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Large deviations for $(1+1)$-dimensional stochastic geometric wave equation

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Abstract

We consider stochastic wave map equation on real line with solutions taking values in a d-dimensional compact Riemannian manifold. We show first that this equation has unique, global, strong in PDE sense, solution in local Sobolev spaces. The main result of the paper is a proof of the Large Deviations Principle for solutions in the case of vanishing noise.

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Contents

1. Introduction

Stochastic PDEs for manifold-valued processes have attracted a great deal of attention due to their wide range of applications in physics, in particular in kinetic theory of phase transitions and quantum field theory, see e.g. Bruned et al. [6], the first and the second named authors [7–9], Carroll [23], Funaki [38] and Röckner et al. [58] and references therein. In this paper we are dealing with a particular stochastic PDE, known as a stochastic geometric wave equation (SGWE), that was introduced and studied by the first and the third named authors in a series of papers [15], [17,19], see also [18].

The aim of this paper is to prove a large deviations principle (LDP) for the one-dimensional stochastic wave equation with solutions taking values in a d-dimensional compact Riemannian manifold M. More precisely we will consider the equation

$$\mathbf{D}_t \partial_t u^\varepsilon = \mathbf{D}_x \partial_x u^\varepsilon + \sqrt{\varepsilon} Y_{u^\varepsilon}(\partial_t u^\varepsilon, \partial_x u^\varepsilon)\, \dot{W}, \tag{1.1}$$

where $\varepsilon \in (0,1]$ approaches zero. Here $\mathbf{D}$ is the connection on the pull-back bundle $u^{-1}TM$ of the tangent bundle over M induced by the Riemannian connection on M, see e.g. [16,60], Y is a non-linearity and W is a spatially homogeneous Wiener process on $\mathbb{R}$. A precise formulation is provided in Section 3. Here we only note that we will work with the extrinsic formulation of (1.1), that is, we assume M to be isometrically embedded into a certain Euclidean space $\mathbb{R}^n$, which holds true due to the celebrated Nash isometric embedding theorem [49]. Then, in view of Remark 2.5 in [15], equation (1.1) can be written in the form

$$\partial_{tt} u^\varepsilon = \partial_{xx} u^\varepsilon + A_{u^\varepsilon}(\partial_t u^\varepsilon, \partial_t u^\varepsilon) - A_{u^\varepsilon}(\partial_x u^\varepsilon, \partial_x u^\varepsilon) + \sqrt{\varepsilon} Y_{u^\varepsilon}(\partial_t u^\varepsilon, \partial_x u^\varepsilon)\, \dot{W}, \tag{1.2}$$

where A is the second fundamental form of the submanifold $M \subseteq \mathbb{R}^n$. More details about the equivalence of extrinsic and intrinsic formulations of stochastic PDEs can be found in Sections 2 and 12 of [15]. Moreover, using the notation $z^\varepsilon = (u^\varepsilon, \partial_t u^\varepsilon)$ equation (1.2) can be formally written as the following stochastic evolution equation in a suitable topological vector space,

$$dz^\varepsilon(t) = (\mathcal{G}z^\varepsilon(t) + \mathbf{F}(z^\varepsilon(t)))dt + \sqrt{\varepsilon}\mathbf{G}(z^\varepsilon(t))dW(t), \quad t \in [0, T], \tag{1.3}$$

with a generator $\mathcal{G}$ of some C_0-group, and the drift and diffusion operators $\mathbf{F}$ and $\mathbf{G}$, see Proposition 3.11 and Remark 4.4 for details.

Due to its importance for applications, LDP for stochastic PDEs has been widely studied by many authors, see e.g. [35] and [36]. However, analysis of large deviations for stochastic PDEs for manifold-valued processes is very little understood. To the best of our knowledge, LDP has only been established for the stochastic Landau-Lifshitz-Gilbert equation with solutions taking values in the two dimensional sphere [9]. Our paper is the first to study LDP for SGWE. One should also mention a PhD thesis by Hussain [42], see also [11], who has established the LDP for stochastic heat equation with codimension one constraint.

If $\varepsilon = 0$ then equation (1.2) reduces to a deterministic wave maps equation. It has been intensively studied in recent years due to its importance in field theory and general relativity, see for example [39] and references therein. It turns out that solutions to the deterministic geometric wave equation can exhibit a very complex behavior including (multiple) blowups and shrinking and expanding bubbles, see [3,4]. In some cases the Soliton Resolution Conjecture has been proved, see [43]. Various concepts of stability of these phenomena, including the stability of soliton solutions has also been intensely studied [29]. It seems natural to investigate stability for wave maps by studying the impact of small random perturbations and this idea leads to equation (1.2). Let us recall that the stability of solitons under the influence of noise has already been studied by means of LDP for the Schrödinger equations, see [28]. LDP, once established, will provide a tool for more precise analysis of the stability of wave maps.

Finally, let us recall that in [46] large deviations techniques are applied to derive a rigorous connection between the Yang-Mills measure and the energy functional. While in our work the problem is much easier because of the assumed regularity of the noise, we believe we provide a starting point for an analogous result in the case of less regular noises. Equations of stochastic flows for harmonic maps with very irregular noise have been recently proposed in [6] and [58].

Another motivation for studying equation (1.2) with $\epsilon > 0$ comes from the Hamiltonian structure of deterministic wave equation. Deterministic Hamiltonian systems may have infinite number of invariant measures and are not ergodic, see the discussion of this problem in [32]. Characterisation of such systems is a long standing problem. The main idea, which goes back to Kolmogorov-Eckmann-Ruelle, is to choose a suitable small random perturbation such that the solution to stochastic system is a Markov process with the unique invariant measure and then one can select a "physical" invariant measure of the deterministic system by taking the limit of vanishing noise, see for example [27], where this idea is applied to wave maps. A finite dimensional toy example was studied in [2].

Our proof of the large deviations principle relies on the weak convergence method introduced in [21] and is based on a variational representation formula for certain functionals of the driving infinite dimensional Brownian motion. However, the approach of [21] can not be directly applied to the SGWE and requires a number of modifications, see Section 5 below.

Recently in [61] the authors have established an LDP for a certain class of Banach space valued stochastic differential equations by a different method, but their argument does not apply to SGWE studied in this paper because, for example the wave operator does not generate a compact C_0-semigroup.

Finally, we note that the approach we developed in this paper can be applied to a number of problems that are open at present, including the beam equation studied in [14], and the nonlinear wave equation with polynomial nonlinearity and spatially homogeneous noise. In particular, this method would generalize the results of [52] and [64]. Our approach would also lead to an extension of the work of Martirosyan [48] who considers a nonlinear wave equations on a bounded domain. We believe that the methods of the present work will allow us to obtain the large deviations principle for the family of stationary measures generated by the flow of stochastic wave equation, with multiplicative white noise, in non-local Sobolev spaces over the full space $\mathbb{R}^d$.

The organisation of the paper is as follows. In Section 2, we introduce our notation and state the definitions used in the paper. Section 3 contains some properties of the nonlinear drift terms and the diffusion coefficient that we need later. In Section 4 we prove the existence of a unique global and strong in PDE sense solution to the skeleton equation associated to (1.2). The proof of Large Deviations Principle, based on weak convergence approach, is provided in Section 5. In Appendix A, we recall the intrinsic and extrinsic formulation of SGWE from [15] and state, without proof, an equivalence result between them. We conclude the paper with Appendices B and C, where we state modified version of the existing results on global well-posedness of (1.2) and energy inequality from [15] that we use frequently in the paper.

Finally, let us point out that the current paper is an expanded and corrected version of paper [10].

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2. Notation

For any two non-negative quantities a and b, we write $a \lesssim b$ if there exists a universal constant $c > 0$ such that $a \le cb$, and we write $a \simeq b$ when $a \lesssim b$ and $b \lesssim a$. In case we want to emphasize the dependence of c on some parameters $a_1, \dots, a_k$, then we write, respectively, $\lesssim_{a_1,\dots,a_k}$ and $\simeq_{a_1,\dots,a_k}$. We will denote by $B_R(a)$, for $a \in \mathbb{R}$ and $R > 0$, the open ball in $\mathbb{R}$ with center at a and we put $B_R = B_R(0)$. Now we list the notation used throughout the whole paper.

- $\mathbb{N} = \{0, 1, \cdots\}$ denotes the set of natural numbers, $\mathbb{R}_+ = [0, \infty)$, Leb denotes the Lebesgue measure.
- Let $I \subseteq \mathbb{R}$ be an open interval. By $L^p(I; \mathbb{R}^n)$, $p \in [1, \infty)$, we denote the classical real Banach space of all (equivalence classes of) $\mathbb{R}^n$-valued p-integrable maps on I. The norm on $L^p(I; \mathbb{R}^n)$ is given by

$$\|u\|_{L^p(I;\mathbb{R}^n)} := \left(\int_I |u(x)|^p \, dx\right)^{\frac{1}{p}}, \qquad u \in L^p(I;\mathbb{R}^n), \tag{2.1}$$

where $|\cdot|$ is Euclidean norm on $\mathbb{R}^n$. For $p=\infty$, we consider the usual modification to essential supremum.

- For any $p \in [1,\infty]$, $L^p_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n)$ stands for a metrizable topological vector space equipped with a natural countable family of seminorms $\{p_j\}_{j\in\mathbb{N}}$ defined by

$$p_j(u) := \|u\|_{L^p(B_j;\mathbb{R}^n)}, \qquad u \in L^2_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n),\ j \in \mathbb{N}. \tag{2.2}$$

- By $H^{k,p}(I;\mathbb{R}^n)$, for $p \in [1,\infty]$ and $k \in \mathbb{N}$, we denote the Banach space of all $u \in L^p(I;\mathbb{R}^n)$ for which $D^j u \in L^p(I;\mathbb{R}^n)$, $j = 0,1,\dots,k$, where D^j is the weak derivative of order j. The norm here is given by

$$\|u\|_{H^{k,p}(I;\mathbb{R}^n)} := \left(\sum_{j=0}^{k} \|D^j u\|^p_{L^p(I;\mathbb{R}^n)}\right)^{\frac{1}{p}}, \qquad u \in H^{k,p}(I;\mathbb{R}^n). \tag{2.3}$$

- We write $H^{k,p}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n)$, for $p \in [1,\infty]$ and $k \in \mathbb{N}$, to denote the space of all elements $u \in L^p_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n)$ whose weak derivatives up to order k belong to $L^p_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n)$. It is relevant to note that $H^{k,p}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n)$ is a metrizable topological vector space equipped with the following natural countable family of seminorms $\{q_j\}_{j\in\mathbb{N}}$,

$$q_j(u) := \|u\|_{H^{k,p}(B_j;\mathbb{R}^n)}, \qquad u \in H^{k,p}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n),\ j \in \mathbb{N}. \tag{2.4}$$

The spaces $H^{k,2}(I;\mathbb{R}^n)$ and $H^{k,2}_{loc}(\mathbb{R};\mathbb{R}^n)$ are usually denoted by $H^k(I;\mathbb{R}^n)$ and $H^k_{loc}(\mathbb{R};\mathbb{R}^n)$ respectively.

- We set

$$\mathcal{H} := H^2(\mathbb{R};\mathbb{R}^n) \times H^1(\mathbb{R};\mathbb{R}^n), \quad \mathcal{H}_{\mathrm{loc}} := H^2_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n) \times H^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n). \tag{2.5}$$

- To shorten the notation in calculation we set the following rules:
  - if the space where function is taking value, for example $\mathbb{R}^n$, is clear then to save the space we will omit $\mathbb{R}^n$, for example $H^k(I)$ instead $H^k(I;\mathbb{R}^n)$;
  - if $I = (0,T)$ or $(-R,R)$ or $B(x,R)$, for some $T, R > 0$ and $x \in \mathbb{R}$, then instead of $L^p(I;\mathbb{R}^n)$ we write, respectively, $L^p(0,T;\mathbb{R}^n)$, $L^p(B_R;\mathbb{R}^n)$, $L^p(B(x,R);\mathbb{R}^n)$. Similarly for H^k and H^k_{loc} spaces.
  - write $\mathcal{H}(B_R)$ or $\mathcal{H}_R$ for $H^2((-R,R);\mathbb{R}^n) \times H^1((-R,R);\mathbb{R}^n)$.
- For any nonnegative integer j, let $\mathcal{C}^j(\mathbb{R})$ be the space of real valued continuous functions whose derivatives up to order j are continuous on $\mathbb{R}$. We also need the family of spaces $\mathcal{C}^j_b(\mathbb{R})$ defined by

$$\mathcal{C}^j_b(\mathbb{R}) := \left\{u \in \mathcal{C}^j(\mathbb{R}); \forall \alpha \in \mathbb{N}, \alpha \le j, \exists K_\alpha, \|D^j u\|_{L^\infty(\mathbb{R})} < K_\alpha\right\}.$$

For $j=0$ we will write $\mathcal{C}_b(\mathbb{R})$ instead $\mathcal{C}^0_b(\mathbb{R})$.

- Given $T > 0$ and a real Banach space E, we denote by $\mathcal{C}([0, T]; E)$ the real Banach space of all E-valued continuous functions $u : [0, T] \to E$ endowed with the norm

$$\|u\|_{\mathcal{C}([0,T];E)} := \sup_{t\in[0,T]} \|u(t)\|_E, \qquad u \in \mathcal{C}([0, T]; E).$$

  By ${}_0\mathcal{C}([0, T]; E)$ we mean the set of elements of $\mathcal{C}([0, T]; E)$ vanishes at origin, that is,

$${}_0\mathcal{C}([0, T]; E) := \{u \in \mathcal{C}([0, T]; E) : u(0) = 0\}.$$

- For given metric space (X, ρ), by $\mathcal{C}(\mathbb{R}; X)$ we mean the space of continuous functions from $\mathbb{R}$ to X which is equipped with the metric

$$(f, g) \mapsto \sum_{j=1}^{\infty} \frac{1}{2^j} \min\{1, \sup_{t\in[-j,j]} \rho(f(t), g(t))\}.$$

- We denote the tangent and the normal bundle of a smooth manifold M by TM and NM, respectively. Let $\mathfrak{F}(M)$ be the set of all smooth $\mathbb{R}$-valued function on M.
- A map $u : \mathbb{R} \to M$ belongs to $H^k_{loc}(\mathbb{R}; M)$ provided that $\theta \circ u \in H^k_{loc}(\mathbb{R}; \mathbb{R})$ for every $\theta \in \mathfrak{F}(M)$. We equip $H^k_{loc}(\mathbb{R}; M)$ with the topology induced by the mappings

$$H^k_{loc}(\mathbb{R}; M) \ni u \mapsto \theta \circ u \in H^k_{loc}(\mathbb{R}; \mathbb{R}), \quad \theta \in \mathfrak{F}(M).$$

  Since the tangent bundle TM of a manifold M is also a manifold, this definition covers Sobolev spaces of TM-valued maps too.
- By $\mathcal{L}(X, Y)$ we denote the space of all linear continuous operators from a topological vector space X to Y. If H_1, H_2 are two separable Hilbert spaces then $\mathscr{L}_2(H_1, H_2) \subset \mathcal{L}(H_1, H_2)$ will denote the space of Hilbert–Schmidt operators acting from H_1 to H_2.
- We denote by $\mathcal{S}(\mathbb{R})$ the space of Schwartz functions on $\mathbb{R}$ and write $\mathcal{S}'(\mathbb{R})$ for its dual, which is the space of tempered distributions on $\mathbb{R}$. By L^2_λ we denote the weighted space $L^2(\mathbb{R}, \lambda)$, where $d\lambda(x) := e^{-x^2} dx$, $x \in \mathbb{R}$.

3. Preliminaries

In this section we discuss all the required preliminaries about the nonlinearity and the diffusion coefficient that we need in Section 4. We are following Sections 3 to 5 of [15] very closely here. Below we use the notation $\mathcal{F}(\cdot)$, along with $\widehat{\cdot}$, to denote the Fourier transform.

3.1. The Wiener process

Let μ be a symmetric Borel measure on $\mathbb{R}$. The random forcing we consider is in the form of a spatially homogeneous Wiener process on $\mathbb{R}$ with a spectral measure μ satisfying

$$\int_{\mathbb{R}} (1 + |x|^2)^2 \, \mu(dx) < \infty. \tag{3.1}$$

By $L^2(\mathbb{R}, \mu, \mathbb{C})$ we denote the Banach space of complex-valued functions that are square integrable with respect to the measure μ.

Definition 3.1. An $\mathcal{S}'(\mathbb{R})$-valued process $W = \{W(t), t \geq 0\}$, on a given stochastic basis $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \mathbb{P})$, is called a spatially homogeneous Wiener process with spectral measure μ provided that

(1) for every $\varphi \in \mathcal{S}(\mathbb{R})$, $\{W(t)(\varphi), t \geq 0\}$ is a real-valued $(\mathfrak{F}_t)$-adapted Wiener process,
(2) $W(t)(a\varphi + \psi) = aW(t)(\varphi) + W(t)(\psi)$ holds almost surely for every $t \geq 0$, $a \in \mathbb{R}$ and $\varphi, \psi \in \mathcal{S}(\mathbb{R})$,
(3) for every $t \geq 0$ and $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R})$, $\mathbb{E}\,[W(t)(\varphi_1)W(t)(\varphi_1)] = t\langle\hat{\varphi}_1, \hat{\varphi}_2\rangle_{L^2(\mu)}$.

It is shown in [56] that the Reproducing Kernel Hilbert Space (RKHS) H_μ of the Gaussian distribution of $W(1)$ is given by

$$H_\mu := \left\{\widehat{\psi\mu} : \psi \in L^2(\mathbb{R}, \mu, \mathbb{C}), \psi(x) = \overline{\psi(-x)}, x \in \mathbb{R}\right\}.$$

Note that H_μ endowed with inner-product

$$\left\langle\widehat{\psi_1\mu}, \widehat{\psi_2\mu}\right\rangle_{H_\mu} := \int_{\mathbb{R}} \psi_1(x)\overline{\psi_2(x)}\,\mu(dx),$$

is a Hilbert space.

Recall from [56,57] that W can be regarded as a cylindrical Wiener process on H_μ and it takes values in any Hilbert space E, such that the embedding $H_\mu \hookrightarrow E$ is Hilbert-Schmidt. Since we explicitly know the structure of H_μ, the next result, whose proof is based on [54, Lemma 2.2] and discussion with Szymon Peszat [55] shows that assumption (3.1) is equivalent to saying that the paths of W belong to $\mathcal{C}([0, T]; H^2_\lambda(\mathbb{R}))$, where the space $H^s_\lambda(\mathbb{R}), s \geq 0$, is defined as the completion of $\mathcal{S}(\mathbb{R})$ with respect to the norm

$$\|u\|_{H^s_\lambda(\mathbb{R})} := \left(\int_{\mathbb{R}} (1 + |x|^2)^s |\mathcal{F}(\lambda^{1/2}u)(x)|^2\,dx\right)^{\frac{1}{2}}, \tag{3.2}$$

where $\mathcal{F}$ denoted the Fourier transform and, with a slight abuse of notation, $\lambda(x) = e^{-x^2}$, $x \in \mathbb{R}$, denotes the density of the measure λ.

Lemma 3.2. *Let us assume that the measure μ satisfies (3.1). Then the identity map from H_μ into $H^2_\lambda(\mathbb{R})$ is a Hilbert-Schmidt operator.*

Proof of Lemma 3.2. To simplify the notation we set

$$L^2_{(s)}(\mathbb{R}, \mu) := \{f \in L^2(\mathbb{R}, \mu; \mathbb{C}) : f(x) = \overline{f(-x)}, \ \ \forall x \in \mathbb{R}\}.$$

Let $\{e_k\}_{k\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R})$ be an orthonormal basis of $L^2_{(s)}(\mathbb{R},\mu)$. Then, by the definition of H_μ, $\{\mathcal{F}(e_k\mu)\}_{k\in\mathbb{N}}$ is an orthonormal basis of H_μ. Invoking the convolution property of the Fourier transform and the Bessel inequality, we obtain,

$$
\begin{aligned}
\sum_{k=1}^{\infty} \|\widehat{e_k\mu}\|^2_{H^2_\lambda} &= \sum_{k=1}^{\infty} \int_{\mathbb{R}} (1+|x|^2)|\mathcal{F}\left(\lambda^{1/2}\mathcal{F}(e_k\mu)\right)(x)|^2\,dx \\
&\le \int_{\mathbb{R}} (1+|x|^2)^2 \left(\sum_{k=1}^{\infty} |\mathcal{F}\left(\lambda^{1/2}\mathcal{F}(e_k\mu)\right)(x)|^2\right) dx \\
&= \int_{\mathbb{R}} (1+|x|^2)^2 \left(\sum_{k=1}^{\infty} \left|\int_{\mathbb{R}} \mathcal{F}\left(\lambda^{1/2}\right)(x-z)e_k(z)\,\mu(dz)\right|^2\right) dx \\
&\le \int_{\mathbb{R}^2} (1+|x|^2)^2 |\mathcal{F}\left(\lambda^{1/2}\right)(x-z)|^2\,\mu(dz)\,dx \\
&= \int_{\mathbb{R}^2} (1+|x+z|^2)^2 |\mathcal{F}\left(\lambda^{1/2}\right)(x)|^2\,\mu(dz)\,dx \\
&\lesssim \|\lambda^{1/2}\|^2_{H^1_\lambda(\mathbb{R})} \int_{\mathbb{R}} (1+|z|^2)^2\,\mu(dz).
\end{aligned}
$$

Hence Lemma 3.2. $\square$

It is relevant to note here that $H^2_\lambda(\mathbb{R})$ is a subset of $H^2_{\mathrm{loc}}(\mathbb{R})$ and the embedding is continuous.

