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A General Omitting Types Theorem in Mathematical Fuzzy Logic

Guillermo Badia and Carles Noguera

***Abstract*—This paper is a contribution to the theoretical study of weighted structures in fuzzy logic. We consider an important item from classical model theory: the construction of models that do not have any collection satisfying certain prescribed properties, that is, an omitting types theorem. We generalize the work done by Cintula and Diaconescu (Omitting Types Theorem for Fuzzy Logics, *IEEE Transactions on Fuzzy Systems* 27(2):273–277, 2019), who solved the problem for standard one-sided types. Instead, we introduce types for fuzzy structures as pairs of sets of formulas with free variables (expressing, respectively, properties to be satisfied and those to be avoided) and prove the corresponding omitting types theorem in the framework of uninorm-based logics.**

***Index Terms*—mathematical fuzzy logic, omitting types theorem, first-order fuzzy logics, uninorms, weighted structures**

I. Introduction

Mathematical fuzzy logic studies graded logics as particular kinds of many-valued inference systems in several formalisms, including first-order predicate languages. Models of such first-order graded logics are variations of classical structures in which predicates are evaluated over wide classes of algebras of truth degrees, beyond the classical two-valued Boolean algebra. They can be seen as a formal rendering of various structures used by fuzzy set theory for its numerous applications (see e.g. [1]) or, more generally, as *weighted structures* used recently in several areas of computer science, e.g., for preference modeling [2], argumentation theory [3], models of description logics [4], or valued constraint satisfaction problems [5]–[7].

Classical model theory is the study of the construction and classification of two-valued structures using Boolean logics (cf. [8]–[10]). Chen Chung Chang and H. Jerome Keisler proposed, in their 1966 monograph [11], to extend classical model theory to capture better some metric notions. Their starting point was the observation that, unlike in algebraic structures, in metric structures the basic relation is not equality but the distance between two objects, which can be seen as a binary relation taking values in the real unit interval. Their idea slowly gained momentum till it reached much popularity in recent years. Indeed, Itaï Ben Yaacov, Alexander Berenstein, C. Ward Henson, Alexander Usvyatsov, and others have developed a full spectrum of model-theoretic techniques and constructions in this setting, thus turning continuous model theory into a prominent branch of model theory; see e.g. [12], [13]. This field is known to be connected to many-valued logic (see [14]).

The study of models of first-order fuzzy logics is based on the corresponding strong completeness theorems [15], [16] and has already addressed several crucial topics such as: characterization of completeness properties with respect to models based on particular classes of algebras [17], models of logics with evaluated syntax [18], [19], study of mappings and diagrams [21], ultraproduct constructions [22], characterization of elementary equivalence in terms of elementary mappings [23], characterization of elementary classes as those closed under elementary equivalence and ultraproducts [24], Löwenheim–Skolem theorems [25], and back-and-forth systems for elementary equivalence [26].

For classical model theory [8]–[10] a type is, roughly speaking, a description of an object (or a tuple of objects) whose existence is consistent with a given theory (i.e., a collection of formulas). Types are the central notion of another important item in the classical agenda: the study of models where many types are omitted, that is, the construction of structures forbidding elements satisfying certain expressible properties. For instance, in the theory of strict partial orders, one might omit the type p:

$$\{x_1 < x_2, x_2 < x_3, x_3 < x_4\} \cup \{x_4 \nless x_1\}.$$

A strict partial order $\langle P, < \rangle$ omits p iff every four-element total order in $\langle P, < \rangle$ forms a cycle.

In continuous model theory, the construction of such models is well known (see [12], [27], [28]). In mathematical fuzzy logic, the problem was originally addressed in [18] for evaluated syntax, and more recently in [29] for the general case. However, previous results only applied to definitions of types precluding the possibility of expressing properties that fail *simpliciter*. By the latter, we mean formulas that take value < 1, not that fail because they lead to some absurdity, e.g. the bottom of the lattice.

The goal of the present paper is to establish a new omitting types theorem, generalizing the main result in [29], where we consider types in the form of pairs that track the properties that hold as well as the properties that do not. This is useful, for example, in cases where we want to consider the analogue of the above described type for fuzzy strict partial orders as opposed to classical strict partial orders. The second set in the union described contains a formula stating that the order relation fails between x_4 and x_1. This, in the fuzzy setting,

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would come to mean that $x_4 < x_1$ takes a value < 1, which is, in general, not the same as taking value 0. But without a way to capture the idea that formulas fail simpliciter, all we can express in the standard language of fuzzy logics would be the formula $\neg(x_4 < x_1)$, which is defined as $(x_4 < x_1) \to \overline{0}$. Hence, our more general result, seems to be more useful in actual applications, such as in the theory of fuzzy orders. For example, when expressing acyclicity conditions of fuzzy graphs, usually one has to use means that are not readily captured by the notion of a theory of the logical language (e.g. [30]). In contrast, we can take care of this problem inside our logical setting thanks to the introduction of a generalization of the notion of theory that keeps track not only of formulas that must hold, but also of what formulas must fail. Of course, one could also add a new connective directly, but our interest is to obtain a general result that stays in the original, simpler, framework of [29].

The omitting types theorem obtained in the present paper stands in clear constrast with the complementary results published in [31], in which we have shown a construction of *saturated* models, that is, models that (instead of omitting types) realize as many types as possible.

The paper is organized as follows. Section II presents the necessary logical preliminaries we need by recalling several semantical notions from mathematical fuzzy logic, namely, the algebraic counterpart of extensions of the uninorm logic UL and fuzzy first-order models based on such algebras. Section III introduces the notion of tableaux (necessary for our treatment of types) as pairs of sets of formulas and proves that each consistent tableau has a model. Furthermore, it defines types as pairs of sets of formulas with some free variables that are consistent with respect to a given tableau and presents a property of certain spaces of such types which is reminiscent of compactness (and will substitute it) in this setting. Section IV provides an omitting types theorem for the setting of tableaux (with a certain technical restriction on the number of constant symbols present in the formulas in question), generalizing the result in [29], and draws some meaningful corollaries. In Section V, we introduce a variant of the central result, using a more complicated notion of unsupported types, which allows us to drop the restriction on the number of constants. Finally, Section VI ends the paper with some concluding remarks.

II. PRELIMINARIES

In this section we introduce the object of our study, fuzzy first-order models, and several necessary related notions for the development of the paper. For comprehensive information on the subject, one may consult the Handbook of Mathematical Fuzzy Logic [32].

We choose, as the underlying propositional basis for the first-order setting, the class of residuated uninorm-based logics [33]. This class contains most of the well-studied particular systems of fuzzy logic that can be found in the literature and has been recently proposed as a suitable framework for reasoning with graded predicates in [20]. These logics retain important properties, such as associativity and commutativity of the residuated conjunction, that will be used to obtain the results of this paper.

The algebraic semantics of such logics is based on UL-*algebras* [32, Definition 2.1.5], that is, algebraic structures in the language $\mathcal{L} = \{\wedge, \vee, \&, \to, \overline{0}, \overline{1}, \bot, \top\}$ of the form $\boldsymbol{A} = \langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \&^{\boldsymbol{A}}, \to^{\boldsymbol{A}}, \overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}}, \bot^{\boldsymbol{A}}, \top^{\boldsymbol{A}} \rangle$ such that

- $\langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \bot^{\boldsymbol{A}}, \top^{\boldsymbol{A}} \rangle$ is a bounded lattice,
- $\langle A, \&^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}} \rangle$ is a commutative monoid,
- for each $a, b, c \in A$, we have:

$$a \mathbin{\&^{\boldsymbol{A}}} b \leq c \quad \text{iff} \quad b \leq a \to^{\boldsymbol{A}} c, \qquad \text{(res)}$$

$$((a \to^{\boldsymbol{A}} b) \wedge \overline{1}^{\boldsymbol{A}}) \vee^{\boldsymbol{A}} ((b \to^{\boldsymbol{A}} a) \wedge^{\boldsymbol{A}} \overline{1}^{\boldsymbol{A}}) = \overline{1}^{\boldsymbol{A}} \qquad \text{(lin)}$$

The presence of constant $\overline{0}$ in the language and its corresponding distinguished element $\overline{0}^{\boldsymbol{A}}$ might look a bit puzzling because it comes without any specific assumption in the definition. However, it plays the role of defining negation (as $\neg^{\boldsymbol{A}} a = a \to^{\boldsymbol{A}} \overline{0}^{\boldsymbol{A}}$) which allows to obtain important extensions of UL, such as those that require an involutive negation. Also, in extensions such as MTL stronger logics $\overline{0}$ and $\overline{1}$ become the smallest and largest truth-values, respectively. Observe that in the extension given by Gödel algebras [34, Definition 1.3.5], the mapping sending $\overline{0}$ to $\overline{0}$ and every $x > 0$ to $\overline{1}$ is a homomorphism of the algebra of truth functions. Hence, there is no formula $\varphi(p)$ of $\mathcal{L}$ which is true iff $p < \overline{1}$. So the property of a formula failing simpliciter is not definable, in general, by a formula of our language.

