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Related Thunsdorff type and Frank–Pick type inequalities for Sugeno integral

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ABSTRACT

The purpose of this paper is to investigate the Thunsdorff's inequality for Sugeno integral. By an example, we show that the classical form of this inequality does not hold for Sugeno integral . Then, by reviewing the initial conditions, we prove two main theorems for this inequality. Finally, by checking the special case of the aforementioned Thunsdorff's inequality, we prove Frank–Pick type inequality for the Sugeno integral and illustrate it by an example.

1. Introduction

In 1974, M. Sugeno introduced fuzzy measures and Sugeno integral for the first time which became an imporant analytical method of uncertain information measuring [14]. Sugeno integral is applied in many fields such as management decision-making, medical decision-making, control engineering and so on. Many authors such as Ralescu and Adams considered equivalent definitions of Sugeno integral [11]. Román-Flores et al. examined level-continuity of Sugeno integral and H-continuity of fuzzy measures [12,13]. For more details of Sugeno integral, we refer readers to Agahi et al. [1,2], Grabisch [7], Pap and Functions [8], Pap [9], Pap and Štrboja [10].

The study of fuzzy integral is attributed to Román–Flores et al. Many inequalities such as Markov's, Chebyshev's, Jensen's, Minkowski's, Hölder's and Hardy's inequalities have been studied by Flores–Franulič and Román–Flores for Sugeno integral (see [5,6] and their references). Recently, in Daraby et al. [4], B. Daraby et al. studied some other inequalities for Sugeno integral.

All these inequalities for Sugeno integral were motivated by the related inequalities known for Riemann or Lebesgue integral. One of inequalities known for Riemann integral but not yet studied in the framework of Sugeno integral, namely Thunsdorff's inequality see [3], is given as follows:

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If f is non-negative and concave on $[a, b]$ and if $0 < r < s$, then

$$\left(\frac{1+s}{b-a}\int_a^b f^s dx\right)^{1/s} \le \left(\frac{1+r}{b-a}\int_a^b f^r dx\right)^{1/r}. \tag{1.1}$$

The case $r = 1$, $s = 2$ is the Frank–Pick inequality.

In this paper, we intend to prove a version of Thunsdorff's and Frank–Pick's inequality for the Sugeno integral.

This paper is organized as follows: in Section 2, some preliminaries are presented. In Sections 3 and 4 we propose the Thunsdorff and Frank–Pick's inequalities for Sugeno integral. Finally, in the last section, we present a short conclusion.

2. Preliminaries

In this section, we will provide some definitions and concepts for the next sections.

Throughout this paper, let X be a non-empty set and Σ be a $\sigma-$algebra of subsets of X.

Definition 2.1 (Ralescu and Adams [11]). A set function $\mu : \Sigma \to [0, +\infty]$ is called a fuzzy measure if the following properties are satisfied:

1. $\mu(\emptyset) = 0$;
2. $A \subseteq B \Rightarrow \mu(A) \le \mu(B)$ (monotonicity);
3. $A_1 \subseteq A_2 \subseteq \ldots \Rightarrow \lim \mu(A_i) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right)$ (continuity from below);
4. $A_1 \supseteq A_2 \supseteq \ldots$ and $\mu(A_1) < \infty \Rightarrow \lim \mu(A_i) = \mu\left(\bigcap_{i=1}^{\infty} A_i\right)$ (continuity from above).

When μ is a fuzzy measure, the triple (X, Σ, μ) is called a fuzzy measure space.

If f is a non-negative real-valued function on X, we will denote $F_\alpha = \{x \in X \mid f(x) \ge \alpha\} = \{f \ge \alpha\}$, the α-level of f, for $\alpha > 0$. The set $F_0 = \overline{\{x \in X \mid f(x) > 0\}} = \mathrm{supp}(f)$ is the support of f.