Remark 3.3. It is important to note that all the results of this paper are valid for any Wiener process which takes values in the space $H^2_\lambda(\mathbb{R})$ not just for the Wiener process which is space homogeneous. However, in the case of space homogeneity, the solution process will be space homogeneous if the initial data is space homogeneous.

The next result, whose detailed proof can be found in [51, Lemma 1], plays very important role in deriving the required estimates for the terms involving diffusion coefficient.

Lemma 3.4. *If the measure μ satisfies (3.1), then H_μ is continuously embedded in $\mathcal{C}^2_b(\mathbb{R})$. Moreover, for given $g \in H^j(B(x,R);\mathbb{R}^n)$, where $x\in\mathbb{R}$, $R>0$ and $j\in\{0,1,2\}$, the multiplication operator*

$$
H_\mu \ni \xi \mapsto g\cdot\xi \in H^j(B(x,R);\mathbb{R}^n),
$$

is Hilbert-Schmidt and $\exists\, c>0$, independent of R, x, g, ξ and j, such that

$$
\|\xi \mapsto g\cdot\xi\|_{\mathscr{L}_2(H_\mu,H^j(B(x,R);\mathbb{R}^n))} \le c\|g\|_{H^j(B(x,R);\mathbb{R}^n)}.
$$

Remark 3.5. Note that the constant c of inequality in Lemma 3.4 does not depend on the size and position of the ball. However, if we consider a cylindrical Wiener process, then c will also depend on the center x but will be bounded on bounded sets with respect to x.

3.2. Extensions of non-linear term

Recall that M is isometrically embedded into a certain Euclidean space $\mathbb{R}^n$ and $T_pM \subseteq \mathbb{R}^n$ and $N_pM \subseteq \mathbb{R}^n$ are the tangent and the normal vector spaces at $p \in M$, respectively. Further recall that A is the second fundamental form tensor of $M \subseteq \mathbb{R}^n$. Thus, for each $p \in M$, $A_p : T_pM \times T_pM \to N_pM$. It is well known, see e.g. [41], that A_p, $p \in M$, is a symmetric bilinear form.

Since we are following the approach of [7], [15], and [40], one of the main steps in the proof of the existence theorem is to consider the problem (1.2) in the ambient space $\mathbb{R}^n$ with an appropriate extension of A and Y from their domain to $\mathbb{R}^n$. In this section we discuss two extensions of A which work fine in the context of stochastic wave map, as displayed in [15].

Remark 3.6. Let us note that we only prove the existence of a global solution to SPDE in the Euclidean space, obtained by considering suitable extensions of A and Y to $\mathbb{R}^n$, for the initial data taking values in the manifold M and not for an arbitrary initial data. On the other hand, it is possible to prove that the approximating equation (4.10) has a local solution for every initial data. But we don't know if this solution is global unless the initial data takes values in M.

The same remarks apply to the skeleton equation (4.10) studied in Section 4.

Let us denote by $\mathcal{E}$ the exponential function

$$T\mathbb{R}^n \ni (p, \xi) \mapsto p + \xi \in \mathbb{R}^n,$$

relative to the Riemannian manifold $\mathbb{R}^n$ equipped with the standard Euclidean metric. The proof of the following proposition about the existence of an open set O containing M, which is called a tubular neighborhood of M, can be found in [53, Proposition 7.26, p. 200].

Proposition 3.7. *There exists an $\mathbb{R}^n$-open neighborhood O around M and an NM-open neighborhood V around the set $\{(p, 0) \in NM : p \in NM\}$ such that the restriction of the exponential map $\mathcal{E}|_V : V \to O$ is a diffeomorphism. Moreover, the neighborhood V can be chosen in such a way that $(p, t\xi) \in V$ whenever $t \in [-1, 1]$ and $(p, \xi) \in V$.*

In case of no ambiguity, we will denote the diffeomorphism $\mathcal{E}|_V : V \to O$ by $\mathcal{E}$. By using the Proposition 3.7, diffeomorphism $i : NM \ni (p, \xi) \mapsto (p, -\xi) \in NM$ and the standard argument of partition of unity, one can construct a function $\Upsilon : \mathbb{R}^n \to \mathbb{R}^n$ which identifies the manifold M as its fixed point set. To be precise we have the following result.

Lemma 3.8 *([15, Corollary 3.4 and Remark 3.5]). There exists a smooth compactly supported function $\Upsilon : \mathbb{R}^n \to \mathbb{R}^n$ which has the following properties:*

(1) *restriction of Υ on O is a diffeomorphism,*
(2) *$\Upsilon\big|_O = \mathcal{E} \circ i \circ \mathcal{E}^{-1} : O \to O$ is an involution on the tubular neighborhood O of M,*
(3) *$\Upsilon(\Upsilon(q)) = q$ for every $q \in O$,*

(4) *if $q \in O$, then $\Upsilon(q) = q$ if and only if $q \in M$,*
(5) *if $p \in M$, then*

$$\Upsilon'(p)\xi = \begin{cases} \xi, & \text{provided } \xi \in T_pM, \\ -\xi & \text{provided } \xi \in N_pM. \end{cases}$$

The following result is the first extension of the second fundamental form that we use in this paper.

Proposition 3.9 *([15, Proposition 3.6]). If we define*

$$B_q(a,b) = \sum_{i,j=1}^{n} \frac{\partial^2 \Upsilon}{\partial q_i \partial q_j}(q) a_i b_j = \Upsilon''_q(a,b), \qquad q \in \mathbb{R}^n, \quad a,b \in \mathbb{R}^n, \tag{3.3}$$

and

$$\mathcal{A}_q(a,b) = \frac{1}{2} B_{\Upsilon(q)}(\Upsilon'(q)a, \Upsilon'(q)b), \qquad q \in \mathbb{R}^n, \quad a,b \in \mathbb{R}^n, \tag{3.4}$$

then, for every $p \in M$,

$$\mathcal{A}_p(\xi,\eta) = A_p(\xi,\eta), \ \xi,\eta \in T_pM,$$

and

$$\mathcal{A}_{\Upsilon(q)}(\Upsilon'(q)a, \Upsilon'(q)b) = \Upsilon'(q)\mathcal{A}_q(a,b) + B_q(a,b), \ q \in O, \ a,b \in \mathbb{R}^n. \tag{3.5}$$

Along with the extension $\mathcal{A}$, defined by formula (3.4), we also need the extension $\mathscr{A}$, defined by formula (3.6), of the second fundamental form tensor A which will be perpendicular to the tangent space.

Proposition 3.10 *([15, Proposition 3.7]). Consider the function*

$$\mathscr{A} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \ni (q,a,b) \mapsto \mathscr{A}_q(a,b) \in \mathbb{R}^n,$$

defined by formula

$$\mathscr{A}_q(a,b) = \sum_{i,j=1}^{n} a_i v_{ij}(q) b_j = A_q(\pi_q(a), \pi_q(b)), \qquad q \in \mathbb{R}^n, \quad a \in \mathbb{R}^n, \quad b \in \mathbb{R}^n, \tag{3.6}$$

where π_p, $p \in M$, is the orthogonal projection of $\mathbb{R}^n$ onto T_pM, and v_{ij}, for $i,j \in \{1,\dots,n\}$, are smooth and symmetric (i.e. $v_{ij} = v_{ji}$) extensions of $v_{ij}(p) := A_p(\pi_p e_i, \pi_p e_j)$ to ambient space $\mathbb{R}^n$. Then $\mathscr{A}$ satisfies the following:

(1) *$\mathscr{A}$ is smooth in (q,a,b) and symmetric in (a,b) for every q,*
(2) *$\mathscr{A}_p(\xi,\eta) = A_p(\xi,\eta)$ for every $p \in M$, $\xi,\eta \in T_pM$,*
(3) *$\mathscr{A}_p(a,b)$ is perpendicular to T_pM for every $p \in M$, $a,b \in \mathbb{R}^n$.*

3.3. The C_0-group and the extension operators

In this subsection we recall some facts on infinitesimal generators of the linear wave equation and on the extension operators in various Sobolev spaces, see [15, Section 5] for details.

Proposition 3.11. *Assume that $k, n \in \mathbb{N}$. The one parameter family of operators (S_t), $t \in \mathbb{R}$ defined by*

$$S_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos[t(-\Delta)^{1/2}]u^1 & + & (-\Delta)^{-1/2}\sin[t(-\Delta)^{1/2}]v^1 \\ & \vdots & \\ \cos[t(-\Delta)^{1/2}]u^n & + & (-\Delta)^{-1/2}\sin[t(-\Delta)^{1/2}]v^n \\ -(-\Delta)^{1/2}\sin[t(-\Delta)^{1/2}]u^1 & + & \cos[t(-\Delta)^{1/2}]v^1 \\ & \vdots & \\ -(-\Delta)^{1/2}\sin[t(-\Delta)^{1/2}]u^n & + & \cos[t(-\Delta)^{1/2}]v^n \end{pmatrix} \tag{3.7}$$

is a C_0-group on

$$\mathcal{H}^k := H^{k+1}(\mathbb{R}; \mathbb{R}^n) \times H^k(\mathbb{R}; \mathbb{R}^n),$$

and its infinitesimal generator is an operator $\mathcal{G}^k = \mathcal{G}$ defined by

$$D(\mathcal{G}^k) = H^{k+2}(\mathbb{R}; \mathbb{R}^n) \times H^{k+1}(\mathbb{R}; \mathbb{R}^n),$$
$$\mathcal{G}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ \Delta u \end{pmatrix}.$$

Remark 3.12. It is enlightening to observe that the C_0 group $(S_t)_{t\in\mathbb{R}}$ on the Hilbert space $\mathcal{H}^k$ defined above in (3.7) has a unique extension to a C_0 group $(\tilde{S}_t)_{t\in\mathbb{R}}$ on the topological vector space $\mathcal{H}^k_{\mathrm{loc}} := H^{k+1}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^n) \times H^k_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^n)$. In particular, on the space $\mathcal{H}_{\mathrm{loc}} = \mathcal{H}^1_{\mathrm{loc}}$.

The following result is well known, see e.g. [47] and [34, Section II.5.4].

Proposition 3.13. *Let $k \in \mathbb{N}$. There exists a linear bounded operator*

$$E^k : H^k((-1,1); \mathbb{R}^n) \to H^k(\mathbb{R}; \mathbb{R}^n),$$

such that

(i) *$E^k f = f$ almost everywhere on $(-1,1)$ whenever $f \in H^k((-1,1); \mathbb{R}^n)$,*
(ii) *$E^k f$ vanishes outside of $(-2,2)$ whenever $f \in H^k((-1,1); \mathbb{R}^n)$,*
(iii) *$E^k f \in \mathcal{C}^k(\mathbb{R}; \mathbb{R}^n))$, if $f \in \mathcal{C}^k([-1,1]; \mathbb{R}^n))$,*
(iv) *if $j \in \mathbb{N}$ and $j < k$, then there exists a unique extension of E^k to a bounded linear operator from $H^j((-1,1); \mathbb{R}^n)$ to $H^j(\mathbb{R}; \mathbb{R}^n)$.*

Definition 3.14. For $k \in \mathbb{N}$, $r > 0$ we define the operators

$$E^k_r : H^j((-r,r); \mathbb{R}^n) \to H^j(\mathbb{R}; \mathbb{R}^n), \qquad j \in \mathbb{N}, j \le k,$$

called as r-scaled E^k operators, by the following formula

$$(E_r^k f)(x) = \{E^k[y \mapsto f(yr)]\}\left(\frac{x}{r}\right), \qquad x \in \mathbb{R}, \tag{3.8}$$

for $r > 0$ and $f \in H^k((-r, r); \mathbb{R}^n)$.

The following remark will be useful in Lemma 4.7.

Remark 3.15. We can rewrite (3.8) as $(E_r^k f)(x) = (E^k f_r)(\frac{x}{r})$, $f \in H^k((-r, r); \mathbb{R}^n)$ where $f_r : (-1, 1) \ni y \mapsto f(yr) \in \mathbb{R}^n$. Also, observe that for $f \in H^1((-r, r); \mathbb{R}^n)$

$$\|f_r\|^2_{H^1((-1,1);\mathbb{R}^n)} \le (r^{-1} + r)\|f\|^2_{H^1((-r,r);\mathbb{R}^n)}.$$

3.4. The diffusion coefficient

In this subsection we discuss the assumptions on diffusion coefficient Y which we only need in Section 4. It is relevant to note that due to a technical issue, which is explained in Section 5, we need to consider stricter conditions on Y in establishing the large deviation principle for (1.2). Here $Y_p : T_pM \times T_pM \to T_pM$, for $p \in M$, is a mapping satisfying,

$$|Y_p(\xi, \eta)|_{T_pM} \le C_Y(1 + |\xi|_{T_pM} + |\eta|_{T_pM}), \qquad p \in M, \quad \xi, \eta \in T_pM,$$

for some constant $C_Y > 0$ which is independent of p. By invoking Lemma 3.8 and [15, Proposition 3.10], we can extend the noise coefficient to map $Y : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \ni (p, a, b) \mapsto Y_p(a, b) \in \mathbb{R}^n$ which satisfies the following:

Y.1 for $q \in O$ and $a, b \in \mathbb{R}^n$,

$$Y_{\Upsilon(q)}\left(\Upsilon'(q)a, \Upsilon'(q)b\right) = \Upsilon'(q)Y_q(a, b), \tag{3.9}$$

Y.2 there exists a compact set $K_Y \subset \mathbb{R}^n$ containing M such that $Y_p(a, b) = 0$, for all $a, b \in \mathbb{R}^n$, whenever $p \notin K_Y$,

Y.3 Y is of $\mathcal{C}^2$-class and there exist positive constants C_{Y_i}, $i \in \{1, 2, 3\}$ such that, with notation $Y(p, a, b) := Y_p(a, b)$, for every $p, a, b \in \mathbb{R}^n$,

$$|Y_p(a, b)| \le C_{Y_0}(1 + |a| + |b|), \tag{3.10}$$

$$\left|\frac{\partial Y}{\partial p_i}(p, a, b)\right| \le C_{Y_1}(1 + |a| + |b|), \quad i = 1, \dots, n, \tag{3.11}$$

$$\left|\frac{\partial Y}{\partial a_i}(p, a, b)\right| + \left|\frac{\partial Y}{\partial b_i}(p, a, b)\right| \le C_{Y_2}, \quad i = 1, \dots, n, \tag{3.12}$$

$$\left|\frac{\partial^2 Y}{\partial x_j \partial y_i}(p, a, b)\right| \le C_{Y_3}, \quad x, y \in \{p, a, b\} \text{ and } i, j \in \{1, \dots, n\}. \tag{3.13}$$

4. Skeleton equation

The purpose of this section is to introduce and study the deterministic equation associated to the stochastic geometric wave equation (1.2). Define a space

$$ {}_0H^{1,2}(0,T,H_\mu) := \left\{ h \in {}_0C([0,T],H_\mu) : \dot{h} \in L^2(0,T;H_\mu) \right\}. \tag{4.1}$$

Note that ${}_0H^{1,2}(0,T,H_\mu)$ is a Hilbert space with norm $\int_0^T \|\dot{h}(t)\|_{H_\mu}^2\,dt$ and the map

$$ L^2(0,T;H_\mu) \ni \dot{h} \mapsto h = \left\{ t \mapsto \int_0^t \dot{h}(s)\,ds \right\} \in {}_0H^{1,2}(0,T,H_\mu), \tag{4.2}$$

is an isometric isomorphism. For $h \in {}_0H^{1,2}(0,T,H_\mu)$, we consider the so called "skeleton equation" associated to problem

$$ \begin{cases} \mathbf{D}_t \partial_t u = \mathbf{D}_x \partial_x u + Y_u(\partial_t u, \partial_x u)\,\dot{h}, \\ u(0,\cdot) = u_0,\ \partial_t u(t,\cdot)_{|t=0} = v_0 \end{cases} \tag{4.3}$$

i.e.,

$$ \begin{cases} \partial_{tt} u = \partial_{xx} u + A_u(\partial_t u, \partial_t u) - A_u(\partial_x u, \partial_x u) + Y_u(\partial_t u, \partial_x u)\,\dot{h}, \\ u(0,\cdot) = u_0,\ \ \partial_t u(0,\cdot) = v_0. \end{cases} \tag{4.4}$$

Recall that M is a compact Riemannian manifold which is isometrically embedded into some Euclidean space $\mathbb{R}^n$, and hence, we can assume that M is a submanifold of $\mathbb{R}^n$. The following main result of this section is closely related to [15, Theorem 11.1].

Theorem 4.1. *Let us assume that $T > 0$, $h \in {}_0H^{1,2}(0,T,H_\mu)$ and $(u_0, v_0) \in H^2_{loc} \times H^1_{loc}(\mathbb{R};TM)$. Then there exists a function $u : [0,T) \times \mathbb{R} \to M$ such that for every $R > T$ the following assertions hold:*

- (i) *u belongs to $\mathcal{C}^1([0,T)\times\mathbb{R};M)$,*
- (ii) *$[0,T) \ni t \mapsto u(t,\cdot) \in H^2((-R,R);M)$ is continuous,*
- (iii) *$[0,T) \ni t \mapsto u(t,\cdot) \in H^1((-R,R);M)$ is continuously differentiable,*
- (iv) *$u(0,x) = u_0(x)$ and $\partial_t u(0,x) = v_0(x,\omega)$ holds for every $x \in \mathbb{R}$,*
- (v) *for every vector field X on M, and every $t \geq 0$ and $R > 0$*

$$ \langle \partial_t u(t), X(u(t)) \rangle_{T_{u(t)}M} = \langle v_0, X(u_0) \rangle_{T_{u(t)}M} + \int_0^t \langle \mathbf{D}_x \partial_x u(s), X(u(s)) \rangle_{T_{u(s)}M}\,ds $$

$$ + \int_0^t \langle \partial_t u(s), \nabla_{\partial_t u(s)} X \rangle_{T_{u(s)}M}\,ds $$

$$+\int_0^t \langle X(u(s)), Y_{u(s)}(\partial_t u(s), \partial_x u(s))\dot{h}(s)\rangle_{T_{u(s)}M}\, ds,$$

holds in $L^2(-R, R)$.

Moreover, if $R > T$ *and* $U : [0, T) \times (-R, R) \to M$ *is a map which satisfies conditions (ii)–(v) and*

(i') *U belongs to* $\mathcal{C}^1([0, T) \times (-R, R); M)$,

then

$$U(t, x) = u(t, x) \quad \textit{for every} \quad |x| \le R - t \quad \textit{and} \quad t \in [0, T). \tag{4.5}$$

Definition 4.2. Assume that $T > 0$, $h \in {}_0H^{1,2}(0, T, H_\mu)$ and $(u_0, v_0) \in H^2_{\mathrm{loc}} \times H^1_{\mathrm{loc}}(\mathbb{R}; TM)$. A function $u : [0, T) \times \mathbb{R} \to M$ satisfying the conditions (i)–(v) in Theorem 4.1 is called an intrinsic solution to problem (4.4).