$\boldsymbol{A}$ is called a UL-*chain* if its underlying lattice is linearly ordered. *Standard* UL-chains are those defined over the real unit interval $[0, 1]$ with its usual order; in that case the operation $\&^{\boldsymbol{A}}$ is a residuated uninorm, that is, a left-continuous binary associative commutative monotonic operation with a neutral element $\overline{1}^{\boldsymbol{A}}$ (which, by the way, need not coincide with the value 1).

Let $Fm_{\mathcal{L}}$ denote the set of propositional formulas written in the language of UL-algebras with a denumerable set of variables and let $\boldsymbol{Fm}_{\mathcal{L}}$ be the absolutely free algebra defined on such set. Given a UL-algebra $\boldsymbol{A}$, we say that an $\boldsymbol{A}$-*evaluation* is a homomorphism from $\boldsymbol{Fm}_{\mathcal{L}}$ to $\boldsymbol{A}$. The logic of all UL-algebras is defined by establishing, for each $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$, $\Gamma \models \varphi$ if and only if, for each UL-algebra $\boldsymbol{A}$ and each $\boldsymbol{A}$-evaluation e, we have $e(\varphi) \geq \overline{1}^{\boldsymbol{A}}$, whenever $e(\psi) \geq \overline{1}^{\boldsymbol{A}}$ for each $\psi \in \Gamma$. The logic UL is, hence, defined as preservation of truth over all UL-algebras, where the notion of truth is given by the set of designated elements, or *filter*, $\mathcal{F}^{\boldsymbol{A}} = \{a \in A \mid a \geq \overline{1}^{\boldsymbol{A}}\}$. The standard completeness theorem of UL proves that the logic is also complete with respect to its intended semantics: the class of UL-chains defined over $[0, 1]$ by residuated uninorms (the standard UL-chains); this justifies the name of UL (uninorm logic).

Most well-known propositional fuzzy logics can be obtained by extending UL with additional axioms and rules (in a possibly expanded language). Important examples are Gödel–Dummett logic G and Łukasiewicz logic Ł.

A *predicate signature* $\mathcal{P}$ is a triple $\langle \mathbf{P}, \mathbf{F}, \mathbf{ar} \rangle$, where $\mathbf{P}$ is a non-empty set of predicate symbols, $\mathbf{F}$ is a set of function symbols, and $\mathbf{ar}$ is a function assigning to each symbol

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a natural number called the *arity* of the symbol. Let us further fix a denumerable set V whose elements are called *object variables*. The sets of $\mathcal{P}$-*terms*, *atomic* $\mathcal{P}$-*formulas*, and $\langle\mathcal{L},\mathcal{P}\rangle$-*formulas* are defined as in classical logic. A $\mathcal{P}$-*structure* $\mathfrak{M}$ is a pair $\langle \boldsymbol{A}, \mathbf{M}\rangle$ where $\boldsymbol{A}$ is a UL-chain and $\mathbf{M} = \langle M, \langle P_{\mathbf{M}}\rangle_{P\in\mathbf{P}}, \langle F_{\mathbf{M}}\rangle_{F\in\mathbf{F}}\rangle$, where M is a non-empty domain; $P_{\mathbf{M}}$ is a function $M^n \to A$, for each n-ary predicate symbol $P \in \mathbf{P}$; and $F_{\mathbf{M}}$ is a function $M^n \to M$ for each n-ary function symbol $F \in \mathbf{F}$. An $\mathfrak{M}$-*evaluation* of the object variables is a mapping $\mathrm{v}\colon V \to M$; by $\mathrm{v}[x{\to}a]$ we denote the $\mathfrak{M}$-evaluation where $\mathrm{v}[x{\to}a](x) = a$ and $\mathrm{v}[x{\to}a](y) = \mathrm{v}(y)$ for each object variable $y \neq x$. We define the *values* of the terms and the *truth values* of the formulas as (where for $\circ$ stands for any n-ary connective in $\mathcal{L}$):

$$\begin{aligned}
\|x\|^{\mathfrak{M}}_{\mathrm{v}} &= \mathrm{v}(x),\\
\|F(t_1,\dots,t_n)\|^{\mathfrak{M}}_{\mathrm{v}} &= F_{\mathbf{M}}(\|t_1\|^{\mathfrak{M}}_{\mathrm{v}},\dots,\|t_n\|^{\mathfrak{M}}_{\mathrm{v}}),\\
\|P(t_1,\dots,t_n)\|^{\mathfrak{M}}_{\mathrm{v}} &= P_{\mathbf{M}}(\|t_1\|^{\mathfrak{M}}_{\mathrm{v}},\dots,\|t_n\|^{\mathfrak{M}}_{\mathrm{v}}),\\
\|\circ(\varphi_1,\dots,\varphi_n)\|^{\mathfrak{M}}_{\mathrm{v}} &= \circ^{\boldsymbol{A}}(\|\varphi_1\|^{\mathfrak{M}}_{\mathrm{v}},\dots,\|\varphi_n\|^{\mathfrak{M}}_{\mathrm{v}}),\\
\|(\forall x)\varphi\|^{\mathfrak{M}}_{\mathrm{v}} &= \inf_{\leq_{\boldsymbol{A}}}\{\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}[x\to m]} \mid m \in M\},\\
\|(\exists x)\varphi\|^{\mathfrak{M}}_{\mathrm{v}} &= \sup_{\leq_{\boldsymbol{A}}}\{\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}[x\to m]} \mid m \in M\}.
\end{aligned}$$

If the infimum or supremum does not exist, the corresponding truth-value is undefined and, hence, the formula would not have one. Since we want to avoid this scenario, we restrict our attention to so called *safe structures*, where this never happens. We say that $\mathfrak{M}$ is *safe* if $\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}}$ is defined for each $\mathcal{P}$-formula φ and each $\mathfrak{M}$-evaluation v. Formulas without free variables are called *sentences* and a set of sentences is called a *theory*. Observe that if φ is a sentence, then its value does not depend on a particular $\mathfrak{M}$-evaluation; we denote its value as $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}}$. If φ has free variables among $\{x_1,\dots,x_n\}$ we will denote it as $\varphi(x_1,\dots,x_n)$; then the value of the formula under a certain evaluation v depends only on the values given to the free variables; if $\mathrm{v}(x_i) = d_i \in M$ we denote $\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}}$ as $\|\varphi(d_1,\dots,d_n)\|^{\boldsymbol{A}}_{\mathbf{M}}$. We say that $\mathfrak{M}$ is a *model* of a theory T, in symbols $\mathfrak{M} \models T$, if it is safe and for each $\varphi \in T$, $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} \geq \overline{1}^{\boldsymbol{A}}$. We can introduce (local) logical consequence between sets of formulas and formulas in the natural way (lifting it from the propositional case): $\Gamma \vDash \varphi$ if for every model $\mathfrak{M}$, $\|\psi\|^{\mathfrak{M}}_{\mathrm{v}} \geq \overline{1}^{\boldsymbol{A}}$ for every $\psi \in \Gamma$ only if $\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}} \geq \overline{1}^{\boldsymbol{A}}$. This notion is denoted as $\vDash^l$ in [29].

Such first-order logics satisfy some important properties (see e.g. [35]), for each theory $T \cup \{\varphi,\psi,\chi\}$ (inductively defining for each formula α: $\alpha^0 = \overline{1}$, and for each natural n, $\alpha^{n+1} = \alpha^n \mathbin{\&} \alpha$):

1) Local deduction theorem: $T,\varphi \models \psi$ if, and only if, there is a natural number n such that $T \models (\varphi \wedge \overline{1})^n \to \psi$.
2) Proof by cases: If $T,\varphi \models \chi$ and $T,\psi \models \chi$, then $T, \varphi \vee \psi \models \chi$.
3) Consequence compactness: If $T \vDash \varphi$, then for some finite $T_0 \subseteq T$, $T_0 \vDash \varphi$.

Observe that, alternatively, we could have introduced calculi and a corresponding notion of deduction $\vdash$ for these logics, but we prefer to keep the focus of the paper on the semantics.