If σ is a sigma-algebra of subsets of X, we define the following:

$$\mathfrak{F}^\sigma = (X) = \{f : X \to [0, \infty) | \quad f \text{ is } \sigma - \text{measurable}\}.$$

Definition 2.2 (Pap [8], Wang and Klir [15]). Let μ be a fuzzy measure on (X, Σ). If $f \in \mathfrak{F}^\sigma(X)$ and $A \in \Sigma$, then the Sugeno integral of f on A is defined by

$$⨍_A f d\mu = \bigvee_{\alpha \ge 0} (\alpha \wedge \mu(A \cap F_\alpha)),$$

where $\vee$ and $\wedge$ denotes the operations sup and inf on $[0, \infty]$. If $A = X$, the fuzzy integral may also be denoted by $⨍ f d\mu$.

The following proposition gives the most elementary properties of the Sugeno integral.

Proposition 2.3 (Pap [8], Wang and Klir [15]). *Let (X, Σ, μ) be a fuzzy measure space, $A, B \in \sum$ and $f, g \in \mathfrak{F}^\sigma(X)$. We have*

1. $⨍_A f d\mu \le \mu(A)$;
2. $⨍_A k d\mu = k \wedge \mu(A)$, *for any constant* $k \in [0, \infty)$;
3. $⨍_A f d\mu < \alpha \Leftrightarrow$ *there exists* $\gamma < \alpha$ *such that* $(A \cap \{f \ge \gamma\}) < \alpha$;
4. $⨍_A f d\mu > \alpha \Leftrightarrow$ *there exists* $\gamma > \alpha$ *such that* $(A \cap \{f \ge \gamma\}) > \alpha$.
5. *Let* $⨍_X f d\mu = k$. *Then* $⨍_X \min(f, k) d\mu = ⨍_X \max(k, f) d\mu = ⨍_X f d\mu = k$.

Remark 2.4. Consider the survival function F associated to f on A, that is to say,

$$F(\alpha) = \mu(A \cap \{f \ge \alpha\}).$$

Then

$$F(\alpha) = \alpha \Rightarrow ⨍_A f d\mu = \alpha.$$

Thus, from a numerical (or computational) point of view, the Sugeno integral can be calculated by solving the equation $F(\alpha) = \alpha$ (if the solution exists).

Lemma 2.5 ([4]). *Let (X, Σ, μ) be a fuzzy measure space, let $A \in \Sigma$ and let $f : X \to \mathbb{R}$ be a measurable function such that $⨍_A f d\mu \le 1$. Then, for any $s \ge 1$, we have*

$$⨍_A f^s d\mu \ge \left(⨍_A f d\mu\right)^s.$$

Lemma 2.6. *Let* (X, Σ, μ) *be a fuzzy measure space, let* $A \in \Sigma$ *and let* $f : X \to \mathbb{R}$ *be a measurable function such that* $⨍_A f d\mu \leq 1$. *Then*

$$⨍_X f d\mu = ⨍_X \min(1, f) d\mu.$$

Proof. The proof follows directly from Proposition 2.3.(5.). □

3. Main results

In this section, we prove Thunsdorff's and Frank–Pick's inequalities for Sugeno integral.

3.1. Thunsdorff's inequality for Sugeno integral

Firstly, by an example, we show that (1.1) is not valid for Sugeno integral.

Example 3.1. Let $f(x) = \sqrt{x}$ and let μ be the Lebesque measure.

(i) Suppose that f is defined from $[0, 1]$ to $[0, 1]$, $r = \frac{1}{3}$ and $s = \frac{1}{2}$. Simple calculations show that

$$\begin{aligned}\left(\frac{1+r}{b-a}⨍_0^1 f^r(x)d\mu\right)^{1/r} &= \left(\left(1+\frac{1}{3}\right)⨍_0^1 \left(\sqrt{x}\right)^{1/3} d\mu\right)^3 \\ &= 1.1166,\end{aligned}$$

and

$$\begin{aligned}\left(\frac{1+s}{b-a}⨍_0^1 f^s(x)d\mu\right)^{1/s} &= \left(\left(1+\frac{1}{2}\right)⨍_0^1 \left(\sqrt{x}\right)^{1/2} d\mu\right)^2 \\ &= 1.1810.\end{aligned}$$

Therefore,

$$1.1166 = \left(\frac{1+r}{b-a}⨍_0^1 f^r(x)d\mu\right)^{1/r} \ngeq \left(\frac{1+s}{b-a}⨍_0^1 f^s(x)d\mu\right)^{1/s} = 1.1810.$$