A function $u : [0, T) \times \mathbb{R} \to M$ is called an extrinsic solution to problem (4.4) if and only if for every $R > T$ the following five conditions hold,

(1) $[0, T) \ni t \mapsto u(t, \cdot) \in H^2((-R, R); \mathbb{R}^n)$ is continuous,
(2) $[0, T) \ni t \mapsto u(t, \cdot) \in H^1((-R, R); \mathbb{R}^n)$ is continuously differentiable,
(3) $u(t, x) \in M$ for every $t \in [0, T)$ and $x \in \mathbb{R}$,
(4) $u(0, x) = u_0(x)$ and $\partial_t u(0, x) = v_0(x)$ for every $x \in \mathbb{R}$,
(5) for every $t \in [0, T)$ the following holds in $L^2((-R, R); \mathbb{R}^n)$,

$$\partial_t u(t) = v_0 + \int_0^t \big[\partial_{xx}u(s) - A_{u(s)}(\partial_x u(s), \partial_x u(s)) + A_{u(s)}(\partial_t u(s), \partial_t u(s))\big]\, ds$$

$$+\int_0^t Y_{u(s)}(\partial_t u(s), \partial_x u(s))\dot{h}(s)\, ds. \tag{4.6}$$

Remark 4.3. Let us observe that due to Theorem A.3 a function $u : [0, T) \times \mathbb{R} \to M$ an intrinsic solution to problem (4.4) iff it is an extrinsic solution to problem (4.4). Hence in what follows we will simply use a notion “solution” to problem (4.4).

Remark 4.4. Using the C_0-group $(\tilde{S}_t)_{t\in\mathbb{R}}$ from Remark 3.12 one could expect that if a function $u : [0, T) \times \mathbb{R} \to M$ is a solution to problem (4.4), then the $\mathcal{H}_{\mathrm{loc}}$ function

$$z : [0, T] \ni t \mapsto \begin{pmatrix} u(t) \\ \partial_t u(t) \end{pmatrix} \in \mathcal{H}_{\mathrm{loc}},$$

satisfies the following mild-integral equation

$$z(t) = S_t z_0 + \int_0^t S_{t-s}\mathbf{F}(z(s))\,ds + \int_0^t S_{t-s}(\mathbf{G}(z(s))\dot{h}(s))\,ds, \quad t \in [0,T], \tag{4.7}$$

where, for $z = (u,v) \in \mathcal{H}_{\text{loc}}$ and $y \in H_\mu$, we put

$$\mathbf{F}(z) = \begin{pmatrix} 0 \\ A_u(v,v) - A_u(\partial_x u, \partial_x u) \end{pmatrix}, \quad \mathbf{G}(z)y = \begin{pmatrix} 0 \\ Y_u(v, \partial_x u)\, y \end{pmatrix}. \tag{4.8}$$

It seems that in some sense the maps $\mathbf{G}$ and $\mathbf{F}$ are locally Lipschitz continuous. However, we have not introduced a metric on the space $H^2_{\text{loc}} \times H^1_{\text{loc}}$ to make this assertion rigorous. Instead we have used some localization and approximation techniques to prove the existence of global solutions to our skeleton equation (4.4). There are two main difficulties in the proof of such a result. The first one is the invariance of the manifold M and the second one is the no-blowup. The former issue is dealt with in Proposition 4.10 while the latter issue is treated in Proposition 4.12. Our proof is motivated by a proof from the paper [15]. However, such a result is not sufficient to establish LDP with a proper rate function because of the lack of compactness. This additional problem is studied in the following Section 5.

The beginning of a proof of the existence part of Theorem 4.1. We begin with an observation that in view of Remark 4.3 it is sufficient to prove that there exists a function $u : [0,T) \times \mathbb{R} \to M$ such that for every $R > T$ the five conditions [1]–[5] from Definition 4.2 are satisfied.

For this purpose let us fix $R > T$ and $r > R + T$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a smooth compactly supported function such that $\varphi(x) = 1$ for $x \in (-r, r)$ and $\varphi(x) = 0$ for $x \notin (-2r, 2r)$. Next, with the convention $z = (u,v) \in \mathcal{H}$ and $u_x = \partial_x u$, we define the following maps

$$\mathbf{F}_r : [0,T] \times \mathcal{H} \ni (t,z) \mapsto \begin{pmatrix} 0 \\ E^1_{r-t}[\mathcal{A}_u(v,v) - \mathcal{A}_u(u_x,u_x)] \end{pmatrix} \in \mathcal{H},$$

$$\mathbf{G}_r : [0,T] \times \mathcal{H} \ni (t,z) \mapsto \begin{pmatrix} 0 \\ (E^1_{r-t} Y_u(v,u_x))\cdot \end{pmatrix} \in \mathscr{L}_2(H_\mu, \mathcal{H}),$$

$$\mathbf{Q}_r : \mathcal{H} \ni z \mapsto \begin{pmatrix} \varphi \cdot \Upsilon(u) \\ \varphi \cdot \Upsilon'(u)v \end{pmatrix} \in \mathcal{H},$$

where for $(u,v) \in \mathcal{H}$, $E^1_{r-t} Y_u(v,u_x) \in H^1_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$ and $(E^1_{r-t} Y_u(v,u_x))\cdot$ is the multiplication operator defined by

$$(E^1_{r-t} Y_u(v,u_x))\cdot : H_\mu \ni \xi \mapsto (E^1_{r-t} Y_u(v,u_x)) \cdot \xi \in H^1_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$$

satisfy Lemma 3.4. Moreover, we define, for $k \in \mathbb{N}$, the following maps

$$\mathbf{F}_{r,k} : [0,T] \times \mathcal{H} \ni (t,z) \mapsto \chi\Big(\frac{|z|_{\mathcal{H}_{r-t}}}{k}\Big)\mathbf{F}_r(t,z) \in \mathcal{H},$$

$$\mathbf{G}_{r,k} : [0,T] \times \mathcal{H} \ni (t,z) \mapsto \chi\Big(\frac{|z|_{\mathcal{H}_{r-t}}}{k}\Big)\mathbf{G}_r(t,z) \in \mathscr{L}_2(H_\mu, \mathcal{H}),$$

where $\chi(s) = \max\{0, \min\{1, 2-s\}\}$, $s \geq 0$, i.e.

$$\chi : [0, \infty) \ni s \mapsto \begin{cases} 1, & \text{if} \quad s \in [0, 1], \\ 2 - s, & \text{if} \quad s \in (1, 2], \\ 0, & \text{if} \quad s \in (2, \infty). \end{cases} \tag{4.9}$$

The following two properties of $\mathbf{Q}_r$, which we state without proof, are taken from [15, Section 7].

Lemma 4.5. *If $z = (u, v) \in \mathcal{H}$ is such that $u(x) \in M$ and $v(x) \in T_{u(x)}M$ for $x \in (-r, r)$, then $\mathbf{Q}_r(z) = z$ on $(-r, r)$.*

Lemma 4.6. *The mapping $\mathbf{Q}_r$ is of $\mathcal{C}^1$-class and its derivative, with $z = (u, v) \in \mathcal{H}$, satisfies*

$$\mathbf{Q}_r'(z)w = \begin{pmatrix} \varphi \cdot \Upsilon'(u)w^1 \\ \varphi \cdot [\Upsilon''(u)(v, w^1) + \Upsilon'(u)w^2] \end{pmatrix}, \quad w = (w^1, w^2) \in \mathcal{H}.$$

In the following arguments, till the end of the proof of Corollary 4.9 we choose and fix $k \in \mathbb{N}$.

Our first objective is to prove the well posedness of the following approximating version of equation (4.7),

$$z(t) = S_t\xi + \int_0^t S_{t-s}\mathbf{F}_{r,k}(s, z(s))\, ds + \int_0^t S_{t-s}(\mathbf{G}_{r,k}(s, z(s))\dot{h}(s))\, ds, \;\; t \in [0, T]. \tag{4.10}$$

This will be achieved in Corollary 4.8 after we have proved the next lemma. The proof of the Theorem will be completed later. □

The next result is about the Lipschitz properties of the localized maps defined above.

Lemma 4.7. *The functions $\mathbf{F}_r$ and $\mathbf{G}_r$ are continuous. Moreover, the functions $\mathbf{F}_{r,k}$ and $\mathbf{G}_{r,k}$ are globally Lipschitz in the second variable, i.e. there exists a constant $C_{r,k} > 0$ such that*

$$\begin{aligned} \|\mathbf{F}_{r,k}(t, z) - \mathbf{F}_{r,k}(t, w)\|_{\mathcal{H}} + \|\mathbf{G}_{r,k}(t, z) - \mathbf{G}_{r,k}(t, w)\|_{\mathscr{L}_2(H_\mu, \mathcal{H})} \\ \le C_{r,k}\|z - w\|_{\mathcal{H}_{r-t}}, \end{aligned} \tag{4.11}$$

for all $t \in [0, T]$ and $z, w \in \mathcal{H}$.

Proof of Lemma 4.7. The continuity of functions $\mathbf{F}_r$ and $\mathbf{G}_r$ is a consequence of the Sobolev embedding $H^1(\mathbb{R}) \subseteq \mathcal{C}_b(\mathbb{R})$ in conjunction with Lemma 3.4.

To prove the Lipschitz property, let us choose and fix $t \in [0, T]$ and $z = (u, v)$, $w = (\tilde{u}, \tilde{v}) \in \mathcal{H}$. Note that due to the definitions of $\mathbf{F}_{r,k}$ and $\mathbf{G}_{r,k}$, it is sufficient to prove (4.11) in the case $\|z\|_{\mathcal{H}_{r-t}}, \|w\|_{\mathcal{H}_{r-t}} \le k$.

Let us set $I_{rt} := (t - r, r - t)$. Since in the chosen case $\mathbf{F}_{r,k}(t, z) = \mathbf{F}_r(t, z)$ and $\mathbf{F}_{r,k}(t, w) = \mathbf{F}_r(t, w)$, by Proposition 3.13 and Remark 3.15, there exists $C_E(r, t) > 0$ such that

$$\begin{aligned} \|\mathbf{F}_{r,k}(t, z) - \mathbf{F}_{r,k}(t, w)\|_{\mathcal{H}} \le C_E(r, t)\big[\|\mathcal{A}_u(v, v) - \mathcal{A}_{\tilde{u}}(\tilde{v}, \tilde{v})\|_{H^1(I_{rt})} \\ + \|\mathcal{A}_u(u_x, u_x) - \mathcal{A}_{\tilde{u}}(\tilde{u}_x, \tilde{u}_x)\|_{H^1(I_{rt})}\big]. \end{aligned} \tag{4.12}$$

Since Υ is smooth and has compact support, see Lemma 3.8, from (3.4) observe that

$$\mathcal{A} : \mathbb{R}^n \ni q \mapsto \mathcal{A}_q \in \mathcal{L}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n),$$

is smooth, compactly supported (in particular bounded) and globally Lipschitz. Recall the following well-known interpolation inequality, refer [9, (2.12)],

$$\|u\|^2_{L^\infty(I)} \le k_e^2 \|u\|_{L^2(I)} \|u\|_{H^1(I)}, \quad u \in H^1(I), \tag{4.13}$$

where I is any open interval in $\mathbb{R}$ and $k_e = 2\max\left\{1, \frac{1}{\sqrt{|I|}}\right\}$. Note that since $r > R + T$ and $t \in [0,T]$, $|I_{rt}| = 2(r-t) > 2R$. Thus, we can choose $k_e = 2\max\left\{1, \frac{1}{\sqrt{|R|}}\right\}$. Consequently, using the above mentioned properties of $\mathcal{A}$ and the interpolation inequality (4.13) we get

$$\begin{aligned}
\|\mathcal{A}_u(v,v) - \mathcal{A}_{\tilde{u}}(\tilde{v},\tilde{v})\|_{L^2(I_{rt})} &\le \|\mathcal{A}_u(v,v) - \mathcal{A}_{\tilde{u}}(v,v)\|_{L^2(I_{rt})} \\
&\quad + \|\mathcal{A}_{\tilde{u}}(v,v) - \mathcal{A}_{\tilde{u}}(\tilde{v},v)\|_{L^2(I_{rt})} \\
&\quad + \|\mathcal{A}_{\tilde{u}}(\tilde{v},v) - \mathcal{A}_{\tilde{u}}(\tilde{v},\tilde{v})\|_{L^2(I_{rt})} \\
&\le L_{\mathcal{A}} \|v\|^2_{L^\infty(I_{rt})} \|u-\tilde{u}\|_{L^2(I_{rt})} \\
&\quad + B_{\mathcal{A}} \left[\|v\|_{L^\infty(I_{rt})} + \|\tilde{v}\|_{L^\infty(I_{rt})}\right] \|v-\tilde{v}\|_{L^2(I_{rt})} \\
&\le C(L_{\mathcal{A}}, B_{\mathcal{A}}, R, k, k_e) \|z-w\|_{\mathcal{H}_{r-t}},
\end{aligned} \tag{4.14}$$

where $L_{\mathcal{A}}$ and $B_{\mathcal{A}}$ are the Lipschitz constants and bound of $\mathcal{A}$, respectively. Next, since $\mathcal{A}$ is smooth and have compact support, if we set $L_{\mathcal{A}'}$ and $B_{\mathcal{A}'}$ are the Lipschitz constants and bound of

$$\mathcal{A}' : \mathbb{R}^n \ni q \mapsto d_q\mathcal{A} \in \mathcal{L}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n),$$

then by adding and subtracting the terms as we did to get (4.14) followed by invoking the properties of $\mathcal{A}'$ and the interpolation inequality (4.13) we have

$$\begin{aligned}
&\|d_x\left[\mathcal{A}_u(v,v) - \mathcal{A}_{\tilde{u}}(\tilde{v},\tilde{v})\right]\|_{L^2(I_{rt})} \\
&\le \|d_u\mathcal{A}(v,v)(u_x) - d_{\tilde{u}}\mathcal{A}(\tilde{v},\tilde{v})(\tilde{u}_x)\|_{L^2(I_{rt})} + 2\|\mathcal{A}_u(v_x,v) - \mathcal{A}_{\tilde{u}}(\tilde{v}_x,\tilde{v})\|_{L^2(I_{rt})} \\
&\le L_{\mathcal{A}'}\|u_x\|_{L^\infty(I_{rt})}\|v\|^2_{L^\infty(I_{rt})}\|u-\tilde{u}\|_{L^2(I_{rt})} + B_{\mathcal{A}'}\|v\|^2_{L^\infty(I_{rt})}\|u_x-\tilde{u}_x\|_{L^2(I_{rt})} \\
&\quad + B_{\mathcal{A}'}\left[\|v\|_{L^\infty(I_{rt})} + \|\tilde{v}\|_{L^\infty(I_{rt})}\right]\|v-\tilde{v}\|_{L^2(I_{rt})}\|\tilde{u}_x\|_{L^\infty(I_{rt})} \\
&\quad + 2\left[L_{\mathcal{A}}\|u-\tilde{u}\|_{L^\infty(I_{rt})}\|v\|_{L^\infty(I_{rt})}\|v_x\|_{L^2(I_{rt})} + B_{\mathcal{A}}\|v_x-\tilde{v}_x\|_{L^2(I_{rt})}\|v\|_{L^\infty(I_{rt})}\right. \\
&\quad \left. + B_{\mathcal{A}}\|v-\tilde{v}\|_{L^\infty(I_{rt})}\|\tilde{v}_x\|_{L^2(I_{rt})}\right] \\
&\lesssim_{L_{\mathcal{A}},B_{\mathcal{A}},L_{\mathcal{A}'},B_{\mathcal{A}'},k_e} \left[\|u-\tilde{u}\|_{H^2(I_{rt})}\|u\|_{H^2(I_{rt})}\|v\|^2_{H^1(I_{rt})} + \|u-\tilde{u}\|_{H^2(I_{rt})}\|v\|^2_{H^1(I_{rt})}\right. \\
&\quad + \|v-\tilde{v}\|_{H^1(I_{rt})}\left[\|v\|_{H^1(I_{rt})} + \|\tilde{v}\|_{H^1(I_{rt})}\right]\|\tilde{u}\|_{H^2(I_{rt})} + \|u-\tilde{u}\|_{H^2(I_{rt})}\|v\|^2_{H^1(I_{rt})} \\
&\quad \left. + \|v-\tilde{v}\|_{H^1(I_{rt})}\left(\|v\|_{H^1(I_{rt})} + \|\tilde{v}\|_{H^1(I_{rt})}\right)\right]
\end{aligned}$$

$$\lesssim_k \|z-w\|_{\mathcal{H}_{r-t}}, \tag{4.15}$$

where the last step holds since $\|z\|_{\mathcal{H}_{r-t}}, \|w\|_{\mathcal{H}_{r-t}} \le k$. By following similar procedure of (4.14) and (4.15) we also get

$$\|\mathcal{A}_u(u_x,u_x)-\mathcal{A}_{\tilde u}(\tilde u_x,\tilde u_x)\|_{H^1(I_{rt})} \lesssim_{L_{\mathcal{A}},B_{\mathcal{A}},L_{\mathcal{A}'},B_{\mathcal{A}'},k_e,k} \|z-w\|_{\mathcal{H}_{r-t}}.$$

Hence by substituting the estimates back in (4.12) we are done with (4.11) for $F_{r,k}$-term.

Next, we consider $G_{r,k}$. As for $F_{r,k}$, it is sufficient to perform the calculations for the case $\|z\|_{\mathcal{H}_{r-t}}, \|w\|_{\mathcal{H}_{r-t}} \le k$. By invoking Lemma 3.4 followed by Remark 3.15 we have

$$\begin{aligned}\|\mathbf{G}_{r,k}(t,z)-\mathbf{G}_{r,k}(t,w)\|^2_{\mathscr{L}_2(H_\mu,\mathcal{H})} &\le \|(E^1_{r-t}Y_u(v,u_x))\cdot -(E^1_{r-t}Y_{\tilde u}(\tilde v,\tilde u_x))\cdot\|^2_{\mathscr{L}_2(H_\mu,H^1(\mathbb{R}))}\\ &\le c_{r,t}\, C_E(r,t)\, \|Y_u(v,u_x)-Y_{\tilde u}(\tilde v,\tilde u_x)\|^2_{H^1(I_{rt})}.\end{aligned}$$

Recall that the 1-D Sobolev embedding gives $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$. Consequently, by the Taylor formula [24, Theorem 5.6.1] and inequalities (3.11)–(3.12) we have

$$\begin{aligned}\|Y_u(v,\partial_x u)-Y_{\tilde u}(\tilde v,\tilde u_x)\|^2_{L^2(I_{rt})} &\le \int_{I_{rt}} |Y_{u(x)}(v(x),u_x(x))-Y_{\tilde u(x)}(v(x),u_x(x))|^2\,dx\\ &\quad+\int_{I_{rt}} |Y_{\tilde u(x)}(v(x),u_x(x))-Y_{\tilde u(x)}(v(x),\tilde u_x(x))|^2\,dx\\ &\quad+\int_{I_{rt}} |Y_{\tilde u(x)}(v(x),\tilde u_x(x))-Y_{\tilde u(x)}(\tilde v(x),\tilde u_x(x))|^2\,dx\\ &\le C_Y^2\left[1+\|v\|^2_{H^1(I_{rt})}+\|u\|^2_{H^1(I_{rt})}\right]\|u-\tilde u\|^2_{H^2(I_{rt})}\\ &\quad+C_{Y_2}^2\left[\|u_x-\tilde u_x\|^2_{H^1(I_{rt})}+\|v-\tilde v\|^2_{H^1(I_{rt})}\right]\\ &\lesssim_{k,C_Y,C_{Y_2}} \|z-w\|^2_{\mathcal{H}_{r-t}}.\end{aligned} \tag{4.16}$$

For homogeneous part of the norm, that is L^2-norm of the derivative, we have

$$\begin{aligned}&\|d_x\left[Y_u(v,u_x)-Y_{\tilde u}(\tilde v,\tilde u_x)\right]\|^2_{L^2(I_{rt})}\\ &\lesssim \int_{I_{rt}}\sum_{i=1}^n\Bigg\{\left|\frac{\partial Y}{\partial p_i}(u(x),v(x),u_x(x))\frac{du^i}{dx}(x)-\frac{\partial Y}{\partial p_i}(\tilde u(x),\tilde v(x),\tilde u_x(x))\frac{d\tilde u^i}{dx}(x)\right|^2\\ &\quad+\left|\frac{\partial Y}{\partial a_i}(u(x),v(x),u_x(x))\frac{dv^i}{dx}(x)-\frac{\partial Y}{\partial a_i}(\tilde u(x),\tilde v(x),\tilde u_x(x))\frac{d\tilde v^i}{dx}(x)\right|^2\\ &\quad+\left|\frac{\partial Y}{\partial b_i}(u(x),v(x),u_x(x))\frac{du^i_x}{dx}(x)-\frac{\partial Y}{\partial b_i}(\tilde u(x),\tilde v(x),\tilde u_x(x))\frac{d\partial_x\tilde u^i}{dx}(x)\right|^2\Bigg\}\,dx\\ &=: Y_1+Y_2+Y_3.\end{aligned} \tag{4.17}$$