Finally, we need a general convention for this paper: we will restrict our study to models that are *filter-witnessed*. A filter-witnessed model is a structure $\mathfrak{M} = \langle \boldsymbol{A}, \mathbf{M}\rangle$ satisfying the following property: if $\|(\exists x)\varphi\|^{\mathfrak{M}}_{\mathrm{v}} \geq \overline{1}^{\boldsymbol{A}}$, then $\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}[x\to m]} \geq \overline{1}^{\boldsymbol{A}}$, for some $m \in M$. Henceforth, by a *model* we will always mean one such model.

Given a model $\mathfrak{M} = \langle \boldsymbol{A}, \mathbf{M}\rangle$ and a collection $D \subseteq M$, we denote by $\mathrm{Th}_D(\mathfrak{M})$ the theory of $\mathfrak{M}$ relative to D, that is, the collection of all sentences φ in a language obtained by augmenting with a list of constants to denote the elements from D such that $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} \geq \overline{1}^{\boldsymbol{A}}$. On the other hand, $\overrightarrow{\mathrm{Th}}_D(\mathfrak{M})$ will simply denote the set-theoretic complement of $\mathrm{Th}_D(\mathfrak{M})$.

III. Tableaux and Spaces of types

A *tableau* is a pair $\langle T, U\rangle$ such that T and U are sets of formulas. A tableau $\langle T_0, U_0\rangle$ is called a *subtableau* of $\langle T, U\rangle$ if $T_0 \subseteq T$ and $U_0 \subseteq U$. If both sets are singletons, we simply write $\langle \varphi, \varphi'\rangle$ instead of $\langle\{\varphi\},\{\varphi'\}\rangle$. $\langle T, U\rangle$ is *satisfied* by a model $\mathfrak{M} = \langle \boldsymbol{A}, \mathbf{M}\rangle$, if there is an $\mathfrak{M}$-evaluation v such that for each $\varphi \in T$, $\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}} \geq \overline{1}^{\boldsymbol{A}}$, and for all $\psi \in U$, $\|\psi\|^{\mathfrak{M}}_{\mathrm{v}} < \overline{1}^{\boldsymbol{A}}$. Also, we write $\langle T, U\rangle \models \varphi$ meaning that for any model and evaluation that satisfies $\langle T, U\rangle$, the model and the evaluation must make φ true as well. A tableau $\langle T, U\rangle$ is said to be *consistent* if $T \models \bigvee U_0$ for no finite $U_0 \subseteq U$. In the extreme case, we define $\bigvee \emptyset$ as $\perp$.

Following [16], we say that a set of sentences T is a $\exists$-*Henkin theory* if, whenever $T \models (\exists x)\varphi(x)$, there is a constant c such that $T \models \varphi(c)$. T is a *Henkin theory* if $T \not\models (\forall x)\varphi(x)$ implies that there is a constant c such that $T \not\models \varphi(c)$. T is *doubly Henkin* if it is both $\exists$-Henkin and Henkin. T is a *linear theory* if for any pair of sentences φ,ψ either $T \models \varphi \to \psi$ or $T \models \psi \to \varphi$.

The following result (proved in [31] using local deduction theorem, proof by cases, and consequence compactness) ensures that each consistent tableau has a model, which will be necessary in the remainder.

Theorem 1. (Model Existence Theorem) *Let $\langle T, U\rangle$ be a consistent tableau. Then there is a model that satisfies $\langle T, U\rangle$. Furthermore, if the language is countable the model is countable as well.*

We can already introduce the general notion of type with respect to a given tableau.

Definition 1. *A tableau $\langle p, p'\rangle$ in some free variables is a* type *of a tableau $\langle T, U\rangle$ if $\langle T \cup p, U \cup p'\rangle$ is satisfiable. We call $\langle p, p'\rangle$ an n-*type *to signify that $p \cup p'$ has n free variables. Finally, $\langle p, p'\rangle$ is called* complete *if for any φ, either $\varphi \in p$ or $\varphi \in p'$.*

Let $S_n(T, U)$ be the collection of all complete n-types of the tableau $\langle T, U\rangle$. This is the space of prime filter-ideal pairs of the n-Lindenbaum algebra of our logic with the quotient algebra constructed by the relation $\varphi \equiv \psi$ iff $\langle T, U\rangle \vDash \varphi \leftrightarrow \psi$.

Given formulas σ and θ, we define $[\langle\sigma,\theta\rangle] = \{\langle p, p'\rangle \in S_n(T, U) \mid \sigma \in p, \theta \in p'\}$. Consider now the collection $B = \{[\langle\phi,\psi\rangle] \mid \phi,\psi \text{ are formulas}\}$. Intuitively, this simply contains all the sets of pairs of theories where ϕ is expected to be true while ψ is expected to fail, for any two formulas ϕ and ψ. B is the base for a topology on $S_n(T,U)$ since it clearly covers the space and it is closed under finite intersections, namely we have that $[\langle\phi,\psi\rangle]\cap[\langle\phi',\psi'\rangle]=[\langle\phi\wedge\phi',\psi\vee\psi'\rangle]\in B$ (here we use that models are evaluated on a UL-chain). Then, there is a topology on $S_n(T,U)$ such that every open set is just the union of a collection of sets from B.

Definition 2. *We will say that a type $\langle p,p'\rangle$ is* generated *if $\{\langle p,p'\rangle\}=[\langle\varphi,\varphi'\rangle]$ for some formulas φ,φ'.*

The next proposition shows that generated types coincide with isolated points of the topology for a particular choice of tableau and, moreover, have a useful characterization.

Given a model $\mathfrak{M}=\langle \boldsymbol{A},\mathbf{M}\rangle$ and a collection $D\subseteq M$, we denote by $\mathrm{Th}_D(\mathfrak{M})$ the theory of $\mathfrak{M}$ relative to D, that is, the collection of all sentences φ in the language augmented with constants to denote the elements from D such that $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}}\geq \overline{1}^{\boldsymbol{A}}$. On the other hand, $\overline{\mathrm{Th}}_D(\mathfrak{M})$ will simply denote the set-theoretic complement of $\mathrm{Th}_D(\mathfrak{M})$.

Proposition 2. *Let $\langle p,p'\rangle\in S_n(\mathrm{Th}_D(\boldsymbol{B},\mathbf{M}),\overline{\mathrm{Th}}_D(\boldsymbol{B},\mathbf{M}))$ for some model $\langle \boldsymbol{B},\mathbf{M}\rangle$ and $D\subseteq M$, with the topology described above. Then, the following are equivalent:*

(i) *$\langle p,p'\rangle$ is an isolated point.*
(ii) *$\langle p,p'\rangle$ is generated.*
(iii) *There are formulas $\varphi\in p$ and $\varphi'\in p'$ such that for each pair of formulas ψ and ψ' we have:*
$\psi\in p$ and $\psi'\in p'$ iff
$\langle \mathrm{Th}_D(\boldsymbol{B},\mathbf{M})\cup\{\varphi\},\overline{\mathrm{Th}}_D(\boldsymbol{B},\mathbf{M})\cup\{\varphi'\}\rangle\vDash\langle\psi,\psi'\rangle$.

Proof. $(i)\implies(ii)$: This is clear since, in any topology, an open singleton belongs to every base.

$(ii)\implies(i)$: Obvious.

$(ii)\implies(iii)$: Suppose that $\{\langle p,p'\rangle\}=[\langle\varphi,\varphi'\rangle]$. Take a model $\langle \boldsymbol{C},\mathbf{N}\rangle$ where some sequence of individuals $\overrightarrow{d}$ satisfies $\langle \mathrm{Th}_D(\boldsymbol{B},\mathbf{M})\cup\{\varphi\},\overline{\mathrm{Th}}_D(\boldsymbol{B},\mathbf{M})\cup\{\varphi'\}\rangle$. But then the complete type pair of $\overrightarrow{d}$ is in $S_n(\mathrm{Th}_D(\boldsymbol{B},\mathbf{M}),\overline{\mathrm{Th}}_D(\boldsymbol{B},\mathbf{M}))$, and moreover, in $[\langle\varphi,\varphi'\rangle]$, so it must be identical to $\langle p,p'\rangle$.

For the other direction suppose that

$$\langle \mathrm{Th}_D(\boldsymbol{B},\mathbf{M})\cup\{\varphi\},\overline{\mathrm{Th}}_D(\boldsymbol{B},\mathbf{M})\cup\{\varphi'\}\rangle\vDash\langle\psi,\psi'\rangle.$$

But, since $\langle p,p'\rangle$ is complete, we have that, indeed, $\psi\in p$ and $\psi'\in p'$.