(ii) If f defined from $[1, 3]$ to $[1, 3]$, $r = 1$ and $s = 2$. We have

$$\begin{aligned}\left(\frac{1+r}{b-a}⨍_a^b f^r(x)d\mu\right)^1 &= \frac{2}{2}⨍_1^3 \left(\sqrt{x}\right)^1 d\mu \\ &= 1.3028.\end{aligned}$$

As the same way

$$\begin{aligned}\left(\frac{1+s}{b-a}⨍_a^b f^s(x)d\mu\right)^{1/s} &= \left(\frac{3}{2}⨍_1^3 \left(\sqrt{x}\right)^2 d\mu\right)^{1/2} \\ &= 1.5.\end{aligned}$$

It follows that

$$1.3028 \ngeq 1.5.$$

In other words, (1.1) is not valid for Sugeno integral.

Remark 3.2. Thunsdorff's inequality (1.1) can be rewritten in the following form:

$$\left((1+s).\int_{[a,b]} f^s d\lambda_{[a,b]}\right)^{\frac{1}{s}} \leq \left((1+r).\int_{[a,b]} f^r d\lambda_{[a,b]}\right)^{\frac{1}{r}},$$

where $\lambda_{[a,b]}$ is the normed Lebesgue measure on Borel subsets of $[a, b[$, i.e.,

$$\lambda_{[a,b]}(E) = \frac{\lambda(E)}{\lambda([a,b])} = \frac{\lambda(E)}{(b-a)}.$$

Also, defining a real function $\phi_{[a,b]}, f :]0, \infty[\to [0, \infty[$ by

$$\phi_{[a,b]}, f(r) = \left((1+r).\int_{[a,b]} f^r d\lambda_{[a,b]}\right)^{\frac{1}{r}},$$

the Thunsdorff inequality is equivalent to the decreasingness of $\phi_{[a,b]}, f$.

In the sequel, we present and prove a fuzzy version of (1.1) if $f : [0,1] \to [0,\infty]$ and $r, s \in (0,\infty)$.

Theorem 3.3. *Let (X, Σ, μ) be a fuzzy measure space with normed fuzzy measure μ, $f \in \mathfrak{F}^{\sigma}(X)$. Then for any positive real constants r, s with $r < s$, it holds*

$$\left(1 + \left(\frac{1}{s}\right) \cdot \fint_X f^s d\mu\right)^{\frac{1}{s}} \le \left(1 + \left(\frac{1}{r}\right) . \fint_X f^r d\mu\right)^{\frac{1}{r}} \tag{3.1}$$

Proof. Based on Proposition 2.3.(1.), $\fint_X f d\mu \le \mu(X) = 1$. Then, due to Lemma 2.6.,

$$\fint_X f d\mu = \fint_X \min(1, f) d\mu.$$

Similarly one can show

$$\fint_X f^s d\mu = \fint_X (\min(1, f^s) d\mu) = \fint_X (\min(1, f))^s d\mu$$

and

$$\fint_X f^r d\mu = \fint_X (\min(1, f^r) d\mu) = \fint_X (\min(1, f))^r d\mu.$$

As far as $r < s$, it is evident that $(\min(1, f))^s \le (\min(1, f))^r$, what ensures

$$\fint_X f^s d\mu = \fint_X (\min(1, f))^s d\mu \le \fint_X (\min(1, f))^r d\mu = \fint_X f^r d\mu.$$

More, $0 < r < s$ implies $0 < \frac{1}{s} < \frac{1}{r}$, and, consequently,

$$\left(\frac{1}{s}\right) . \fint_X f^s d\mu \le \left(\frac{1}{r}\right) . \fint_X f^r d\mu \tag{3.2}$$

Adding 1 to both parts of the last inequality one obtain

$$1 \le 1 + \left(\frac{1}{s}\right) . \fint_X f^s d\mu \le 1 + \left(\frac{1}{r}\right) . \fint_X f^r d\mu$$

and therefore

$$\left(1 + \left(\frac{1}{s}\right) . \fint_X f^s d\mu\right)^{\frac{1}{s}} \le \left(1 + \left(\frac{1}{r}\right) . \fint_X f^r d\mu\right)^{\frac{1}{r}},$$

proving the validity of (3.1). □

Remark 3.4.