We will estimate each term separately by using the 1-D Sobolev embedding, the Taylor formula and inequalities (3.11)–(3.13) as follows:

$$
\begin{aligned}
Y_1 \lesssim \int_{I_{rt}} \sum_{i=1}^{n} \Bigg\{ &\left| \frac{\partial Y}{\partial p_i}(u(x), v(x),\ u_x(x)) \frac{du^i}{dx}(x) - \frac{\partial Y}{\partial p_i}(\tilde{u}(x), v(x), u_x(x)) \frac{du^i}{dx}(x) \right|^2 \\
&+ \left| \frac{\partial Y}{\partial p_i}(\tilde{u}(x), v(x), u_x(x)) \frac{du^i}{dx}(x) - \frac{\partial Y}{\partial p_i}(\tilde{u}(x), v(x), u_x(x)) \frac{d\tilde{u}^i}{dx}(x) \right|^2 \\
&+ \left| \frac{\partial Y}{\partial p_i}(\tilde{u}(x), v(x), u_x(x)) \frac{d\tilde{u}^i}{dx}(x) - \frac{\partial Y}{\partial p_i}(\tilde{u}(x), \tilde{v}(x), u_x(x)) \frac{d\tilde{u}^i}{dx}(x) \right|^2 \\
&+ \left| \frac{\partial Y}{\partial p_i}(\tilde{u}(x), \tilde{v}(x), u_x(x)) \frac{d\tilde{u}^i}{dx}(x) - \frac{\partial Y}{\partial p_i}(\tilde{u}(x), \tilde{v}(x), \tilde{u}_x(x)) \frac{d\tilde{u}^i}{dx}(x) \right|^2 \Bigg\}\, dx \\
\lesssim\ & C_{Y_3}^2 \|u - \tilde{u}\|_{L^2(I_{rt})}^2 \|u_x\|_{H^1(I_{rt})}^2 + C_{Y_1}^2 \left[1 + \|v\|_{H^1(I_{rt})}^2 + \|u_x\|_{H^1(I_{rt})}^2 \right] \|u_x - \tilde{u}_x\|_{L^2(I_{rt})}^2 \\
&+ C_{Y_3}^2 \|v - \tilde{v}\|_{L^2(I_{rt})}^2 \|\tilde{u}_x\|_{H^1(I_{rt})}^2 + C_{Y_3}^2 \|u_x - \tilde{u}_x\|_{L^2(I_{rt})}^2 \|\tilde{u}_x\|_{H^1(I_{rt})}^2 \\
\lesssim_{k, C_{Y_2}, C_{Y_3}, C_{Y_1}}\ & \|z - w\|_{\mathcal{H}_{r-t}}^2.
\end{aligned} \tag{4.18}
$$

Terms Y_2 and Y_3 are quite similar so it is enough to estimate only one. For Y_2 we have the following calculation

$$
\begin{aligned}
Y_2 \lesssim \int_{I_{rt}} \sum_{i=1}^{n} \Bigg\{ &\left| \frac{\partial Y}{\partial a_i}(u(x), v(x), u_x(x)) \frac{dv^i}{dx}(x) - \frac{\partial Y}{\partial a_i}(\tilde{u}(x), v(x), u_x(x)) \frac{dv^i}{dx}(x) \right|^2 dx \\
&+ \left| \frac{\partial Y}{\partial a_i}(\tilde{u}(x), v(x), u_x(x)) \frac{dv^i}{dx}(x) - \frac{\partial Y}{\partial a_i}(\tilde{u}(x), \tilde{v}(x), u_x(x)) \frac{dv^i}{dx}(x) \right|^2 dx \\
&+ \left| \frac{\partial Y}{\partial a_i}(\tilde{u}(x), \tilde{v}(x), u_x(x)) \frac{dv^i}{dx}(x) - \frac{\partial Y}{\partial a_i}(\tilde{u}(x), \tilde{v}(x), \tilde{u}_x(x)) \frac{dv^i}{dx}(x) \right|^2 dx \\
&+ \left| \frac{\partial Y}{\partial a_i}(\tilde{u}(x), \tilde{v}(x), \tilde{u}_x(x)) \frac{dv^i}{dx}(x) - \frac{\partial Y}{\partial a_i}(\tilde{u}(x), \tilde{v}(x), \tilde{u}_x(x)) \frac{d\tilde{v}^i}{dx}(x) \right|^2 dx \Bigg\} \\
\lesssim\ & C_{Y_3}^2 \|u - \tilde{u}\|_{H^1(I_{rt})}^2 \|v_x\|_{L^2(I_{rt})}^2 + C_{Y_3}^2 \|v - \tilde{v}\|_{H^1(I_{rt})}^2 \|v_x\|_{L^2(I_{rt})}^2 \\
&+ C_{Y_3}^2 \|u_x - \tilde{u}_x\|_{H^1(I_{rt})}^2 \|v_x\|_{L^2(I_{rt})}^2 + C_{Y_3}^2 C_{r,t} \|v_x - \tilde{v}_x\|_{L^2(I_{rt})}^2 \\
\lesssim_{k, C_{r,t} C_{Y_3}}\ & \|z - w\|_{\mathcal{H}_{r-t}}^2.
\end{aligned} \tag{4.19}
$$

Hence by substituting (4.18)–(4.19) into (4.17) we get

$$
\|d_x \left[Y_u(v, u_x) - Y_{\tilde{u}}(\tilde{v}, \tilde{u}_x) \right]\|_{L^2(I_{rt})}^2 \lesssim_{k, C_{r,t}, C_{Y_2}, C_{Y_3}, C_{Y_1}} \|z - w\|_{\mathcal{H}_{r-t}}^2,
$$

which together with (4.16) gives $G_{r,k}$ part of (4.11). Hence the Lipschitz property from Lemma 4.7 follows. □

The following result follows directly from Lemma 4.7 and the standard theory of PDE via semigroup approach, refer [1] and [45] for a detailed proof.

Corollary 4.8. *For all $\xi \in \mathcal{H}$ and $h \in {}_0H^{1,2}(0,T,H_\mu)$, there exists a unique $z = z_k$ in $\mathcal{C}([0,T];\mathcal{H})$ which is a solution of equation* (4.10).

From now on, for each $r > R+T$ and $k \in \mathbb{N}$, the solution from Corollary 4.8 will be denoted by $z_{r,k}$ and called the *approximate solution*. To proceed further we define the following two auxiliary functions

$$\widetilde{F}_{r,k} : [0,T] \times \mathcal{H} \ni (t,z) \mapsto \begin{pmatrix} 0 \\ \varphi \cdot \Upsilon'(u)\mathbf{F}^2_{r,k}(t,z) + \varphi B_u(v,v) - \varphi B_u(u_x,u_x) \end{pmatrix} - \begin{pmatrix} 0 \\ \Delta\varphi \cdot h(u) + 2\varphi_x \cdot h'(u)u_x \end{pmatrix} \in \mathcal{H},$$

and

$$\widetilde{G}_{r,k} : [0,T] \times \mathcal{H} \ni (t,z) \mapsto \begin{pmatrix} 0 \\ \varphi \cdot \Upsilon'(u)\mathbf{G}^2_{r,k}(t,z) \end{pmatrix} \in \mathcal{H}.$$

Here $\mathbf{F}^2_{r,k}(s,z_{r,k}(s))$ and $\mathbf{G}^2_{r,k}(s,z_{r,k}(s))$ denote the second components of the vectors $\mathbf{F}_{r,k}(s,z_{r,k}(s))$ and $\mathbf{G}_{r,k}(s,z_{r,k}(s))$, respectively. The following corollary relates the solution $z_{r,k}$ with its transformation under the map $\mathbf{Q}_r$ and allows to understand the need of the functions $\widetilde{F}_{r,k}$ and $\widetilde{G}_{r,k}$.

Corollary 4.9. *Let us assume that $\xi := (E^2_r u_0, E^1_r v_0)$ and that $z_{r,k} \in \mathcal{C}([0,T];\mathcal{H})$ is a solution of equation* (4.10)*. Then the $\widetilde{z}_{r,k} = \mathbf{Q}_r(z_{r,k})$ satisfies,*

$$\widetilde{z}_{r,k}(t) = S_t\mathbf{Q}_r(\xi) + \int_0^t S_{t-s}\widetilde{F}_{r,k}(s,z_{r,k}(s))\,ds + \int_0^t S_{t-s}(\widetilde{G}_{r,k}(s,z_{r,k}(s))\dot{h}(s))\,ds, \quad t \in [0,T].$$

Proof of Corollary 4.9. First observe that by the action of $\mathbf{Q}_r'$ and $\mathcal{G}$ on the elements of $\mathcal{H}$ from Lemma 4.6 and (3.11), respectively, we get

$$\begin{aligned} &\mathbf{Q}_r'(z_{r,k}(s))\left(\mathbf{F}_{r,k}(s,z_{r,k}(s)) + \mathbf{G}_{r,k}(s,z_{r,k}(s))\dot{h}(s)\right) \\ &\quad = \begin{pmatrix} 0 \\ \varphi \cdot \left\{[\Upsilon'(u_{r,k}(s))](\mathbf{F}^2_{r,k}(s,z_{r,k}(s))) + [\Upsilon'(u_{r,k}(s))](\mathbf{G}^2_{r,k}(s,z_{r,k}(s))\dot{h}(s))\right\} \end{pmatrix}. \end{aligned} \tag{4.20}$$

Moreover, since by applying Lemma 4.6 and (3.11) to $z = (u,v) \in \mathcal{H}$ we have

$$\begin{aligned} F(z) := \mathbf{Q}_r'\mathcal{G}z - \mathcal{G}\mathbf{Q}_r z = &\begin{pmatrix} \varphi \cdot [\Upsilon'(u)](v) \\ \varphi \cdot \left\{[\Upsilon''(u)](v,v) + [\Upsilon'(u)](u'')\right\} \end{pmatrix} \\ &- \begin{pmatrix} \varphi \cdot [\Upsilon'(u)](v) \\ \varphi'' \cdot \Upsilon(u) + 2\varphi' \cdot [\Upsilon'(u)](u') + \varphi \cdot [\Upsilon'(u)](u'') + \varphi \cdot [\Upsilon''(u)](u',u') \end{pmatrix}, \end{aligned} \tag{4.21}$$

substitution $z = z_{r,k}(s) = (u_{r,k}(s), v_{r,k}(s)) \in \mathcal{H}$ in (4.21) with (4.20) followed by definition (3.3) gives, for $s \in [0, T]$,

$$
\begin{aligned}
&\mathbf{Q}_r'(z_{r,k}(s))\left(\mathbf{F}_{r,k}(s, z_{r,k}(s)) + \mathbf{G}_{r,k}(s, z_{r,k}(s))\right) + F(z_{r,k}(s)) \\
&= \begin{pmatrix} 0 \\ \varphi \cdot [\Upsilon'(u_{r,k}(s))](\mathbf{F}^2_{r,k}(s, z_{r,k}(s))) + \varphi \cdot [\Upsilon''(u_{r,k}(s))](v_{r,k}(s), v_{r,k}(s)) \\ -\varphi \cdot [\Upsilon''(u_{r,k}(s))](\partial_x u_{r,k}(s), \partial_x u_{r,k}(s)) \end{pmatrix} \\
&- \begin{pmatrix} 0 \\ -\varphi'' \cdot \Upsilon(u_{r,k}(s)) + 2\varphi' \cdot [\Upsilon'(u_{r,k}(s))](\partial_x u_{r,k}(s)) + \varphi \cdot [\Upsilon'(u_{r,k}(s))](\mathbf{G}^2_{r,k}(s, z_{r,k}(s))) \end{pmatrix} \\
&= \widetilde{F}_{r,k}(s, z_{r,k}(s)) + \widetilde{G}_{r,k}(s, z_{r,k}(s)).
\end{aligned}
$$

Hence, if we have

$$
\int_0^T \left[\|\mathbf{F}_{r,k}(s, z_{r,k}(s))\|_{\mathcal{H}} + \|\mathbf{G}_{r,k}(s, z_{r,k}(s))\dot{h}(s)\|_{\mathcal{H}}\right] ds < \infty, \tag{4.22}
$$

then by invoking [15, Lemma 6.4] with

$$
L = \mathbf{Q}_r,\ K = U = \mathcal{H},\ A = B = \mathcal{G},\ g(s) = 0,\ f(s) = \mathbf{F}_{r,k}(s, z_{r,k}(s)) + \mathbf{G}_{r,k}(s, z_{r,k}(s))\dot{h}(s),
$$

we are done with the proof here. But (4.22) follows by Lemma 4.7, because $h \in {}_0H^{1,2}(0, T, H_\mu)$ and the following holds due to the Hölder inequality

$$
\begin{aligned}
\int_0^T \|\mathbf{G}_{r,k}(s, z_{r,k}(s))\dot{h}(s)\|_{\mathcal{H}}\, ds &= \int_0^T \|\mathbf{G}^2_{r,k}(s, z_{r,k}(s))\dot{h}(s)\|_{H^1(\mathbb{R})}\, ds \\
&\le \left(\int_0^T \|(\mathbf{G}^2_{r,k}(s, z_{r,k}(s)))\cdot\|^2_{\mathscr{L}_2(H_\mu, H^1(\mathbb{R}))}\, ds\right)^{\frac{1}{2}} \left(\int_0^T \|\dot{h}(s)\|^2_{H_\mu}\, ds\right)^{\frac{1}{2}}. \qquad \square
\end{aligned}
$$

Next we prove that the approximate solution $z_{r,k}$ stays on the manifold. Define the following three positive reals: for each $r > R + T$ and $k \in \mathbb{N}$,

$$
\begin{cases}
\tau_k^1 := \inf\{t \in [0, T] : \|z_{r,k}(t)\|_{\mathcal{H}_{r-t}} \ge k\}, \\
\tau_k^2 := \inf\{t \in [0, T] : \|\widetilde{z}_{r,k}(t)\|_{\mathcal{H}_{r-t}} \ge k\}, \\
\tau_k^3 := \inf\{t \in [0, T] : \exists x,\ |x| \le r - t,\ u_{r,k}(t, x) \notin O\}, \\
\tau_k := \tau_k^1 \wedge \tau_k^2 \wedge \tau_k^3.
\end{cases} \tag{4.23}
$$

Also, define the following $\mathcal{H}$-valued functions of time $t \in [0, T]$

$$
\begin{aligned}
a_k(t) &= S_t\xi + \int_0^t S_{t-s}\mathbb{1}_{[0,\tau_k)}(s)\mathbf{F}_{r,k}(s, z_{r,k}(s))\,ds \\
&\quad + \int_0^t S_{t-s}(\mathbb{1}_{[0,\tau_k)}(s)\mathbf{G}_{r,k}(s, z_{r,k}(s))\dot{h}(s))\,ds, \\
\widetilde{a}_k(t) &= S_t\mathbf{Q}_r(\xi) + \int_0^t S_{t-s}\mathbb{1}_{[0,\tau_k)}(s)\widetilde{F}_{r,k}(s, z_{r,k}(s))\,ds \\
&\quad + \int_0^t S_{t-s}(\mathbb{1}_{[0,\tau_k)}(s)\widetilde{G}_{r,k}(s, z_{r,k}(s))\dot{h}(s))\,ds.
\end{aligned}
\tag{4.24}
$$

Proposition 4.10. *For each $k \in \mathbb{N}$ and $\xi := (E_r^2 u_0, E_r^1 v_0)$, the functions a_k, $\widetilde{a}_k$, $z_{r,k}$ and $\widetilde{z}_{r,k}$ coincide on $[0, \tau_k)$. In particular, $u_{r,k}(t, x) \in M$ for $|x| \le r - t$ and $t \le \tau_k$. Consequently, $\tau_k = \tau_k^1 = \tau_k^2 \le \tau_k^3$.*

Proof of Proposition 4.10. Let us fix k. First note that, due to indicator function,

$$
a_k = z_{r,k} \qquad \text{and} \qquad \widetilde{a}_k = \widetilde{z}_{r,k} \text{ on } [0, \tau_k). \tag{4.25}
$$

Next, since $E_{r-s}^1 f = f$ on $|x| \le r - s$, see Proposition 3.13, and $\varphi = 1$ on $(-r, r)$, by Lemma 4.5 followed by (3.5) we infer that

$$
\begin{cases}
\mathbb{1}_{[0,\tau_k)}(s)[\widetilde{F}_{r,k}(s, z_{r,k}(s))](x) = \mathbb{1}_{[0,\tau_k)}(s)[\mathbf{F}_{r,k}(s, \widetilde{z}_{r,k}(s))](x), \\
\mathbb{1}_{[0,\tau_k)}(s)[\widetilde{G}_{r,k}(s, z_{r,k}(s))e](x) = \mathbb{1}_{[0,\tau_k)}(s)[\mathbf{G}_{r,k}(s, \widetilde{z}_{r,k}(s))e](x), \quad e \in K,
\end{cases}
\tag{4.26}
$$

holds for every $|x| \le r - s$, $0 \le s \le T$. Now we claim that if we denote

$$
p(t) := \frac{1}{2}\|a_k(t) - \widetilde{a}_k(t)\|^2_{\mathcal{H}_{r-t}},
$$

then the map $s \mapsto p(s \wedge \tau_k)$ is continuous and uniformly bounded. Indeed, since, by Proposition 3.13, $\xi(x) = (u_0(x), v_0(x)) \in TM$ for $|x| \le r$, the uniform boundedness is an easy consequence of bound property of C_0-group, Lemmata 4.5 and 4.7. Continuity of $s \mapsto p(s \wedge \tau_k)$ follows from the following:

(1) for every $z \in \mathcal{H}$, the map $t \mapsto \|z\|^2_{\mathcal{H}_{r-t}}$ is continuous;
(2) for each t, the map

$$
L^2(\mathbb{R}) \ni u \mapsto \int_0^t |u(s)|^2\,ds \in \mathbb{R},
$$

is locally Lipschitz.