$(iii)\implies(ii)$: We claim that $\{\langle p,p'\rangle\}=[\langle\varphi,\varphi'\rangle]$. Let $\langle q,q'\rangle\in[\langle\varphi,\varphi'\rangle]$. If $\psi\in p,\psi'\in p'$, then

$$\langle \mathrm{Th}_D(\boldsymbol{B},\mathbf{M})\cup\{\varphi\},\overline{\mathrm{Th}}_D(\boldsymbol{B},\mathbf{M})\cup\{\varphi'\}\rangle\vDash\langle\psi,\psi'\rangle.$$

Thus, in the model for $\langle q,q'\rangle$ we must also have that $\langle\psi,\psi'\rangle$, which by the completeness of $\langle q,q'\rangle$, gives that $\psi\in q,\psi'\in q'$. On the other hand, if $\psi\notin p$ and $\psi'\notin p'$, by the completeness of $\langle p,p'\rangle$, in fact, $\psi'\in p,\psi\in p'$. Consequently, as before, $\psi'\in q$ and $\psi\in q'$, which, by the completeness of $\langle q,q'\rangle$ means that $\psi\notin q$ and $\psi'\notin q'$. □

A topological space is said to be *strongly S-closed* if every family of open sets with the finite intersection property has a non-empty intersection [36]. Moreover, we will say that a space is *almost strongly S-closed* if every family of *basic* open sets with the finite intersection property has a non-empty intersection.

Using the model existence theorem (Theorem 1), we can easily establish the following.

Proposition 3. (Tableaux almost strong S-closedness) *Let $\langle T,U\rangle$ be a tableau. If every $\langle T_0,U_0\rangle$, with $|T_0|,|U_0|$ finite and $T_0\subseteq T$ and $U_0\subseteq U$, is satisfiable, then $\langle T,U\rangle$ is satisfied in some model. Furthermore, if the language is countable the model is countable as well.*

Corollary 4. *$S_n(T,U)$ is almost strongly S-closed.*

In fact, we have that tableaux almost strong S-closedness is equivalent to the almost strong S-closedness of our spaces of types. It turns out that defining an appropriate subbase, by Alexander Subbase Theorem, we can also establish the compactness of the space. However, in general, given that strong S-closedness implies almost strong S-closedness, and that S-closedness is known to be independent of compactness ([36]), we cannot claim that almost strong S-closedness implies compactness as a matter of topology.

We need to emphasize that almost strong S-closedness for tableaux and compactness of the usual topological space where we consider simply theories instead of tableaux are not the same. The usual compactness property would state that any set of formulas T that fails to have a model has a finite $T_0\subseteq T$ that also fails to have a model. If we look at Ł with the standard algebra $[0,1]_{Ł}$, it has compactness in this weaker sense. However, it is well known that tableaux almost strong S-closedness would fail (the reader can adapt the argument from Remark 3.2.14 from [37]).

Now we will explore a generalization of the notion of generation to possibly incomplete types. First, we will begin by quoting some definitions from [29] (pp. 274-275):

> Let T be a theory and Σ a set of formulas with the free variables $\overrightarrow{x}=\langle x_1,\dots,x_n\rangle$. We call Σ an *n-type* over T if $T\cup\Sigma$ is satisfiable, i.e., $\Sigma\nvDash^l\overline{0}$. We say that a model $\mathfrak{M}$ of T
>
> 1) *realizes* Σ if $\mathfrak{M}\models\Sigma(\overrightarrow{m})$ for some $\overrightarrow{m}\in M^n$;
> 2) *omits* Σ if $\mathfrak{M}$ does not realize Σ.
>
> *Definition 4*: An *n*-type Σ is *isolated* in T if there are formulas $\varphi(\overrightarrow{x},\overrightarrow{y})$ and $\tau(\overrightarrow{x},\overrightarrow{y})$ such that
>
> 1) $T,\varphi(\overrightarrow{x},\overrightarrow{y})\nvDash^l\tau(\overrightarrow{x},\overrightarrow{y})$;
> 2) $T,\varphi(\overrightarrow{x},\overrightarrow{y})\vDash^l\sigma(\overrightarrow{x})\vee\tau(\overrightarrow{x},\overrightarrow{y})$ for all $\sigma(\overrightarrow{x})\in\Sigma$.

These definitions can be rewritten in our setting as follows. First, Σ is a type $\langle\Sigma,\emptyset\rangle$ and T is a tableau $\langle T,\emptyset\rangle$. 1) means that $\langle T\cup\{\varphi(\overrightarrow{x},\overrightarrow{y})\},\{\tau(\overrightarrow{x},\overrightarrow{y})\cup\emptyset\}\rangle$ is satisfiable. Moreover, 2) means that $\langle T\cup\{\varphi(\overrightarrow{x},\overrightarrow{y})\},\{\tau(\overrightarrow{x},\overrightarrow{y})\}\cup\emptyset\rangle\vDash\langle\{\sigma(\overrightarrow{x})\},\emptyset\rangle$ for every $\sigma(\overrightarrow{x})\in\Sigma$. Next, we generalize this idea by allowing to have sets of formulas where we used to have the empty set.

Definition 3. *A type $\langle p,p'\rangle$ over $\langle T,U\rangle$ in the free variables $\overrightarrow{x}=\langle x_1,\dots,x_n\rangle$ is* supported *if there are formulas $\varphi(\overrightarrow{x},\overrightarrow{y})$, $\varphi'(\overrightarrow{x},\overrightarrow{y})$ such that*

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$\langle T \cup \{\varphi(\overrightarrow{x}, \overrightarrow{y})\}, U \cup \{\varphi'(\overrightarrow{x}, \overrightarrow{y})\}\rangle$ is satisfiable[1] *and, for every $\psi(\overrightarrow{x}) \in p$ and $\psi'(\overrightarrow{x}) \in p'$, we have that $\langle T \cup \{\varphi(\overrightarrow{x}, \overrightarrow{y})\}, U \cup \{\varphi'(\overrightarrow{x}, \overrightarrow{y})\}\rangle \vDash \langle \psi(\overrightarrow{x}), \psi'(\overrightarrow{x})\rangle$.*

As in [29] (Remark 5, p. 275), we can observe that this notion collapses with the standard notion in the case of classical first-order logic. This means, *a fortiori*, that any examples of the classical definitions are also examples of ours.

We end this section with the two additional useful definitions of principal and unsuppported type.

Definition 4. *A supported type $\langle p, p'\rangle$ is* principal *if there are φ, φ' as previously described such that $\exists \overrightarrow{y} \varphi(\bar{x}, \overrightarrow{y}) \in p$ and $\forall \overrightarrow{y} \varphi'(\bar{x}, \overrightarrow{y}) \in p'$.*

Definition 5. *A type $\langle p, p'\rangle$ of $\langle T, U\rangle$ in the free variables $\overrightarrow{x} = \langle x_1, \ldots, x_n\rangle$ is* unsupported *if for any formulas $\varphi(\overrightarrow{x}, \overrightarrow{y})$, $\varphi'(\overrightarrow{x}, \overrightarrow{y})$ (possibly with new variables) such that $\langle T \cup \{\varphi(\overrightarrow{x}, \overrightarrow{y})\}, U \cup \{\varphi'(\overrightarrow{x}, \overrightarrow{y})\}\rangle$ is satisfiable, there are $\psi(\overrightarrow{x}) \in p, \psi'(\overrightarrow{x}) \in p'$ such that $\langle T \cup \{\varphi(\overrightarrow{x}, \overrightarrow{y})\}, U \cup \{\varphi'(\overrightarrow{x}, \overrightarrow{y})\}\rangle \nvDash \psi(\overrightarrow{x})$ or $\langle T \cup \{\varphi(\overrightarrow{x}, \overrightarrow{y}), \psi'(\overrightarrow{x})\}, U \cup \{\varphi'(\overrightarrow{x}, \overrightarrow{y})\}\rangle$ is satisfiable.*

Observe that the definition of an unsupported type is simply a convenient restatement of the negation of the definition of a supported type.