(i) As a by-product, a new integral inequality (3.2) for Sugeno integral was obtained.

(ii) Define two real functions τ_X, f, μ and $\eta_X, f, \mu :]0, \infty[\to [0, \infty[$ by

$$\tau_X, f, \mu(r) = \left(1 + \left(\frac{1}{r}\right) . \fint_X f^r d\mu\right)^{\frac{1}{r}}$$

and

$$\eta_X, f, \mu(r) = \left(\frac{1}{r}\right) . \fint_X f^r d\mu.$$

Then our version of Thunsdorff's inequality (3.1) is equivalent to the decreasingness in r of the function τ_X, f, μ. Similarly, integral inequality (3.2) is equivalent to the decreasingness of the function η_X, f, μ (for all $f \in \mathfrak{F}^{\sigma}(X)$) and any normed fuzzy measure μ on Σ.

Example 3.5.

(i) Let $f : [0,1] \to [0,1]$ be defined as $f(x) = \frac{1}{x+1}$, $r = \frac{1}{3}$ and $s = \frac{1}{2}$. With simple calculations, we have

$$\begin{aligned}\left(1 + \left(\frac{1}{s}\right) \fint_0^1 f^s(x) d\mu\right)^{1/s} &= \left(1 + \left(\frac{1}{\frac{1}{2}}\right) \fint_0^1 \left(\frac{1}{x+1}\right)^{1/2} d\mu\right)^2 \\ &= 6.3001,\end{aligned}$$

and

$$\begin{aligned}\left(1 + \left(\frac{1}{r}\right) \fint_0^1 f^r(x) d\mu\right)^{1/r} &= \left(1 + \left(\frac{1}{\frac{1}{3}}\right) \fint_0^1 \left(\frac{1}{x+1}\right)^{1/3} d\mu\right)^3 \\ &= 41.318.\end{aligned}$$

Therefore, (3.1) (also (3.2)) is valid.

(ii) Let $f(x) = x^2$, $r = 1$ and $s = 2$. Then, we get

$$\left(1 + \left(\frac{1}{s}\right) \fint_0^1 f^s(x) d\mu\right)^{1/s} = \left(1 + \left(\frac{1}{2}\right) \fint_0^1 (x^2)^2 d\mu\right)^{1/2} = 1.0666,$$

and

$$\left(1 + \left(\frac{1}{r}\right) \fint_0^1 f^r(x) d\mu\right)^{1/r} = 1 + \fint_0^1 x^2 d\mu = 1.382.$$

The proof of Theorem 3.8 can be applied also in the case when the fuzzy measure μ is not normed, only then we have some constraint on the original Sugeno integral.

Corollary 3.6. *Let (X, Σ, μ) be a fuzzy measure space and $f \in \mathfrak{F}^\sigma(X)$ be such that $\fint_X f d\mu \leq 1$. Then for any positive real constants $r, s, r < s$, it holds*

$$\left(1 + \left(\frac{1}{s}\right) . \fint_X f^s d\mu\right)^{\frac{1}{s}} \leq \left(1 + \left(\frac{1}{r}\right) . \fint_X f^r d\mu\right)^{\frac{1}{r}}$$

and

$$\left(\frac{1}{s}\right) . \fint_X f^s d\mu \leq \left(\frac{1}{r}\right) . \fint_X f^r d\mu.$$

Proof. The proof follows from Proposition 2.3.(5.), using similar argumentation as in the proof of Theorem 3.8. □

Also the next result can be shown, considering Proposition 2.3.(5.) and similar reasoning as in Theorem 3.8.

Corollary 3.7. *Let (X, Σ, μ) be a fuzzy measure space and $f \in \mathfrak{F}^\sigma(X)$ be such that $\fint_X f d\mu \geq 1$. Then*

$$r . \fint_X f^r d\mu \leq s . \fint_X f^s d\mu. \tag{3.3}$$

Note that if $\fint_X f d\mu = 1$, we can consider both Corollaries 3.6 and 3.7, i.e., both inequalities (3.2) and (3.3) should be valid.