Now observe that by applying Proposition C.1 for

$$k = 1, L = I, T = r, x = 0 \quad \text{and} \quad z(t) = (u(t), v(t)) := a_k(t) - \widetilde{a}_k(t),$$

we get $\mathbf{e}(t, r; 0, z(t)) = p(t)$, and the following

$$\mathbf{e}(t, r; 0, (t)) \leq \mathbf{e}(0, r; 0, z_0) + \int_0^t V(s, z(s))\, ds. \tag{4.27}$$

Here

$$\begin{aligned} V(t, z(t)) := {}& \langle u(t), v(t)\rangle_{L^2(B_{r-t})} + \langle v(t), f(t)\rangle_{L^2(B_{r-t})} + \langle \partial_x v(t), \partial_x f(t)\rangle_{L^2(B_{r-t})} \\ & + \langle v(t), g(t)\rangle_{L^2(B_{r-t})} + \langle \partial_x v(t), \partial_x g(t)\rangle_{L^2(B_{r-t})}, \end{aligned}$$

and

$$\begin{aligned} \begin{pmatrix} 0 \\ f(t) \end{pmatrix} &:= \mathbb{1}_{[0,\tau_k)}(t)[\mathbf{F}_{r,k}(t, z_{r,k}(t)) - \widetilde{F}_{r,k}(t, z_{r,k}(t))], \\ \begin{pmatrix} 0 \\ g(t) \end{pmatrix} &:= \mathbb{1}_{[0,\tau_k)}(t)[\mathbf{G}_{r,k}(t, z_{r,k}(t))\dot{h}(t) - \widetilde{G}_{r,k}(t, z_{r,k}(t))\dot{h}(t)]. \end{aligned}$$

Due to the extension operators E_r^2 and E_r^1 the initial data ξ in the definition (4.24) satisfies the assumption of Lemma 4.5, $S_t\mathbf{Q}_r(\xi) = S_t\xi$, and so $\mathbf{e}(0, 0; 0, z(0)) = p(0) = 0$. Next observe that by the Cauchy-Schwarz inequality we have

$$\begin{aligned} V(t, z(t)) \leq {}& \frac{1}{2}\|u(t)\|^2_{L^2(B_{r-t})} + \frac{3}{2}\|v(t)\|^2_{L^2(B_{r-t})} + \frac{1}{2}\|f(t)\|^2_{L^2(B_{r-t})} + \|\partial_x v(t)\|^2_{L^2(B_{r-t})} \\ & + \frac{1}{2}\|\partial_x f(t)\|^2_{L^2(B_{r-t})} + \frac{1}{2}\|g(t)\|^2_{L^2(B_{r-t})} + \frac{1}{2}\|\partial_x g(t)\|^2_{L^2(B_{r-t})} \\ \leq {}& 3p(t) + \frac{1}{2}\|f(t)\|^2_{H^1(B_{r-t})} + \frac{1}{2}\|g(t)\|^2_{H^1(B_{r-t})}. \end{aligned}$$

By using above into (4.27) and, then, by invoking equalities (4.26) and (4.25), definition (4.23), Lemma 3.4 and Lemma 4.7 we have the following calculation, for every $t \in [0, T]$,

$$\begin{aligned} p(t) \leq {}& \int_0^t 3p(s)\, ds + \frac{1}{2}\int_0^t \mathbb{1}_{[0,\tau_k)}(s)\|\mathbf{F}^2_{r,k}(s, z_{r,k}(s)) - \mathbf{F}^2_{r,k}(s, \tilde{z}_{r,k}(s))\|^2_{H^1(B_{r-s})}\, ds \\ & + \frac{1}{2}\int_0^t \mathbb{1}_{[0,\tau_k)}(s)\|\mathbf{G}^2_{r,k}(s, z_{r,k}(s)) - \mathbf{G}^2_{r,k}(s, \tilde{z}_{r,k}(s))\|^2_{\mathscr{L}_2(H_\mu, H^1(B_{r-s}))}\|\dot{h}(s)\|^2_{H_\mu}\, ds \\ \leq {}& 3\int_0^t p(s)\, ds + \frac{1}{2}C^2_{r,k}\int_0^t \mathbb{1}_{[0,\tau_k)}(s)\|z_{r,k}(s) - \widetilde{z}_{r,k}(s)\|^2_{\mathcal{H}_{r-s}}\, ds \end{aligned}$$

$$+\frac{1}{2}C_{r,k}^2\int_0^t \mathbb{1}_{[0,\tau_k)}(s)\|z_{r,k}(s)-\widetilde{z}_{r,k}(s)\|^2_{\mathcal{H}_{r-s}}\|\dot{h}(s)\|^2_{H_\mu}\,ds$$

$$\leq (3+C_{r,k}^2)\int_0^t p(s)(1+\|\dot{h}(s)\|^2_{H_\mu})\,ds. \tag{4.28}$$

Consequently by the Gronwall Lemma, for $t\in[0,\tau_k]$,

$$p(t)\lesssim_{C_{r,k}} p(0)\exp\left[\int_0^t (1+\|\dot{h}(s)\|^2_{H_\mu})\,ds\right]. \tag{4.29}$$

Note that the right hand side in (4.29) is finite because $h\in {}_0H^{1,2}(0,T,H_\mu)$. Since we know that $p(0)=0$ we arrive to $p(t)=0$ on $t\in[0,\tau_k]$. This further implies that $a_k(t,x)=\widetilde{a}_k(t,x)$ hold for $|x|\leq r-t$ and $t\leq\tau_k$. Consequently, $z_{r,k}(t,x)=\widetilde{z}_{r,k}(t,x)$ hold for $|x|\leq r-t$ and $t\leq\tau_k$. So, because $\widetilde{z}_{r,k}(t,x)=\mathbf{Q}_r(z_{r,k}(t))$ and $\varphi=1$ on $(-r,r)$,

$$u_{r,k}(t,x)=\Upsilon(u_{r,k}(t,x)),\qquad \text{for } |x|\leq r-t,\quad t\leq\tau_k. \tag{4.30}$$

Since, by definition (4.23) of τ_k, $u_{r,k}(t,x)\in O$, equality (4.30) and Lemma 3.8, gives $u_{r,k}(t,x)\in M$ for $|x|\leq r-t$ and $t\leq\tau_k$. This suggests that $\tau_k\leq\tau_k^3$ and hence $\tau_k=\tau_k^1\wedge\tau_k^2$. It remains to show that $\tau_k^1=\tau_k^2$. But suppose it does not hold and without loss of generality we assume that $\tau_k^1>\tau_k^2$. Then by definition (4.23) and the continuity of $z_{r,k}$ and $\tilde{z}_{r,k}$ in time we have

$$\|z_{r,k}(\tau_k^2,\cdot)\|_{\mathcal{H}_{r-\tau_k^2}}<k\quad\text{ but }\quad\|\widetilde{z}_{r,k}(\tau_k^2,\cdot)\|_{\mathcal{H}_{r-\tau_k^2}}\geq k,$$

which contradicts the above mentioned consequence of $p=0$ on $[0,\tau_k]$. Hence we conclude that $\tau_k^1=\tau_k^2$ and this finishes the proof of Proposition 4.10. □

Next in the ongoing proof of Theorem 4.1 we show that the approximate solutions extend each other. Recall that $r>R+T$ is fixed for given $T>0$.

Lemma 4.11. *Let $k\in\mathbb{N}$ and $\xi=(E_r^2u_0,E_r^1v_0)$. Then $z_{r,k+1}(t,x)=z_{r,k}(t,x)$ on $|x|\leq r-t$, $t\leq\tau_k$, and $\tau_k\leq\tau_{k+1}$.*

Proof of Lemma 4.11. Define

$$p(t):=\frac{1}{2}\|a_{k+1}(t)-a_k(t)\|^2_{H^1(B_{r-t})\times L^2(B_{r-t})}.$$

As an application of Proposition C.1, by performing the computation based on (4.27)–(4.28), with $k=0$ and rest the same, we obtain

$$p(t) \le 2\int_0^t p(s)\,ds + \frac{1}{2}\int_0^t \|\mathbb{1}_{[0,\tau_{k+1})}(s)\mathbf{F}_r^2(s, z_{r,k+1}(s)) - \mathbb{1}_{[0,\tau_k)}(s)\mathbf{F}_r^2(s, z_{r,k}(s))\|^2_{L^2(B_{r-s})}\,ds$$
$$+ \frac{1}{2}\int_0^t \|\mathbb{1}_{[0,\tau_{k+1})}(s)\mathbf{G}_r^2(s, z_{r,k+1}(s))\dot{h}(s) - \mathbb{1}_{[0,\tau_k)}(s)\mathbf{G}_r^2(s, z_{r,k}(s))\dot{h}(s)\|^2_{L^2(B_{r-s})}\,ds. \tag{4.31}$$

Then, since $\mathbf{F}_r$ and $\mathbf{G}_r$ depends on $u_{r,k}(s)$, $u_{r,k+1}(s)$ and their first partial derivatives, with respect to time t and space x, which are actually bounded on the interval $(-(r-s), r-s)$ by some constant C_r for every $s < \tau_{k+1} \wedge \tau_k$, by evaluating (4.31) on $t \wedge \tau_{k+1} \wedge \tau_k$ following the use of Lemmata 4.7 and 3.4 we get

$$\begin{aligned} p(t \wedge \tau_{k+1} \wedge \tau_k) &\le 2\int_0^t p(s \wedge \tau_{k+1} \wedge \tau_k)\,ds \\ &\quad + \frac{1}{2}\int_0^{t\wedge\tau_{k+1}\wedge\tau_k} \|\mathbf{F}_r^2(s, z_{r,k+1}(s)) - \mathbf{F}_r^2(s, z_{r,k}(s))\|^2_{L^2(B_{r-s})}\,ds \\ &\quad + \frac{1}{2}\int_0^{t\wedge\tau_{k+1}\wedge\tau_k} \|\mathbf{G}_r^2(s, z_{r,k+1}(s))\zeta(s) - \mathbf{G}_r^2(s, z_{r,k}(s))\dot{h}(s)\|^2_{L^2(B_{r-s})}\,ds \\ &\lesssim_k \int_0^t p(s \wedge \tau_{k+1} \wedge \tau_k)(1 + \|\dot{h}(s)\|^2_{H_\mu})\,ds. \end{aligned}$$

Hence by the Gronwall Lemma we infer that $p = 0$ on $[0, \tau_{k+1} \wedge \tau_k]$.

Consequently, we claim that $\tau_k \le \tau_{k+1}$. We divide the proof of our claim in the following three exhaustive subcases. Due to (4.23), the subcases when $\|\xi\|_{\mathcal{H}_r} > k+1$ and $k < \|\xi\|_{\mathcal{H}_r} \le k+1$ are trivial. In the last subcase when $\|\xi\|_{\mathcal{H}_r} \le k$ we prove the claim $\tau_k \le \tau_{k+1}$ by the method of contradiction, and so assume that $\tau_k > \tau_{k+1}$ is true. Then, because of continuity in time of $z_{r,k}$ and $z_{r,k+1}$, by (4.23) we have

$$\|z_{r,k}(\tau_{k+1})\|_{\mathcal{H}_{r-\tau_{k+1}}} < k \quad \text{and} \quad \|z_{r,k+1}(\tau_{k+1})\|_{\mathcal{H}_{r-\tau_{k+1}}} \ge k. \tag{4.32}$$

However, since $p(t) = 0$ for $t \in [0, \tau_{k+1} \wedge \tau_k]$ and $(u_0(x), v_0(x)) \in TM$ for $|x| < r$, by argument based on the one made after (4.29), in the Proposition 4.10, we get $z_{r,k}(t,x) = z_{r,k+1}(t,x)$ for every $t \in [0, \tau_{k+1}]$ and $|x| \le r - t$. But this contradicts (4.32) and we finish the proof of our claim and, in result, the proof of Lemma 4.11. □

Since by definition (4.23) and Lemma 4.11 the sequence of stopping times $\{\tau_k\}_{k\ge1}$ is bounded and non-decreasing, it makes sense to denote by τ the limit of $\{\tau_k\}_{k\ge1}$. Now by using [15, Lemma 10.1], we prove that the approximate solutions do not explode which is same as the following in terms of τ.

Proposition 4.12. *For τ_k defined in (4.23), $\tau := \lim_{k\to\infty} \tau_k = T$.*

Proof of Proposition 4.12. We first notice that by a particular case of the Chojnowska-Michalik Theorem [26], when the diffusion coefficient is absent, we have that for each k the approximate solution $z_{r,k}$, as a function of time t, is $H^1(\mathbb{R};\mathbb{R}^n)\times L^2(\mathbb{R};\mathbb{R}^n)$-valued and satisfies

$$z_{r,k}(t) = \xi + \int_0^t \mathcal{G}z_{r,k}(s)\,ds + \int_0^t \mathbf{F}_{r,k}(s, z_{r,k}(s))\,ds + \int_0^t \mathbf{G}_{r,k}(s, z_{r,k}(s))\dot{h}(s)\,ds, \tag{4.33}$$

for $t \le T$. In particular,

$$u_{r,k}(t) = \xi_1 + \int_0^t v_{r,k}(s)\,ds,$$

for $t \le T$, where $\xi_1 = E_r^2 u_0$ and the integral converges in $H^1(\mathbb{R};\mathbb{R}^n)$. Hence

$$\partial_t u_{r,k}(s,x) = v_{r,k}(s,x), \qquad \text{for all} \quad s \in [0,T], x \in \mathbb{R}.$$

Next, by keeping in mind the Proposition 4.10, we set

$$l(t) := \|a_k(t)\|^2_{H^1(B_{r-t})\times L^2(B_{r-t})} \qquad \text{and} \qquad q(t) := \log(1 + \|a_k(t)\|^2_{\mathcal{H}_{r-t}}).$$

By applying Proposition C.1, respectively, with $k = 0, 1$ and $L(x) = x, \log(1+x)$, followed by the use of Lemma 4.7 we get

$$\begin{aligned} l(t) \le l(0) &+ \int_0^t l(s)\,ds + \int_0^t \mathbb{1}_{[0,\tau_k]}(s)\langle v_{r,k}(s), \varphi(s)\rangle_{L^2(B_{r-s})}\,ds \\ &+ \int_0^t \mathbb{1}_{[0,\tau_k]}(s)\langle v_{r,k}(s), \psi(s)\rangle_{L^2(B_{r-s})}\,ds, \end{aligned} \tag{4.34}$$

and

$$\begin{aligned} q(t) \le q(0) &+ \int_0^t \frac{\|a_k(s)\|^2_{\mathcal{H}_{r-s}}}{1 + \|a_k(s)\|^2_{\mathcal{H}_{r-s}}}\,ds \\ &+ \int_0^t \mathbb{1}_{[0,\tau_k]}(s)\frac{\langle v_{r,k}(s), \varphi(s)\rangle_{L^2(B_{r-s})}}{1 + \|a_k(s)\|^2_{\mathcal{H}_{r-s}}}\,ds + \int_0^t \mathbb{1}_{[0,\tau_k]}(s)\frac{\langle \partial_x v_{r,k}(s), \partial_x[\varphi(s)]\rangle_{L^2(B_{r-s})}}{1 + \|a_k(s)\|^2_{\mathcal{H}_{r-s}}}\,ds \end{aligned}$$

$$+\int_0^t \mathbb{1}_{[0,\tau_k]}(s)\frac{\langle v_{r,k}(s),\psi(s)\rangle_{L^2(B_{r-s})}}{1+\|a_k(s)\|^2_{\mathcal{H}_{r-s}}}\,ds+\int_0^t \mathbb{1}_{[0,\tau_k]}(s)\frac{\langle \partial_x v_{r,k}(s),\partial_x[\psi(s)]\rangle_{L^2(B_{r-s})}}{1+\|a_k(s)\|^2_{\mathcal{H}_{r-s}}}\,ds. \tag{4.35}$$

Here

$$\varphi(s):=\mathcal{A}_{u_{r,k}(s)}(v_{r,k}(s),v_{r,k}(s))-\mathcal{A}_{u_{r,k}(s)}(\partial_x u_{r,k}(s),\partial_x u_{r,k}(s)),$$
$$\psi(s):=Y_{u_{r,k}(s)}(\partial_t u_{r,k}(s),\partial_x u_{r,k}(s))\dot{h}(s).$$

Since by Proposition 4.10 $u_{r,k}(s,x)\in M$ for $|x|\le r-s$ and $s\le\tau_k$, we have

$$u_{r,k}(s,x)\in M \quad \text{and} \quad \partial_t u_{r,k}(s,x)=v_{r,k}(s,x)\in T_{u_{r,k}(s,x)}M,$$

on the mentioned domain of s and x. Consequently, by Proposition 3.9, we get

$$\begin{aligned}\mathcal{A}_{u_{r,k}(s,x)}(v_{r,k}(s,x),v_{r,k}(s,x))&=A_{u_{r,k}(s,x)}(v_{r,k}(s,x),v_{r,k}(s,x)),\\ \mathcal{A}_{u_{r,k}(s,x)}(\partial_x u_{r,k}(s,x),\partial_x u_{r,k}(s,x))&=A_{u_{r,k}(s,x)}(\partial_x u_{r,k}(s,x),\partial_x u_{r,k}(s,x)),\end{aligned} \tag{4.36}$$

on $|x|\le r-s$ and $s\le\tau_k$. Hence, since $v_{r,k}(s,x)\in T_{u_{r,k}(s,x)}M$, and by definition, $A_{u_{r,k}(s,x)}\in N_{u_{r,k}(s,x)}M$, the L^2-inner product on domain B_{r-s} vanishes and, in result, the second integrals in (4.34) and (4.35) are equal to zero.

Next, to deal with the integral containing terms ψ, we follow Lemma 4.7 and we invoke Lemma 3.4, estimate (3.10), and Proposition 4.10 to get

$$\begin{aligned}&\langle v_{r,k}(s),Y_{u_{r,k}(s)}(\partial_t u_{r,k}(s),\partial_x u_{r,k}(s))\dot{h}(s)\rangle_{L^2(B_{r-s})}\\ &\lesssim \|v_{r,k}(s)\|^2_{L^2(B_{r-s})}+\|Y_{u_{r,k}(s)}(\partial_t u_{r,k}(s),\partial_x u_{r,k}(s))\dot{h}(s)\|^2_{L^2(B_{r-s})}\\ &\le \|v_{r,k}(s)\|^2_{L^2(B_{r-s})}+C_{Y_0}^2C_r^2\left(1+\|v_{r,k}(s)\|^2_{L^2(B_{r-s})}+\|\partial_x u_{r,k}(s)\|^2_{L^2(B_{r-s})}\right)\|\dot{h}(s)\|^2_{H_\mu}\\ &\lesssim (1+l(s))(1+\|\dot{h}(s)\|^2_{H_\mu}),\end{aligned} \tag{4.37}$$

for some $C_r>0$, and estimates (3.11)–(3.12) yields

$$\begin{aligned}&\langle v_{r,k}(s),Y_{u_{r,k}(s)}(\partial_t u_{r,k}(s),\partial_x u_{r,k}(s))\dot{h}(s)\rangle_{L^2(B_{r-s})}\\ &\qquad+\langle \partial_x v_{r,k}(s),\partial_x[Y_{u_{r,k}(s)}(\partial_t u_{r,k}(s),\partial_x u_{r,k}(s))\dot{h}(s)]\rangle_{L^2(B_{r-s})}\\ &\lesssim \|v_{r,k}(s)\|^2_{H^1(B_{r-s})}+\|Y_{u_{r,k}(s)}(\partial_t u_{r,k}(s),\partial_x u_{r,k}(s))\dot{h}(s)\|^2_{H^1(B_{r-s})}\\ &\le \|v_{r,k}(s)\|^2_{H^1(B_{r-s})}+\|\dot{h}(s)\|^2_{H_\mu}\Big[C_{Y_0}^2C_r^2\left(1+\|v_{r,k}(s)\|^2_{L^2(B_{r-s})}+\|\partial_x u_{r,k}(s)\|^2_{L^2(B_{r-s})}\right)\\ &\quad+C_{Y_1}^2\left(1+\|v_{r,k}(s)\|^2_{H^1(B_{r-s})}+\|\partial_x u_{r,k}(s)\|^2_{H^1(B_{r-s})}\right)\|u_{r,k}(s)\|^2_{H^1(B_{r-s})}\\ &\quad+C_{Y_2}^2\left(\|v_{r,k}(s)\|^2_{L^2(B_{r-s})}+\|\partial_x u_{r,k}(s)\|^2_{L^2(B_{r-s})}\right)\Big]\\ &\lesssim_{C_r,C_{Y_i}}\ (1+l(s))\,(1+\|a_k(s)\|^2_{\mathcal{H}_{r-s}})(1+\|\dot{h}(s)\|^2_{H_\mu}),\quad i=0,1,2.\end{aligned} \tag{4.38}$$

By substituting the estimates (4.36) and (4.37) in the inequality (4.34) we get

$$l(t) \lesssim l(0) + \int_0^t \mathbb{1}_{[0,\tau_k]}(s)(1 + l(s))\ (1 + \|\dot{h}(s)\|^2_{H_\mu})\, ds. \tag{4.39}$$

Now we define S_j as the set of initial data whose norm under extension is bounded by j, more precisely,

$$S_j := \{(u_0, v_0) \in \mathcal{H}_{\rm loc} :\ \|\xi\|_{\mathcal{H}_r} \le j \text{ where } \xi := (E^2_r u_0, E^1_r v_0)\}.$$

Then, for the initial data belonging to S_j, the Gronwall Lemma on (4.39) yields

$$1 + l_j(t \wedge \tau_k) \le K_{r,j}, \qquad t \le T, \quad j \in \mathbb{N}, \tag{4.40}$$

where the constant $K_{r,j}$ also depends on $\|\dot{h}\|_{L^2(0,T;H_\mu)}$ and l_j stands to show that (4.40) holds under S_j only.

