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A constructive framework to define fusion functions with floating domains in arbitrary closed real intervals

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A B S T R A C T

Fusion functions and their most important subclass, aggregation functions, have been successfully applied in fuzzy modeling. However, there are practical problems, such as classification via Convolutional Neural Networks (CNNs), where the data to be aggregated are not modeling membership degrees in the unit interval. In this scenario, systems could benefit from the application of operators defined in domains different from $[0,1]$, although, presenting similar behavior of some aggregation functions whose subclasses are currently defined only in the fuzzy context (e.g., overlap functions and t-norms). So, the main objective of this paper is to present a general framework to characterize classes of fusion functions with floating domains, called (a,b)-fusion functions, defined on any closed real interval $[a,b]$, based on classes of core fusion functions defined on $[0,1]$. The fundamental aspect of this framework is that the properties of a core fusion function are preserved in the context of the analogous (a,b)-fusion function. Construction methods are presented, and some properties are studied. We also introduce a framework to define fusion functions in which the inputs come from an interval $[a,b]$ but the output is mapped on a possibly different interval $[c,d]$. Finally, we present an illustrative example in image classification via CNNs.

1. Introduction

Fusion functions are operators defined to combine/fuse several numerical values from the unit interval $[0,1]$ into a single representative one, also from this same interval [37]. The most known and studied class of fusion functions is that of aggregation functions [18], which are increasing fusion functions with some boundary conditions. Aggregation functions, in fact,

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can be defined on any interval $[a, b]$, with $a, b \in \mathbb{R}$ and $a < b$, such as the ordered weighted averaging (OWA) [48] operator and the Choquet integral [7]. However, most of its subclasses (e.g., that of overlap functions [6], t-norms [27], t-conorms [27] and uninorms [49]) were defined specifically on $[0, 1]$, as they are mostly used to model fuzzy logic operations over membership degrees or truth-values.

For that reason, aggregation functions and their subclasses have been successfully employed in a plethora of theoretical and applied fields that involves some sort of fuzzy modeling. For instance, overlap functions and their generalizations (such as general overlap functions [9]) show good results when applied as a fuzzy conjunction operator in problems where associativity of the applied aggregation function is not required, such as in image processing [26], fuzzy rule-based classification [35], decision making [13] and forest fire detection [16].

Some problems, though, may have imperfect information [50], meaning that there may be uncertainty in the process of assigning the membership degrees or defining the membership functions to be applied in the fuzzy modeling [36]. Several works tackled this challenge in different ways accordingly to their perspective on uncertainty [4], by using, for example, interval-valued fuzzy sets [19] or intuitionistic fuzzy sets. Naturally, aggregation functions (and many of their subclasses) were extended to be applied in each one of those contexts (e.g., interval-valued aggregation functions [1] and intuitionistic aggregations [47]). Such generalizations can also be studied through the lens of lattice theory[1]. Recently, it was observed in the literature the development of many classes of aggregation functions on lattices, such as t-norms and t-conorms [14], uninorms [8], overlap and grouping functions [40]. Although some of those defined functions could operate with inputs that are not from the unit interval, there has not been an interest in applying such generalizations of aggregation functions in applications that are not fuzzy in nature.

We point out that the necessity of defining aggregation functions in intervals that are not the unit interval may be observed in the literature, even in the fuzzy context. For example, the ordinal sums of t-norms (t-conorms) [14] and overlap (grouping) functions [46] acting on $[0, 1]$ are defined on the basis of t-norms (t-conorms) and overlap (grouping) functions acting on a family of non-empty, pairwise disjoint open subintervals (x, y), which, although included in $[0, 1]$, are not equal to $[0, 1]$.

Still, there are practical problems where the data to be aggregated are not modeling membership degrees, truth values or some extension of them considering uncertainty modeling, which could benefit from the application of functions with similar behaviour of aggregation functions that are currently defined only to operate in the fuzzy context. That is the case, for example, with the pooling process in Convolutional Neural Networks (CNNs) [30], which are widely applied in image processing [39], and with recurrent neural networks [20], such as Long Short-Term Memory [24], which are used in several machine learning problems with sequential information [21].

Then, the main objective of this paper is to present a framework to characterize extended classes of fusion functions on a floating domain $[a, b]^n$, which we call (a, b)-fusion functions, based on core classes of fusion functions defined on $[0, 1]^n$. The fundamental aspect of this framework is that the properties of the core fusion function, defined in the context of the unit interval, are preserved in the context of an arbitrary interval $[a, b]$ when defining an analogous (a, b)-fusion function. We point out that this property preservation is not trivial, since there are a multitude of ways of characterizing properties that are equivalent in the context of the unit interval, but that can lead to different concepts when defined in another interval $[a, b]$.

Since the motivation comes from an application standpoint, we present some construction methods for these newly defined (a, b)-aggregation functions, based on some core known aggregation functions (e.g., n-dimensional overlap functions [22], t-norms [27], t-conorms [27] and uninorms [49]), guaranteeing that the constructed function behaves in $[a, b]$ in a similar manner as the core function does in $[0, 1]$. Furthermore, the presented construction methods are based on the choice of a core aggregation function and an increasing bijective function, both able to be defined with parameters that can be manipulated/adapted/learned, accordingly to the application at hand, without sacrificing the main properties of the desired constructed function. Then, we proceed to study some interesting properties of aggregation functions, namely, idempotency, a kind of generalized migrativity (introduced here) and abstract homogeneity [43], and how such properties are preserved when our construction methods for (a, b)-aggregation functions are applied.

Following that, we present the main concepts to develop a similar framework to define fusion functions whose the inputs come from an interval $[a, b]$ but the output is mapped on a possibly different interval $[c, d]$. We call them (a, b, c, d)-fusion functions. Then, based on this framework, subclasses of (a, b, c, d)-fusion functions are defined and construction methods for them are presented. We show that, under some constraints, when a constructed (a, b, c, d)-aggregation function is based on an (a, b)-aggregation function, which, in turn, is based on a core aggregation function defined on $[0, 1]^n$, then, it is equivalent to the (a, b, c, d)-aggregation function obtained directly from the same core aggregation function defined in $[0, 1]$.

Finally, to highlight the applicability of the developed theoretical concepts, we present an illustrative example in which (a, b)-fusion functions (in particular, (a, b)-aggregation functions) are applied as the pooling operator of a CNN, in a practical image classification problem.

The paper is organized as follows: in Section 2, important preliminary concepts are presented. Then, in Section 3, we introduce and discuss the notion of property shifting, which is how we denominate the action of properly transpose a given property from one domain to another, and develop a general framework for defining classes of (a, b)-fusion functions based

[1] For more on lattice theory, see [17].

on classes of fusion functions, showing examples. Section 4 is dedicated to the introduction of construction methods for different classes of (a,b)-aggregation functions. The study of some properties of aggregation functions and their counterparts in the context of (a,b)-aggregation functions, with particular interest in the study of (a,b)-aggregation functions obtained by our construction methods, is presented in Section 5. Following that, in Section 6, the main concepts of (a,b,c,d)-fusion concepts are developed, focusing on different ways to construct them. An illustrative example in which (a,b)-fusion functions are applied in a CNN to deal with a practical image classification problem is shown in Section 7. Our concluding remarks are presented in Section 8, where we review the main contributions of the paper and propose some possible future lines of work.

2. Preliminary concepts

In this section, we recall some preliminary concepts that are relevant for the development of the paper.

For the remainder of this work, consider $a, b \in \mathbb{R}$, such that $a < b$.

Consider a function $F : [a,b]^2 \rightarrow [a,b]$. Then, F is said to be *symmetric* if, for all $x, y \in [a,b]$, it holds that $F(x,y) = F(y,x)$, meaning that the value of the function does not depend on the order of the arguments. Also, F is said to be *associative* if, for all $x,y,z \in [a,b]$, it holds that $F(x, F(y,z)) = F(F(x,y),z)$. Moreover, F is said to have a *neutral element* e if, for all $x \in [a,b]$, it holds that $F(x,e) = F(e,x) = x$. Symmetry, associativity and neutral element properties can be generalized for n-ary functions as shown in [18] Section 2.2.3, 2.3.1 and 2.5.1.

Let us denote $\vec{x} = (x_1, \ldots, x_n) \in [a,b]^n$, where $n > 1$.

A function $F : [a,b]^n \rightarrow [a,b]$ is said to be *increasing* if, for any $\vec{x_1}, \vec{x_2} \in [a,b]^n$ such that $\vec{x_1} \leqslant \vec{x_2}$, it holds that $F(\vec{x_1}) \leqslant F(\vec{x_2})$ [18].

Definition 2.1. [27] A function $N : [0,1] \rightarrow [0,1]$ is a fuzzy negation if the following conditions hold:

(N1) $N(0) = 1$ and $N(1) = 0$;
(N2) If $x \leqslant y$ then $N(y) \leqslant N(x)$, for all $x, y \in [0,1]$.

If N also satisfies the involutive property,

(N3) $N(N(x)) = x$, for all $x \in [0,1]$,

then it is said to be a strong fuzzy negation.

Example 2.1. The Zadeh negation given, for all $x \in [0,1]$, by

$$N_Z(x) = 1 - x,$$

is a strong fuzzy negation.

The concept of fusion function [37] was originally defined in the context of the unit interval as an arbitrary function $F : [0,1]^n \rightarrow [0,1]$.

Definition 2.2. [27] Given a strong fuzzy negation $N : [0,1] \rightarrow [0,1]$ and a fusion function $F : [0,1]^n \rightarrow [0,1]$, then the fusion function $F^N : [0,1]^n \rightarrow [0,1]$ defined, for all $\vec{x} \in [0,1]^n$, by

$$F^N(\vec{x}) = N(F(N(x_1), \ldots, N(x_n))), \tag{1}$$

is the N-dual of F.

When it is clear by the context, the N_Z-dual function (dual with respect to the Zadeh negation) of F is just called dual of F, and is denoted by F^d. Observe that $\left(F^N\right)^N = F$, since N is a strong negation.

Here we recall the representation, introduced by Asmus et al. [2], of a class of fusion functions through its set of sufficient and necessary properties, which we denominate as *constitutive properties*. Let $\mathscr{F}$ be a subclass of fusion functions $F : [0,1]^n \rightarrow [0,1]$ and $P_{\mathscr{F}}$ be a set of constitutive properties of the functions from $\mathscr{F}$, such that it includes: (i) boundary conditions for any $F \in \mathscr{F}$, (ii) some kind of monotonicity and (iii) possibly other constraints not related to neither (i) nor (ii). Such subclass of functions is given by:

$$\mathscr{F} = \left\{F : [0,1]^n \rightarrow [0,1] \mid F \text{ satisfies all the properties in } P_{\mathscr{F}}\right\}. \tag{2}$$

We present the same style of representation for the definition of aggregation functions, which is the most important subclass of fusion functions, as follows:

Definition 2.3. [18] An aggregation function is any function $A \in \mathscr{A}$, where:

$$\mathscr{A} = \{A : [0,1]^n \rightarrow [0,1] |\ A \text{ satisfies all the properties in } P_{\mathscr{A}}\}$$

with

$$P_{\mathscr{A}} = \{(\mathbf{A1}), (\mathbf{A2})\},$$

and

(A1)A is increasing;
(A2)$A(0,\ldots,0) = 0$ and $A(1,\ldots,1) = 1$.

Example 2.2.

i)The function $AM : [0,1]^n \rightarrow [0,1]$ (arithmetic mean), given, for all $\vec{x} \in [0,1]^n$, by

$$AM(\vec{x}) = \frac{\sum_{i=1}^{n} x_i}{n}, \tag{3}$$

is an aggregation function.
ii)The function $AW : [0,1]^n \rightarrow [0,1]$ (weighted arithmetic mean), given, for all $\vec{x}, \vec{w} \in [0,1]^n$, by

$$AW(\vec{x}) = \sum_{i=1}^{n} x_i \cdot w_i, \tag{4}$$

such that $\sum_{i=1}^{n} w_i = 1$, is an aggregation function.

There are many subclasses of aggregation functions defined in the literature. Here we highlight some of them that are going to be of importance on this work.

Definition 2.4. [22] An n-dimensional overlap function is any fusion function $O \in \mathscr{O}$, such that:

$$\mathscr{O} = \{O : [0,1]^n \rightarrow [0,1] |\ O \text{ satisfies all the properties in } P_{\mathscr{O}}\}$$

where

$$P_{\mathscr{O}} = \{(\mathbf{O1}), (\mathbf{O2}), (\mathbf{O3}), (\mathbf{O4}), (\mathbf{O5})\},$$

and

(O1) O is symmetric;
(O2) $O(\vec{x}) = 0 \iff \prod_{i=1}^{n} x_i = 0$;
(O3) $O(\vec{x}) = 1 \iff \prod_{i=1}^{n} x_i = 1$;
(O4) O is increasing;
(O5) O is continuous.

A 2-dimensional overlap function is just called overlap function [6].

Remark 2.1. Taking into consideration Definitions 2.3 and 2.4, one can observe that conditions **(A1)** and **(O4)** are the same one (increasingness). However, we decide to label them differently so that each condition is associated with one respective class of functions, to aid the readability of the mathematical proofs in this paper.

Example 2.3.

i)The function $O_P : [0,1]^n \rightarrow [0,1]$ (product overlap), given, for all $\vec{x} \in [0,1]^n$, by

$$GM(\vec{x}) = \prod_{i=1}^{n} x_i, \tag{5}$$

is an n-dimensional overlap function.
ii)The function $GM : [0,1]^n \rightarrow [0,1]$ (geometric mean), given, for all $\vec{x} \in [0,1]^n$, by

$$GM(\vec{x}) = \sqrt[n]{\prod_{i=1}^{n} x_i}, \quad (6)$$

is an n-dimensional overlap function.

Theorem 2.1. [22] *Consider a continuous aggregation function* $A : [0,1]^m \to [0,1]$, *such that*

(PA) $A(\vec{x}) = 0$ *if and only if* $x_i = 0$, *for some* $i \in \{1,\dots,m\}$;
(PB) $A(\vec{x}) = 1$ *if and only if* $x_i = 1$, *for all* $i \in \{1,\dots,m\}$;

and a tuple of n-dimensional overlap functions $\vec{O} = (O_1,\dots,O_m)$. *Then, the mapping* $A_{\vec{O}} : [0,1]^n \to [0,1]$, *defined, for all* $\vec{x} \in [0,1]^n$, *by*

$$A_{\vec{O}}(\vec{x}) = A(O_1(\vec{x}),\dots,O_m(\vec{x})), \quad (7)$$

is an n-dimensional overlap function.

Corollary 2.1. [22] *Consider an m-dimensional overlap function* $OC : [0,1]^m \to [0,1]$ *and a tuple of n-dimensional overlap functions* $\vec{O} = (O_1,\dots,O_m)$. *Then, the mapping* $OC_{\vec{O}} : [0,1]^n \to [0,1]$, *defined for all* $\vec{x} \in [0,1]^n$, *by*

$$OC_{\vec{O}}(\vec{x}) = OC(O_1(\vec{x}),\dots,O_m(\vec{x})), \quad (8)$$

is an n-dimensional overlap function.

By Corollary 2.1, one can observe that the class of n-dimensional overlap functions is self closed with respect to the generalized composition.

Definition 2.5. [27] A t-norm is any bivariate fusion function $T \in \mathscr{T}$, such that:

$$\mathscr{T} = \left\{T : [0,1]^2 \to [0,1] \mid T \text{ satisfies all the properties in } P_{\mathscr{T}}\right\}$$

where

$$P_{\mathscr{T}} = \{(\mathbf{T1}), (\mathbf{T2}), (\mathbf{T3}), (\mathbf{T4})\},$$

and

(T1) T is symmetric;
(T2) T is associative;
(T3) T has 1 as its neutral element;
(T4) T is increasing.