Now we provide an example (Example 7.2.1 from [8]) of the definitions in the setting of classical two-valued logic (the particular case when the UL-algebra is simply the two-element chain). We consider the tableau $\langle T, \emptyset\rangle$ in a language with only the unary relations P_n ($n \in \omega$) where T is the set of sentences $\exists x P_0(x), \exists x(P_0(x) \to \bot), \exists x(P_0(x) \wedge P_1(x)), \exists x(P_0(x) \wedge (P_1(x) \to \bot)), \exists x(P_1(x) \wedge (P_0(x) \to \bot))$, etc., for all possible combinations of the predicates. All models of this tableau make exactly the same sentences true (i.e. take value $\geq \overline{1}^{\boldsymbol{A}}$) in the language under consideration [8, Exercise 16, p. 341]. For any $s \subseteq \omega$, we can define the tableau $\langle \{P_i(x) \mid i \in s\}, \{P_i(x) \mid i \notin s\}\rangle$, which is a type of $\langle T, \emptyset\rangle$ (this follows by an application of Proposition 3). Clearly, since $\langle T, \emptyset\rangle$ also has a countable model by Proposition 3, in this model, for some $s' \subseteq \omega$, the tableau $\langle \{P_i(x) \mid i \in s'\}, \{P_i(x) \mid i \notin s'\}\rangle$ will be omitted. But then the tableau for s' must be unsupported. Otherwise, there are formulas $\varphi(x, \overrightarrow{y})$, $\varphi'(x, \overrightarrow{y})$ such that $\langle T \cup \{\varphi(x, \overrightarrow{y})\}, \{\varphi'(x, \overrightarrow{y})\}\rangle$ is satisfiable and, for every $\psi(x) \in \{P_i(x) \mid i \in s'\}$ and $\psi'(x) \in \{P_i(x) \mid i \notin s'\}$, we have that $\langle T \cup \{\varphi(x), \overrightarrow{y})\}, \{\varphi'(x, \overrightarrow{y})\}\rangle \vDash \langle \psi(x), \psi'(x)\rangle$. Since we are working with the two-element chain as background, $\langle T \cup \{\varphi(x, \overrightarrow{y})\}, \{\varphi'(x, \overrightarrow{y})\}\rangle$ can be rewritten equivalently as $\langle T \cup \{\exists x \overrightarrow{y}(\varphi(x, \overrightarrow{y}) \wedge (\varphi'(x, \overrightarrow{y}) \to \bot))\}\rangle$. But then any model of $\langle T, \emptyset\rangle$ would also have to satisfy $\langle T \cup \{\varphi(x, \overrightarrow{y})\}, \{\varphi'(x, \overrightarrow{y})\}\rangle$, which is a contradiction.

IV. Omitting types

Before providing our theorem, we recall the main result from [29]:

[1]Observe that if we dispense with this hypothesis in the definition, every type would be supported because certainly $\langle T \cup \{\bot\}, U \cup \{\top\}\rangle \vDash \psi$ and $\langle T \cup \{\bot, \psi'\}, U \cup \{\top\}\rangle$ is not satisfiable.

Theorem 7 (Omitting types theorem): Let $\mathcal{P}$ be a countable predicate language, T be a consistent theory such that at most finitely many of its elements involve object constants, and Σ be a nonisolated n-type over T. Then, there is a countable model of T that omits Σ.

Our more general version of this result is next. The reason for the hypothesis in the theorem regarding the number of constants involved will be explained at the appropriate point in the proof.

Theorem 5. (Omitting types) *Let $\langle T, U\rangle$ be a tableau, such that at most finitely many of its elements involve object constants, realized by some model and $\langle p, p'\rangle$ an unsupported n-type of $\langle T, U\rangle$. Then there is a model satisfying $\langle T, U\rangle$ which omits $\langle p, p'\rangle$.*

Proof. We start by getting rid of the constants. Let T^* and U^* be the subsets of T and U, respectively, that contain the finitely many formulas involving constants. Let us say that O is the collection of constants used in T^* and U^*. We can form sentences $\bigwedge T^*$ and $\bigvee U^*$ by the finiteness of T^* and U^*. Replacing all constants by variables in a uniform way we can obtain formulas $\bigwedge T^*(\overrightarrow{x})$ and $\bigvee U^*(\overrightarrow{y})$, possibly with some common free variables.

We add a countable set C of new constants to the language. Let us enumerate as $\varphi_0, \varphi_1, \varphi_2, \ldots$ all the formulas of the expanded language, as $\langle \theta_0, \psi_0\rangle, \langle \theta_1, \psi_1\rangle, \langle \theta_2, \psi_2\rangle, \ldots$ all pairs of formulas and as $\overrightarrow{d_0}, \overrightarrow{d_1}, \overrightarrow{d_2}, \ldots$ all n-tuples of new constants. The strategy is to build a sequence $\langle T_0, U_0\rangle, \langle T_1, U_1\rangle, \ldots$ of tableaux such that $T_0 \subseteq T_1 \subseteq \ldots$ and $U_0 \subseteq U_1 \subseteq \ldots$, and for which $T_i \nvDash U_i$, $\bigcup_{i<\omega} T_i$ is a linear doubly Henkin theory, $\bigcup_{i<\omega} U_i$ is directed and for each $i < \omega$, $\langle T_i, U_i\rangle$ is satisfiable. Moreover, we need to guarantee that for each $\overrightarrow{d_i}$ there is some $\varphi \in p$ that will be false for $\overrightarrow{d_i}$ in the canonical model of the tableau $\langle \bigcup_{i<\omega} T_i, \bigcup_{i<\omega} U_i\rangle$. We will imitate the proof of the model existence theorem obtained in [31].

Stage 0 : Simply let $T_0 = (T \setminus T^*) \cup \{\bigwedge T^*(\overrightarrow{c})\}$ and $U_0 = (U \setminus U^*) \cup \{\bigvee U^*(\overrightarrow{d})\}$, where $\overrightarrow{c}$ and $\overrightarrow{d}$ are fresh sequences of constants from C, possibly with some elements in common. In the final (canonical) model that we will build, constants from O will be interpreted using the fresh constants just mentioned.

Stage $s+1 = 4i+1$: At this stage, we make sure that our final theory will be Henkin. If φ_i is not of the form $\forall x \chi(x)$, then let $T_{s+1} = T_s$ and $U_{s+1} = U_s$. So suppose that $\varphi_i = \forall x \chi(x)$. There are two cases to consider:

(i) There is $U'_s \subseteq U_s$ such that $T_s \vDash (\bigvee U'_s) \vee \forall x \chi(x)$, then we let $T_{s+1} = T_s \cup \{\forall x \chi(x)\}$ and $U_{s+1} = U_s$.
(ii) Otherwise, let $T_{s+1} = T_s$ and $U_{s+1} = U_s \cup \{\chi(c)\}$ (where c is the first unused constant from C).

The fact that in case (i), $\langle T_{s+1}, U_{s+1}\rangle$ has a model follows easily. Take any model of $\langle T_s, U_s\rangle$. Obviously, $(\bigvee U'_s) \vee \forall x \chi(x)$ for some finite $U'_s \subseteq U_s$ would have to hold in that model, but this means that $\forall x \chi(x)$ must hold there (that is, it must get value $\geq \overline{1}^{\boldsymbol{A}}$) given that $\bigvee U'_s$ does not (i.e., it gets some value $< \overline{1}^{\boldsymbol{A}}$).

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Now we have to show that in case (ii), $\langle T_{s+1}, U_{s+1}\rangle$ has a model as well. This is obtained by tableaux almost strong S-closedness. For take any finite $U'_{s+1} \subseteq U_{s+1}$. Then we must have that $T_s \not\models (\bigvee U'_{s+1}) \vee \forall x\chi(x)$. Take some model $\langle \boldsymbol{A}, \mathbf{M}\rangle$ witnessing this fact. Since in such model

$$\left\|(\bigvee U'_{s+1}) \vee \forall x\chi(x)\right\|^{\boldsymbol{A}}_{\mathbf{M}} < \bar{1}^{\boldsymbol{A}},$$

we get that in fact it is a model of $\langle T_{s+1}, U'_{s+1} \cup \{\chi(c)\}\rangle$ interpreting c appropriately.

STAGE $s+1 = 4i+2$: At this stage we make sure that we will eventually get an $\exists$-Henkin theory. If φ_i is not of the form $\exists x\chi(x)$, then let $T_{s+1} = T_s$ and $U_{s+1} = U_s$. Otherwise, as in Lemma 2 (2) from [16] we have two cases to consider:

(i) There is $U'_s \subseteq U_s$ such that $T_s \cup \{\varphi_i\} \models \bigvee U'_s$, then we let $T_{s+1} = T_s$ and $U_{s+1} = U_s$.
(ii) Otherwise, let $T_{s+1} = T_s \cup \{\chi(c)\}$ (where c is the first unused constant from C) and $U_{s+1} = U_s$.

In case (i), $\langle T_{s+1}, U_{s+1}\rangle$ has a model by inductive hypothesis. Now, in case (ii), we again use tableaux almost strong S-closedness. For take any finite $U'_{s+1} \subseteq U_{s+1}$. Then $T_s \cup \varphi_i \not\models \bigvee U'_{s+1}$, so there is a Henkin model witnessing this fact, which in turn models $\langle T_{s+1}, U'_{s+1}\rangle$ by interpreting c appropriately.