Next to deal with the third integral in (4.35), denote by O its integrand, we recall the following celebrated Gagliardo-Nirenberg inequalities, see e.g. [37],

$$|\psi|^2_{L^\infty(r-s)} \le |\psi|^2_{L^2(B_{r-s})} + 2|\psi|_{L^2(B_{r-s})}|\dot{\psi}|_{L^2(B_{r-s})}, \qquad \psi \in H^1(B_{r-s}). \tag{4.41}$$

Then by applying [15, Lemma 10.1] followed by the generalized Hölder inequality and (4.41) we infer

$$\begin{aligned} |O(s)| &\lesssim \mathbb{1}_{[0,\tau_k)}(s) \frac{\int_{B_{r-s}} \{|\partial_x v_{r,k}||\partial_x u_{r,k}||v_{r,k}|^2 + |\partial_{xx} u_{r,k}||\partial_x u_{r,k}|^2|v_{r,k}| + |\partial_x v_{r,k}||\partial_x u_{r,k}|^3\}\, dx}{1 + \|a_k(s)\|^2_{\mathcal{H}_{r-s}}} \\ &\lesssim \mathbb{1}_{[0,\tau_k)}(s) \frac{l(s)\|a_k(s)\|^2_{\mathcal{H}_{r-s}}}{1 + \|a_k(s)\|^2_{\mathcal{H}_{r-s}}} \le \mathbb{1}_{[0,\tau_k)}(s)(1 + l(s)). \end{aligned} \tag{4.42}$$

So, by substituting (4.36), (4.37) and (4.42) in (4.35) we have

$$q(t) \lesssim 1 + q(0) + \int_0^t \mathbb{1}_{[0,\tau_k)}(s)(1 + l(s))\ (1 + \|\dot{h}(s)\|^2_{H_\mu})\, ds.$$

Consequently, by applying (4.40), we obtain on S_j,

$$\begin{aligned} q_j(t \wedge \tau_k) &\lesssim 1 + q_j(0) + \int_0^t [1 + l_j(s \wedge \tau_k)]\ (1 + \|\dot{h}(s)\|^2_{H_\mu})\, ds \\ &\le C_{r,j}\ \|\dot{h}\|_{L^2(0,T;H_\mu)}, \qquad j \in \mathbb{N}, t \in [0, T], \end{aligned} \tag{4.43}$$

for some $C_{r,j} > 0$, where in the last step we have used that $r > T$ and on set S_j the quantity $q_j(0)$ is bounded by $\log(1 + j)$.

To complete the proof let us fix $t < T$. Then, by Proposition 4.10,

$$|a_k(\tau_k)|_{\mathcal{H}_{r-\tau_k}} = |z_{r,k}(\tau_k)|_{\mathcal{H}_{r-\tau_k}} \geq k \quad \text{whenever} \quad \tau_k \leq t.$$

So for every k such that $\tau_k \leq t$ we have

$$\log(1+k^2) \leq q(\tau_k) = q(t \wedge \tau_k).$$

Thus by restricting to S_j and using inequality (4.43), we obtain

$$\log(1+k^2) \leq q_j(t \wedge \tau_k) \lesssim C_{r,j} \|\dot{h}\|_{L^2(0,T;H_\mu)}. \tag{4.44}$$

In this way, if $\lim\limits_{k\to\infty} \tau_k = t_0$ for any $t_0 < T$, then by taking $k \to \infty$ in (4.44) we get $C_{r,j}\|\dot{h}\|_{L^2(0,T;H_\mu)} \geq \infty$ which is absurd. Since this holds for every $j \in \mathbb{N}$ and $t_0 < T$, we infer that $\tau = T$. Hence, the proof of Proposition 4.12 is complete. □

The conclusion of the proof of the existence part of Theorem 4.1. Now we have all the machinery required to finish the proof of Theorem 4.1. Define

$$w_{r,k}(t) := \begin{pmatrix} E^2_{r-t} u_{r,k}(t) \\ E^1_{r-t} v_{r,k}(t) \end{pmatrix},$$

and observe that $w_{r,k} : [0,T) \to \mathcal{H}$ is continuous. If we set

$$z_r(t) := \lim_{k\to\infty} w_{r,k}(t), \qquad t < T, \tag{4.45}$$

then by Lemma 4.11 and Proposition 4.12 it is straightforward to verify that, for every $t < T$, the sequence $\{w_{r,k}(t)\}_{k\in\mathbb{N}}$ is Cauchy in $\mathcal{H}$. But since $\mathcal{H}$ is complete, the limit in (4.45) converges in $\mathcal{H}$. Moreover, since by Proposition 4.12 $z_{r,k}(t) = z_{r,k_1}(t)$ for every $k_1 \geq k$ and $t \leq \tau_k$, we have that $z_r(t) = w_{r,k}(t)$ for $t \leq \tau_k$. In particular, $[0,T) \ni t \mapsto z_r(t) \in \mathcal{H}$ is continuous and $z_r(t,x) = z_{r,k}(t,x)$ for $|x| \leq r - t$ if $t \leq \tau_k$.

Hence, if we write $z_r(t) = (u_r(t), v_r(t))$, then we have shown that u_r satisfy the first conclusion of the Theorem B.1. In the remaining proof of the existence part we will show that the z_r, defined in (4.45), will satisfy all the remaining conclusions. Evaluation of (4.33) at $t \wedge \tau_k$ together applying the result from previous paragraph gives

$$z_{r,k}(t\wedge\tau_k) = \xi + \int\limits_0^{t\wedge\tau_k} \mathcal{G} z_{r,k}(s)\,ds + \int\limits_0^{t\wedge\tau_k} \mathbf{F}_r(s, z_{r,k}(s))\,ds + \int\limits_0^{t\wedge\tau_k} \mathbf{G}_r(s, z_{r,k}(s))\dot{h}(s)\,ds, \tag{4.46}$$

and this equality holds in $H^1(\mathbb{R};\mathbb{R}^n) \times L^2(\mathbb{R};\mathbb{R}^n)$. Restricting to the interval $(-R,R)$, (4.46) becomes

$$z_r(t\wedge\tau_k) = \xi + \int\limits_0^{t\wedge\tau_k} \mathcal{G} z_r(s)\,ds + \int\limits_0^{t\wedge\tau_k} \mathbf{F}_r(s, z_r(s))\,ds + \int\limits_0^{t\wedge\tau_k} \mathbf{G}_r(s, z_r(s))\dot{h}(s)\,ds,$$

under the action of natural projection from $H^1(\mathbb{R};\mathbb{R}^n) \times L^2(\mathbb{R};\mathbb{R}^n)$ to $H^1((-R,R);\mathbb{R}^n) \times L^2((-R,R);\mathbb{R}^n)$. Here the integrals converge in $H^1((-R,R);\mathbb{R}^n) \times L^2((-R,R);\mathbb{R}^n)$. Taking the limit $k \to \infty$ on both the sides, the dominated convergence theorem yields

$$z_r(t) = \xi + \int_0^t \mathcal{G} z_r(s)\,ds + \int_0^t \mathbf{F}_r(s, z_r(s))\,ds + \int_0^t \mathbf{G}_r(s, z_r(s))\dot{h}(s)\,ds, \qquad t < T,$$

in $H^1((-R,R);\mathbb{R}^n) \times L^2((-R,R);\mathbb{R}^n)$. In particular, by looking to each component separately we have, for every $t < T$,

$$u_r(t) = u_0 + \int_0^t v_r(s)\,ds, \tag{4.47}$$

in $H^1((-R,R);\mathbb{R}^n)$, and

$$\begin{aligned} v_r(t) = v_0 &+ \int_0^t \left[\partial_{xx} u_r(s) + A_{u_r(s)}(v_r(s), v_r(s)) - A_{u_r(s)}(\partial_x u_r(s), \partial_x u_r(s))\right] ds \\ &+ \int_0^t Y_{u_r(s)}(v_r(s), \partial_x u_r(s))\dot{h}(s)\,ds, \end{aligned} \tag{4.48}$$

holds in $L^2((-R,R);\mathbb{R}^n)$. It is relevant to note that in the formula above, we have replaced $\mathcal{A}$ by A which makes sense because due to Proposition 4.10 and Proposition 4.12, $u_r(t,x) = u_{r,k}(t,x) \in M$ for $|x| \le r - t$ and $t < T$. Hence we are done with the proof of existence part. The proof of the existence part of Theorem 4.1 is now complete. □

Proof of the uniqueness part of Theorem 4.1. For the uniqueness part let us fix R and T such that $R > T$ and a map $U : [0,T) \times (-R,R) \to M$ which satisfies conditions (ii)–(v) and (i') mentioned in the statement of Theorem 4.1. We will show (4.5) with $u : [0,T) \times \mathbb{R} \to M$ as constructed in the existence part.

Since we seek solutions that take values in the Fréchet space $H^2_{\text{loc}}(\mathbb{R};\mathbb{R}^n) \times H^1_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$, we localize the problem using a sequence of non-linear wave equations. Before doing this let us point out that the skeleton equation (4.6) can be equivalently written in the following form

$$z(t) = S_t \xi + \int_0^t S_{t-s} \mathbf{F}_{r,k}(s, z(s))\,ds + \int_0^t S_{t-s}(\mathbf{G}_{r,k}(s, z(s))\dot{h}(s))\,ds, \tag{4.49}$$

where

$$z(t) = (u(t), \partial_t u(t)) \in \mathcal{H}, \;\; t \in [0,T). \tag{4.50}$$

Concerning the uniqueness, let us define a function $Z : [0,T) \to \mathcal{H}$

$$Z(t) := \begin{pmatrix} E_R^2 U(t) \\ E_R^1 \partial_t U(t) \end{pmatrix}, \qquad t \in [0, T)$$

and observe that it is a continuous function. Let us also define a stopping time σ_k by

$$\sigma_k := \tau_k \wedge \inf\{t < T : \|Z(t)\|_{\mathcal{H}_{r-t}} \geq k\},$$

and a function $\beta : [0, T) \to \mathcal{H}$ as, for $t \in [0, T)$,

$$\beta(t) := S_t \xi + \int_0^t S_{t-s} \mathbb{1}_{[0,\sigma_k)}(s) \mathbf{F}_{r,k}(s, Z(s))\, ds + \int_0^t S_{t-s} \mathbb{1}_{[0,\sigma_k)}(s) \mathbf{G}_{r,k}(s, Z(s)) \dot{h}(s)\, ds.$$

In the same vein as in the existence part of the proof, as an application of the Chojnowska-Michalik Theorem and projection operator, the restriction of β on $\mathcal{H}_R$, which we denote by b, satisfies

$$b(t) = \xi + \int_0^t \mathcal{G} b(s)\, ds + \int_0^t \begin{pmatrix} 0 \\ \mathcal{A}_{U(s)}(\partial_t U(s), \partial_t U(s)) - \mathcal{A}_{U(s)}(\partial_x U(s), \partial_x U(s)) \end{pmatrix} ds$$

$$+ \int_0^t \begin{pmatrix} 0 \\ Y_{U(s)}(\partial_t U(s), \partial_x U(s)) \dot{h}(s) \end{pmatrix} ds, \qquad t \leq \sigma_k,$$

where the integrals converge in $H^1((-R, R); \mathbb{R}^n) \times L^2((-R, R); \mathbb{R}^n)$. Then since $U(t)$ and $\partial_t U(t)$ have similar form, respectively to (4.47) and (4.48), by direct computation we deduce that function p defined as

$$p(t) := b(t) - \begin{pmatrix} U(t) \\ \partial_t U(t) \end{pmatrix},$$

satisfies

$$p(t) = \int_0^t \mathcal{G} p(s)\, ds, \qquad t \leq \sigma_k.$$

Since the above implies that p satisfies the linear homogeneous wave equation with null initial data, by [15, Remark 6.2],

$$p(t, x) = 0 \quad \text{for} \quad |x| \leq R - t, t \leq \sigma_k. \tag{4.51}$$

Next we set

$$q(t) := \|\beta(t) - a_k(t)\|^2_{\mathcal{H}_{R-t}},$$

and apply Proposition C.1, with $k = 1$, $T = r$, $L = I$, to obtain

$$
\begin{aligned}
q(t\wedge\sigma_k) \le 2\int_0^{t\wedge\sigma_k} q(s)\,ds + \int_0^{t} \|\mathbf{F}_{r,k}(s, Z(s)) - \mathbf{F}_{r,k}(s, a_k(s))\|_{\mathcal{H}}^2\,ds \\
+ \int_0^{t\wedge\sigma_k} \|\mathbf{G}_{r,k}(s, Z(s))\dot{h}(s) - \mathbf{G}_{r,k}(s, a_k(s))\dot{h}(s)\|_{\mathcal{H}}^2\,ds.
\end{aligned}
\tag{4.52}
$$

But we know that $r-t > R-t$, and by definition $\sigma_k \le \tau_k$ which implies

$$
\mathbf{F}_{r,k}(t,z) = \mathbf{F}_{R,k}(t,z), \qquad \mathbf{G}_{r,k}(t,z) = \mathbf{G}_{R,k}(t,z) \quad \text{on} \quad (t-R, R-t),
$$

whenever $\|z\|_{\mathcal{H}_{r-t}} \le k$. Consequently, the estimate (4.52) becomes

$$
\begin{aligned}
q(t\wedge\sigma_k) \le 2\int_0^{t\wedge\sigma_k} q(s)\,ds + \int_0^{t\wedge\sigma_k} \|\mathbf{F}_{R,k}(s, Z(s)) - \mathbf{F}_{R,k}(s, a_k(s))\|_{\mathcal{H}}^2]\,ds \\
+ \int_0^{t\wedge\sigma_k} \|\mathbf{G}_{R,k}(s, Z(s))\dot{h}(s) - \mathbf{G}_{R,k}(s, a_k(s))\dot{h}(s)\|_{\mathcal{H}}^2\,ds.
\end{aligned}
$$

Invoking Lemmata 4.7 and 3.4 followed by (4.51) yields

$$
q(t\wedge\sigma_k) \le C_R \int_0^{t\wedge\sigma_k} q(s)(1+\|\dot{h}(s)\|_{H_\mu}^2)\,ds.
$$

Therefore, we get $q=0$ on $[0,\sigma_k)$ by the Gronwall Lemma. Since in the limit $k\to\infty$, σ_k goes to T as τ_k, by taking k to infinity, by Proposition 4.10 we obtain that $u_r(t,x) = U(t,x)$ for each $t<T$ and $|x| \le R-t$. The proof of the uniqueness part of Theorem 4.1 and hence the whole proof is now complete. □

5. Large deviation principle

In this section we establish a large deviation principle (LDP) for system (1.2) via a weak convergence approach developed in [21] and [22] which is based on variational representations of infinite-dimensional Wiener processes.

First, let us recall the general criteria for LDP obtained in [21]. Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space with an increasing family $\mathbb{F} := \{\mathfrak{F}_t, t\ge 0\}$ of the sub-σ-fields of $\mathfrak{F}$ satisfying the usual conditions. Let $\mathscr{B}(E)$ denote the Borel σ-field of the Polish space E (i.e. complete separable metric space). Since we are interested in the large deviations of continuous stochastic processes, we follow [25] and consider the following definition of large deviations principle given in terms of random variables.

Definition 5.1. The $(E, \mathscr{B}(E))$-valued random family $\{X^\varepsilon\}_{\varepsilon>0}$, defined on $(\Omega, \mathfrak{F}, \mathbb{P})$, is said to satisfy a large deviation principle on E with the good rate function $\mathcal{I}$ if the following conditions hold:

(1) $\mathcal{I}$ **is a good rate function**: The function $\mathcal{I}: E \to [0, \infty]$ is such that for each $\mathcal{M} \in [0, \infty)$ the level set $\{\phi \in E : \mathcal{I}(\phi) \leq \mathcal{M}\}$ is a compact subset of E.
(2) **Large deviation upper bound**: For each closed subset F of E

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}\left[X^\varepsilon \in F\right] \leq -\inf_{u \in F} \mathcal{I}(u).$$

(3) **Large deviation lower bound**: For each open subset G of E

$$\liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P}\left[X^\varepsilon \in G\right] \geq -\inf_{u \in G} \mathcal{I}(u),$$

where by convention the infimum over an empty set is $+\infty$.

Assume that K, H are separable Hilbert spaces such that the embedding $K \hookrightarrow H$ is Hilbert-Schmidt. Let $W := \{W(t), t \geq 0\}$ be a cylindrical Wiener process on K defined on $(\Omega, \mathfrak{F}, \mathbb{F}, \mathbb{P})$. Hence the paths of W take values in $\mathcal{C}([0, \infty); H)$.

Let us, for the whole section, fix a number $T > 0$. Note that the RKHS linked to W restricted to the time interval $[0, T]$ is equal to ${}_0H^{1,2}(0, T; K)$. Let $\mathscr{S}$ be the class of K-valued $\mathbb{F}$-predictable processes ϕ which trajectories belong to ${}_0H^{1,2}(0, T; K)$, $\mathbb{P}$-almost surely. For $\mathcal{M} > 0$, we set

$$S_{\mathcal{M}} := \left\{ h \in {}_0H^{1,2}(0, T; K) : \int_0^T \|\dot{h}(s)\|_K^2 \, ds \leq \mathcal{M} \right\}. \tag{5.1}$$

The set $S_{\mathcal{M}}$ endowed with the weak topology from ${}_0H^{1,2}(0, T; K)$, is metrizable by the following metric

$$d_1(h, k) := \sum_{i=1}^{\infty} \frac{1}{2^i} \left| \int_0^T \langle \dot{h}(s) - \dot{k}(s), e_i \rangle_K \, ds \right|,$$

where $\{e_i\}_{i \in \mathbb{N}}$ is a complete orthonormal basis for $L^2(0, T; K)$, is a Polish space, see [22]. Define $\mathscr{S}_{\mathcal{M}}$ as the set of bounded stochastic controls by

$$\mathscr{S}_{\mathcal{M}} := \{\phi \in \mathscr{S} : \phi(\omega) \in S_{\mathcal{M}}, \mathbb{P}\text{-a.s.}\}.$$

Note that $\cup_{M>0} \mathscr{S}_{\mathcal{M}}$ is a proper subset of $\mathscr{S}$. Next, consider a family indexed by $\varepsilon \in (0, 1]$ of Borel measurable maps

$$J^\varepsilon : {}_0\mathcal{C}([0, T]; H) \to E.$$

We denote by μ^ε the "image" measure on E of $\mathbb{P}$ by J^ε, that is,

$$\mu^\varepsilon = J^\varepsilon(\mathbb{P}), \quad i.e. \quad \mu^\varepsilon(A) = \mathbb{P}\left((J^\varepsilon)^{-1}(A)\right), \quad A \in \mathscr{B}(E).$$

We have the following result.

Theorem 5.2 *([21, Theorem 4.4]). Suppose that there exists a measurable map $J^0 : {}_0\mathcal{C}([0,T]; H) \to E$ such that*

BD1: *if $\mathcal{M} > 0$ and a family $\{h_\varepsilon\} \subset \mathscr{S}_{\mathcal{M}}$ converges in law as $S_{\mathcal{M}}$-valued random elements to $h \in \mathscr{S}_{\mathcal{M}}$ as $\varepsilon \to 0$, then the random variables*

$$
{}_0\mathcal{C}([0,T]; H) \ni \omega \mapsto J^\varepsilon\left(\omega + \frac{1}{\sqrt{\varepsilon}} \int_0^{\cdot} \dot{h}_\varepsilon(s)\, ds\right) \in E,
$$

converges in law, as $\varepsilon \searrow 0$, to the random variable $J^0\left(\int_0^{\cdot} \dot{h}(s)\, ds\right)$,
BD2: *for every $\mathcal{M} > 0$, the set*

$$
\left\{ J^0\left(\int_0^{\cdot} \dot{h}(s)\, ds\right) : h \in S_{\mathcal{M}} \right\},
$$

is a compact subset of E.

Then the family of measures μ^ε satisfies the large deviation principle (LDP) with the rate function defined by

$$
\mathcal{I}(u) := \inf\left\{ \frac{1}{2} \int_0^T \|\dot{h}(s)\|_K^2\, ds : {}_0H^{1,2}(0,T;K) \text{ and } u = J^0\left(\int_0^{\cdot} \dot{h}(s)\, ds\right)\right\}, \tag{5.2}
$$

with the convention $\inf\{\emptyset\} = +\infty$.

5.1. Main result

It is important to note that in transferring the general theory argument from Theorem 5.2 in our setting we require some information about the difference of solutions at two different times. Hence we need to strengthen the assumptions on diffusion coefficient. In the remaining part of this paper, we assume that

$$
Y : M \ni p \mapsto Y(p) \in T_pM,
$$

is a smooth vector field on compact Riemannian manifold M, which can be considered as a submanifold of $\mathbb{R}^n$, such that its extension, denote again by Y, on the ambient space $\mathbb{R}^n$ is smooth and satisfies

Y.4 there exists a compact set $K_Y \subset \mathbb{R}^n$ such that $Y(p) = 0$ if $p \notin K_Y$,
Y.5 for $q \in O$, $Y(\Upsilon(q)) = \Upsilon'(q)Y(q)$,
Y.6 for some $C_Y > 0$

$$
|Y(p)| \leq C_Y(1 + |p|), \qquad \left|\frac{\partial Y}{\partial p_i}(p)\right| \leq C_Y, \text{ and } \left|\frac{\partial^2 Y}{\partial p_i \partial p_j}(p)\right| \leq C_Y,
$$

for $p \in K_Y, i, j = 1, \ldots, n$.