Example 2.4.

i)The function $T_L : [0,1]^2 \to [0,1]$ (Łukasiewicz t-norm), given, for all $x, y \in [0,1]$, by

$$T_L(x,y) = \max\{x + y - 1, 0\}, \quad (9)$$

is a t-norm.

ii)The function $T_H : [0,1]^2 \to [0,1]$ (Hamacher product), given, for all $x, y \in [0,1]$, by

$$T_H(x,y) = \begin{cases} 0 & \text{if } x = y = 0, \\ \frac{xy}{x+y-xy} & \text{otherwise}, \end{cases} \quad (10)$$

is a t-norm.

Definition 2.6. [27] A t-conorm is any bivariate fusion function $S \in \mathscr{S}$, such that:

$$\mathscr{S} = \left\{S : [0,1]^2 \to [0,1] | \, S \text{ satisfies all the properties in } P_{\mathscr{S}}\right\}$$

where

$$P_{\mathscr{S}} = \{(\mathbf{S1}), (\mathbf{S2}), (\mathbf{S3}), (\mathbf{S4})\},$$

and

(S1) S is symmetric;
(S2) S is associative;
(S3) S has 0 as its neutral element;
(S4) S is increasing.

Example 2.5. The function $S_p : [0,1]^2 \to [0,1]$ (probabilistic sum), given, for all $x, y \in [0,1]$, by

$$S_p(x,y) = x + y - xy, \tag{11}$$

is a t-conorm.

Definition 2.7. [49] An uninorm is any bivariate fusion function $U \in \mathscr{U}$, such that:

$$\mathscr{U} = \left\{U : [0,1]^2 \to [0,1] | \, U \text{ satisfies all the properties in } P_{\mathscr{U}}\right\}$$

where

$$P_{\mathscr{U}} = \{(\mathbf{U1}), (\mathbf{U2}), (\mathbf{U3}), (\mathbf{U4})\},$$

and

(U1) U is symmetric;
(U2) U is associative;
(U3) U has a neutral element;
(U4) U is increasing.

Example 2.6.

i)Consider $e \in [0,1]$. Then, the function $U_C : [0,1]^2 \to [0,1]$, given, for all $x, y \in [0,1]$, by

$$U_C(x,y) = \begin{cases} \max\{x,y\} & \text{if } (x,y) \in [e,1]^2, \\ \min\{x,y\} & \text{otherwise,} \end{cases} \tag{12}$$

is an uninorm with e as its neutral element;

ii)The function $U_P : [0,1]^2 \to [0,1]$, given, for all $x, y \in [0,1]$, by

$$U_P(x,y) = \begin{cases} 0 & \text{if } (x,y) \in \{(1,0),(0,1)\}, \\ \frac{xy}{(1-x)(1-y)+xy} & \text{otherwise,} \end{cases} \tag{13}$$

is an uninorm with $\frac{1}{2}$ as its neutral element.

3. $\mathcal{F}$-shifted $(\boldsymbol{a}, \boldsymbol{b})$-fusion functions

The main goal of this section is to introduce a general framework to define new classes of functions with similar behaviour as some known subclasses of fusion/aggregation functions, but that are not limited to the unit interval. The idea is to define those new classes of functions (acting on an interval $[a,b]$) through a sets of properties that mirrors the ones from the known functions (acting on $[0,1]$).

Definition 3.1. An (a,b)-fusion function is an arbitrary function $F^{a,b} : [a,b]^n \to [a,b]$.

It is clear that every fusion function is an (a,b)-fusion function for $a = 0$ and $b = 1$. Then, henceforward, every $(0,1)$-fusion function is called here just as fusion function.

We denote by $\mathscr{F}^{a,b}$ a subclass of (a,b)-fusion functions determined by a set of constitutive properties $P_{\mathscr{F}^{a,b}}$.

The action of *shifting* a property **(P1)** of a function $F_1 : [a_1, b_1]^n \to [a_1, b_1]$ from $[a_1, b_1]$ to $[a_2, b_2]$ is to "rewrite" **(P1)** so that it conveys the same concept in the context of $[a_2, b_2]$, resulting in a property **(P2)** of a function $F_2 : [a_2, b_2]^n \to [a_2, b_2]$. In other words, **(P2)** is the counterpart in $[a_2, b_2]$ for the property **(P1)** (see Example 3.1). Some properties can be shifted without any rewriting (e.g., monotonicity, continuity, associativity and idempotency). However, boundary conditions, in general, have to be rewritten when shifted.

Example 3.1. Suppose that we intend to define a property **(A2′)** that conveys the boundary conditions of a function $F : [-10, 10]^n \to [-10, 10]$ by shifting the property **(A2)** of aggregation functions (Definition 2.3). It is clear that **(A2)** is written taking into consideration the boundaries of $[0, 1]$, since aggregation functions are defined on the unit interval. So, a natural way to shift **(A2)** from $[0, 1]$ to $[-10, 10]$ is to rewrite it by changing the lower and upper boundaries accordingly, resulting in **(A2′)** as follows:

(A2′) $A(-10, \ldots, -10) = -10$ and $A(10, \ldots, 10) = 10$.

Remark 3.1. A given property in the context of the interval $[0, 1]$ can be defined for a general interval $[a, b]$ in different ways, so that it coincides with the original definition when $a = 0$ and $b = 1$. This is the case of the 1-Lipschitz property [18]. A bivariate fusion function $F : [0, 1]^2 \to [0, 1]$ has this property if, for all $x_1, x_2, y_1, y_2 \in [0, 1]$, one has that:

$$|F(x_1, y_1) - F(x_2, y_2)| \leqslant |x_1 - x_2| + |y_1 - y_2|. \tag{14}$$

Observe that this property, expressed by Inequality (14), can be defined without modifications for (a, b)-fusion functions. Now, consider the following expression for a property of a bivariate (a, b)-fusion function $F^{a,b} : [a, b]^2 \to [a, b]$:

$$|F^{a,b}(x_1, y_1) - F^{a,b}(x_2, y_2)| \leqslant \frac{|x_1 - x_2| + |y_1 - y_2|}{(b - a)^k}, k \in [0, +\infty), \tag{15}$$

for all $x_1, x_2, y_1, y_2 \in [a, b]$. The property expressed by Eq. (15) coincides with the 1-Lipschitz property in the particular case when $k = 0$, or when $b - a = 1$. However, it is clear that the properties expressed by Eqs. (14) and (15) are not equivalent, that is, they do not convey the same concept. That is why, when shifting the 1-Lipschitz property from $[0, 1]$ to $[a, b]$, one should express it by Eq. (14), without rewriting it, in order to avoid introducing a different concept. The above example shows that not all properties of fusion functions on $[0, 1]$ and those on $[a, b]$ can be related by an isomorphism. In general, an isomorphism between $[0, 1]$ and $[a, b]$ preserves algebraic properties, such as symmetry, associativity and idempotency, but not analytical properties.

Definition 3.2. Let $\mathscr{F}$ be the subclass of fusion functions $F : [0, 1]^n \to [0, 1]$ determined by the set of constitutive properties $P_{\mathscr{F}}$, defined in Eq. (2). Then, a set of constitutive properties $P_{\mathscr{F}^{a,b}}$ of a class of (a, b)-fusion functions $\mathscr{F}^{a,b}$ is said to be $\mathscr{F}$-shiftable if $P_{\mathscr{F}}$ coincides with the set composed of all the properties obtained by shifting each property of $P_{\mathscr{F}^{a,b}}$ from $[a, b]$ to $[0, 1]$.

Definition 3.3. Let $P_{\mathscr{F}}$ be the set of constitutive properties of a class of fusion functions $\mathscr{F}$. Then, $\mathscr{F}^{a,b}$, given by

$$\mathscr{F}^{a,b} = \left\{ F^{a,b} : [a, b]^n \to [a, b] | \, F^{a,b} \text{ satisfies all the properties in } P^{a,b}_{\mathscr{F}} \right\}, \tag{16}$$

is said to be $\mathscr{F}$-shifted if $P^{a,b}_{\mathscr{F}}$ is $\mathscr{F}$-shiftable.

A $\mathscr{F}$-shifted class of (a, b)-fusion functions $\mathscr{F}^{a,b}$ is a counterpart (in $[a, b]$) of a class of fusion function $\mathscr{F}$ (in $[0, 1]$).

Example 3.2.

i) Consider a subclass of $(-10, 10)$-fusion functions $\mathscr{F}\mathscr{A}^{-10,10}$, with its set of constitutive properties $P_{\mathscr{F}\mathscr{A}^{-10,10}}$ given by:

$P_{\mathscr{F}\mathscr{A}^{-10,10}} = \{ (\mathbf{A1'}), (\mathbf{A2'}) \}$,

where, for all $FA^{-10,10} \in \mathscr{F}\mathscr{A}^{-10,10}$, it holds that:

(A1')

$FA^{-10,10}$ is increasing;

(A2')

$FA^{-10,10}(-10, \ldots, -10) = -10$ and $FA^{-10,10}(10, \ldots, 10) = 10$.

Then, $P_{\mathscr{F}\mathscr{A}^{-10,10}}$ is $\mathscr{A}$-shiftable, since we obtain **(A1)** and **(A2)** (Definition 2.3), which are the defining properties of $\mathscr{A}$, by shifting **(A1')** and **(A2')** from $[-10, 10]$ to $[0, 1]$. Thus, $\mathscr{F}\mathscr{A}^{-10,10}$ is an $\mathscr{A}$-shifted class of (a, b)-fusion functions.

ii) Consider the class of n-dimensional overlap functions $\mathscr{O}$ and a subclass of (a, b)-fusion functions $\mathscr{H}^{a,b}$ with its set of constitutive properties $P_{\mathscr{H}^{a,b}}$, given by:

$$P_{\mathscr{H}^{a,b}} = \{(\mathbf{H1}), (\mathbf{H2})\},$$

where, for all $H^{a,b} \in \mathscr{H}^{a,b}$, it holds that:
(H1)

$H^{a,b}$ is symmetric;
(H2)

$H^{a,b}$ is associative.

Clearly, $P_{\mathscr{H}^{a,b}}$ is not $\mathscr{O}$-shiftable, since we cannot transpose their properties to the context of the unit interval so that they coincide with the properties from $P_{\mathscr{O}}$ (Definition 2.4). Thus, $\mathscr{H}^{a,b}$ is not an $\mathscr{O}$-shifted class of (a, b)-fusion functions. However, if we consider the class $\mathscr{H}$ of symmetric and associative fusion functions, then it is immediate that $\mathscr{H}^{a,b}$ is $\mathscr{H}$-shifted.

In [18], aggregation functions were already defined in the context of a domain $[a, b]^n$. But here, to avoid confusion, we call them aggregation functions only when $a = 0$ and $b = 1$ (Definition 2.3). Otherwise, we call them (a, b)-aggregation functions, just to standardize the notation. The definition of the class of (a, b)-aggregation functions is given as follows:

Definition 3.4. [18] An (a, b)-aggregation function is any function $A^{a,b} \in \mathscr{A}^{a,b}$, such that:

$$\mathscr{A}^{a,b} = \left\{A^{a,b} : [a, b]^n \to [a, b] |\ A^{a,b} \text{ satisfies all the properties in } P^{a,b}_{\mathscr{A}}\right\}$$

where

$$P^{a,b}_{\mathscr{A}} = \{(\mathbf{A1}*), (\mathbf{A2}*)\},$$

and

(A1*)$A^{a,b}$ is increasing;
(A2*)$A^{a,b}(a, \ldots, a) = a$ and $A^{a,b}(b, \ldots, b) = b$.

Example 3.3.

i) The arithmetic mean $AM : [a, b]^n \to [a, b]$, given by Eq. (3), is an (a, b)-aggregation function for any arbitrary $a, b \in \mathbb{R}$, such that $a < b$;
ii) The product operation is a $(0, b)$-fusion function with $b \leqslant 1$ and an (a, b)-fusion function when $a < 0, b \leqslant 1$ and $a^2 \leqslant b$ (e.g., $[-1, 1]$). It is only considered an (a, b)-aggregation function in the particular case where $a = 0$ and $b = 1$. However, in Section 4 we present a construction method in which one can obtain an (a, b)-aggregation function based on the product (or any other aggregation function, for that matter) for any arbitrary $a, b \in \mathbb{R}$, such that $a < b$.

The following results are immediate:

Proposition 3.1. *Consider the class of aggregation functions $\mathscr{A}$ and its set of constitutive properties $P_{\mathscr{A}}$ (from Definition 2.3). Then the set of properties $P^{a,b}_{\mathscr{A}}$ (from Definition 3.4) is $\mathscr{A}$-shiftable.*

Corollary 3.1. *The class $\mathscr{A}^{a,b}$ of (a, b)-aggregation functions (Definition 3.4) is $\mathscr{A}$-shifted.*

Here we study some $\mathscr{A}$-shifted subclasses of (a, b)-aggregation functions.

Analogous to Definition 3.3 of $\mathscr{F}$-shifted subclasses of (a, b)-fusion functions, one can define $\mathscr{A}$-shifted subclasses of (a, b)-aggregation functions, as follows:

Definition 3.5. Let $P_{\mathscr{A}'}$ be the set of constitutive properties of a subclass of aggregation functions $\mathscr{A}'$. Then, $\mathscr{A}'^{a,b}$, given by

$$\mathscr{A}'^{a,b} = \left\{A'^{a,b} : [a, b]^n \to [a, b] |\ A'^{a,b} \text{ satisfies all the properties in } P^{a,b}_{\mathscr{A}'}\right\}, \quad (17)$$

is said to be $\mathscr{A}'$-shifted if $P^{a,b}_{\mathscr{A}'}$ is $\mathscr{A}'$-shiftable.

Observe that any $\mathscr{A}$-shifted subclass of (a,b)-aggregation functions is also an $\mathscr{A}$-shifted subclass of (a,b)-fusion functions.

Now, let us define different $\mathscr{A}'$-shifted subclasses of (a,b)-aggregation functions $\mathscr{A}'^{a,b} \subseteq \mathscr{A}^{a,b}$, based on a subclass of aggregation functions $\mathscr{A}' \subseteq \mathscr{A}$. First, for a given subclass $\mathscr{A}'^{a,b}$, one must define its set of constitutive properties $P^{a,b}_{\mathscr{A}'}$ in a way for it to be $\mathscr{A}'$-shiftable.

Example 3.4. Suppose that we intend to define an $\mathscr{O}$-shifted subclass $\mathscr{O}^{a,b}$ of (a,b)-aggregation functions as the counterpart in $[a,b]$ for the class of n-dimensional overlap functions $\mathscr{O}$ (Definition 2.4). For that, we have to define the set of constitutive properties $P_{\mathscr{O}^{a,b}}$ in a way for it to be $\mathscr{O}$-shiftable, that is, so that $P_{\mathscr{O}^{a,b}} = P_{\mathscr{O}}$ when shifting the properties of $P_{\mathscr{O}^{a,b}}$ from $[a,b]$ to $[0,1]$.

From Definition 2.4, we see that the set $P_{\mathscr{O}}$ has three properties that can be shifted without rewriting them: **(O1)**, **(O4)** and **(O5)**. So, these three properties can be part of the set $P_{\mathscr{O}^{a,b}}$. However, properties **(O2)** and **(O3)** are the lower and upper boundary conditions, respectively, and, thus, they depend on the values of such boundaries (0 and 1). Also, they are defined by means of the product operation which, in the context of the interval $[0,1]$, has the lower boundary as its annihilator element and the upper boundary as its neutral element. This characteristic is not carried when defining such boundary conditions on a different interval $[a,b]$.