STAGE $s+1 = 4i+3$: At this stage we need to guarantee that our resulting model will omit $\langle p, p'\rangle$. So given $\overrightarrow{d_i}$, we may write $T_s \backslash T$ as a conjunction of formulas θ (recall that we have only added finitely many formulas to T at each stage of our construction). Let θ' come from replacing the sequence $\overrightarrow{d_i}$ in θ by a sequence of free variables $\overrightarrow{v}$, and replacing all constants appearing in θ not in $\overrightarrow{d_i}$ by new variables. Similarly, express $U_s \setminus U$ as a disjunction τ (this can be done since we only added finitely many members to U during our construction) and get τ' from τ using new variables replacing constants not among $\overrightarrow{d_i}$ but in a way consistent with what we did in θ'. Any variables in τ' and θ' not replacing elements in the intersection of the witnesses appearing in θ and τ can be bounded by existential quantifiers and universal quantifiers respectively. For instance, say that say that $\overrightarrow{c}$ is a sequence of constants from C, and

$$\theta = \bigwedge_{j \leq n} \varphi_j(\bar{d}_i, \bar{c}, \bar{f})$$

and

$$\tau = \bigvee_{j \leq m} \psi_j(\bar{d}_i, \bar{c}, \bar{e}),$$

we can then say that

$$\theta' = \exists y \bigwedge_{j \leq n} \varphi_j(\bar{v}_i, \bar{x}, \bar{y})$$

and

$$\tau' = \forall y \bigvee_{j \leq m} \psi_j(\bar{v}_i, \bar{x}, \bar{y}).$$

Since $\langle T_s, U_s\rangle$ has a model by construction, $\langle T \cup \{\theta'\}, U \cup \{\tau'\}\rangle$ certainly has a model.

Then, since $\langle p, p'\rangle$ is an unsupported type of $\langle T, U\rangle$, for any formulas ψ, ψ' (possibly with new variables) such that $\langle T \cup \{\psi\}, U \cup \{\psi'\}\rangle$ is satisfiable, there are $\varphi \in p$ and $\varphi' \in p'$ such that $\langle T \cup \{\psi\}, U \cup \{\psi'\}\rangle \not\models \varphi$ or $\langle T \cup \{\psi, \varphi'\}, U \cup \{\psi'\}\rangle$ is satisfiable. In particular, we must have some $\varphi \in p, \varphi' \in p'$ such that

(i) $\langle T \cup \{\theta'\}, U \cup \{\tau'\}\rangle \not\models \varphi$, or
(ii) $\langle T \cup \{\theta', \varphi'\}, U \cup \{\tau'\}\rangle$ is satisfiable.

If (i) holds, we let $T_{s+1} = T_s$ and $U_{s+1} = U_s \cup \{\varphi(\overrightarrow{d_i})\}$. $\langle T_{s+1}, U_{s+1}\rangle$ thus defined is consistent (in fact, it has a model). The model $\langle \boldsymbol{B}, \mathbf{M}\rangle$ provided by the fact $\langle T \cup \{\theta'\}, U \cup \{\tau'\}\rangle \not\models \varphi$ is a model of $\langle T_{s+1}, U_{s+1}\rangle$ interpreting constants in the appropriate way by looking at the elements that witness the quantifiers. Here we use the assumption that our models are filter-witnessed.

On the other hand, if (ii) holds, we let $T_{s+1} = T_s \cup \{\varphi'(\overrightarrow{d_i})\}$ and $U_{s+1} = U_s$. Everything we need to verify for this case follows easily as well.

Observe that if we had allowed infinitely many constants in our original tableau, then, of course, they would have had to be listed in our enumeration of n-tuples of constants $\overrightarrow{d_0}, \overrightarrow{d_1}, \overrightarrow{d_2}, \ldots$. Then, if either θ or τ contained any of these old constants, the argument offered above for the consistency of $\langle T_{s+1}, U_{s+1}\rangle$ would break down. The problem is that the interpretation of such constant might be fixed by $\langle T, U\rangle$ in a given model. Hence, we could not have managed to turn, say, the model $\langle \boldsymbol{B}, \mathbf{M}\rangle$ provided by the fact $\langle T \cup \{\theta'\}, U \cup \{\tau'\}\rangle \not\models \varphi$ into a model of $\langle T_{s+1}, U_{s+1}\rangle$. In the presence of equality, this would not become a problem because stage $s+1 = 4i+2$ would guarantee that every term of the old language is identical to one of the new constants, which means that it would have sufficed to list only the n-tuples of new constants to make sure that the type is omitted in the final model.

STAGE $s+1 = 4i+4$: At this stage we work to ensure that our final theory will be linear. So given the pair $\langle \theta_i, \psi_i\rangle$ proceed as in Lemma 2 (3) from [16]. That is, we start from the assumption that $\langle T_s, U_s\rangle$ is consistent and letting $U_{s+1} = U_s$ we look to add one of $\theta_i \to \psi_i$ or $\psi_i \to \theta_i$ to T_s to obtain T_{s+1} while making the resulting tableau $\langle T_{s+1}, U_{s+1}\rangle$ consistent. Note that if $T_s \cup \{\theta_i \to \psi_i\} \models \bigvee U'_{s+1}$ and $T_s \cup \{\psi_i \to \theta_i\} \models \bigvee U''_{s+1}$, then $T_s \cup \{\theta_i \to \psi_i\} \models (\bigvee U'_{s+1}) \vee (\bigvee U''_{s+1})$ and $T_s \cup \{\psi_i \to \theta_i\} \models (\bigvee U'_{s+1}) \vee (\bigvee U''_{s+1})$. Hence, $T_s \cup \{(\psi_i \to \theta_i) \vee (\theta_i \to \psi_i)\} \models (\bigvee U'_{s+1}) \vee (\bigvee U''_{s+1})$ by proof by cases, and since $\models (\psi_i \to \theta_i) \vee (\theta_i \to \psi_i)$, we obtain that $T_s \models (\bigvee U'_{s+1}) \vee (\bigvee U''_{s+1})$, a contradiction. □

Theorem 6. (Omitting countably many types) *Let $\langle T, U\rangle$ be a tableau, such that at most finitely many of its elements involve object constants, realized by some model and $\langle p_i, p'_i\rangle (i < \omega)$ a sequence of unsupported n-types of $\langle T, U\rangle$. Then there is a model satisfying $\langle T, U\rangle$ which omits $\langle p_i, p'_i\rangle (i < \omega)$.*

Proof. First consider a bijection $f\colon \omega \times \omega \longrightarrow \omega$ and all pairs $\langle \overrightarrow{d}_k, \langle p_j, p'_j\rangle\rangle (k, j < \omega)$. Now, at the stage $4i+3$ in the proof of the omitting types we look at the pair $f^{-1}(i)$, say it is $\langle \overrightarrow{d}_k, \langle p_j, p'_j\rangle\rangle$. We make sure that $\overrightarrow{d}_k$ fails to satisfy $\langle p_j, p'_j\rangle$ in our final canonical model. □

The countability of the language is necessary provided that the language has some binary predicate R. To see this, we adapt a classical counterexample. Take disjoint sets of unary

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predicates C and D such that $|C| > \omega$ and $|D| = \omega$. Now consider the tableaux

$$\langle \{\forall x R(x,x)\} \cup$$
$$\{\exists x, y((Px \wedge Qy) \wedge (R(x,y) \to \bot)) \mid P, Q \in C, P \neq Q\}, \emptyset\rangle$$

and

$$\langle \{\exists x(Tx \wedge (R(v,x) \to \bot)) \mid T \in D\}, \emptyset\rangle.$$

The latter is a type of the former because any model of the former (in particular, the model of this tableau where R is the real crisp equality) will have uncountably many individuals and, hence, the tableau

$$\langle \{\exists x(Tx \wedge (R(v,x) \to \bot)) \mid T \in D\}, \emptyset\rangle.$$

would certainly be satisfiable on that model. Next take any φ, φ' and model $\langle \boldsymbol{B}, \mathbf{M}\rangle$ where some $e \in M$ satisfies

$$\langle \{\forall x R(x,x)\} \cup$$
$$\{\exists x, y((Px \wedge Qy) \wedge (R(x,y) \to \bot)) \mid P, Q \in C, P \neq Q\} \cup$$
$$\{\varphi(v)\}, \{\varphi'(v)\}\rangle.$$

By changing the extension of the first $T \in D$ not appearing in either φ nor φ' to the crisp set $\{e\}$ we obtain a model where $\langle T, U\rangle$ is satisfied by e for

$$T = \{\forall x R(x,x)\} \cup \{\varphi(v)\} \cup$$
$$\{\exists x, y((Px \wedge Qy) \wedge (R(x,y) \to \bot)) \mid P, Q \in C, P \neq Q\}$$
$$U = \{\varphi'(v)\} \cup \{\exists x(Tx \wedge (R(v,x) \to \bot)) \mid T \in D\}.$$

Hence,

$$\langle \{\exists x(Tx \wedge (R(v,x) \to \bot)) \mid T \in D\}, \emptyset\rangle$$

is unsupported but there are no models of

$$\langle \{\forall x R(x,x)\} \cup$$
$$\{\exists x, y((Px \wedge Qy) \wedge (R(x,y) \to \bot)) \mid P, Q \in C, P \neq Q\}, \emptyset\rangle$$

omitting it!