Remark 5.3.

(1) Since K_Y is compact, there exists a C_K such that $|Y(p)| \le C_K$ for $p \in \mathbb{R}^n$.
(2) For $M = \mathbb{S}^2$ case, $Y(p) = p \times e$, $p \in M$, for some fixed vector $e \in \mathbb{R}^3$ satisfies above assumptions.

Since, due to the above assumptions, Y and its first order partial derivatives are Lipschitz, by 1-D Sobolev embedding we easily get the next result.

Lemma 5.4. *For any $R > 0$, there exists a constant $C_{Y,R} > 0$ such that the extension Y defined above satisfies*

$$\begin{aligned}
&(1) \quad \|Y(u)\|_{H^j(B_R)} \le C_{Y,R}(1 + \|u\|_{H^j(B_R)}), \quad j = 0, 1, 2,\\
&(2) \quad \|Y(u) - Y(v)\|_{L^2(B_R)} \le C_{Y,R}\|u - v\|_{L^2(B_R)},\\
&(3) \quad \|Y(u) - Y(v)\|_{H^1(B_R)} \le C_{Y,R}\|u - v\|_{H^1(B_R)}\left(1 + \|u\|_{H^1(B_R)} + \|v\|_{H^1(B_R)}\right).
\end{aligned}$$

Let $(\mathfrak{F}_t^{W,0})$ be the $\mathbb{P}$-augmented filtration generated by the Wiener process W. Now we state the main result of this section for the following small noise Cauchy problem

$$\begin{cases} \partial_{tt} u^\varepsilon = \partial_{xx} u^\varepsilon + A_{u^\varepsilon}(\partial_t u^\varepsilon, \partial_t u^\varepsilon) - A_{u^\varepsilon}(\partial_x u^\varepsilon, \partial_x u^\varepsilon) + \sqrt{\varepsilon} Y(u^\varepsilon)\dot{W}, \\ \left(u^\varepsilon(0), \partial_t u^\varepsilon(0)\right) = (u_0, v_0), \end{cases} \tag{5.3}$$

with the hypothesis that (u_0, v_0) is $\mathfrak{F}_0$-measurable $H^2_{\text{loc}} \times H^1_{\text{loc}}(\mathbb{R}, TM)$-valued random variable, such that $u_0(x,\omega) \in M$ and $v_0(x,\omega) \in T_{u_0(x,\omega)}M$ hold for every $\omega \in \Omega$ and $x \in \mathbb{R}$. Since the small noise problem (5.3), with initial data $(u_0, v_0) \in \mathscr{H}_{\text{loc}}(\mathbb{R}; M)$, is a particular case of Theorem B.1, for given $\varepsilon > 0$ and $T > 0$, there exists a unique global strong $(\mathfrak{F}_t^{W,0})$-adapted solution to (5.3), which we denote by $z^\varepsilon := (u^\varepsilon, \partial_t u^\varepsilon)$, with values in the Polish space

$$\begin{aligned}
\mathcal{X}_T &:= \mathcal{C}\left([0,T]; H^2_{\text{loc}}(\mathbb{R}; \mathbb{R}^n)\right) \times \mathcal{C}\left([0,T]; H^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^n)\right),\\
&\cong \mathcal{C}\left([0,T]; \mathcal{H}_{\text{loc}}\right),
\end{aligned} \tag{5.4}$$

where $\mathcal{H}_{\text{loc}}$ has been defined in (2.5), and satisfy the properties mentioned in Appendix B.

Below, let H_μ be embedded in a separable Hilbert space E via a Hilbert-Schmidt inclusion $\mathbf{i}: H_\mu \hookrightarrow E$ as in Example 3.2, define a filtration

$$\mathcal{G}_t = \sigma(\pi_s : s \le t), \qquad t \in [0,T]$$

on ${}_0\mathcal{C}([0,T]; E)$ where $\pi_s(f) = f(s)$, denote by $\mathfrak{w}$ the Wiener measure with the covariance operator $\mathbf{ii}^*$ on ${}_0\mathcal{C}([0,T]; E)$ and denote by $\mathbf{B}$ the identity mapping on ${}_0\mathcal{C}([0,T]; E)$.

Lemma 5.5. *Let $(u_0, v_0) \in \mathscr{H}_{loc}(\mathbb{R}; M)$. Then there exists a Borel measurable mapping $J^\varepsilon = (U^\varepsilon, V^\varepsilon)$*

$$J^\varepsilon : {}_0\mathcal{C}([0,T]; E) \to \mathcal{X}_T, \tag{5.5}$$

such that

(a) $U^\varepsilon(t,x)$, $V^\varepsilon(t,x)$ *are* $\mathcal{G}_t^{\mathfrak{w}}$*-adapted for every* $(t,x) \in [0,T] \times \mathbb{R}$,
(b) $U^\varepsilon(t,x) : {}_0\mathcal{C}([0,T];E) \to M$ *for every* $(t,x) \in [0,T] \times \mathbb{R}$,
(c) $t \mapsto U^\varepsilon(t) \in H^1_{loc}(\mathbb{R};\mathbb{R}^n)$ *is continuously differentiable and*

$$\frac{dU^\varepsilon(t)}{dt} = V^\varepsilon(t), \quad t \in [0,T],$$

(d) $(U^\varepsilon(0), V^\varepsilon(0)) = (u_0, v_0)$,
(e) $(U^\varepsilon, \mathbf{B})$ *is a solution of* (5.3) *in the sense of Theorem B.1 for the probability measure* $\mathfrak{w}$,
(f) *if* $\tilde{W}$ *is an* E*-valued Wiener process with covariance operator* $\mathbf{ii}^*$ *on some stochastic basis then* $(U^\varepsilon(\tilde{W}), \tilde{W})$ *is a solution of* (5.3) *in the sense of Theorem B.1.*

Proof of Lemma 5.5. For $t \in [0,T]$, define a stopping operator

$$L_t : {}_0\mathcal{C}([0,T];E) \to {}_0\mathcal{C}([0,T];E) : f \mapsto f(\cdot \wedge t)$$

and observe that $\mathcal{G}_t = \sigma(L_t)$ and $\mathfrak{F}_t^W = \sigma(L_t(W))$. The Doob-Dynkin Lemma yields the existence of a Borel measurable mapping J^ε such that $z^\varepsilon = J^\varepsilon(W)$ a.s., and since z^ε is $(\mathfrak{F}_t^{W,0})$-adapted, the same lemma yields the existence of a Borel measurable map

$$l_t : {}_0\mathcal{C}([0,T];E) \to H^2_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n) \times H^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n)$$

such that $z^\varepsilon(t) = l_t(L_t(W))$ a.s. Hence $\mathfrak{w}(J_t^\varepsilon = l_t \circ L_t) = 1$ and we conclude that J_t^ε is $\mathcal{G}_t^{\mathfrak{w}}$-measurable for every $t \in [0,T]$. In particular, we have proved (a). Since $U^\varepsilon(t,x)(W) = u^\varepsilon(t,x) \in M$ a.s. for every $(t,x) \in [0,T] \times \mathbb{R}$ by definition, we get that, $\mathfrak{w}$-a.s., $U^\varepsilon(t,x) \in M$ for every $(t,x) \in [0,T] \times \mathbb{R}$ since paths of U^ε are jointly continuous. Thus (b) holds $\mathfrak{w}$-a.s. Next,

$$u^\varepsilon(t,x) = u_0(x) + \int_0^t \partial_t u^\varepsilon(s,x)\,ds$$

holds a.s. for every $(t,x) \in [0,T] \times \mathbb{R}$ so, as in the previous step, $\mathfrak{w}$-a.s.,

$$U^\varepsilon(t,x) = u_0(x) + \int_0^t V^\varepsilon(s,x)\,ds$$

holds for every $(t,x) \in [0,T] \times \mathbb{R}$ since paths of U^ε and V^ε are jointly continuous. In particular, (c) holds $\mathfrak{w}$-a.s. Moreover, it is obvious that (d) holds $\mathfrak{w}$-a.s. To deal with the $\mathfrak{w}$-exceptional set, denote by γ the smooth geodesic flow on $\mathbb{R} \times TM$ and redefine, on this exceptional set,

$$J^\varepsilon(t,x) = (\gamma(t,u_0(x),v_0(x)), \dot{\gamma}(t,u_0(x),v_0(x)))$$

which satisfies (b), (c) and (d) as well. Finally, if we define $(\tilde{u}^\varepsilon, \tilde{v}^\varepsilon) = (\tilde{U}^\varepsilon(\tilde{W}), \tilde{V}^\varepsilon(\tilde{W}))$ then the finite-dimensional distributions of the processes

$$(V^\varepsilon, \partial_{xx}U^\varepsilon, A_{U^\varepsilon}(\partial_x U^\varepsilon, \partial_x U^\varepsilon), A_{U^\varepsilon}(V^\varepsilon, V^\varepsilon), Y(U^\varepsilon), \mathbf{B})$$
$$(\partial_t u^\varepsilon, \partial_{xx}u^\varepsilon, A_{u^\varepsilon}(\partial_x u^\varepsilon, \partial_x u^\varepsilon), A_{u^\varepsilon}(\partial_t u^\varepsilon, \partial_t u^\varepsilon), Y(u^\varepsilon), W)$$
$$(\tilde{v}^\varepsilon, \partial_{xx}\tilde{u}^\varepsilon, A_{\tilde{u}^\varepsilon}(\partial_x \tilde{u}^\varepsilon, \partial_x \tilde{u}^\varepsilon), A_{\tilde{u}^\varepsilon}(\tilde{v}^\varepsilon, \tilde{v}^\varepsilon), Y(\tilde{u}^\varepsilon), \tilde{W})$$

coincide in every in $L^2((-R, R; \mathbb{R}^n))$. Hence we obtain (e) and (f) e.g. by [50, Theorem 8.3 and Theorem 8.6]. Let us just point out that the measurability and qualitative properties of $\tilde{u}^\varepsilon$ and $\tilde{v}^\varepsilon = \frac{d\tilde{u}^\varepsilon}{dt}$ are guaranteed by (a)–(d). $\square$

Recall from Section 3 that the random perturbation W we consider is a cylindrical Wiener process on H_μ and there exists a separable Hilbert space E such that the embedding of H_μ in E is Hilbert-Schmidt. Hence we can apply the general theory from previous section with the notations defined by taking H_μ instead of K.

Let us define a Borel map

$$J^0 : {}_0\mathcal{C}([0, T]; E) \to \mathcal{X}_T. \tag{5.6}$$

Note that it is well-defined due to Lemma 5.5. If $h \in {}_0\mathcal{C}([0, T]; E) \setminus {}_0H^{1,2}(0, T, H_\mu)$, then we set $J^0(h) = 0$. If $h \in {}_0H^{1,2}(0, T, H_\mu)$ then by Theorem 4.1 there exists a function in $\mathcal{X}_T$, say z_h, that solves

$$\begin{cases} \partial_{tt}u = \partial_{xx}u + A_u(\partial_t u, \partial_t u) - A_u(\partial_x u, \partial_x u) + Y(u)\dot{h}, \\ u(0, \cdot) = u_0, \partial_t u(0, \cdot) = v_0, \end{cases} \tag{5.7}$$

uniquely and we set $J^0(h) = z_h$.

Remark 5.6. At some places in the paper we denote $J^0(h)$ by $J^0\left(\int_0^\cdot \dot{h}(s)\,ds\right)$ to make it clear that the considered differential equation is controlled by $\dot{h}$ not by h.

The main result of this section is as follows.

Theorem 5.7. *The family of laws $\{\mathscr{L}(z^\varepsilon) : \varepsilon \in (0, 1]\}$ on $\mathcal{X}_T$, where $z^\varepsilon := (u^\varepsilon, \partial_t u^\varepsilon)$ is the unique solution to (5.3), satisfies the large deviation principle with rate function $\mathcal{I}$ defined in (5.2).*

Remark 5.8. It is relevant to note that we believe that it is possible to establish the well-posedness result and the LDP for problem (5.3) in a suitable weighted Sobolev spaces but we have decided to use local Sobolev spaces for our analysis because the wave equation structure allows localization. We have adopted the approach of local Sobolev spaces because it has been used in all the previous papers on the stochastic geometric wave equations, see [15,17–19].

Note that, in light of Theorem 5.2, in order to prove the Theorem 5.7 it is sufficient to show the following two statements.

Statement 1: For each $\mathcal{M} > 0$, the set $K_\mathcal{M} := \{J^0(h) : h \in S_\mathcal{M}\}$ is a compact subset of $\mathcal{X}_T$, where $S_\mathcal{M} \subset {}_0H^{1,2}(0, T, H_\mu)$ is the centered closed ball of radius $\mathcal{M}$ endowed with the weak topology.

Statement 2: Assume that $\mathcal{M} > 0$, that $\{\varepsilon_n\}_{n\in\mathbb{N}}$ is an $(0,1]$-valued sequence convergent to 0, that $\{h_n\}_{n\in\mathbb{N}} \subset \mathscr{S}_\mathcal{M}$ converges in law to $h \in \mathscr{S}_\mathcal{M}$ as $\varepsilon \to 0$. Then, the processes

$$ {}_0\mathcal{C}([0,T];E) \ni W(\cdot) \mapsto J^{\varepsilon_n}\left(W(\cdot) + \frac{1}{\sqrt{\varepsilon_n}}\int_0^{\cdot} \dot{h}_n(s)\,ds\right) \in \mathcal{X}_T, \tag{5.8} $$

converge in law on $\mathcal{X}_T$ to $J^0\left(\int_0^{\cdot} \dot{h}(s)\,ds\right)$.

Remark 5.9. By combining the proofs of Theorem B.1 and Theorem 4.1 we infer that the map (5.8) is well-defined and $J^{\varepsilon_n}\left(W(\cdot) + \frac{1}{\sqrt{\varepsilon_n}}\int_0^{\cdot} \dot{h}_n(s)\,ds\right)$ solves the following stochastic control Cauchy problem

$$ \begin{cases} \partial_{tt}u^{\varepsilon_n} = \partial_{xx}u^{\varepsilon_n} + A_{u^{\varepsilon_n}}(\partial_t u^{\varepsilon_n}, \partial_t u^{\varepsilon_n}) - A_{u^{\varepsilon_n}}(\partial_x u^{\varepsilon_n}, \partial_x u^{\varepsilon_n}) + Y(u^{\varepsilon_n})\dot{h}_n \\ \qquad\qquad + \sqrt{\varepsilon_n}Y(u^{\varepsilon_n})\dot{W}, \\ \left(u^{\varepsilon_n}(0), \partial_t u^{\varepsilon_n}(0)\right) = (u_0, v_0), \end{cases} \tag{5.9} $$

for the initial data $(u_0, v_0) \in H^2_{\mathrm{loc}} \times H^1_{\mathrm{loc}}(\mathbb{R}; TM)$.

Remark 5.10. It is clear by now that verification of the LDP reduces to proving two convergence results, see [13,12,20,25,63]. As it was shown first in [9], the second convergence result follows from the first one via the Jakubowski version of the Skorokhod representation theorem. Therefore, establishing LDP reduces, de facto, to proving one convergence result for deterministic controlled problem called also the skeleton equation. This convergence result is specific to the stochastic PDE in question and requires techniques related to the considered equation. Thus, for instance, the proof in [9, Lemma 6.3] for the stochastic Landau-Lifshitz-Gilbert equation, is different from the proof, for stochastic Navier-Stokes equation, of [25, Proposition 3.5]. On technical level, the proof of corresponding result, i.e. **Statement 1**, is the main contribution of our work.

5.2. *Proof of Statement 1*

Let us fix $\mathcal{M} > 0$ and consider a sequence of controls $\{h_n\}_{n\in\mathbb{N}} \subset S_\mathcal{M}$. Let $z_n = (u_n, v_n) := J^0(h_n)$, for $n \in \mathbb{N}$, be a solution to problem (5.7), corresponding to control h_n. Since $S_\mathcal{M}$ is the closed unit ball in the Hilbert space ${}_0H^{1,2}(0,T,H_\mu)$, by the Banach-Alaoglu Theorem [59, Theorem 3.15] or [5, Theorem 3.16], $S_\mathcal{M}$ is weakly compact. Consequently there exists a subsequence of $\{h_n\}_{n\in\mathbb{N}}$, we still denote this by $\{h_n\}_{n\in\mathbb{N}}$, which converges weakly to a limit $h \in S_\mathcal{M}$. Hence in order to complete the proof of **Statement 1** we only need to show that the subsequence $\{z_n\}_{n\in\mathbb{N}}$ converges to $z_h = (u_h, v_h)$ which, by definition, is the unique solution to the Cauchy problem of the skeleton equation (5.7) with the control h.

Before delving into the proof of this claim we establish the following a priori estimate which is a preliminary step required to prove, Proposition 5.16, the main result of this section. Let us recall that $T > 0$ is fixed for the whole section and $\mathcal{M} > 0$ is chosen and fixed in this subsection.

Lemma 5.11. *If $x \in \mathbb{R}$, then there exists a constant $\mathcal{B} > 0$, which depends on $\|(u_0, v_0)\|_{\mathcal{H}(B(x,T))}$, $\mathcal{M}$ and T, such that*

$$\sup_{h\in S_{\mathcal{M}}}\sup_{t\in[0,\frac{T}{2}]} \boldsymbol{e}(t,T;x,z_h(t)) \le \mathcal{B}, \tag{5.10}$$

where z_h is the unique global strong solution to problem (5.7) and

$$\boldsymbol{e}(t,T;x,z) := \frac{1}{2}\|z\|^2_{\mathcal{H}_{B(x,T-t)}} = \frac{1}{2}\Big\{\|u\|^2_{L^2(B(x,T-t))} + \|\partial_x u\|^2_{L^2(B(x,T-t))} + \|v\|^2_{L^2(B(x,T-t))} + \|\partial_{xx} u\|^2_{L^2(B(x,T-t))} + \|\partial_x v\|^2_{L^2(B(x,T-t))}\Big\}, \quad z=(u,v)\in\mathcal{H}_{loc}.$$

Moreover, if we restrict x on an interval $[-a,a]\subset\mathbb{R}$, then the constant $\mathcal{B} := \mathcal{B}(\mathcal{M},T,a)$, which also depends on 'a' now, can be chosen such that

$$\sup_{x\in[-a,a]}\sup_{h\in S_{\mathcal{M}}}\sup_{t\in[0,\frac{T}{2}]} \boldsymbol{e}(t,T;x,z_h(t)) \le \mathcal{B}.$$

Proof of Lemma 5.11. Let us choose and fix $x\in\mathbb{R}$. First note that the last part follows from the first one because by assumptions, $(u_0,v_0)\in\mathcal{H}_{\rm loc}$, in particular, $\|(u_0,v_0)\|_{\mathcal{H}(-a-T,a+T)}<\infty$ and therefore,

$$\sup_{x\in[-a,a]}\|(u_0,v_0)\|_{\mathcal{H}(B(x,T))} \le \|(u_0,v_0)\|_{\mathcal{H}(-a-T,a+T)} < \infty.$$

The procedure to prove (5.10) is based on the proof of Proposition 4.12. Let us fix h in $S_{\mathcal{M}}$ and denote the corresponding solution $z_h := (u_h, v_h)$ which exists due to Theorem 4.1.