So, it is clear that we cannot simply exchange 0 for the left endpoint (a) in condition **(O2)** and 1 for the right endpoint (b) in condition **(O3)** to obtain the analogous boundary conditions for $P_{\mathscr{O}^{a,b}}$. There is more than one way to define such boundary conditions so that they are equivalent to **(O2)** and **(O3)** when $a = 0$ and $b = 1$. Here we present a viable alternative. Considering an (a,b)-fusion function $O^{a,b} : [a,b]^n \rightarrow [a,b]$, the following properties complete the set $P_{\mathscr{O}^{a,b}}$:

(OAB1)
$O^{a,b}$ is symmetric;

(OAB2)
$O^{a,b}(x_1,\ldots,x_n) = a$ if and only if $\prod_{i=1}^{n}(x_i - a) = 0$;

(OAB3)
$O^{a,b}(x_1,\ldots,x_n) = b$ if and only if $\prod_{i=1}^{n}(\frac{x_i - a}{b-a}) = 1$;

(OAB4)
$O^{a,b}$ is increasing;

(OAB5)
$O^{a,b}$ is continuous.

One can observe that **(OAB2)** and **(OAB3)** are equivalent to **(O2)** and **(O3)**, respectively, when $a = 0$ and $b = 1$, since the relevant properties of the product operation are respected in $[0,1]$. The other three properties were just relabelled to not mix the notation. Thus, the set of properties $P_{\mathscr{O}^{a,b}} = \{(\textbf{OAB1}), (\textbf{OAB2}), (\textbf{OAB3}), (\textbf{OAB4}), (\textbf{OAB5})\}$ is $\mathscr{O}$-shiftable.

Based on the set of properties $P_{\mathscr{O}^{a,b}}$ defined in Example 3.4, one can define the class of n-dimensional (a,b)-overlap functions.

Definition 3.6. The class $\mathscr{O}^{a,b}$ of n-dimensional (a,b)-overlap functions $O^{a,b}$ is given by:

$$\mathscr{O}^{a,b} = \left\{ O^{a,b} : [a,b]^n \rightarrow [a,b] |\ O^{a,b} \text{ satisfies all the properties in } P_{\mathscr{O}^{a,b}} \right\} \tag{18}$$

where $P_{\mathscr{O}^{a,b}} = \{(\textbf{OAB1}), (\textbf{OAB2}), (\textbf{OAB3}), (\textbf{OAB4}), (\textbf{OAB5})\}$.

Proposition 3.2. *Consider the class of n-dimensional (a,b)-overlap functions $\mathscr{O}^{a,b}$ (Definition 3.6). Then, $\mathscr{O}^{a,b}$ is $\mathscr{O}$-shifted.*

Proof. Immediate, since $\mathscr{O} \subseteq \mathscr{A}$ and, as shown in Example 3.4, $P_{\mathscr{O}^{a,b}}$ is $\mathscr{O}$-shiftable. □

Example 3.5.

i) The function $MIN : [a,b]^n \rightarrow [a,b]$, given, for all $\vec{x} \in [a,b]^n$, by

$$MIN(\vec{x}) = \min\{x_1, \ldots, x_n\}, \tag{19}$$

is an n-dimensional (a,b)-overlap function;

ii) The geometric mean, given by Eq. (6), is only an n-dimensional (a,b)-overlap function when $a = 0$ and $b > 0$. In Section 4, we present a construction method to obtain an n-dimensional (a,b)-overlap function $O^{a,b}$ based on a given n-dimensional overlap function O (e.g., the geometric mean), for any arbitrary $a, b \in \mathbb{R}$, such that $a < b$.

Since n-dimensional (a,b)-overlap functions are defined by shifting the properties of Definition 2.4 from $[0,1]$ to $[a,b]$, then some other properties of n-dimensional overlap functions that are not explicitly stated on their definition can also be shifted in a similar manner. The next two results exemplify that the properties expressed by Theorem 2.1 and Corollary 2.1 can be shifted from $[0,1]$ to $[a,b]$:

Theorem 3.1. *Consider a continuous (a,b)-aggregation function $A^{a,b} : [a,b]^m \to [a,b]$, such that*

(PA*) $A^{a,b}(\vec{x}) = a$ *if and only if* $x_i = a$, *for some* $i \in \{1, \ldots, m\}$;
(PB*) $A^{a,b}(\vec{x}) = b$ *if and only if* $x_i = b$, *for all* $i \in \{1, \ldots, m\}$;

and a tuple $\overrightarrow{O^{a,b}} = \left(O_1^{a,b}, \ldots, O_m^{a,b}\right)$ *of n-dimensional (a,b)-overlap functions. Then, the mapping* $A^{a,b}_{\overrightarrow{O^{a,b}}} : [a,b]^n \to [a,b]$, *defined for all* $\vec{x} \in [a,b]^n$, *by*

$$A^{a,b}_{\overrightarrow{O^{a,b}}}(\vec{x}) = A^{a,b}\left(O_1^{a,b}(\vec{x}), \ldots, O_m^{a,b}(\vec{x})\right), \tag{20}$$

is an n-dimensional (a,b)-overlap function.

Proof. It is immediate that $A^{a,b}_{\overrightarrow{O^{a,b}}}$ is well defined. Then, by **(O1)**, **(O4)** and **(O5)**, we have that $A^{a,b}_{\overrightarrow{O^{a,b}}}$ respects conditions **(OAB1)**, **(OAB4)** and **(OAB5)**. Now, let us prove that $A^{a,b}_{\overrightarrow{O^{a,b}}}$ respects the remaining conditions of Definition 3.6:

(OAB2)
Suppose that $A^{a,b}_{\overrightarrow{O^{a,b}}}(\vec{x}) = a$, for some $\vec{x} \in [a,b]^n$. Then, by Eq. (20) and **(PA*)**, we have that:

$$O_j^{a,b}\left(\vec{x}\right) = a \text{ for some } j \in \{1, \ldots, m\} \iff x_i = a \text{ for some } i \in \{1, \ldots, n\} \text{ by}(\textbf{OAB2}).$$

On the other hand, if we take $\vec{x} \in [a,b]^n$, such that $\vec{x} = (x_1, \ldots, x_i, \ldots, x_n)$ with $x_i = a$ for some $i \in \{1, \ldots, n\}$, then, by **(OAB2)**, **(PA*)** and Eq. (20), we have that $A^{a,b}_{\overrightarrow{O^{a,b}}}(\vec{x}) = a$.

(OAB3)
Suppose that $A^{a,b}_{\overrightarrow{O^{a,b}}}(\vec{x}) = b$, for all $\vec{x} \in [a,b]^n$. Then, by Eq. (20) and **(PB*)**, it follows that:

$$O_j^{a,b}\left(\vec{x}\right) = b \text{ for all } j \in \{1, \ldots, m\} \iff \vec{x} = (b, \ldots, b), \quad \text{by}(\textbf{OAB3}).$$

Conversely, if $\vec{x} = (b, \ldots, b)$, then, by **(OAB3)**, **(A2*)** and Eq. (20), we have that $A^{a,b}_{\overrightarrow{O^{a,b}}}(\vec{x}) = b$.

□

Corollary 3.2. *Consider an m-dimensional (a,b)-overlap function* $OC^{a,b} : [a,b]^m \to [a,b]$ *and a tuple* $\overrightarrow{O^{a,b}} = \left(O_1^{a,b}, \ldots, O_m^{a,b}\right)$ *of n-dimensional (a,b)-overlap functions. Then, the mapping* $OC_{\overrightarrow{O^{a,b}}} : [a,b]^n \to [a,b]$, *defined for all* $\vec{x} \in [a,b]^n$, *by*

$$OC_{\overrightarrow{O^{a,b}}}(\vec{x}) = OC^{a,b}\left(O_1^{a,b}(\vec{x}), \ldots, O_m^{a,b}(\vec{x})\right), \tag{21}$$

is an n-dimensional (a,b)-overlap function.

Proof. Immediate, since $OC^{a,b}$ is a continuous (a,b)-aggregation function that respects **(PA*)** and **(PB*)**. □

Corollary 3.3. *Consider the weighted arithmetic mean* $AW^{a,b} : [a,b]^m \to [a,b]$ *given, for all* $\vec{x} \in [a,b]^n$*, by Eq. (4), with* $\vec{w} \in [0,1]^m$*, such that* $\sum_{i=1}^m w_i = 1$*, and a tuple* $\overrightarrow{O^{a,b}} = \left(O_1^{a,b}, \ldots, O_m^{a,b}\right)$ *of n-dimensional* (a,b)*-overlap functions. Then, the mapping* $AW_{\overrightarrow{O^{a,b}}} : [a,b]^n \to [a,b]$*, defined, for all* $\vec{x} \in [a,b]^n$*, by*

$$\begin{aligned} AW_{\overrightarrow{O^{a,b}}}(\vec{x}) =& AW^{a,b}\left(O_1^{a,b}(\vec{x}), \ldots, O_m^{a,b}(\vec{x})\right) \\ =& O_1^{a,b}(\vec{x}) \cdot w_1 + \ldots + O_m^{a,b}(\vec{x}) \cdot w_m, \end{aligned} \tag{22}$$

is an n-dimensional (a,b)*-overlap function.*

Proof. Immediate, since $AW^{a,b}$ is a continuous (a,b)-aggregation function that respects **(PA*)** and **(PB*)**. □

Remark 3.2. Notice that, by Corollary 3.2, one can state that the class of (a,b)-overlap functions is self closed with respect to the generalized composition, and, by Corollary 3.3, one can observe that the convex sum of n-dimensional (a,b)-overlap functions is also an n-dimensional (a,b)-overlap function. These properties are especially useful in practical applications, since one can combine different (a,b)-overlap functions to obtain new functions with the same behaviour.

Remark 3.3. In a similar manner in which n-dimensional (a,b)-overlap functions were defined as a counterpart for n-dimensional overlap functions, one could define n-dimensional (a,b)-grouping functions as a counterpart for n-dimensional grouping functions. Since n-dimensional grouping functions are the dual notion of n-dimensional overlap functions, properties such as the one expressed in Corollary 3.2 can also be obtained in the context of n-dimensional (a,b)-grouping functions.

Other $\mathscr{A}'$-shifted classes of (a,b)-aggregation functions can be defined in a similar manner as presented in Example 3.4 and Definition 3.6. To exemplify that, in the following we define (a,b)-t-norms, (a,b)-t-conorms and (a,b)-uninorms.

Definition 3.7. Consider a bivariate (a,b)-fusion function $T^{a,b} : [a,b]^2 \to [a,b]$ and the following properties:

(TAB1)
$T^{a,b}$ is symmetric;

(TAB2)
$T^{a,b}$ is associative;

(TAB3)
$T^{a,b}$ has b as its neutral element;

(TAB4)
$T^{a,b}$ is increasing.

Then, the class $\mathscr{T}^{a,b}$ of (a,b)-t-norms $T^{a,b}$ is given by:

$$\mathscr{T}^{a,b} = \left\{T^{a,b} : [a,b]^2 \to [a,b] \mid T^{a,b} \text{ satisfies all the properties in } P_{\mathscr{T}^{a,b}}\right\} \tag{23}$$

where $P_{\mathscr{T}^{a,b}} = \{(\mathbf{TAB1}), (\mathbf{TAB2}), (\mathbf{TAB3}), (\mathbf{TAB4})\}$.

The following result is immediate:

Proposition 3.3. Consider the class of t-norms $\mathscr{T}$ (Definition 2.5) and the class of (a,b)-t-norms $\mathscr{T}^{a,b}$ (Definition 3.7). Then, the class $\mathscr{T}^{a,b}$ is $\mathscr{T}$-shifted.

Example 3.6.

i) The (a,b)-fusion function $T_L^{a,b} : [a,b]^2 \to [a,b]$, given, for all $x, y \in [a,b]$, by

$$T_L^{a,b}(x,y) = \max\{x+y-b,a\}, \tag{24}$$

is an (a,b)-t-norm. When $a=0$ and $b=1, T_L^{a,b}=T_L$, which is the Łukasiewicz t-norm, given in Eq. (9);

ii) The function $T_H : [a,b]^2 \to [a,b]$, such that $b>1$, given by

$$T_H(x,y) = \begin{cases} a & \text{if } x=y=a, \\ \frac{xy}{x+y-xy} & \text{otherwise}, \end{cases}$$

is inspired by the Hamacher product t-norm, defined in Eq. (51), but cannot be an (a,b)-t-norm, since it is not well defined. It is not trivial to define an "Hamacher product-like" (a,b)-t-norm, so we show in Section 4 a construction method to obtain an (a,b)-t-norm $T^{a,b}$ based on any given core t-norm T.

Remark 3.4. Observe that there is not an analogous result for (a,b)-t-norms as the ones stated in Theorem 3.1 and Corollary 3.2 for n-dimensional (a,b)-overlap functions. Those results derive from the fact that the generalized composition of n-dimensional overlap functions provides an n-dimensional overlap function (Theorem 2.1 and Corollary 2.1), but the same property does not necessarily hold for t-norms.

Definition 3.8. Consider a bivariate (a,b)-fusion function $S^{a,b} : [a,b]^2 \to [a,b]$ and the following properties:

(SAB1)
$S^{a,b}$ is symmetric;

(SAB2)
$S^{a,b}$ is associative;

(SAB3)
$S^{a,b}$ has a as its neutral element;

(SAB4)
$S^{a,b}$ is increasing.

Then, the class $\mathscr{S}^{a,b}$ of (a,b)-t-conorms $S^{a,b}$ is given by:

$$\mathscr{S}^{a,b} = \left\{ S^{a,b} : [a,b]^2 \to [a,b] \mid S^{a,b} \text{ satisfies all the properties in } P_{\mathscr{S}^{a,b}} \right\} \tag{25}$$

where $P_{\mathscr{S}^{a,b}} = \{(\mathbf{SAB1}), (\mathbf{SAB2}), (\mathbf{SAB3}), (\mathbf{SAB4})\}$.

The following result is immediate:

Proposition 3.4. *Consider the class of t-conorms $\mathscr{S}$ (Definition 2.6) and the class of (a,b)-t-conorms $\mathscr{S}^{a,b}$ (Definition 3.8). Then, the class $\mathscr{S}^{a,b}$ is $\mathscr{S}$-shifted.*

Example 3.7.

i) The function $MAX : [a,b]^2 \to [a,b]$, given, for all $x,y \in [a,b]$, by

$$MAX(\vec{x}) = \max\{x,y\}, \tag{26}$$

is an (a,b)-t-conorm;

ii) The probabilistic sum, given by Eq. (11), is only an (a,b)-t-conorm function when $a=0$ and $b=1$. In Section 4, we present a construction method to obtain an n-dimensional (a,b)-t-conorm $S^{a,b}$ based on a given t-conorm S, for any arbitrary $a,b \in \mathbb{R}$, such that $a<b$.

Remark 3.5. Observe that (a,b)-t-norms and (a,b)-t-conorms, constructed via our general framework, can be seen as a particular cases of t-norms and t-conorms, respectively, defined on a bounded poset (see [14]). Also, if considering continuous (a,b)-t-norms and (a,b)-t-conorms, then they are, in fact, I-semigroups introduced and discussed in [38].

Definition 3.9. Consider a bivariate (a,b)-fusion function $U^{a,b} : [a,b]^2 \rightarrow [a,b]$ and the following properties:

(UAB1)
$U^{a,b}$ is symmetric;

(UAB2)
$U^{a,b}$ is associative;

(UAB3)
$U^{a,b}$ has a neutral element;

(UAB4)
$U^{a,b}$ is increasing.

Then, the class $\mathscr{U}^{a,b}$ of (a,b)-uninorms $U^{a,b}$ is given by:

$$\mathscr{U}^{a,b} = \left\{ U^{a,b} : [a,b]^2 \rightarrow [a,b] | \; U^{a,b} \text{ satisfies all the properties in } P_{\mathscr{U}^{a,b}} \right\} \tag{27}$$

where $P_{\mathscr{U}^{a,b}} = \{(\textbf{UAB1}), (\textbf{UAB2}), (\textbf{UAB3}), (\textbf{UAB4})\}$.