This is obviously true, as then $\fint_X f^r d\mu = 1$ for any $r \in (0, \infty)$, and (3.2) turns into $\frac{1}{s} \leq \frac{1}{r}$, while (3.3) turns into $r \leq s$, which are equivalent inequalities.

3.2. Frank–Pick's inequality for Sugeno integral

In the following, we state and prove a fuzzy version of Frank–Pick's inequality if $f : [0, 1] \to [0, \infty]$ and $r, s \in (0, \infty)$.

Theorem 3.8. *Let (X, Σ, μ) be a fuzzy measure space with normed fuzzy measure μ and $f \in \mathfrak{F}^\sigma(X)$. Then*

$$\left(1 + \left(\frac{1}{2}\right) \cdot \fint_X f^2 d\mu\right)^{\frac{1}{2}} \leq 1 + \fint_X f d\mu \tag{3.4}$$

holds.

Proof. Based on Proposition 2.3.(1.), $\fint_X f d\mu \leq \mu(X) = 1$. Then, due to Lemma 2.6.,

$$\fint_X f d\mu = \fint_X \min(1, f) d\mu.$$

Similarly one can show

$$\fint_X f^2 d\mu = \fint_X \left(\min\left(1, f^2\right) d\mu\right) = \fint_X (\min(1, f))^2 d\mu$$

and

$$\fint_X f d\mu = \fint_X (\min(1, f) d\mu) = \fint_X (\min(1, f)) d\mu.$$

It is evident that $(\min(1, f))^2 \leq (\min(1, f))$, what ensures

$$\fint_X f^2 d\mu = \fint_X (\min(1, f))^2 d\mu \leq \fint_X (\min(1, f)) d\mu = \fint_X f d\mu.$$

Consequently,

$$\left(\frac{1}{2}\right) . \fint_X f^2 d\mu \leq \fint_X f d\mu \tag{3.5}$$

Adding 1 to both parts of the last inequality one obtain

$$1 \leq 1 + \left(\frac{1}{2}\right).\int_X f^2 d\mu \leq 1 + \int_X f d\mu$$

and therefore

$$\left(1 + \left(\frac{1}{2}\right).\int_X f^2 d\mu\right)^{\frac{1}{2}} \leq \left(1 + \int_X f d\mu\right),$$

proving the validity of (3.4). □

Now, by an example, we illustrate validity of theorem.

Example 3.9. Let $f : [0,1] \to [0,1]$ be defined as $f(x) = \frac{x+1}{2}$. A simple calculation shows that

$$\begin{aligned}\left(1 + \left(\frac{1}{2}\right) \cdot \int_X f^2 d\mu\right)^{\frac{1}{2}} &= \left(1 + \left(\frac{1}{2}\right) \cdot \int_X \left(\frac{x+1}{2}\right)^2 d\mu\right)^{\frac{1}{2}} \\ &= 1.1267,\end{aligned}$$

and similarly,

$$\begin{aligned}1 + \int_X f d\mu &= 1 + \int_X \frac{x+1}{2} d\mu \\ &= 1.\overline{6}.\end{aligned}$$

Finally the relations (3.4) and (3.5) are valid.

4. Conclusion

In this paper, we prove the Thunsdorff's and Frank–Pick's inequalities for Sugeno integral. By considering the initial conditions for the Thunsdorff's inequality, we proved this inequality. Indeed, we showed that:

If $f : [0,1] \to [0,\infty]$, $r, s \in (0,\infty)$ and $r < s$ then:

$$\left(1 + \left(\frac{1}{s}\right) \cdot \int_X f^s d\mu\right)^{\frac{1}{s}} \leq \left(1 + \left(\frac{1}{r}\right).\int_X f^r d\mu\right)^{\frac{1}{r}},$$

holds, and if f is defined as $f : [0,1] \to [0,1]$, then

$$\left(1 + \left(\frac{1}{2}\right) \cdot \int_X f^2 d\mu\right)^{\frac{1}{2}} \leq 1 + \int_X f d\mu,$$

holds. Also, by examples, we show the validity of theorems.

In the future works, we aim to discuss these inequalities for pseudo and Choquet integrals.

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