Next we provide an application of the proof of the omitting types theorem. We will show the existence of certain pairs of different models whose only realized complete types are principal.

Theorem 7. *Fix a countable predicate language $\mathcal{P}$. Let $\langle T, U\rangle$ be a tableau such that at most finitely many of its elements involve object constants, with an infinite model. Then there are two different countable models $\langle \boldsymbol{B}_1, \mathbf{M}_1\rangle$ and $\langle \boldsymbol{B}_2, \mathbf{M}_2\rangle$ of $\langle T, U\rangle$ with the property that if $\langle p, p'\rangle$ is a complete type which is realized by tuples in M_1 and M_2, then $\langle p, p'\rangle$ is principal.*

Proof. This is as the proof of the omitting types theorem with two changes. First, we build simultaneously two tableaux that will give two different models in the end. To this purpose, we build sequences $\langle T_0, T_0'\rangle, \langle T_1, T_1'\rangle, \ldots$ and $\langle S_0, S_0'\rangle, \langle S_1, S_1'\rangle, \ldots$, the first one corresponding to the model $\langle \boldsymbol{B}_1, \mathbf{M}_1\rangle$ and the second one to $\langle \boldsymbol{B}_2, \mathbf{M}_2\rangle$. Second, we leave out the stage of the construction guaranteeing that a type will be omitted and replace it with a stage making sure that for all sequences of distinct witnesses $\bar{c}, \bar{d}$, if $\bar{a}, \bar{b}$ are the tuples denoted by $\bar{c}, \bar{d}$ in $\langle \boldsymbol{B}_1, \mathbf{M}_1\rangle$ and $\langle \boldsymbol{B}_2, \mathbf{M}_2\rangle$ respectively, and $\langle p, p'\rangle, \langle q, q'\rangle$ are the complete types of $\bar{a}$ and $\bar{b}$ in $\langle \boldsymbol{B}_1, \mathbf{M}_1\rangle$ and $\langle \boldsymbol{B}_2, \mathbf{M}_2\rangle$ respectively, then either $\langle p, p'\rangle \neq \langle q, q'\rangle$ or $\langle p, p'\rangle$ is principal.

STAGE $s + 1 = 4i + 3$: Consider

$$\langle \bigwedge (T_s \setminus T)(\bar{c}, \bar{e}), \bigvee (T_s' \setminus U)(\bar{c}, \bar{e})\rangle$$

and

$$\langle \bigwedge (S_s \setminus T)(\bar{d}, \bar{f}), \bigvee (S_s' \setminus U)(\bar{d}, \bar{f})\rangle,$$

where $\bar{e}$ denotes the witnesses different from $\bar{c}$ in the construction of $\langle T_s, T_s'\rangle$ and similarly with $\bar{f}, \bar{d}$ and $\langle S_s, S_s'\rangle$.

We have to consider two possibilities. First, suppose that

$$\langle \bigwedge (T_s \setminus T)(\bar{c}, \bar{x}), \bigvee (T_s' \setminus U)(\bar{c}, \bar{y})\rangle$$

is a support over $\langle T, U\rangle$ of a complete type $\langle p, p'\rangle$. Then $\langle p, p'\rangle$ is principal (because if $\langle p, p'\rangle$ is complete, $\langle \theta, \theta'\rangle$ is a support of $\langle p, p'\rangle$ iff it generates (p, p')). So we can put in our construction

$$T_{s+1} = T_s \qquad T_{s+1}' = T_s'$$
$$S_{s+1} = S_s \qquad S_{s+1}' = S_s'.$$

Second, suppose that

$$\langle \bigwedge (T_s \setminus T)(\bar{c}, \bar{x}), \bigvee (T_s' \setminus U)(\bar{c}, \bar{y})\rangle$$

is not a support of a type over $\langle T, U\rangle$. Then there are at least two complete types $\langle p_1, p_1'\rangle, \langle p_2, p_2'\rangle$ containing the above as a subtableau. Take an arbitrary complete type $\langle q, q'\rangle$ containing the following as a subtableau

$$\langle \bigwedge (S_s \setminus T)(\bar{d}, \bar{x}), \bigvee (S_s' \setminus U)(\bar{d}, \bar{y})\rangle.$$

One of $\langle p_1, p_1'\rangle, \langle p_2, p_2'\rangle$ has to be different from $\langle q, q'\rangle$, call it $\langle p, p'\rangle$. Then there is a formula $\theta(\bar{x})$ that lives in the left-hand side of one of these two types and in the right-hand side of the other. Say that $\theta(\bar{x}) \in p$ and $\theta(\bar{x}) \in q'$ (the other possibilities are symmetric). Then we put

$$T_{s+1} = T_s \cup \{\theta(\bar{c})\} \qquad T_{s+1}' = T_s'$$
$$S_{s+1} = S_s \qquad S_{s+1}' = S_s' \cup \{\theta(\bar{d})\}. \quad \square$$

Strengthening the property of the previous result, we will say that a model is *atomic* if every type it realizes is principal, i.e. every type is reducible to some of its own formulas. Next, we provide a sufficient condition for a tableau to have an atomic model, which, under certain conditions, can be also be seen to be necessary.

Theorem 8. *Let $\mathcal{P}$ be a countable predicate language and $\langle T, U\rangle$ be a tableau, such that at most finitely many of its elements involve object constants, with infinite models. Then $(i) \implies (ii)$ for*

(i) *For every finite tableau $\langle S_0, S_1\rangle$ which has model in common with $\langle T, U\rangle$, there is a principal type of $\langle T, U\rangle$ containing $\langle S_0, S_1\rangle$ as a subtableau.*
(ii) *$\langle T, U\rangle$ has an atomic model.*

Moreover, if all models of $\langle T, U\rangle$ satisfy the same finite tableaux, then (i) and (ii) are equivalent.

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Proof. $(i) \implies (ii)$: This is again a modification of the proof of the omitting types theorem.

STAGE $s+1 = 4i+3$: Consider

$$\langle \bigwedge (T_s \setminus T)(\bar{c}), \bigvee (T'_s \setminus U)(\bar{c})\rangle$$

where $\bar{c}$ is the sequence of finitely many new witnesses appearing so far in the construction of $\langle T_s, T'_s\rangle$. Now by (i), we can find a type $\langle p, p'\rangle$ of $\langle T, U\rangle$ containing

$$\langle \bigwedge (T_s \setminus T)(\bar{x}), \bigvee (T'_s \setminus U)(\bar{x})\rangle$$

generated by a pair $\langle \varphi(\bar{x}, \bar{y}), \varphi'(\bar{x}, \bar{y})\rangle$.

So we can put in our construction

$$T_{s+1} = T_s \cup \{\varphi(\bar{c}, \bar{d})\} \qquad T'_{s+1} = T'_s \cup \{\varphi'(\bar{c}, \bar{d})\},$$

where $\bar{d}$ is a finite sequence of new constants. The tableau $\langle T_{s+1}, T'_{s+1}\rangle$ is consistent given the consistency of $\langle p \cup \{\varphi(\bar{x}, \bar{y})\}, p' \cup \{\varphi'(\bar{x}, \bar{y})\}\rangle$. This step of the construction guarantees that $\bar{c}$ will realize the type $\langle p, p'\rangle$ in the final Henkin model. Since any sequence $\bar{d}$ of elements of the Henkin model will live in some sequence $\bar{c}$ appearing at some stage $4i+3$ of the construction, $\bar{d}$ will also realize a principal type in the final model (the same $\langle \varphi(\bar{x}, \bar{y}), \varphi'(\bar{x}, \bar{y})\rangle$ which generates the type $\langle p, p'\rangle$ of $\bar{c}$ will *a fortiori* generate the type of $\bar{d}$).

For the second part of the result, it suffices to establish the converse under the assumption that all models of $\langle T, U\rangle$ satisfy the same finite tableaux.