The following result is immediate:

Proposition 3.5. *Consider the class of uninorms $\mathscr{U}$ (Definition 2.7) and the class of (a,b)-uninorms $\mathscr{U}^{a,b}$ (Definition 3.9). Then, the class $\mathscr{U}^{a,b}$ is $\mathscr{U}$-shifted.*

Example 3.8.

i)Consider $q \in [a,b]$. Then, the function $U_C^{a,b} : [a,b]^2 \rightarrow [a,b]$, given, for all $x, y \in [a,b]$, by

$$U_C(x,y) = \begin{cases} \max\{x,y\} & \text{if } x,y \in [q,b], \\ \min\{x,y\} & \text{otherwise}, \end{cases} \tag{28}$$

is an (a,b)-uninorm with q as its neutral element. One may observe that $U_C^{a,b}$ is a counterpart on $[a,b]$ for the uninorm U_C (Eq. (12));

ii)As discussed on Examples 3.5 and 3.6, some aggregation functions are not trivially transposed to obtain an analogous definition on $[a,b]$. That is the case of the U_P uninorm, given by Eq. (13). So, in Section 4, we present a construction method to obtain (a,b)-unimorms, based on a choice of any core unimorm, such as U_P.

Remark 3.6. In the same manner that uninorms can be seen as a generalization of t-norms and t-conorms, it is immediate that (a,b)-uninorms are a generalization of (a,b)-t-norms and (a,b)-t-conorms.

4. Construction methods for $\mathcal{F}$-shifted (a,b)-fusion functions

In [45], Wang et al. introduced a construction method for overlap functions on a lattice L based on a "generator triple" composed of an overlap function (which is bivariate) on some lattice M and two complete homomorphisms from L to M, under several constraints. Here, we develop construction methods for any n-dimensional (a,b)-fusion function, with focus on (a,b)-aggregation functions and their subclasses, based on a core fusion function and an increasing bijective function, without imposing any additional constraints.

Consider a fusion function $F : [0,1]^n \rightarrow [0,1]$ and an increasing and bijective function $\phi : [a,b] \rightarrow [0,1]$ and the (a,b)-fusion function $F_\phi^{a,b} : [a,b]^n \rightarrow [a,b]$ given, for all $x_1, \ldots, x_n \in [a,b]$, by

$$F_\phi^{a,b}(x_1, \ldots, x_n) = \phi^{-1}(F(\phi(x_1), \ldots, \phi(x_n))). \tag{29}$$

Then, F is said to be the core function of $F_\phi^{a,b}$. Eq. (29) plays an important role in the following construction methods. In the remainder of the paper, we denote $F_\phi^{a,b}$ simply by $F^{a,b}$.

Theorem 4.1. *Consider a fusion function $A : [0,1]^n \rightarrow [0,1]$, an increasing and bijective function $\phi : [a,b] \rightarrow [0,1]$ and an (a,b)-fusion function $A^{a,b} : [a,b]^n \rightarrow [a,b]$ given, for all $x_1, \ldots, x_n \in [a,b]$, by*

$$A^{a,b}(x_1,\dots,x_n) = \phi^{-1}(A(\phi(x_1),\dots,\phi(x_n))). \quad (30)$$

Then, $A^{a,b}$ is an (a,b)-aggregation function if and only if A is an aggregation function.

Proof. It is immediate that $A^{a,b}$ is well defined.($\Leftarrow$) Suppose that A is an aggregation function. Then, let us prove that $A^{a,b}$ has all properties from $P^{a,b}_{\mathcal{A}}$:

(A1*)Let $\vec{x},\vec{y} \in [a,b]^n$ be such that $\vec{x} \leqslant \vec{y}$. Since ϕ and A are increasing, then it follows that

$$\vec{x} \leqslant \vec{y} \Rightarrow A^{a,b}(\vec{x}) \leqslant A^{a,b}(\vec{y});$$

(A2*)Consider $\vec{a} = (a,\dots,a)$ and $\vec{b} = (b,\dots,b)$. Then:

$$\begin{aligned} A^{a,b}\left(\vec{a}\right) &= \phi^{-1}(A(\phi(a),\dots,\phi(a))) \\ &= \phi^{-1}(A(0,\dots,0)), \text{ since } \phi \text{ is bijective and increasing} \\ &= \phi^{-1}(0), \text{ by}(\mathbf{A2}) = a, \end{aligned}$$

and

$$\begin{aligned} A^{a,b}\left(\vec{b}\right) &= \phi^{-1}(A(\phi(b),\dots,\phi(b))) \\ &= \phi^{-1}(A(1,\dots,1)), \text{ since } \phi \text{ is bijective and increasing} \\ &= \phi^{-1}(1) \text{ by } (\mathbf{A2}) \\ &= b. \end{aligned}$$

($\Rightarrow$) Suppose that $A^{a,b}$ is an (a,b)-aggregation function. Now, let us prove that A respects all conditions from Definition 2.3:

(A1) Let $\vec{x},\vec{y} \in [0,1]^n$ be such that $\vec{x} \leqslant \vec{y}$. Then, it holds that $\phi^{-1}(x_i) \leqslant \phi^{-1}(y_i)$, for all $i \in \{1,\dots,n\}$, since ϕ^{-1} is increasing. From **(A1*)**, one has that:

$$\begin{aligned} & A^{a,b}\left(\phi^{-1}(x_1),\dots,\phi^{-1}(x_n)\right) \leqslant A^{a,b}\left(\phi^{-1}(y_1),\dots,\phi^{-1}(y_n)\right) \\ \Rightarrow & \phi^{-1}\left(A\left(\phi\left(\phi^{-1}(x_1)\right),\dots,\phi\left(\phi^{-1}(y_1)\right)\right)\right) \leqslant \phi^{-1}\left(A\left(\phi\left(\phi^{-1}(x_1)\right),\dots,\phi\left(\phi^{-1}(y_1)\right)\right)\right), \text{ by Eq. (A2)} \\ \Rightarrow & A(x_1,\dots,x_n) \leqslant A(y_1,\dots,y_n), \text{ since } \phi^{-1} \text{ is bijective and increasing.} \end{aligned}$$

(A2) From **(A2*)**, one has that:

$$\begin{aligned} & A^{a,b}(a,\dots,a) = a \\ \Rightarrow \quad & \phi^{-1}(A(\phi(a),\dots,\phi(a))) = a \\ \Rightarrow \quad & \phi^{-1}(A(0,\dots,0)) = a \\ \Rightarrow \quad & A(0,\dots,0) = 0, \end{aligned}$$

and

$$\begin{aligned} & A^{a,b}(b,\dots,b) = b \\ \Rightarrow \quad & \phi^{-1}(A(\phi(b),\dots,\phi(b))) = b \\ \Rightarrow \quad & \phi^{-1}(A(1,\dots,1)) = b \\ \Rightarrow \quad & A(1,\dots,1) = 1. \end{aligned}$$

□

Example 4.1. A basic increasing bijection $\phi_A : [a,b] \to [0,1]$ is the only affine transform between $[a,b]$ and $[0,1]$, defined, for all $x \in [a,b]$, by

$$\phi_A(x) = \left(\frac{x-a}{b-a}\right). \quad (31)$$

More generally, one may consider $\phi_A^p : [a,b] \to [0,1]$, defined, for all $x \in [a,b]$ and for $p > 0$, by

$$\phi_A^p(x) = \left(\frac{x-a}{b-a}\right)^p. \quad (32)$$

Then, let $GM : [0,1]^n \to [0,1]$ be the geometric mean, given by Eq. (6). Thus, the (a,b)-fusion function $GM^{a,b} : [a,b]^n \to [a,b]$, given, for all $x_1,\dots,x_n \in [a,b]$, by

$$GM^{a,b}(x_1,\dots,x_n) = \left(\phi_A^p\right)^{-1}\left(GM\left(\phi_A^p(x_1),\dots,\phi_A^p(x_n)\right)\right) = \phi_A^{-1}(GM(\phi_A(x_1),\dots,\phi_A(x_n))), \tag{33}$$

is an (a,b)-aggregation function. We can rewrite Eq. (33) as follows:

$$GM^{a,b}(x_1,\dots,x_n) = GM\left(\frac{x_1-a}{b-a},\dots,\frac{x_n-a}{b-a}\right)\cdot(b-a)+a.$$

Remark 4.1. It is immediate that any aggregation function $A : [0,1]^n \to [0,1]$ can be the core function of the construction method presented in Theorem 4.1, as it was the case with the geometric mean in Example 4.1. By applying the construction method, one can obtain an analogous (a,b)-aggregation function for any given aggregation function.

Remark 4.2. In the context of Theorem 4.1, when considering the basic increasing bijection ϕ_A, shown in Example 4.1, some (a,b)-aggregation functions and their respective core aggregation functions share the same formula. This is the case for positively homogeneous and shift invariant aggregation functions [18], like the Choquet integral [7]. In fact, the (a,b)-Choquet integral, constructed by this method, corresponds to the asymmetric Choquet integral introduced by Denneberg [10]. Hence, all the special instances of this function, such as the minimum, maximum, arithmetic mean, weighted mean and OWA [48], preserve their formulas when applied as the core of the construction method for defining analogous (a,b)-aggregation functions.

Remark 4.3. In Theorem 4.1, one could also obtain an (a,b)-aggregation function by considering ϕ as a decreasing bijection. However, for this and the following construction methods, we focus only on applying increasing bijections to facilitate the shifting of properties of the core aggregation function from $[0,1]$ to $[a,b]$.

Remark 4.4. More complex ways could be considered for constructing (a,b)-fusion functions based on increasing (or decreasing) bijections, instead of just ϕ and ϕ^{-1}, as in Eq. (29). For instance, one could consider the monotonic bijections $\eta,\phi_1,\dots,\phi_n : [a,b] \to [0,1]$ and a fusion function $F : [0,1]^n \to [0,1]$ to construct an (a,b)-fusion $F^{a,b} : [a,b]^n \to [a,b]$, defined, for all $\vec{x} \in [a,b]^n$, by

$$F^{a,b}(\vec{x}) = \eta^{-1}F(\phi_1(x_1),\dots,\phi_n(x_n)).$$

With this approach, some shifted properties from F are preserved for $F^{a,b}$ (e.g., $F^{a,b}$ is an (a,b)-aggregation function if and only if F is an aggregation function) but others properties may not be preserved (e.g., symmetry and associativity).

Similar construction methods from the one in Theorem 4.1 can be obtained for different subclasses of (a,b)-aggregation functions.

Theorem 4.2. *Consider a fusion function $O : [0,1]^n \to [0,1]$, an increasing and bijective function $\phi : [a,b] \to [0,1]$ and an (a,b)-fusion function $O^{a,b} : [a,b]^n \to [a,b]$ given, for all $x_1,\dots,x_n \in [a,b]$, by*

$$O^{a,b}(x_1,\dots,x_n) = \phi^{-1}(O(\phi(x_1),\dots,\phi(x_n))), \tag{34}$$

Then, $O^{a,b}$ is an n-dimensional (a,b)-overlap function if and only if O is an n-dimensional overlap function.

Proof. $(\Rightarrow)$ Suppose that $O^{a,b}$ is an n-dimensional (a,b)-overlap function. Then, it is immediate that O is increasing, symmetric and continuous. Let us prove that O respects the remaining conditions of Definition 2.4:

(O2)

$$\begin{aligned}
& O(x_1,\dots,x_n) = 0 \\
\iff & O\left(\phi\left(\phi^{-1}(x_1)\right),\dots,\phi\left(\phi^{-1}(x_n)\right)\right) = 0, \text{ since } \phi \text{ is bijective} \\
\iff & \phi^{-1}\left(O\left(\phi\left(\phi^{-1}(x_1)\right),\dots,\phi\left(\phi^{-1}(x_n)\right)\right)\right) = \phi^{-1}(0) \\
\iff & O^{a,b}\left(\phi^{-1}(x_1),\dots,\phi^{-1}(x_n)\right) = a, \text{ by Eq.(34)} \\
\iff & \phi^{-1}(x_i) = a, \text{ for some } i \in \{1,\dots,n\}, \text{ by } (\mathbf{OAB2}) \\
\iff & x_i = 0, \text{ for some } i \in \{1,\dots,n\}.
\end{aligned}$$

(O3)

$$\begin{aligned}
& O(x_1,\ldots,x_n) = 1 \\
\iff & O\left(\phi\left(\phi^{-1}(x_1)\right),\ldots,\phi\left(\phi^{-1}(x_n)\right)\right) = 1, \text{ since } \phi \text{ is bijective} \\
\iff & \phi^{-1}\left(O\left(\phi\left(\phi^{-1}(x_1)\right),\ldots,\phi\left(\phi^{-1}(x_n)\right)\right)\right) = \phi^{-1}(1) \\
\iff & O^{a,b}\left(\phi^{-1}(x_1),\ldots,\phi^{-1}(x_n)\right) = b, \text{ by Eq.(34)} \\
\iff & \phi^{-1}(x_i) = a, \text{ for all } i \in \{1,\ldots,n\}, \text{ by } (\textbf{OAB3}) \\
\iff & x_i = 1, \text{ for all } i \in \{1,\ldots,n\}.
\end{aligned}$$

($\Leftarrow$) Suppose that O is an n-dimensional overlap function. From **(O1)**, **(O4)** and **(O5)**, we also have that $O^{a,b}$ is symmetric, increasing and continuous. Now, let us prove that it respects the remaining conditions of Definition 3.6:

(OAB2) Suppose that $O^{a,b}(\vec{x}) = a$, for some $\vec{x} \in [a,b]^n$. Then, from Eq. (34), we have that:

$$a = \phi^{-1}(O(\phi(x_1),\ldots,\phi(x_n))) \iff 0 = O(\phi(x_1),\ldots,\phi(x_n)),$$

since ϕ is increasing and bijective. From **(O2)**, it follows that:

$$\phi(x_i) = 0 \text{ for some } i \in \{1,\ldots,n\} \iff x_i = a \text{ for some } i \in \{1,\ldots,n\}.$$

(OAB3) Suppose that $O^{a,b}(\vec{x}) = b$, for all $\vec{x} \in [a,b]^n$. Then, from Eq. (34), we have that:

$$b = \phi^{-1}(O(\phi(x_1),\ldots,\phi(x_n))) \iff 1 = O(\phi(x_1),\ldots,\phi(x_n)),$$

since ϕ is increasing and bijective. From **(O3)**, it follows that:

$$\phi(x_i) = 1 \text{ for all } i \in \{1,\ldots,n\} \iff x_i = b \text{ for all } i \in \{1,\ldots,n\}.$$

□

Example 4.2. The (a,b)-aggregation function $GM^{a,b} : [a,b]^n \to [a,b]$ defined in Example 4.1 is an n-dimensional (a,b)-overlap function.

In the next theorem, we show that one can obtain the same n-dimensional (a,b)-overlap function from two distinct methods, both based on a tuple of core n-dimensional overlap functions $\overrightarrow{O} = (O_1,\ldots,O_m)$. One method consists in first obtaining the n-dimensional overlap function $A_{\overrightarrow{O}}$ by the generalized composition of the core n-dimensional overlap functions by an aggregation function A (as in Theorem 2.1), followed by the application of the construction method of Theorem 4.2 taking $A_{\overrightarrow{O}}$ as the core function. The other method consists in first applying both the construction method of Theorem 4.2 m times, one for each core overlap function from $\overrightarrow{O}$, as well as the construction method of Theorem 4.1 with an aggregation function A as the core function, followed by the generalized composition of the m resulting n-dimensional (a,b)-overlap functions $\left(O_1^{a,b},\ldots,O_m^{a,b}\right)$ by the resulting (a,b)-aggregation function $A^{a,b}$.

