ has a model. Now, choose α such that for every c_β not appearing in $\Delta(x)$, we have that $\beta > \alpha$. Then, all the c_βs can be interpreted by some element bigger than the interpretation of x. Hence, $\Delta(x)\cup S\cup\{c_{\alpha+1} \leq x \to \overline{a}\}$ has a model. But then, by the κ-omitting types property, we have a model $\mathfrak{M}$ of S which omits the above described type, and this forces the cofinality of the ordering

$$\langle\{d \in M \mid \mathfrak{M} \models U[d]\}, \{\langle d, e\rangle \in M^2 \mid \mathfrak{M} \models d < e\}\rangle$$

to be κ, since for every d such that $\mathfrak{M} \models U[d]$ there will be a c_α such that $\mathfrak{M} \models d < c_\alpha$. Hence, we have the model $\mathfrak{M}$ of $\varphi(<, P, I, U, G, V, W)$ that fails to be such that $\langle\omega, <\rangle$ is isomorphic to $\mathfrak{M} \restriction \{<\}|U^{\mathfrak{M}}$.

Then, the Karp property must fail, so we have models $\mathfrak{M}, \mathfrak{N}$ such that

$$\mathfrak{M} \cong_f \mathfrak{N} \qquad \mathfrak{M} \models \varphi \leftrightarrow \overline{a} \quad \mathfrak{N} \not\models \varphi \leftrightarrow \overline{a}.$$

Expand the signature by adding the set of symbols $\{P, G, V, W\}$, where P, V and W are unary predicates, while G is a ternary predicate, and constants $p_\alpha, d_\alpha, c_\alpha$ for all $\alpha < \kappa$. We now consider the theory Δ, defined as follows (for simplicity we assume that all symbols in $Pred_{\tau_0}$ are binary):

"P, G, V, W are all crisp"
$\varphi^{\{y|Vy\}}$, $(\varphi \to \overline{a})^{\{y|Wy\}}$
"The crisp extensions of V, W are disjoint"

Pp_α, for all $\alpha < \kappa$.
$(\exists v, w)(Gp_\alpha c_\alpha v \wedge Gp_\alpha w d_\alpha)$, for all $\alpha < \kappa$.
$(\forall y, z, v, w)((Px \to \overline{0}) \vee (((Gxyz \wedge Gxvw) \to \overline{0}) \vee (y = v \leftrightarrow z = w)))$
$(\forall y, z)((Gp_\alpha yz \to \overline{0}) \vee (Vy \wedge Wz)))$, for all $\alpha < \kappa$.
$(\forall y, z, v, w)(((Gp_\alpha yz \wedge Gp_\alpha vw) \to \overline{0}) \vee (Ryv \leftrightarrow Rzw)) \quad (R \in Pred_{\tau_0}, \alpha < \kappa)$
$(\forall p)((Pp \to \overline{0}) \vee (\forall x)(\exists q, y)(Pq \wedge Gqxy \wedge (\forall z, w)((Gpzw \to \overline{0}) \vee Gqzw))$ (forth property)
$(\forall p)((Pp \to \overline{0}) \vee (\forall x)(\exists q, y)(Pq \wedge Gqyx \wedge (\forall z, w)((Gpzw \to \overline{0}) \vee Gqzw))$ (back property)
$(\forall z, w)((Gp_\alpha zw \to \overline{0}) \vee Gp_\beta zw))$ for all $\alpha < \beta < \kappa$.

Consider the collection of formulas $\Sigma(x)$:

$$\{Vx \vee Wx\} \cup \{(x = c_\alpha \to \overline{0}) \wedge (x = d_\beta \to \overline{0}) \mid \alpha, \beta < \kappa\}.$$

We observe that $\Sigma(x)$ is an unsupported κ-type of Δ. Suppose that $|\Gamma(x)| < \kappa$ and $\Gamma(x) \cup \Delta$ is satisfied by an element d in a model $\mathfrak{M}$. Consider the case that $d \notin \{e \in M \mid \mathfrak{M} \vDash V[e]\} \cup \{e \in M \mid \mathfrak{M} \vDash W[e]\}$, so $\Gamma(x) \cup \Delta \cup \{(Vx \vee Wx) \to \overline{a}\}$ has a model. Otherwise, if $d \in \{e \in M \mid \mathfrak{M} \vDash V[e]\} \cup \{e \in M \mid \mathfrak{M} \vDash W[e]\}$, then we can see that either $\Gamma(x) \cup \Delta \cup \{x = c_\alpha\}$ or $\Gamma(x) \cup \Delta \cup \{x = d_\alpha\}$ have models for some $\alpha < \kappa$. Simply pick $\beta < \kappa$ such that for any $\alpha > \beta$ neither p_α, c_α, nor d_α appear in $\Gamma(x)$. By the description of Δ above (particularly, by the back-and-forth properties), we can find some partial isomorphism q extending p_β such that either d is in its domain or in its range, depending on whether $d \in \{e \in M \mid \mathfrak{M} \vDash V[e]\}$ or $d \in \{e \in M \mid \mathfrak{M} \vDash W[e]\}$. Hence, we can change the interpretations of every p_α, c_α, and d_α $(\alpha > \beta)$ using q, d and $q(d)$ in a way that either $\Gamma(x) \cup \Delta \cup \{x = c_\alpha\}$ or $\Gamma(x) \cup \Delta \cup \{x = d_\alpha\}$ will be satisfied. Then, by the κ-omitting types property applied to $\Sigma(x)$ and Δ, we can conclude that there is a model $\mathfrak{M}$ such that $\mathfrak{M}|\{d \mid \mathfrak{M} \models V[d]\}$ and $\mathfrak{M}|\{d \mid \mathfrak{M} \models W[d]\}$ are isomorphic while one of them satisfies φ whereas the other does not. □

5. Conclusion

In this paper we have shown that an abstract model theory in the context of mathematical fuzzy logic is perfectly viable, at least under certain necessary technical restrictions. It should be stressed once more that all the notions used in this line of research are generalizations of the classical ones, in the precise sense that when restricted to the case $\boldsymbol{A} \cong \boldsymbol{B}_2$ all definitions turn out to coincide with their classical counterparts. However, one should realize that this is neither unmotivated nor a straightforward exercise in generalization. Indeed, on the one hand, there is a whole industry of non-classical predicate logics (not only in the literature of mathematical fuzzy logic, but also in other families of logics) that can benefit from systematical, unified and abstract approaches like the one proposed here. On the other hand, as argued above, the generalization of classical definitions and results to wider frameworks is far from easy, as different formulations of one classical notion (i.e. properties that were equivalent for classical logic) may give rise to non-equivalent notions when considered in a general setting. Choosing the right definitions and convenient design choices is a crucial delicate task for the success of this enterprise. Moreover, as illustrated by this paper, general results may require new mathematically involved proofs.

Since, classical logic and its (abstract) model theory always remain under the scope of our results, we regard this investigation as a contribution to the classical theory too, for it shows how much of the classical assumptions (logical and metalogical) are actually needed for important well-known results to hold. Moreover, we hope to have contributed to showing that mathematical fuzzy logic and many-valued logics in general provide a rich mathematical domain of inquiry where a classical model-theorist could also feel at home.

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