$(ii) \implies (i)$: If $\langle T \cup \{\varphi\}, U \cup \{\varphi'\}\rangle$ has a model, then $\langle \{\varphi\}, \{\varphi'\}\rangle$ is realized in the atomic model of $\langle T, U\rangle$ as well, say by a finite sequence of elements $\bar{a}$. Now consider the complete type in the atomic model of $\bar{a}$: it contains $\langle \{\varphi\}, \{\varphi'\}\rangle$ and is principal by definition. Hence, we have established what we desired. □

V. REMOVING THE RESTRICTION ON CONSTANTS

One might wonder about the possibility of lifting the restriction on the number of constants in the main result of Section IV. Due to the difficulties of equality-free settings, this can be done only by complicating matters in regards to the notion of unsupported type (see [38] for the classical account). In this section, we briefly deal with this problem.

Following [38, Definition 4.1], we may modify the notion of unsupported type as follows:

Definition 6. *A type $\langle p, p'\rangle$ of $\langle T, U\rangle$ in the free variables $\overrightarrow{x} = \langle x_1, \ldots, x_n\rangle$ is* Keisler–Miller unsupported *if for any formulas $\varphi(\overrightarrow{z}, \overrightarrow{y})$, $\varphi'(\overrightarrow{z}, \overrightarrow{y})$ in at least the m free variables $\overrightarrow{z}$ (possibly with new variables $\overrightarrow{y}$) and n-tuple of terms $\overrightarrow{\sigma}$ in at most the variables $\overrightarrow{z}$ (so any other term in $\overrightarrow{\sigma}$ would be a constant) such that $\langle T \cup \{\varphi(\overrightarrow{z}, \overrightarrow{y})\}, U \cup \{\varphi'(\overrightarrow{z}, \overrightarrow{y})\}\rangle$ is satisfiable, there are $\psi(\overrightarrow{x}) \in p$ and $\psi'(\overrightarrow{x}) \in p'$ such that $\langle T \cup \{\varphi(\overrightarrow{x}, \overrightarrow{y})\}, U \cup \{\varphi'(\overrightarrow{x}, \overrightarrow{y})\}\rangle \not\models \psi(\overrightarrow{\sigma})$ or $\langle T \cup \{\varphi(\overrightarrow{x}, \overrightarrow{y}), \psi'(\overrightarrow{\sigma})\}, U \cup \{\varphi'(\overrightarrow{x}, \overrightarrow{y})\}\rangle$ is satisfiable.*

Now we can modify the proof of the omitting types theorem above to obtain the next result.

Theorem 9. (Keisler–Miller Omitting types) *Let $\langle T, U\rangle$ be a tableau realized by some model and $\langle p, p'\rangle$ a Keisler–Miller unsupported n-type of $\langle T, U\rangle$. Then there is a model satisfying $\langle T, U\rangle$ which omits $\langle p, p'\rangle$.*

Proof. The proof mostly follows the general lines of the previous theorem. Now, however, the enumeration $\overrightarrow{d_0}, \overrightarrow{d_1}, \overrightarrow{d_2}, \ldots$ contains all n-tuples of constants (new and old). The only new action will happen at stage $s+1 = 4i+3$.

STAGE $s+1 = 4i+3$: At this stage we need to guarantee that our resulting model will omit $\langle p, p'\rangle$. So, given $\overrightarrow{d_i}$, we may write $T_s \setminus T$ as a conjunction of formulas θ (recall that we have only added finitely many formulas to T at each stage of our construction). The tuple $\overrightarrow{d_i}$ can be partitioned into two parts, $\overrightarrow{d_i}'\overrightarrow{d_i}''$, where the first has all the constants of $\overrightarrow{d_i}$ from the new list C, whereas the second has all the constants of $\overrightarrow{d_i}$ that were in the original language. Let θ' come from replacing the sequence $\overrightarrow{d_i}'$ in θ by a sequence of free variables $\overrightarrow{v}$, and replacing all constants from C appearing in θ but not in $\overrightarrow{d_i}'$ by new variables. Similarly, let τ be the disjunction of all formulas in $U_s \setminus U$ (this can be done since we only added finitely many members to U during our construction). Then, we get τ' from τ by introducing the sequence of free variables $\overrightarrow{v}$ to replace $\overrightarrow{d_i}'$, and using new variables replacing constants in C not among $\overrightarrow{d_i}'$ but in a way consistent with what we did in θ'. Any variables in τ' and θ' not replacing elements in the intersection of the witnesses appearing in θ and τ can be bounded by existential quantifiers and universal quantifiers respectively. For instance, say that $\overrightarrow{c}$ is a sequence of constants from C, and

$$\theta = \bigwedge_{j \leq n} \varphi_j(\overrightarrow{d_i}', \overrightarrow{d_i}'', \bar{c}, \bar{f})$$

and

$$\tau = \bigvee_{j \leq m} \psi_j(\overrightarrow{d_i}', \overrightarrow{d_i}'', \bar{c}, \bar{e}),$$

we can then say that

$$\theta' = \exists y \bigwedge_{j \leq n} \varphi_j(\bar{v}_i, \overrightarrow{d_i}'', \bar{x}, \bar{y})$$

and

$$\tau' = \forall y \bigvee_{j \leq m} \psi_j(\bar{v}_i, \overrightarrow{d_i}'', \bar{x}, \bar{y}).$$

Since $\langle T_s, U_s\rangle$ has a model by construction, $\langle T \cup \{\theta'\}, U \cup \{\tau'\}\rangle$ certainly has a model.

But $\langle p, p'\rangle$ is a Keisler–Miller unsupported type of $\langle T, U\rangle$. So for any formulas $\psi(\overrightarrow{z}, \overrightarrow{y})$, $\psi'(\overrightarrow{z}, \overrightarrow{y})$ in at least the m free variables $\overrightarrow{z}$ (possibly with new variables $\overrightarrow{y}$) and n-tuple of terms $\overrightarrow{\sigma}$ in at most the variables $\overrightarrow{z}$ (so any other term in $\overrightarrow{\sigma}$ would be a constant) such that

$$\langle T \cup \{\psi(\overrightarrow{z}, \overrightarrow{y})\}, U \cup \{\psi'(\overrightarrow{z}, \overrightarrow{y})\}\rangle$$

is satisfiable, there are $\varphi(\overrightarrow{x}) \in p, \varphi'(\overrightarrow{x}) \in p'$ such that

$$\langle T \cup \{\psi(\overrightarrow{x}, \overrightarrow{y})\}, U \cup \{\psi'(\overrightarrow{x}, \overrightarrow{y})\}\rangle \not\models \varphi(\overrightarrow{\sigma})$$

or

$$\langle T \cup \{\psi(\overrightarrow{x}, \overrightarrow{y}), \varphi'(\overrightarrow{\sigma})\}, U \cup \{\psi'(\overrightarrow{x}, \overrightarrow{y})\}\rangle$$

is satisfiable.

In particular, there must be $\varphi \in p$ and $\varphi' \in p'$ such that

(i) $\langle T \cup \{\theta'\}, U \cup \{\tau'\}\rangle \not\models \varphi(\bar{v}_i, \overrightarrow{d_i}'')$, or

(ii) $\langle T \cup \{\theta', \varphi'(\bar{v}_i, \overrightarrow{d_i}'')\}, U \cup \{\tau'\}\rangle$ is satisfiable.

If (i) holds, we let $T_{s+1} = T_s$ and $U_{s+1} = U_s \cup \{\varphi(\overrightarrow{d_i})\}$. $\langle T_{s+1}, U_{s+1}\rangle$ thus defined is consistent (in fact, it has a model). The model $\langle \boldsymbol{B}, \mathbf{M}\rangle$ provided by the fact $\langle T \cup \{\theta'\}, U \cup \{\tau'\}\rangle \not\models \varphi(\bar{v}_i, \overrightarrow{d_i}'')$ is a model of $\langle T_{s+1}, U_{s+1}\rangle$ interpreting constants in the appropriate way by looking at the elements that witness the quantifiers. Here we use the assumption that our models are filter-witnessed.

On the other hand, if (ii) holds, we let $T_{s+1} = T_s \cup \{\varphi'(\overrightarrow{d_i})\}$ and $U_{s+1} = U_s$. Everything we need to verify for this case follows easily as well. □

VI. Conclusion

In this paper we have shown the existence of models realizing very few types, which, as in classical model theory, is accomplished by means of an omitting types theorem. The result has been presented for the wide class of logics based on uninorms. We proved it in the general framework of UL logic but it follows for any of its axiomatic extensions such as MTL or BL since they are based on particular kinds of UL-chains. Some work had already been started along these lines in the context of mathematical fuzzy logic in [18], [27], [29], that have focused on one-sided types with respect to a theory. We have provided a generalization of the result in [29] to the context of tableaux with types as pairs, with the clear benefit of higher expressivity. Our immediate future research plans will concentrate on going the opposite direction by constructing saturated models that satisfy as many (two-sided) types as possible, building on our preliminary results published in [31].

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