Theorem 4.3. *Consider a continuous aggregation function $A : [0,1]^m \to [0,1]$, such that*

(PA) *$A(\vec{x}) = 0$ if and only if $x_i = 0$, for some $i \in \{1,\ldots,m\}$;*
(PB) *$A(\vec{x}) = 1$ if and only if $x_i = 1$, for all $i \in \{1,\ldots,m\}$;*

a tuple $\overrightarrow{O} = (O_1,\ldots,O_m)$ of n-dimensional overlap functions and the n-dimensional overlap function $A_{\overrightarrow{O}} : [0,1]^n \to [0,1]$, defined, for all $\vec{x} \in [0,1]^n$, by

$$A_{\overrightarrow{O}}(\vec{x}) = A(O_1(\vec{x}),\ldots,O_m(\vec{x})). \tag{35}$$

Also, consider an increasing and bijective function $\phi : [a,b] \to [0,1]$, the n-dimensional (a,b)-overlap function $A^{a,b}_{\overrightarrow{O}} : [a,b]^n \to [a,b]$ given, for all $\vec{y} \in [a,b]^n$, by

$$A^{a,b}_{\overrightarrow{O}}(\vec{y}) = \phi^{-1}\left(A_{\overrightarrow{O}}(\phi(y_1),\ldots,\phi(y_n))\right), \tag{36}$$

the (a,b)*-aggregation function* $A^{a,b} : [a,b]^m \to [a,b]$*, given, for all* $\vec{z} \in [a,b]^m$*, by*

$$A^{a,b}(\vec{z}) = \phi^{-1}(A(\phi(z_1),\dots,\phi(z_m))), \tag{37}$$

the n-dimensional (a,b)*-overlap functions* $O_1^{a,b},\dots,O_m^{a,b} : [a,b]^n \to [a,b]$*, given, for all* $\vec{y} \in [a,b]^n$*, by*

$$O_i^{a,b}(\vec{y}) = \phi^{-1}(O_i(\phi(y_1),\dots,\phi(y_n))), i \in \{1,\dots,m\}, \tag{38}$$

and the n-dimensional (a,b)*-overlap function* $OC^{a,b} : [a,b]^n \to [a,b]$*, defined, for all* $\vec{y} \in [a,b]^n$ *by*

$$OC^{a,b}(\vec{y}) = A^{a,b}\left(O_1^{a,b}(\vec{y}),\dots,O_m^{a,b}(\vec{y})\right). \tag{39}$$

Then, it holds that $A^{a,b}_{\vec{O}} = OC^{a,b}$*.*

Proof. Consider $\vec{x} \in [0,1]^n$ and $\vec{y} \in [a,b]^n$ such that $x_i = \phi(y_i)$ for all $i \in \{1,\dots,n\}$. As ϕ is bijective, it is immediate that $y_i = \phi^{-1}(x_i)$ for all $i \in \{1,\dots,n\}$. Then, it follows that:

$$\begin{aligned}
A^{a,b}_{\vec{O}}\left(\vec{y}\right) &= \phi^{-1}\left(A_{\vec{O}}(\phi(y_1),\dots,\phi(y_n))\right), \quad \text{by Eq.}(36)\\
&= \phi^{-1}\left(A_{\vec{O}}(x_1,\dots,x_n)\right)\\
&= \phi^{-1}\left(A\left(O_1\left(\vec{x}\right),\dots,O_m\left(\vec{x}\right)\right)\right), \quad \text{by Eq.}(5)\\
&= \phi^{-1}\left(A\left(\phi\left(\phi^{-1}\left(O_1\left(\vec{x}\right)\right)\right),\dots,\phi\left(\phi^{-1}\left(O_m\left(\vec{x}\right)\right)\right)\right)\right), \quad \text{since } \phi \text{ is bijective}\\
&= A^{a,b}\left(\phi^{-1}\left(O_1\left(\vec{x}\right)\right),\dots,\phi^{-1}\left(O_m\left(\vec{x}\right)\right)\right), \quad \text{by Eq.}(37)\\
&= A^{a,b}(\phi^{-1}(O_1(\phi(y_1),\dots,\phi(y_n))),\dots,\phi^{-1}(O_m(\phi(y_1),\dots,\phi(y_n))))\\
&= A^{a,b}\left(O_1^{a,b}(y_1,\dots,y_n),\dots,O_m^{a,b}(y_1,\dots,y_n)\right), \quad \text{by Eq.}(38)\\
&= OC^{a,b}\left(\vec{y}\right), \quad \text{by Eq.}(39).
\end{aligned}$$

□

Theorem 4.3 shows that the diagram of Fig. 1 commutes, where $\vec{O} = (O_1,\dots,O_m)$ and $\overrightarrow{O^{a,b}} = \left(O_1^{a,b},\dots,O_m^{a,b}\right)$.

Theorem 4.4. *Consider a bivariate fusion function* $T : [0,1]^2 \to [0,1]$*, an increasing and bijective function* $\phi : [a,b] \to [0,1]$ *and a bivariate* (a,b)*-fusion function* $T^{a,b} : [a,b]^2 \to [a,b]$ *given, for all* $x,y \in [a,b]$*, by*

$$T^{a,b}(x,y) = \phi^{-1}(T(\phi(x),\phi(y))), \tag{40}$$

Then, $T^{a,b}$ *is an* (a,b)*-t-norm if and only if T is a t-norm.*

Proof. $(\Rightarrow)$ Suppose that $T^{a,b}$ is an (a,b)-t-norm. Then, it is immediate that T is symmetric **(T1)** and increasing **(T4)**. Let us prove the remaining conditions:

(T2) From **(TAB2)**, one has that, for all $x,y,z \in [0,1]$:

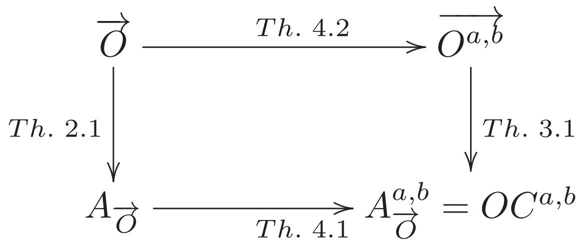

Fig. 1. Commutative diagram of the construction methods of an n-dimensional (a,b)-overlap function based on a tuple of n-dimensional overlap functions.

$$
\begin{aligned}
& T^{a,b}\left(T^{a,b}(\phi^{-1}(x),\phi^{-1}(y)),\phi^{-1}(z)\right) = T^{a,b}\left(\phi^{-1}(x),T^{a,b}(\phi^{-1}(y),\phi^{-1}(z))\right) \\
\Rightarrow\, & T^{a,b}(\phi^{-1}(T(\phi(\phi^{-1}(x)),\phi(\phi^{-1}(y)))),\phi^{-1}(z)) = T^{a,b}(\phi^{-1}(x),\phi^{-1}(T(\phi(\phi^{-1}(y)),\phi(\phi^{-1}(z))))), \\
& \text{by Eq.}(42) \\
\Rightarrow\, & \phi^{-1}(T(T(x,y),z)) = \phi^{-1}(T(x,T(y,z))), \quad \text{since } \phi \text{ is bijective} \\
\Rightarrow\, & T(T(x,y),z) = T(x,T(y,z)),
\end{aligned}
$$

which means that T is associative.

(T3) From **(TAB3)**, one has that, for all $x \in [0,1]$:

$$
\begin{aligned}
& T^{a,b}(\phi^{-1}(x),b) = T^{a,b}(b,\phi^{-1}(x)) = \phi^{-1}(x) \\
\Rightarrow\, & \phi^{-1}(T(\phi(\phi^{-1}(x)),\phi(b))) = \phi^{-1}(x), \quad \text{by Eq.}(42) \\
\Rightarrow\, & \phi^{-1}(T(x,1)) = \phi^{-1}(x), \quad \text{since } \phi \text{ is bijective} \\
\Rightarrow\, & T(x,1) = x,
\end{aligned}
$$

which implies that T has 1 as its neutral element. Since T is symmetric and increasing, the result follows.

Thus, T is a t-norm.

($\Leftarrow$) Suppose that T is a t-norm. From **(T1)** and **(T4)**, we also have that $T^{a,b}$ is symmetric and increasing. Now, let us prove the remaining conditions:

(TAB2)
For all $x, y, z \in [a,b]$, one has that:

$$
\begin{aligned}
T^{a,b}\left(T^{a,b}(x,y),z\right) &= \phi^{-1}\left(T\left(\phi\left(T^{a,b}(x,y)\right),\phi(z)\right)\right), \quad \text{by Eq.}(42) \\
&= \phi^{-1}(T(T(\phi(x),\phi(y)),\phi(z))), \quad \text{since } \phi \text{ is bijective} \\
&= \phi^{-1}(T(\phi(x),T(\phi(y),\phi(z)))), \quad \text{by } (\mathbf{T2}) \\
&= \phi^{-1}\left(T\left(\phi(x),\phi\left(T^{a,b}(y,z)\right)\right)\right) \\
&= T^{a,b}\left(x,T^{a,b}(y,z)\right),
\end{aligned}
$$

showing that $T^{a,b}$ is associative.

(TAB3)
For all $x \in [a,b]$, it holds that:

$$
\begin{aligned}
T^{a,b}(x,b) &= \phi^{-1}(T(\phi(x),\phi(b))), \quad \text{by Eq. } (42) \\
&= \phi^{-1}(T(\phi(x),1)), \quad \text{since } \phi \text{ is bijective} \\
&= \phi^{-1}(\phi(x)), \quad \text{by } (\mathbf{T3}) \\
&= x.
\end{aligned}
$$

Since $T^{a,b}$ is symmetric, it follows that b is its neutral element.

□

Example 4.3. Consider the Hamacher product $T_H : [0,1]^2 \to [0,1]$, given by Eq. (51), and $\phi_A^p : [a,b] \to [0,1]$, defined in Eq. (32). Then, the (a,b)-fusion function $T_H^{a,b} : [a,b]^2 \to [a,b]$, given, for all $x, y \in [a,b]$, by

$$
T_H^{a,b}(x,y) = \left(\phi_A^p\right)^{-1}\left(T_H\left(\phi_A^p(x),\phi_A^p(y)\right)\right), \tag{41}
$$

is an (a,b)-t-norm. By taking $p = 1$, we can rewrite Eq. (41) as follows:

$$
T_H^{a,b}(x,y) = T_H\left(\frac{x-a}{b-a},\frac{y-a}{b-a}\right)\cdot(b-a)+a.
$$

Remark 4.5. It is clear that, in the context of Theorem 4.4, when $a = 0$ and $b = 1$, Eq. (42) provides a t-norm. In this case, if $T = T_P$ (the product t-norm), then the constructed t-norm $T^{0,1}$ is a continuous strict t-norm (strictly increasing in $(0,1]$). If $T = T_L$ (Łukasiewicz t-norm, given in Eq. (9)), then the constructed $T^{0,1}$ is a continuous nilpotent t-norm.

Theorem 4.5. *Consider a bivariate fusion function $S : [0,1]^2 \to [0,1]$, an increasing and bijective function $\phi : [a,b] \to [0,1]$ and a bivariate (a,b)-fusion function $S^{a,b} : [a,b]^2 \to [a,b]$ given, for all $x, y \in [a,b]$, by*

$$S^{a,b}(x,y) = \phi^{-1}(S(\phi(x),\phi(y))), \tag{42}$$

Then, $S^{a,b}$ is an (a,b)-t-conorm if and only if S is a t-conorm.

Proof. Analogous to the proof of Theorem 4.4. □

Example 4.4. Consider the probabilistic sum $S_p : [0,1]^2 \to [0,1]$, given by Eq. (11), and $\phi_A^p : [a,b] \to [0,1]$, defined in Eq. (32). Then, the (a,b)-fusion function $S_p^{a,b} : [a,b]^2 \to [a,b]$, given, for all $x, y \in [a,b]$, by

$$S_p^{a,b}(x,y) = \left(\phi_A^p\right)^{-1}\left(S_p\left(\phi_A^p(x),\phi_A^p(y)\right)\right), \tag{43}$$

is an (a,b)-t-conorm. By taking $p = 1$, we can rewrite Eq. (43) as follows:

$$S_p^{a,b}(x,y) = S_p\left(\frac{x-a}{b-a},\frac{y-a}{b-a}\right)\cdot(b-a)+a.$$

Theorem 4.6. *Consider $e \in [0,1], q \in [a,b]$, a bivariate fusion function $U : [0,1]^2 \to [0,1]$, an increasing and bijective function $\phi : [a,b] \to [0,1]$, such that $\phi(q) = e$, and a bivariate (a,b)-fusion function $U^{a,b} : [a,b]^2 \to [a,b]$ given, for all $x, y \in [a,b]$, by*

$$U^{a,b}(x,y) = \phi^{-1}(U(\phi(x),\phi(y))), \tag{44}$$

Then, $U^{a,b}$ is an (a,b)-uninorm with q as its neutral element if and only if U is an uninorm with e as its neutral element.

Proof. Analogous to the proof of Theorem 4.4. □

Example 4.5. Consider the unimorm $U_P : [0,1]^2 \to [0,1]$, given by Eq. (13), and $\phi_A^p : [a,b] \to [0,1]$, defined in Eq. (32). Then, the (a,b)-fusion function $U_P^{a,b} : [a,b]^2 \to [a,b]$, given, for all $x, y \in [a,b]$, by

$$U_P^{a,b}(x,y) = \left(\phi_A^p\right)^{-1}\left(U_P\left(\phi_A^p(x),\phi_A^p(y)\right)\right), \tag{45}$$

is an (a,b)-uninorm. By taking $p = 1$, we can rewrite Eq. (45) as follows:

$$U_P^{a,b}(x,y) = U_P\left(\frac{x-a}{b-a},\frac{y-a}{b-a}\right)\cdot(b-a)+a.$$

5. Study of some properties of (a,b)-aggregation functions

In this section, we analyze some properties of (a,b)-aggregation functions, in particular, the cases in which the properties of the core aggregation functions are preserved/shifted when constructing an analogous (a,b)-aggregation functions via the previously introduced construction methods.

5.1. Idempotency and averaging properties

A fusion function $F : [0,1]^n \to [0,1]$ is idempotent [18] if, for all $x \in [0,1]$, it holds that:

$$F(x,\ldots,x) = x. \tag{46}$$

Clearly, idempotency can be analogously defined for (a,b)-fusion functions.

Proposition 5.1. *Let $A^{a,b} \in \mathscr{A}^{a,b}$ be an (a,b)-aggregation function. Then, $A^{a,b}|_{[c,d]}$ is a (c,d)-aggregation function for all $[c,d] \subseteq [a,b]$ if and only if $A^{a,b}$ is idempotent.*

Proof. ($\Rightarrow$) Suppose that $A^{a,b}|_{[c,d]}$ is a (c,d)-aggregation function for all $[c,d] \subseteq [a,b]$. Then, for all $[c,d] \subseteq [a,b]$, it holds that $A^{a,b}(c,\ldots,c) = c$ and $A^{a,b}(d,\ldots,d) = d$, meaning that $A^{a,b}(x,\ldots,x) = x$, for all $x \in [a,b]$;

($\Leftarrow$) Now, suppose that $A^{a,b}$ is idempotent. Then, it is immediate that $A^{a,b}|_{[c,d]}$ is increasing and idempotent. Moreover, for all $[c,d] \subseteq [a,b]$, it follows that $A^{a,b}|_{[c,d]}(c,\ldots,c) = c$ and $A^{a,b}|_{[c,d]}(d,\ldots,d) = d$, meaning that $A^{a,b}|_{[c,d]}$ is a (c,d)-aggregation function. □

Theorem 5.1. *Let $A:[0,1]^n \to [0,1]$ be an aggregation function, $\phi:[a,b] \to [0,1]$ an increasing bijective function and $A^{a,b}:[a,b]^n \to [a,b]$ an (a,b)-aggregation function defined, for all $\vec{x} \in [a,b]^n$, by $A^{a,b}(\vec{x}) = \phi^{-1}(A(\phi(x_1),\ldots,\phi(x_n)))$. Then, $A^{a,b}$ is idempotent if and only if A is idempotent.*

Proof. $(\Rightarrow)$ Suppose that $A^{a,b}$ is idempotent. So, for all $x \in [0,1]$, it follows that:

$$\begin{aligned} & A^{a,b}\left(\phi^{-1}(x),\ldots,\phi^{-1}(x)\right) = \phi^{-1}(x) \\ \Rightarrow\ & \phi^{-1}\left(A\left(\phi\left(\phi^{-1}(x)\right),\ldots,\phi\left(\phi^{-1}(x)\right)\right)\right) = \phi^{-1}(x), \quad \text{by Eq.(30)} \\ \Rightarrow\ & A(x,\ldots,x) = x, \quad \text{since } \phi \text{ is bijective,} \end{aligned}$$

showing that A is idempotent.

$(\Leftarrow)$ Suppose that A is idempotent. Thus, for all $x \in [a,b]^n$, it holds that:

$$\begin{aligned} A^{a,b}(x,\ldots,x) &= \phi^{-1}(A(\phi(x),\ldots,\phi(x))), \quad \text{by Eq. (30)} \\ &= \phi^{-1}(\phi(x)), \quad \text{since } A \text{ is idempotent,} \\ &= x, \quad \text{since } \phi \text{ is bijective,} \end{aligned}$$

which means that $A^{a,b}$ is idempotent. □

A fusion function $F:[0,1]^n \to [0,1]$ is averaging when, for all $\vec{x} \in [0,1]^n$, it holds that:

$$\min\{\vec{x}\} \leqslant F(\vec{x}) \leqslant \max\{\vec{x}\}.$$

In the context of aggregation functions, since they are increasing, the idempotency and averaging properties are equivalent [18]. The same holds for (a,b)-aggregation functions, since they are also increasing, and the averaging property can be naturally shifted from $[0,1]$ to $[a,b]$ (the same holds for idempotency). Therefore, the following result is immediate.

Corollary 5.1. *Let $A:[0,1]^n \to [0,1]$ be an aggregation function, $\phi:[a,b] \to [0,1]$ an increasing bijective function and $A^{a,b}:[a,b]^n \to [a,b]$ the (a,b)-aggregation function defined, for all $\vec{x} \in [a,b]^n$, by $A^{a,b}(\vec{x}) = \phi^{-1}(A(\phi(x_1),\ldots,\phi(x_n)))$. Then, $A^{a,b}$ is averaging if and only if A is averaging.*

Example 5.1.

i)The arithmetic mean is an idempotent and averaging (a,b)-aggregation function;

ii)The n-dimensional (a,b)-overlap function $GM^{a,b}$, given by Eq. (33), is also idempotent and averaging.

5.2. Generalized migrativity

Consider $\alpha \in [0,1]$. A bivariate fusion function $F:[0,1]^2 \to [0,1]$ is said to be α-migrative [12] if, for all $x,y \in [0,1]$, it holds that:

$$F(\alpha \cdot x, y) = F(x, \alpha \cdot y). \tag{47}$$

In [15], α-migrativity was generalized by replacing both product operations on Eq. (47) by a t-norm T, obtaining the concept of (α,T)-migrativity. Humberto et al. [5] generalized this concept by considering an aggregation function B, instead of a t-norm, introducing the (α,B)-migrativity. Qiao and Hu [41] studied the migrativity property for an overlap function O, rewriting Eq. (47), with $F = O$ and replacing the first product operation by an overlap function O_1 and the second product operation by an overlap function O_2, resulting in the concept of (α,O_1,O_2)-migrativity for overlap functions. More recently, Qiao [40] introduced a similar definition of migrativity for overlap functions on lattices, where O_1 and O_2 are replaced, respectively, by binary operators A,B on a lattice L, with $\alpha \in L$, named (α,A,B)-migrativity of overlap functions. Inspired by such developments, here we introduce the concept of (α,F_1,F_2)-migrativity of a fusion function F, as follows:

Definition 5.1. Consider $\alpha \in [0,1]$ and two fusion functions $F_1,F_2:[0,1]^n \to [0,1]$. A fusion function $F:[0,1]^n \to [0,1]$ is said to be (k,α,F_1,F_2)-migrative if, for all $\vec{x} \in [0,1]^n$, it holds that:

$$F(F_1(\alpha,x_1),x_2,\ldots,x_n) = F(x_1,\ldots,F_2(\alpha,x_k),\ldots,x_n), \tag{48}$$

for some $k \in \{2,\ldots,n\}$. Whenever, F is (k,α,F_1,F_2)-migrative for all $k \in \{2,\ldots,n\}$, then it is said to be (α,F_1,F_2)-migrative.

However, when constructing an (a,b)-aggregation function as a counterpart of a (generalized) migrative aggregation function, the constructed function, most likely, does not respect any definitions of migrativity that are made in the context of the unit interval. So, here we shift the property of (α,F_1,F_2)-migrativity (Definition 5.1) from $[0,1]$ to $[a,b]$, which results in the following definition:

Definition 5.2. Consider $\delta \in [a,b]$ and two (a,b)-fusion functions $F_1^{a,b}, F_2^{a,b} : [a,b]^n \to [a,b]$. An (a,b)-fusion function $F^{a,b} : [a,b]^n \to [0,1]$ is said to be $\left(k,\delta,F_1^{a,b},F_2^{a,b}\right)$-migrative if, for all $\vec{x} \in [a,b]^n$, it holds that:

$$F^{a,b}\left(F_1^{a,b}(\delta,x_1),x_2,\ldots,x_n\right) = F^{a,b}\left(x_1,\ldots,F_2^{a,b}(\delta,x_k),\ldots,x_n\right), \tag{49}$$

for some $k \in \{2,\ldots,n\}$. Whenever, $F^{a,b}$ is $\left(k,\delta,F_1^{a,b},F_2^{a,b}\right)$-migrative for all $k \in \{2,\ldots,n\}$, then it is said to be $\left(\delta,F_1^{a,b},F_2^{a,b}\right)$-migrative.

Theorem 5.2. *Let $\phi : [a,b] \to [0,1]$ be an increasing bijective function, $F_1^{a,b}, F_2^{a,b} : [a,b]^2 \to [a,b]$ be two bivariate (a,b)-fusion functions defined, for all $\vec{x} \in [a,b]^2$, by Eq. (29), with $F_1, F_2 : [0,1]^2 \to [0,1]$ as their respective core fusion functions, and $A^{a,b} : [a,b]^n \to [a,b]$ be an (a,b)-aggregation function defined, for all $\vec{y} \in [a,b]^n$, by Eq. (30), with $A : [0,1]^n \to [0,1]$ as its core aggregation function. Then, for $\delta \in [a,b], A^{a,b}$ is $\left(\delta,F_1^{a,b},F_2^{a,b}\right)$-migrative if and only if A is $(\phi(\delta),F_1,F_2)$-migrative.*

Proof. $(\Rightarrow)$ Suppose that $A^{a,b}$ is $\left(\delta,F_1^{a,b},F_2^{a,b}\right)$-migrative. So, for all $\delta \in [a,b], \vec{x} \in [0,1]^n$ and $i \in \{2,\ldots,n\}$, by Definition 5.2, it follows that:

$$\begin{aligned}
&A^{a,b}\left(F_1^{a,b}\left(\delta,\phi^{-1}(x_1)\right),\phi^{-1}(x_2),\ldots,\phi^{-1}(x_n)\right) = A^{a,b}\left(\phi^{-1}(x_1),\ldots,F_2^{a,b}\left(\delta,\phi^{-1}(x_i)\right),\ldots,\phi^{-1}(x_n)\right)\\
\Rightarrow &A^{a,b}\left(\phi^{-1}\left(F_1\left(\phi(\delta),\phi\left(\phi^{-1}(x_1)\right)\right)\right),\phi^{-1}(x_2),\ldots,\phi^{-1}(x_n)\right) =\\
&A^{a,b}\left(\phi^{-1}(x_1),\ldots,\phi^{-1}\left(F_2\left(\phi(\delta),\phi\left(\phi^{-1}(x_i)\right)\right)\right),\ldots,\phi^{-1}(x_n)\right), \text{ by Eq.(29)}\\
\Rightarrow &\phi^{-1}\left(A\left(\phi\left(\phi^{-1}\left(F_1\left(\phi(\delta),\phi\left(\phi^{-1}(x_1)\right)\right)\right)\right),\phi\left(\phi^{-1}(x_2)\right),\ldots,\phi\left(\phi^{-1}(x_n)\right)\right)\right) =\\
&\phi^{-1}\left(A\left(\phi\left(\phi^{-1}(x_1)\right),\ldots,\phi\left(\phi^{-1}\left(F_2\left(\phi(\delta),\phi\left(\phi^{-1}(x_i)\right)\right)\right)\right),\ldots,\phi\left(\phi^{-1}(x_n)\right)\right)\right), \text{ by Eq.(30)}\\
\Rightarrow &A(F_1(\phi(\delta),x_1),x_2,\ldots,x_n) = A(x_1,\ldots,F_2(\phi(\delta),x_i),\ldots,x_n), \text{ since } \phi^{-1} \text{ is bijective,}
\end{aligned}$$

showing that A is $(\phi(\delta),F_1,F_2)$-migrative.$(\Leftarrow)$ Suppose that A is $(\phi(\delta),F_1,F_2)$-migrative. Thus, for all $\delta,x,\ldots,x_n \in [a,b]$ and $i \in \{2,\ldots,n\}$, it holds that:

$$\begin{aligned}
A^{a,b}\left(F_1^{a,b}(\delta,x_1),x_2,\ldots,x_n\right) &= A^{a,b}(\phi^{-1}(F_1(\phi(\delta),\phi(x_1))),x_2,\ldots,x_n), \quad \text{by Eq.(29)}\\
&= \phi^{-1}(A(\phi(\phi^{-1}(F_1(\phi(\delta),\phi(x_1)))),\phi(x_2),\ldots,\phi(x_n))),\\
&\quad \text{by Eq.(30)}\\
&= \phi^{-1}(A(F_1(\phi(\delta),\phi(x_1)),\phi(x_2),\ldots,\phi(x_n))), \quad \text{since } \phi \text{ is bijective,}\\
&= \phi^{-1}(A(\phi(x_1),\ldots,F_2(\phi(\delta),\phi(x_i)),\ldots,\phi(x_n))),\\
&\quad \text{since } A \text{ is } (\phi(\delta),F_1,F_2)-\text{migrative,}\\
&= \phi^{-1}(A(\phi(x_1),\ldots,\phi(\phi^{-1}(F_2(\phi(\delta),\phi(x_i)))),\ldots,\phi(x_n))),\\
&\quad \text{since } \phi \text{ is bijective,}\\
&= A^{a,b}(x_1,\ldots,\phi^{-1}(F_2(\phi(\delta),\phi(x_i))),\ldots,x_n), \quad \text{by Eq.(30)}\\
&= A^{a,b}\left(x_1,\ldots,F_2^{a,b}(\delta,x_i),\ldots,x_n\right), \quad \text{by Eq.(29)}
\end{aligned}$$

which means that $A^{a,b}$ is $\left(\delta,F_1^{a,b},F_2^{a,b}\right)$-migrative. □

Corollary 5.2. *Let $\phi : [a,b] \to [0,1]$ be an increasing bijective function, $F_1^{a,b}, F_2^{a,b} : [a,b]^2 \to [a,b]$ be two bivariate (a,b)-fusion functions defined, for all $\vec{x} \in [a,b]^2$, by Eq. (29), with $F_1, F_2 : [0,1]^2 \to [0,1]$ as their respective core fusion functions, and $A^{a,b} : [a,b]^n \to [a,b]$ be an (a,b)-aggregation function defined, for all $\vec{y} \in [a,b]^n$, by Eq. (30), with $A : [0,1]^n \to [0,1]$ as its core aggregation function. Then, for $\delta \in [a,b]$ and $k \in \{2,\ldots,n\}, A^{a,b}$ is $\left(k,\delta,F_1^{a,b},F_2^{a,b}\right)$-migrative if and only if A is $(k,\phi(\delta),F_1,F_2)$-migrative.*

Example 5.2.

i) Let $\mathfrak{A} = \{A_k : [0,1]^n \to [0,1] \,|\, k \in \{2,\ldots,n\}\}$, where A_k is defined, for all $\vec{x} \in [0,1]^n$, by

$$A_k(\vec{x}) = \begin{cases} 0, & \text{if } x_k = 0, \\ \frac{\prod_{i=1}^{n} x_i^2}{x_k}, & \text{otherwise}, \end{cases} \tag{50}$$

be a family of aggregation functions and $F_1, F_2 : [0,1]^2 \to [0,1]$ be two bivariate fusion functions, defined, for all $x, y \in [0,1]$, respectively, by

$$F_1(x,y) = x \cdot y,$$

and

$$F_2(x,y) = x^2 \cdot y,$$

It is immediate that each aggregation function $A_k \in \mathfrak{A}$ is (k, α, F_1, F_2)-migrative, with $\alpha \in [0,1]$. Now, considering an increasing and bijective function $\phi : [a,b] \to [0,1]$, define the (a,b)-fusion functions $F_1^{a,b}, F_2^{a,b} : [a,b]^2 \to [a,b]$, through Eq. (29), with F_1 and F_2 as their respective core aggregation functions. Also, define the (a,b)-aggregation functions $A_k : [a,b]^n \to [a,b]$, through Eq. (30), with A_k as their core aggregation functions and $k \in \{2,\ldots,n\}$. Thus, for $\delta = \phi^{-1}(\alpha)$, one has that every $A_k^{a,b}$ is a $\left(k, \delta, F_1^{a,b}, F_2^{a,b}\right)$-migrative function. Observe that this result does not imply that, for some specific $k \in \{2,\ldots,n\}, A_k^{a,b}$ is $\left(\delta, F_1^{a,b}, F_2^{a,b}\right)$-migrative.

ii) Consider $\delta \in [a,b]$, the product overlap O_P, given by Eq. (5), and let $O_P^{a,b} : [a,b]^n \to [a,b]$ be the n-dimensional (a,b)-overlap function obtained by Theorem 4.2, based on O_P as its core n-dimensional overlap function and an increasing and bijective function $\phi : [a,b] \to [0,1]$. Then, one has that $O_P^{a,b}$ is $\left(\delta, O_P^{a,b}, O_P^{a,b}\right)$-migrative. If $n = 2$, $\left(\delta, O_P^{a,b}, O_P^{a,b}\right)$-migrativity is the result of shifting the traditional α-migrativity property from $[0,1]$ to $[a,b]$;

iii) Consider $\alpha \in (0,1)$, the overlap function $O_q : [0,1]^2 \to [0,1]$, defined, for all $x, y \in [0,1]$, by $O_q(x,y) = x^q \cdot y^q$, with $q > 0$, and the aggregation function $A : [0,1]^n \to [0,1]$, given, for $\vec{x} \in [0,1]^n$, by

$$A\left(\vec{x}\right) = \begin{cases} \prod_{i=1}^{n} x_i, & \text{if } x_j \in [0,\alpha] \text{ for some } j \in \{1,\ldots,n\} \\ 1, & \text{otherwise}. \end{cases} \tag{51}$$

Then, A is (α, O_q, O_q)-migrative. Observe that, if $n = 2, A$ coincides with the function $O^{(\alpha)}$, presented in [41] (Example 3.1), which is an example of a function that is (α, O_1, O_2) migrative, with $O_1 = O_2 = O_q$. Considering an increasing and bijective function $\phi : [a,b] \to [0,1]$, define the (a,b)-overlap function $O_q^{a,b} : [a,b]^2 \to [a,b]$, through Eq. (34), with O_q as its core overlap function. Also, define the (a,b)-aggregation function $A^{a,b} : [a,b]^n \to [a,b]$, through Eq. (30), with A as its core aggregation function. Then, for $\delta = \phi^{-1}(\alpha), A^{a,b}$ is a $\left(\delta, O_q^{a,b}, O_q^{a,b}\right)$-migrative function;

iv) Consider $\alpha \in [0,1]$, the projection function $PROJ_2 : [0,1]^2 \to [a,b]$, given, for all $x, y \in [0,1]$, by $F_2(x,y) = y$, the bivariate arithmetic mean $BAM : [0,1]^2 \to [0,1]$ given, for all $x, y \in [0,1]$, by $BAM^{a,b}(x,y) = \frac{x+y}{2}$ and the projection function $PROJ_1 : [0,1]^n \to [0,1]$, given, for all $\vec{x} \in [0,1]^n$, by $PROJ_1(\vec{x}) = x_1$. Then, one has that $PROJ_1$ is $(\alpha, PROJ_2, BAM)$-migrative. Considering $\delta \in [a,b]$, if we define the functions $PROJ_1^{a,b} : [a,b]^n \to [a,b]$ and $PROJ_2^{a,b}, BAM^{a,b} : [a,b]^2 \to [a,b]$ analogously, then, it is immediate that $PROJ_1^{a,b}$ is $\left(\delta, PROJ_2^{a,b}, BAM^{a,b}\right)$-migrative.

5.3. Generalized homogeneity

A fusion function $F : [0,1]^n \to [0,1]$ is said to be homogeneous of order $\gamma \in [0,+\infty)$ if, for any $\alpha, x_1, \ldots, x_n \in [0,1]$, it holds that:

$$F(\alpha \cdot x_1, \ldots, \alpha \cdot x_n) = \alpha^\gamma \cdot F(x_1, \ldots, x_n), \tag{52}$$

considering $0^0 = 0$. This property was generalized in [43], in the form of abstract homogeneity of order 1, by replacing the product operations in Eq. (52) by another bivariate fusion function g and applying an automorphism on the parameter α, with $\gamma = 1$. When this automorphism is the identity function, we obtain the g-homogeneity property, defined as follows:

Definition 5.3. [43] Consider a bivariate fusion function $g : [0,1]^2 \to [0,1]$. A fusion function $F : [0,1]^n \to [0,1]$ is said to be g-homogeneous if, for any $\alpha, x_1, \ldots, x_n \in [0,1]$, it holds that:

$$F(g(\alpha, x_1), \ldots, g(\alpha, x_n)) = g(\alpha, F(x_1, \ldots, x_n)). \tag{53}$$

As discussed for the generalized migrativity property, (a,b)-aggregation functions constructed based on g-homogeneous aggregation functions are not expected to be g-homogeneous, since g is not an (a,b)-fusion function. So, let us shift the g-homogeneity property from $[0,1]$ to $[a,b]$, as follows:

Definition 5.4. Consider a bivariate (a,b)-fusion function $g^{a,b}:[a,b]^2\to[a,b]$. An (a,b)-fusion function $F^{a,b}:[a,b]^n\to[a,b]$ is said to be $g^{a,b}$-homogeneous if, for any $\delta,x_1,\ldots,x_n\in[a,b]$, it holds that:

$$F^{a,b}\left(g^{a,b}(\delta,x_1),\ldots,g^{a,b}(\delta,x_n)\right)=g^{a,b}\left(\delta,F^{a,b}(x_1,\ldots,x_n)\right). \tag{54}$$

Theorem 5.3. *Let $\phi:[a,b]\to[0,1]$ be an increasing bijective function, $g^{a,b}:[a,b]^n\to[a,b]$ be an (a,b)-fusion function defined, for all $\vec{x}\in[a,b]^n$, by Eq. (29), with $g:[0,1]^2\to[0,1]$ as its core fusion function, and $A^{a,b}:[a,b]^n\to[a,b]$ be an (a,b)-aggregation function defined, for all $\vec{x}\in[a,b]^n$, by Eq. (30), with $A:[0,1]^n\to[0,1]$ as its core aggregation function. Then, $A^{a,b}$ is $g^{a,b}$-homogeneous if and only if A is g-homogeneous.*

Proof. $(\Rightarrow)$ Suppose that $A^{a,b}$ is $g^{a,b}$-homogeneous. So, for all $\delta\in[a,b]$ and $\vec{x}\in[0,1]^n$, it follows that:

$$\begin{aligned}
&A^{a,b}\left(g^{a,b}(\delta,\phi^{-1}(x_1)),\ldots,g^{a,b}(\delta,\phi^{-1}(x_n))\right)=g^{a,b}\left(\delta,A^{a,b}\left(\phi^{-1}(x_1),\ldots,\phi^{-1}(x_n)\right)\right)\\
\Rightarrow&A^{a,b}\left(\phi^{-1}\left(g\left(\phi(\delta),\phi\left(\phi^{-1}(x_1)\right)\right)\right),\ldots,\phi^{-1}\left(g\left(\phi(\delta),\phi\left(\phi^{-1}(x_1)\right)\right)\right)\right)\\
&\quad=\phi^{-1}\left(g\left(\phi(\delta),\phi\left(A^{a,b}\left(\phi^{-1}(x_1),\ldots,\phi^{-1}(x_n)\right)\right)\right)\right),\quad\text{by Eq.(29)}\\
\Rightarrow&\phi^{-1}\left(A\left(\phi\left(\phi^{-1}\left(g\left(\phi(\delta),\phi\left(\phi^{-1}(x_1)\right)\right)\right)\right),\ldots,\phi\left(\phi^{-1}\left(g\left(\phi(\delta),\phi\left(\phi^{-1}(x_1)\right)\right)\right)\right)\right)\right)\\
&\quad=\phi^{-1}\left(g\left(\phi(\delta),\phi\left(\phi^{-1}\left(A\left(\phi\left(\phi^{-1}(x_1)\right),\ldots,\phi\left(\phi^{-1}(x_n)\right)\right)\right)\right)\right)\right),\quad\text{by Eq.(30)}\\
\Rightarrow&A(g(\phi(\delta),x_1),\ldots,g(\phi(\delta),x_n))=g(\phi(\delta),A(x_1,\ldots,x_n)),\quad\text{since }\phi\text{ is bijective},
\end{aligned}$$

showing that A is g-homogeneous.

$(\Leftarrow)$ Suppose that A is g-homogeneous. Thus, for all $\delta,x_1,\ldots,x_n\in[a,b]$, it holds that:

$$\begin{aligned}
A^{a,b}\left(g^{a,b}(\delta,x_1),\ldots,g^{a,b}(\delta,x_n)\right)&=A^{a,b}\left(\phi^{-1}(g(\phi(\delta),\phi(x_1))),\ldots,\phi^{-1}(g(\phi(\delta),\phi(x_n)))\right),\\
&\quad\text{by Eq.(29)}\\
&=\phi^{-1}\left(A\left(\phi\left(\phi^{-1}(g(\phi(\delta),\phi(x_1)))\right),\ldots,\phi\left(\phi^{-1}(g(\phi(\delta),\phi(x_n)))\right)\right)\right),\\
&\quad\text{by Eq.(30)}\\
&=\phi^{-1}(A(g(\phi(\delta),\phi(x_1)),\ldots,g(\phi(\delta),\phi(x_n)))),\quad\text{since }\phi\text{ is bijective},\\
&=\phi^{-1}(g(\phi(\delta),A(\phi(x_1),\ldots,\phi(x_n)))),\quad\text{since }A\text{ is }g-\text{homogeneous},\\
&=\phi^{-1}\left(g\left(\phi(\delta),\phi\left(\phi^{-1}(A(\phi(x_1),\ldots,\phi(x_n)))\right)\right)\right),\quad\text{since }\phi\text{ is bijective},\\
&=\phi^{-1}\left(g\left(\phi(\delta),\phi\left(A^{a,b}(x_1,\ldots,x_n)\right)\right)\right),\quad\text{by Eq.(30)}\\
&=g^{a,b}\left(\delta,A^{a,b}(x_1,\ldots,x_n)\right),\quad\text{by Eq.(29)}
\end{aligned}$$

which means that $A^{a,b}$ is $g^{a,b}$-homogeneous. □

Example 5.3.

i)Consider the bivariate arithmetic mean $BAM^{a,b}:[a,b]^2\to[a,b]$ given, for all $x,y\in[a,b]$ by $BAM^{a,b}(x,y)=\frac{x+y}{2}$. Then, the ($n$-ary) arithmetic mean, given by Eq. (3), is a $BAM^{a,b}$-homogeneous (a,b)-aggregation function;

ii)Consider the (a,b)-overlap function $BGM^{a,b}$, constructed via Theorem 4.2 with the overlap function $BGM:[0,1]^2\to[0,1]$, given, for all $x,y\in[0,1]$, by $BGM(x,y)=\sqrt{x\cdot y}$, as its core function. Also, consider the (a,b)-aggregation functions $MIN:[a,b]^n\to[a,b]$, given by Eq. (26) (minimum operator), and $MAX:[a,b]^n\to[a,b]$, given, for all $\vec{x}\in[a,b]^n$, by

$$MAX(\vec{x})=\max\{x_1,\ldots,x_n\}.$$

Then, MIN and MAX are $BGM^{a,b}$-homogeneous (a,b)-aggregation functions.

6. Towards $\mathcal{F}$-shifted (a, b, c, d)-fusion functions

In Section 3, we presented a framework to define new classes of functions with domain $[a,b]^n$ and codomain $[a,b]$ based on functions with domain $[0,1]^n$ and codomain $[0,1]$. That is, we showed how to define (a,b)-fusion functions based on fusion functions, by shifting their defining properties. Here, we discuss the concepts necessary to develop a similar framework to define classes of functions with domain $[a,b]^n$ and codomain $[c,d]$, such that $c,d \in \mathbb{R}$ and $c < d$. We call those functions as (a,b,c,d)-fusion functions.

Definition 6.1. An (a,b,c,d)-fusion function is an arbitrary function $F_{a,b}^{c,d} : [a,b]^n \to [c,d]$.

It is immediate that every fusion function is an (a,b,c,d)-fusion function for $a = c = 0$ and $b = d = 1$. Also, every (a,b)-fusion function is an (a,b,c,d)-fusion function when $a = c$ and $b = d$. So, every $(0,1,0,1)$-fusion function is called just as fusion function and every (a,b,a,b)-fusion function is called just as (a,b)-fusion function.

Properties from either fusion functions or (a,b)-fusion functions can be shifted to the context of (a,b,c,d)-fusion functions, by taking into consideration the domain $[a,b]^n$ and codomain $[c,d]$.

Example 6.1. Suppose that we intend to shift the property **(A2')** (see Example 3.1) that conveys the boundary conditions of an $(-10,10)$-aggregation function $F : [-10,10]^n \to [-10,10]$ to obtain an analogous property for a $(-10,10,0,10)$-fusion function $H : [-10,10]^n \to [0,10]$. The shifted property **(A2†)** is defined as follows:

(A2†) $A(-10,\ldots,-10) = 0$ and $A(10,\ldots,10) = 10$.

Based on Definition 3.4, we define (a,b,c,d)-aggregation functions in the following.

Definition 6.2. An (a,b,c,d)-aggregation function is any function $A_{a,b}^{c,d} \in \mathscr{A}_{a,b}^{c,d}$, such that:

$$\mathscr{A}_{a,b}^{c,d} = \left\{ A_{a,b}^{c,d} : [a,b]^n \to [c,d] \,|\, A_{a,b}^{c,d} \text{ satisfies all the properties in } P_{\mathscr{A}}^{\dagger} \right\}$$

where

$$P_{\mathscr{A}}^{\dagger} = \left\{ \left(\mathbf{A1}^{\dagger}\right), \left(\mathbf{A2}^{\dagger}\right) \right\},$$

and

(A1†)$A_{a,b}^{c,d}$ is increasing;
(A2†)$A_{a,b}^{c,d}(a,\ldots,a) = c$ and $A_{a,b}^{c,d}(b,\ldots,b) = d$.

Example 6.2. The bivariate $(-10,10,0,10)$-fusion function $H : [-10,10]^2 \to [0,10]$, given, for all $x, y \in [-10,10]$, by

$$H(x,y) = \frac{x + y + 20}{4}$$

is a bivariate $(-10,10,0,10)$-aggregation function.

The construction method for (a,b)-aggregation functions presented in Theorem 4.1 can be adapted to obtain a construction method for (a,b,c,d)-aggregation functions based on a core aggregation function.

Theorem 6.1. *Consider a fusion function $A : [0,1]^n \to [0,1]$, an increasing and bijective function $\phi : [a,b] \to [0,1]$, an increasing and bijective function $\psi : [0,1] \to [c,d]$ and an (a,b,c,d)-fusion function $A_{a,b}^{c,d} : [a,b]^n \to [c,d]$ given, for all $x_1,\ldots,x_n \in [a,b]$, by*

$$A_{a,b}^{c,d}(x_1,\ldots,x_n) = \psi(A(\phi(x_1),\ldots,\phi(x_n))). \tag{55}$$

Then, $A_{a,b}^{c,d}$ is an (a,b,c,d)-aggregation function if and only if A is an aggregation function.

Proof. Analogous to the proof of Theorem 4.1. □

Remark 6.1. Observe that Eq. (55) is more general than Eq. (30), even in the particular case when $[a,b] = [c,d]$, since ψ does not need to be the inverse of ϕ.

Example 6.3. Consider the geometric mean $GM : [0,1]^n \to [0,1]$, given by Eq. (6), an increasing and bijective function $\phi : [a,b] \to [0,1]$, defined, for all $x \in [a,b]$, by

$$\phi(x) = \left(\frac{x-a}{b-a}\right)^p,\ p > 0,$$

and an increasing and bijective function $\psi : [0,1] \to [c,d]$, defined, for all $y \in [0,1]$, by

$$\psi(y) = y^{\frac{1}{q}} \cdot (d-c) + c,\ q > 0.$$

Then, the (a,b,c,d)-fusion function $GM^{c,d}_{a,b} : [a,b]^n \to [c,d]$, given, for all $x_1,\ldots,x_n \in [a,b]$, by

$$GM^{c,d}_{a,b}(x_1,\ldots,x_n) = \psi(GM(\phi(x_1),\ldots,\phi(x_n))), \tag{56}$$

is an (a,b,c,d)-aggregation function. By taking $p = q = 1$, we can rewrite Eq. (56) as follows:

$$GM^{c,d}_{a,b}(x_1,\ldots,x_n) = GM\left(\frac{x_1-a}{b-a},\ldots,\frac{x_n-a}{b-a}\right) \cdot (d-c) + c. \tag{57}$$

In the following, we present a construction method for (a,b,c,d)-aggregation function based on a core (a,b)-aggregation function.

Theorem 6.2. *Consider an (a,b)-fusion function $A^{a,b} : [a,b]^n \to [a,b]$, an increasing and bijective function $\theta : [a,b] \to [c,d]$ and an (a,b,c,d)-fusion function $A^{c,d}_{a,b} : [a,b]^n \to [c,d]$ given, for all $x_1,\ldots,x_n \in [a,b]$, by*

$$A^{c,d}_{a,b}(x_1,\ldots,x_n) = \theta\Big(A^{a,b}(x_1,\ldots,x_n)\Big). \tag{58}$$

Then, $A^{c,d}_{a,b}$ is an (a,b,c,d)-aggregation function if and only if $A^{a,b}$ is an (a,b)-aggregation function.

Proof. Analogous to the proof of Theorem 4.1. □

Example 6.4. Consider the aggregation function $GM : [0,1]^n \to [0,1]$, given by Eq. (6), the (a,b)-aggregation function $GM^{a,b} : [a,b]^n \to [a,b]$, given by Eq. (33), and increasing and bijective function $\theta : [a,b] \to [c,d]$, defined, for all $x \in [a,b]$, by

$$\theta(x) = \left(\frac{x-a}{b-a}\right) \cdot (d-c) + c. \tag{59}$$

Then, the (a,b,c,d)-fusion function $GM^{c,d}_{a,b} : [a,b]^n \to [c,d]$, given, for all $x_1,\ldots,x_n \in [a,b]$, by

$$GM^{c,d}_{a,b}(x_1,\ldots,x_n) = \theta\Big(GM^{a,b}(x_1,\ldots,x_n)\Big), \tag{60}$$

is an (a,b,c,d)-aggregation function. From Eqs. (33), (59) and (60), one has that:

$$GM^{c,d}_{a,b}(x_1,\ldots,x_n) = GM\left(\frac{x_1-a}{b-a},\ldots,\frac{x_n-a}{b-a}\right) \cdot (d-c) + c. \tag{61}$$

One can observe that Eqs. (57) and (61) coincide. This fact is derived from the following theorem.

Theorem 6.3. *Let $AR^{c,d}_{a,b} : [a,b]^n \to [c,d]$ be an (a,b,c,d)-aggregation function constructed via Theorem 6.1 using increasing and bijective functions $\phi : [a,b] \to [0,1]$ and $\psi : [0,1] \to [c,d]$, and a core aggregation function $A : [0,1]^n \to [0,1]$. Let $AS^{c,d}_{a,b} : [a,b]^n \to [c,d]$ be an (a,b,c,d)-aggregation function constructed via Theorem 6.2 using an increasing and bijective function $\theta : [a,b] \to [c,d]$ and the core (a,b)-aggregation function $A^{a,b} : [a,b]^n \to [a,b]$, which, in turn, is constructed via Theorem 4.1 using ϕ and the core aggregation function A. Thus, if $\psi = \theta \circ \phi^{-1}$ then $AR^{c,d}_{a,b} = AS^{c,d}_{a,b}$.*

Proof. For all $\vec{x} \in [a,b]^n$, one has that:

$$\begin{aligned}
&\psi = \theta \circ \phi^{-1}\\
\Rightarrow &\psi(A(\phi(x_1),\ldots,\phi(x_n))) = (\theta \circ \phi^{-1})(A(\phi(x_1),\ldots,\phi(x_n)))\\
\Rightarrow &\psi(A(\phi(x_1),\ldots,\phi(x_n))) = \theta(\phi^{-1}(A(\phi(x_1),\ldots,\phi(x_n))))\\
\Rightarrow &\psi(A(\phi(x_1),\ldots,\phi(x_n))) = \theta\Big(A^{a,b}(x_1,\ldots,x_n)\Big), \quad \text{by Eq.}(30)\\
\Rightarrow &AR^{c,d}_{a,b} = AS^{c,d}_{a,b}, \quad \text{by Eqs.}(55) \text{ and } (58).
\end{aligned}$$

□

Theorem 6.3 shows that the diagram presented in Fig. 2 commutes, whenever $\psi = \theta \circ \phi^{-1}$.

7. An illustrative example

In this section, we show an illustrative example where (a, b)-fusion functions, which are constructed by the introduced construction methods and whose classes are defined through our presented framework, are applied as the pooling operator of a CNN to deal with an image classification problem.

First, we recall the main aspects of CNNs and point out how (a, b)-fusion functions (particularly, (a, b)-aggregation functions) are incorporated on their architecture.

7.1. Convolutional neural networks

CNNs have established themselves as the state-of-the-art technique for computer vision related tasks during the previous decade, since the breakthrough of Krizhevsky et al. [29]. Equipped with a sequential structure that extracts progressively more fine-grained features, as well as with a strong an efficient optimization algorithm, CNNs have achieved impressive results in image classification [23] and segmentation [3], among others tasks. In the context of image classification, CNNs are usually comprised of a feature extraction block, followed by a classifier (usually a Multilayer Perceptron).

The steps performed by a CNN are logically divided in sequential layers. The feature extraction process is taking care of in convolution layers. Each of these layers receives, either an input image or a "feature matrix" generated by a previous convolution layer, and computes the response of each local region of the input to a series of convolution "filters". The values of these filters represent possible visual features that may, or may not, be present in the input. They are randomly initialized and iteratively optimized in a supervised way using the gradient descent algorithm [42]. The convolution of each of these filters generates a "feature image" which represents the presence or absence of the represented feature in all parts of the input. All the feature images are concatenated with each other, generating a multidimensional feature matrix that is outputted to the following layer. Additionally, a non-linear function (or activation function) is applied to all the values of the feature matrix, similarly to other neural network architectures.

The number of filters in a convolution layer can be high, which would mean an increase in the dimensionality of the output feature matrix with respect to its input. This, in turn, would make the task of the network's classifier difficult, contradicting the objective of the feature extractor. In order to solve this problem, pooling layers are applied after convolution layers.

7.1.1. Pooling functions

Pooling layers reduce the dimensionality of a feature image via traditional image downsampling. That is, the feature image is separated in disjoint $k \times k$ windows (or submatrices) channelwise, and their values are aggregated using some function $f : \mathbb{R}^n \to \mathbb{R}$, where $n = k \cdot k$ is the number of values in each window to be aggregated.

Although some alternatives have been proposed [11], current implementations of CNNs usually employ the arithmetic mean or the maximum to aggregate these values. Other aggregation functions would be a direct alternative for these functions, but the fact that the data to be aggregated is real valued poses a challenge. However, if we take into account that, in practice these values are typically bound to an interval $[a, b] \in \mathbb{R}$, we can see how (a, b)-aggregation functions are a perfect candidate to substitute these classic pooling functions, with the intent to analyze their effect on the classification acuity of the system.

Thus, in this example, we apply (a, b)-t-norms (Definition 3.7), (a, b)-t-conorms (Definition 3.8) and (a, b)-uninorms (Definition 3.9) as the aggregation operator. Since all of these functions are associative, we can directly apply them to the multidimensional setup and aggregate the n values of each feature image window.

It is noteworthy that the classes of (a, b)-t-norms, (a, b)-t-conorms and (a, b)-uninorms were defined through the presented framework to preserve the constitutive properties of t-norms, t-conorms and unimors, respectively. Each of those classes of functions has a different nature in fuzzy logic, with t-norms acting in a conjunctive manner, t-conorms in a disjunctive one and uninorms combining both behaviours. An additional objective we pursue with this experiment is finding out if either of these behaviours is beneficial for the pooling process of a CNN.

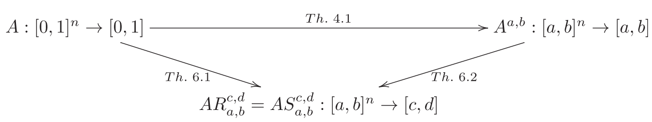

Fig. 2. Commutative diagram of the construction methods of an n-dimensional (a, b, c, d)-aggregation function based on a core aggregation function A.

7.2. Experimental setup

- **CNN architectures**: In order to exemplify the suitability of (a,b)-aggregation functions to CNNs, we have performed experiments replacing the pooling functions used by two different CNN architectures. The first one is the LeNet-5 architecture presented in [31], which represents the standard CNN model. LeNet-5 uses two convolutional layers followed by their respective average pooling layers, as well as a two layer perceptron classifier. We have additionally added batch normalization layers [25] after each pooling layer in order to soften the loss landscape and ease the learning process [44]. The second model is a "deeper" model presented in [32], which replaces convolution filters by multilayer perceptrons and the final classifier by a Global Average Pooling layer [34]. In order to mitigate the "vanishing gradient" problem that deep neural networks tend to face, hidden layer supervision [33] is used, which appends different classifiers at different levels of the model and combines the loss associated with the different predictions.
  We have set the ReLU activation function $f(x) = \max(0,x)$ as non-linearity for both models.
- **Dataset**: For our experimentation we have considered the CIFAR10 dataset [28], a fully balanced dataset composed of 60000 small 32×32 pixel color images. Images are divided into 10 classes corresponding to objects and animals of the real world, such as planes and birds. The dataset is divided into a 50000 image train partition, and a 10000 test one.
- **Pooling functions**: We have worked with different examples of (a,b)-aggregation functions of different nature. In particular, we have considered (a,b)-t-norms, (a,b)-t-conorms and (a,b)-uninorms, each one obtained by the construction methods presented in Theorems 4.4, 4.5 and 4.6, respectively. In every constructed function, we considered the basic bijection ϕ_A, given by Eq. (31). We set a and b to the minimum and maximum value of the input feature matrix, respectively. All the tested functions are presented in Table 1. In the "Core Function" column of this Table, we present the expression of each core aggregation function applied in the construction methods to obtain each correspondent (a,b)-aggregation function.
  We compare our results with the ones obtained using average and maximum pooling, since they are the most common pooling functions used in the literature.

7.3. Results

The testing results offered for each of the different trained models are presented in Table 2. We report the mean and standard deviation accuracy obtained after training 5 equivalent models with different random initializations.

Firstly, it becomes apparent that (a,b)-t-norms are poor candidates for pooling functions. We believe that their conjunctive nature, which results in smaller outputs, results counterproductive for preserving the higher activations obtained by convolution filters. Additionally, some functions result in classifiers with a 10% accuracy rate, which is equivalent for this dataset to a random classifier.

On the other hand, (a,b)-t-conorms achieve the best general results, with several of them clearly outperforming both average and maximum pooling (which is also an example of (a,b)-t-conorm). This leads us to believe that the disjunctive behaviour of these functions can be beneficial for CNNs. In particular, we point out that the best performing function on both architectures, $S_p^{a,b}$, was constructed with the probabilistic sum S_p as its core function, showing that our constructive framework can produce competitive functions for non-fuzzy applications based on well established functions that are used in fuzzy modeling.

Table 1
(a,b)-aggregation functions considered for substituting the pooling operation.

	Name	Core Function
(a,b)-t-norms	$T_p^{a,b}$	$T_P(x,y) = x \cdot y$
	$T_L^{a,b}$	$T_L(x,y) = \max\{x+y-b,0\}$
	$T_H^{a,b}$	$T_H(x,y) = \begin{cases} 0 & \text{if } x=y=0 \\ \frac{xy}{x+y-xy} & \text{otherwise} \end{cases}$
(a,b)-t-conorms	$S_p^{a,b}$	$S_p(x,y) = x+y-xy$
	$S_L^{a,b}$	$S_L(x,y) = \min\{x+y,1\}$
	$S_H^{a,b}$	$S_H(x,y) = \begin{cases} 1 & \text{if } xy=1 \\ \frac{2xy-x-y}{xy-1} & \text{otherwise} \end{cases}$
(a,b)-uninorms	$U_{min,max}^{a,b}$	$U_{min,max} = \begin{cases} \min(x,y) & \text{if } (x,y) \in [0,0.5]^2 \\ \max(x,y) & \text{otherwise} \end{cases}$
	$U_{T_p,S_p}^{a,b}$	$U_{T_p,S_p} = \begin{cases} 2xy & \text{if } (x,y) \in [0,0.5]^2 \\ 2x+2y-1-2xy & \text{if } (x,y) \in [0.5,1]^2 \\ \max(x,y) & \text{otherwise} \end{cases}$
	$U_{L,L}^{a,b}$	$U_{L,L} = \begin{cases} \max\{x+y-\frac{1}{2},0\} & \text{if } (x,y) \in [0,0.5]^2 \\ \min\{x+y-\frac{1}{2},1\} & \text{if } (x,y) \in [0.5,1]^2 \\ \max(x,y) & \text{otherwise} \end{cases}$

Table 2
Results obtained over two different architectures when using the different functions as pooling function.

	Function	Architecture 1	Architecture 2
	Avg	75.29 ± 0.56	86.24 ± 0.31
	Max	74.15 ± 0.11	86.31 ± 0.41
(a,b)-t-norms	$T_p^{a,b}$	53.05 ± 0.53	10.00 ± 0.00
	$T_L^{a,b}$	13.25 ± 4.29	10.00 ± 0.00
	$T_H^{a,b}$	65.33 ± 0.39	80.07 ± 1.32
(a,b)-t-conorms	$S_p^{a,b}$	$\mathbf{75.36 \pm 0.24}$	$\mathbf{87.92 \pm 0.29}$
	$S_L^{a,b}$	74.92 ± 0.41	87.71 ± 0.31
	$S_H^{a,b}$	74.94 ± 0.31	87.86 ± 0.32
(a,b)-uninorms	$U_{min,max}^{a,b}$	64.18 ± 0.27	85.86 ± 0.20
	$U_{T_p,S_p}^{a,b}$	50.25 ± 17.28	20.90 ± 9.08
	$U_{L,L}^{a,b}$	24.95 ± 15.72	23.56 ± 3.29

Finally, (a,b)-uninorms show a mixed but overall poor results, indicating that a pure disjunctive behaviour is preferable when working with this family of models.

8. Conclusion

In this paper, we sought to provide a theoretical tool set to support the definition of new classes of fusion operators that can aggregate data from any real closed interval, based on analogous known classes of such operators that are defined, specifically, on the unit interval. There are many practical applications that can benefit from the developed concepts, in particular with the assurance that the advantageous properties of known aggregation functions can be preserved (shifted) when applying the newly developed functions, even on problems that do not necessarily involve fuzzy modeling, such as image classification via CNNs.

Here, we review our main theoretical contributions:

- The introduction of the concept property shifting, which is a novel denomination for the action of properly transposing properties from one domain to another without sacrificing their fundamental characteristics;
- The development of a general framework for defining (a,b)-fusion functions, possibly in intervals other than $[0,1]$, by shifting the defining properties of known fusion functions;
- The introduction of construction methods for different subclasses of (a,b)-fusion functions, based on choices of a core fusion function and an increasing bijective function, which makes them highly adaptable and prone to be applied in different practical problems;
- The study of both known and newly defined properties of aggregation functions, along with their shifted counterparts in $[a,b]$, and how they are related when we construct (a,b)-aggregation functions via our construction methods;
- The development of a general framework for defining (a,b,c,d)-fusion functions, which is designed in an analogous manner as the one for (a,b)-fusion functions;
- The introduction of construction methods for (a,b,c,d)-aggregation functions, highlighting the different ways one can obtain a given (a,b,c,d)-aggregation function.

To showcase the applicability of the developed theoretical concepts, we presented an illustrative example in which (a,b)-aggregation functions ((a,b)-t-norms, (a,b)-t-conorms and (a,b)-uninorms) carried out the pooling process of a CNN applied in an image processing problem. We observed that (a,b)-t-conorms produced the best results, even surpassing the classic pooling operators, which motivate us to further develop this type of neural network with other types of (a,b)-fusion functions with similar behaviour. Moreover, the experiment showed that there is promise in applying known fuzzy operators (such as the probabilistic sum) as the core functions of constructed (a,b)-fusion functions in practical problems that do not necessarily involve fuzzy modeling.

Future works, regarding the theoretical standpoint, may include a deeper study of particular classes of (a,b)-fusion functions, defined through our framework, with special interest in cases in which the shifting of properties may not be trivial.

CRediT authorship contribution statement

Tiago da Cruz Asmus: Writing – original draft, Conceptualization, Methodology, Investigation, Writing – review & editing. **Graçaliz Pereira Dimuro:** Conceptualization, Methodology, Investigation, Writing – review & editing. **Benjamín Bedre-**

gal: Conceptualization, Methodology, Investigation, Writing – review & editing. **José Antonio Sanz:** Conceptualization, Methodology, Writing – review & editing. **Javier Fernandez:** Conceptualization, Methodology, Writing – review & editing. **Iosu Rodriguez-Martinez:** Investigation, Software, Writing – review & editing. **Radko Mesiar:** Conceptualization, Methodology, Writing – review & editing. **Humberto Bustince:** Conceptualization, Writing – review & editing, Supervision, Project administration.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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