Since x is fixed, we will avoid writing it explicitly in the norm. Define

$$l(t,T;x) := \frac{1}{2}\|(u_h(t),v_h(t)\|^2_{H^1(B_{T-t})\times L^2(B_{T-t})}, \quad t\in[0,T].$$

To shorten the notation we will write $l(t)$ in place of $l(t,T;x)$. Thus, invoking Proposition C.1, with $k=0$ and $L=I$, implies, for $t\in[0,T]$,

$$l(t) \le l(0) + \int_0^t \langle u_h(r), v_h(s)\rangle_{L^2(B_{T-s})}\,ds + \int_0^t \langle v_h(s), f_h(s)\rangle_{L^2(B_{T-s})}\,ds + \int_0^t \langle v_h(s), Y(u_h(s))\dot{h}(s)\rangle_{L^2(B_{T-s})}\,ds, \tag{5.11}$$

where

$$f_h(r) := A_{u_h(r)}(v_h(r), v_h(r) - A_{u_h(r)}(\partial_x u_h(r), \partial_x u_h(r).$$

Since $v_h(r)\in T_{u_h(r)}M$ and by definition $A_{u_h(r)}(\cdot,\cdot)\in N_{u_h(r)}M$, the second integral in (5.11) vanishes. Because $u_h(r)\in M$, invoking the Cauchy-Schwartz inequality, Lemmata 3.4 and 5.4 implies

$$l(t) \le l(0) + \left(\frac{C_Y^2 C_T^2}{2} + 2\right) \int_0^t (1 + l(s))(1 + \|\dot{h}(s)\|_{H_\mu}^2)\, ds.$$

Consequently, by applying the Gronwall Lemma and using $h \in S_{\mathcal{M}}$ we get

$$l(t) \lesssim_{C_Y, C_T} (1 + l(0)) \left[T + \|\dot{h}\|^2_{L^2(0,T;H_\mu)}\right] \le (T + \mathcal{M})(1 + l(0)). \tag{5.12}$$

Next we define

$$q(t) := \log(1 + \|z_h(t)\|^2_{\mathcal{H}_{T-t}}).$$

Then Proposition C.1, with $k = 1$ and $L(x) = \log(1 + x)$, gives, for $t \in [0, \frac{T}{2}]$,

$$\begin{aligned} q(t) \le q(0) &+ \int_0^t \frac{\|z_h(s)\|^2_{\mathcal{H}_{T-s}}}{1 + \|z_h(s)\|^2_{\mathcal{H}_{T-s}}}\, ds \\ &+ \int_0^t \frac{\langle v_h(s), f_h(s)\rangle_{L^2(B_{T-s})}}{1 + \|z_h(s)\|^2_{\mathcal{H}_{T-s}}}\, ds + \int_0^t \frac{\langle \partial_x v_h(s), \partial_x[f_h(s)]\rangle_{L^2(B_{T-s})}}{1 + \|z_h(s)\|^2_{\mathcal{H}_{T-s}}}\, ds \\ &+ \int_0^t \frac{\langle v_h(s), Y(u_h(s))\dot{h}(s)\rangle_{L^2(B_{T-s})}}{1 + \|z_k(s)\|^2_{\mathcal{H}_{T-s}}}\, ds + \int_0^t \frac{\langle \partial_x v_h(s), \partial_x[Y(u_h(s))\dot{h}(s)]\rangle_{L^2(B_{T-s})}}{1 + \|z_h(s)\|^2_{\mathcal{H}_{T-s}}}\, ds. \end{aligned}$$

Since by perpendicularity the second integral in above vanishes, by doing the calculation based on (4.38) and (4.42) we deduce

$$\begin{aligned} q(t) &\lesssim_T 1 + q(0) + \int_0^t \frac{l(s)\|z_h(s)\|^2_{\mathcal{H}_{T-s}}}{1 + \|z_h(s)\|^2_{\mathcal{H}_{T-s}}}\, ds \\ &\quad + \int_0^t \frac{(1 + l(s))\,(1 + \|z_h(s)\|^2_{\mathcal{H}_{T-s}})(1 + \|\dot{h}(s)\|^2_{H_\mu})}{1 + \|z_k(s)\|^2_{\mathcal{H}_{T-s}}}\, ds \\ &\le 1 + q(0) + \int_0^t (1 + l(s))(1 + \|\dot{h}(s)\|^2_{H_\mu})\, ds, \end{aligned}$$

which further implies, due to (5.12) and $h \in S_{\mathcal{M}}$,

$$q(t) \lesssim 1 + q(0) + (T + \mathcal{M})^2(1 + l(0)).$$

In terms of z_h, that is, for each $x \in \mathbb{R}$ and $t \in [0, \frac{T}{2}]$,

$$\|z_h(t)\|^2_{\mathcal{H}_{B(x,T-t)}} \lesssim \exp\left[\|(u_0, v_0)\|^2_{\mathcal{H}_{B(x,T)}} (T + \mathcal{M})^2\right].$$

Since above holds for every $t \in [0, \frac{T}{2}]$, $h \in S_{\mathcal{M}}$, by taking supremum on t and h we get (5.10), and hence the proof of Lemma 5.11. □

Remark 5.12. Since $B(x, \frac{T}{2}) \subseteq B(x, T-t)$ for every $t \in [0, \frac{T}{2}]$, Lemma 5.11 implies also that for $R = \frac{T}{2}$,

$$\sup_{x\in[-a,a]} \sup_{h\in S_{\mathcal{M}}} \sup_{t\in[0,\frac{T}{2}]} \frac{1}{2}\left\{\|u_h(t)\|^2_{H^2(B(x,R))} + \|v_h(t)\|^2_{H^1(B(x,R))}\right\} \le \mathcal{B}(\mathcal{M}, T, a).$$

Recall that, in the current subsection 5.2, we have the sequence $\{h_n\}_{n\in\mathbb{N}}$ which converges weakly to a limit $h \in S_{\mathcal{M}}$. Now we prove the main result of this subsection which will allow to complete the proof of **Statement 1**.

Proposition 5.13. *Let $z_n = (u_n, v_n) := J^0(h_n)$, for $n \in \mathbb{N}$, be a solution to problem (5.7), corresponding to control h_n and similarly let $z_h = (u_h, v_h) := J^0(h)$. Then the sequence $\{z_n\}_{n\in\mathbb{N}}$ converges to z_h in the space $\mathcal{X}_T$.*

In particular, the map

$$S_{\mathcal{M}} \in h \mapsto J^0(h) \in \mathcal{X}_T,$$

is Borel measurable.

Proof of Proposition 5.13. Let us first note that the second part of the Proposition follows from first one because continuous maps are Borel measurable.

Towards proving the first conclusion let us consider the objects as in the assumptions of Proposition 5.13. In particular, $z_h = (u_h, v_h)$ and $z_n = (u_n, v_n)$, are the unique global strong solutions, respectively, to

$$\begin{cases} \partial_{tt}u_h = \partial_{xx}u_h + A_{u_h}(\partial_t u_h, \partial_t u_h) - A_{u_h}(\partial_x u_h, \partial_x u_h) + Y(u_h)\dot{h}, \\ (u_h(0), v_h(0)) = (u_0, v_0), \qquad \text{where } v_n h := \partial_t u_h, \end{cases} \tag{5.13}$$

and

$$\begin{cases} \partial_{tt}u_n = \partial_{xx}u_n + A_{u_n}(\partial_t u_n, \partial_t u_n) - A_{u_n}(\partial_x u_n, \partial_x u_n) + Y(u_n)\dot{h}_n, \\ (u_n(0), v_n(0)) = (u_0, v_0), \qquad \text{where } v_n := \partial_t u_n. \end{cases} \tag{5.14}$$

Hence $\mathfrak{z}_n := (\mathfrak{u}_n, \mathfrak{v}_n) = z_h - z_n$ is the unique global strong solution to, with null initial data,

$$\begin{aligned} \partial_{tt}\mathfrak{u}_n = \partial_{xx}\mathfrak{u}_n &- A_{u_h}(\partial_x u_h, \partial_x u_h) + A_{u_n}(\partial_x u_n, \partial_x u_n) + A_{u_h}(\partial_t u_h, \partial_t u_h) \\ &- A_{u_n}(\partial_t u_n, \partial_t u_n) + Y(u_h)\dot{h} - Y(u_n)\dot{h}_n, \end{aligned} \tag{5.15}$$

where $\mathfrak{v}_n := \partial_t \mathfrak{u}_n$. This implies that

$$\mathfrak{z}_n(t) = \int_0^t S_{t-s}\begin{pmatrix} 0 \\ f_n(s) \end{pmatrix} ds + \int_0^t S_{t-s}\begin{pmatrix} 0 \\ g_n(s) \end{pmatrix} ds, \quad t \in [0,T].$$

Here

$$f_n(s) := -A_{u_h(s)}(\partial_x u_h(s), \partial_x u_h(s)) + A_{u_n(s)}(\partial_x u_n(s), \partial_x u_n(s)) + A_{u_h(s)}(\partial_t u_h(s), \partial_t u_h(s)) \\ - A_{u_n(s)}(\partial_t u_n(s), \partial_t u_n(s)),$$

and

$$g_n(s) := Y(u_h(s))\dot{h}(s) - Y(u_n(s))\dot{h}_n(s).$$

We aim to show that

$$\mathfrak{z}_n \xrightarrow[n\to\infty]{} 0 \quad \text{in} \quad \mathcal{C}\Big([0,T], H^2_{\text{loc}}(\mathbb{R};\mathbb{R}^n)\Big) \times \mathcal{C}\Big([0,T], H^1_{\text{loc}}(\mathbb{R};\mathbb{R}^n)\Big),$$

that is, for every $R > 0$ and $x \in \mathbb{R}$,

$$\sup_{t\in[0,T]}\Big[\|\mathfrak{u}_n(t)\|^2_{H^2(B(x,R))} + \|\mathfrak{v}_n(t)\|^2_{H^1(B(x,R))}\Big] \to 0 \text{ as } n \to \infty. \tag{5.16}$$

Without loss of generality we assume $x = 0$. Since a compact set in $\mathbb{R}$ can be covered by a finite number of any given closed interval of non-zero length, it is sufficient to prove above for a fixed $R > 0$ whose value we set to T.

Let φ be a bump function which takes value 1 on B_R and vanishes outside $\overline{B_{2R}}$. Define

$$\bar{u}_n(t,x) := u_n(t,x)\varphi(x) \quad \text{and} \quad \bar{u}_h(t,x) := u_h(t,x)\varphi(x),$$

so

$$\bar{v}_n(t,x) = \varphi(x)v_n(t,x), \qquad \bar{v}_h(t,x) = \varphi(x)v_h(t,x),$$

and with notation $\bar{\mathfrak{u}}_n := \bar{u}_n - \bar{u}_h$,

$$\partial_{tt}\bar{\mathfrak{u}}_n - \partial_{xx}\bar{\mathfrak{u}}_n = \big[A_{u_n}(\partial_t u_n, \partial_t u_n) - A_{u_n}(\partial_x u_n, \partial_x u_n) - A_{u_h}(\partial_t u_h, \partial_t u_h) \\ + A_{u_h}(\partial_x u_h, \partial_x u_h)\big]\varphi - (u_n - u_h)\partial_{xx}\varphi - 2(\partial_x u_n - \partial_x u_h)\partial_x\varphi + \big[Y(u_n)\dot{h}_n - Y(u_h)\dot{h}\big]\varphi \\ =: \bar{f}_n + \bar{g}_n.$$

Here

$$\bar{f}_n(s) := \big[A_{u_n(s)}(\partial_t u_n(s), \partial_t u_n(s)) - A_{u_n(s)}(\partial_x u_n(s), \partial_x u_n(s)) - A_{u_h(s)}(\partial_t u_h(s), \partial_t u_h(s)) \\ + A_{u_h(s)}(\partial_x u_h(s), \partial_x u_h(s))\big]\varphi - (u_n(s) - u_h(s))\partial_{xx}\varphi - 2(\partial_x u_n(s) - \partial_x u_h(s))\partial_x\varphi,$$

and

$$\bar{g}_n(s) := \big[Y(u_n(s))\dot{h}_n(s) - Y(u_h(s))\dot{h}(s)\big]\varphi, \quad s \in [0,T].$$

Next, by direct computation we can find constants $C_\varphi, \bar{C}_\varphi > 0$, depend on $\varphi, \varphi', \varphi''$, such that, for all $t \in [0,T]$ and $n \in \mathbb{N}$,

$$\begin{aligned} \|\bar{\mathfrak{u}}_n(t)\|^2_{H^2(-R,R)} + \|\bar{\mathfrak{v}}_n(t)\|^2_{H^1(-R,R)} &\leq C_\varphi\big[\|\mathfrak{u}_n(t)\|^2_{H^2(-R,R)} + \|\mathfrak{v}_n(t)\|^2_{H^1(-R,R)}\big] \\ &\leq \bar{C}_\varphi\big[\|\bar{\mathfrak{u}}_n(t)\|^2_{H^2(-R,R)} + \|\bar{\mathfrak{v}}_n(t)\|^2_{H^1(-R,R)}\big]. \end{aligned} \tag{5.17}$$

Hence, in order to prove assertion (5.16) it is enough to prove the following

$$\sup_{t\in[0,T]}\Big[\|\bar{\mathfrak{u}}_n(t)\|^2_{H^2(-R,R)} + \|\bar{\mathfrak{v}}_n(t)\|^2_{H^1(-R,R)}\Big] \to 0 \text{ as } n \to \infty. \tag{5.18}$$

Using the time dependent balls in the space $\mathbb{R}$, what is more natural in the context of the wave equations, we observe that claim (5.18) is a consequence of the following one.

$$\sup_{t\in[0,R]}\Big[\|\bar{\mathfrak{u}}_n(t)\|^2_{H^2(B_{\mathcal{T}-t})} + \|\bar{\mathfrak{v}}_n(t)\|^2_{H^1(B_{\mathcal{T}-t})}\Big] \to 0 \text{ as } n \to \infty, \tag{5.19}$$

where $\mathcal{T} := 4T$. Indeed, because for every $t \in [0,R]$, $\mathcal{T} - t > 2R$ and consequently, we have

$$\begin{aligned} \|\bar{\mathfrak{u}}_n(t)\|^2_{H^2(B_R)} + \|\bar{\mathfrak{v}}_n(t)\|^2_{H^1(B_R)} &\leq \|\bar{\mathfrak{u}}_n(t)\|^2_{H^2(B_{2R})} + \|\bar{\mathfrak{v}}_n(t)\|^2_{H^1(B_{2R})} \\ &\leq \sup_{t\in[0,R]}\Big[\|\bar{\mathfrak{u}}_n(t)\|^2_{H^2(B_{\mathcal{T}-t})} + \|\bar{\mathfrak{v}}_n(t)\|^2_{H^1(B_{\mathcal{T}-t})}\Big]. \end{aligned}$$

So we conclude that in order to prove Proposition 5.13 it is enough to show (5.19).

Proof of claim (5.19). Let us set $l(t,z) := \frac{1}{2}\|z\|^2_{\mathcal{H}_{\mathcal{T}-t}}$, for $z = (u,v) \in \mathcal{H}_{\text{loc}}$ and $t \in [0,R]$. Invoking Proposition C.1, with null diffusion part and $k = 1, L = I, x = 0$, gives, for every $t \in [0,R]$,

$$l(t, \bar{\mathfrak{z}}_n(t)) \leq \int_0^t \mathbb{V}(r, \bar{\mathfrak{z}}_n(r))\, dr, \tag{5.20}$$

where $\bar{\mathfrak{z}}_n(t) = (\bar{\mathfrak{u}}_n(t), \bar{\mathfrak{v}}_n(t))$ and

$$\begin{aligned} \mathbb{V}(t, \bar{\mathfrak{z}}_n(t)) &= \langle \bar{\mathfrak{u}}_n(t), \bar{\mathfrak{v}}_n(t)\rangle_{L^2(B_{\mathcal{T}-t})} + \langle \bar{\mathfrak{v}}_n(t), \bar{f}_n(t)\rangle_{L^2(B_{\mathcal{T}-t})} \\ &\quad + \langle \partial_x \bar{\mathfrak{v}}_n(t), \partial_x \bar{f}_n(t)\rangle_{L^2(B_{\mathcal{T}-t})} + \langle \bar{\mathfrak{v}}_n(t), \bar{g}_n(t)\rangle_{L^2(B_{\mathcal{T}-t})} \\ &\quad + \langle \partial_x \bar{\mathfrak{v}}_n(t), \partial_x \bar{g}_n(t)\rangle_{L^2(B_{\mathcal{T}-t})} \\ &=: \mathbb{V}_f(t, \bar{\mathfrak{z}}_n(t)) + \mathbb{V}_g(t, \bar{\mathfrak{z}}_n(t)). \end{aligned}$$

We estimate $\mathbb{V}_f(t, \bar{\mathfrak{z}}_n(t))$ and $\mathbb{V}_g(t, \bar{\mathfrak{z}}_n(t))$ separately as follows. Since $\mathcal{T} - t > 2R$, for every $t \in [0,R]$ and $\varphi(y), \varphi'(y) = 0$ for $y \notin \overline{B_{2R}}$, we have

$$\int_0^t \mathbb{V}_f(r,\bar{\mathfrak{z}}(r))\,dr = \int_0^t \Bigg[\int_{B_{2R}} \{\varphi(y)\mathfrak{u}_n(r,y)\varphi(y)\mathfrak{v}_n(r,y) + \varphi(y)\mathfrak{v}_n(r,y)\bar{f}_n(r,y)$$

$$+\varphi'(y)\mathfrak{v}_n(r,y)\partial_x \bar{f}_n(r,y) + \varphi(y)\partial_x\mathfrak{v}_n(r,y)\partial_x\bar{f}_n(r,y)\}\,dy\Bigg]dr$$

$$\lesssim_{\varphi,\varphi'} \int_0^t l(r,\bar{\mathfrak{z}}_n(r))\,dr + \int_0^t \|\bar{f}_n(r)\|^2_{H^1(B_{2R})}\,dr,$$

and

$$\int_0^t \mathbb{V}_g(r,\bar{\mathfrak{z}}(r))\,dr = \int_0^t \Big(\langle\bar{\mathfrak{v}}_n(r),\bar{g}_n(r)\rangle_{L^2(B_{\mathcal{T}-r})} + \langle\partial_x\bar{\mathfrak{v}}_n(r),\partial_x\bar{g}_n(r)\rangle_{L^2(B_{\mathcal{T}-r})}\Big)\,dr$$

$$= \int_0^t \big(\langle\bar{\mathfrak{v}}_n(r),\bar{g}_n(r)\rangle_{L^2(B_{2R})} + \langle\partial_x\bar{\mathfrak{v}}_n(r),\partial_x\bar{g}_n(r)\rangle_{L^2(B_{2R})}\big)\,dr.$$

Let us estimate the terms involving $\bar{f}_n$ first. Since u_n, u_h takes values on manifold M, by using the properties of φ and invoking interpolation inequality (4.13), as pursued in Lemma 4.7, followed by Lemma 5.11 we deduce that

$$\begin{aligned}
\|\bar{f}_n(r)\|^2_{L^2(B_{2R})} &\lesssim_{\varphi,\varphi',\varphi''} \|A_{u_n(r)}(v_n(r),v_n(r)) - A_{u_h(r)}(v_n(r),v_n(r))\|^2_{L^2(B_{2R})}\\
&\quad + \|A_{u_h(r)}(v_n(r),v_n(r)) - A_{u_h(r)}(v_n(r),v_h(r))\|^2_{L^2(B_{2R})}\\
&\quad + \|A_{u_h(r)}(v_n(r),v_h(r)) - A_{u_h(r)}(v_h(r),v_h(r))\|^2_{L^2(B_{2R})}\\
&\quad + \|A_{u_n(r)}(\partial_x u_n(r),\partial_x u_n(r)) - A_{u_h(r)}(\partial_x u_n(r),\partial_x u_n(r))\|^2_{L^2(B_{2R})}\\
&\quad + \|A_{u_h(r)}(\partial_x u_n(r),\partial_x u_n(r)) - A_{u_h(r)}(\partial_x u_n(r),\partial_x u_h(r))\|^2_{L^2(B_{2R})}\\
&\quad + \|A_{u_h(r)}(\partial_x u_n(r),\partial_x u_h(r)) - A_{u_h(r)}(\partial_x u_h(r),\partial_x u_h(r))\|^2_{L^2(B_{2R})}\\
&\quad + \|u_n(r)-u_h(r)\|^2_{L^2(B_{2R})} + 2\|\partial_x u_n(r) - \partial_x u_h(r)\|^2_{L^2(B_{2R})}\\
&\lesssim_{L_A,B_A,R} \|u_n(r)-u_h(r)\|^2_{L^2(B_{2R})}\|v_n(r)\|^4_{L^\infty(B_{2R})}\\
&\quad + \|v_n(r)-v_h(r)\|^2_{L^2(B_{2R})}\Big(\|v_n(r)\|^2_{L^\infty(B_{2R})} + \|v_h(r)\|^2_{L^\infty(B_{2R})}\Big)\\
&\quad + \|u_n(r)-u_h(r)\|^2_{L^2(B_{2R})}\|\partial_x u_n(r)\|^4_{L^\infty(B_{2R})}\\
&\quad + \|\partial_x u_n(r)-\partial_x u_h(r)\|^2_{L^2(B_{2R})}\Big(\|\partial_x u_n(r)\|^2_{L^\infty(B_{2R})} + \|\partial_x u_h(r)\|^2_{L^\infty(B_{2R})}\Big)\\
&\quad + \|u_n(r)-u_h(r)\|^2_{L^2(B_{2R})} + 2\|\partial_x u_n(r) - \partial_x u_h(r)\|^2_{L^2(B_{2R})}\\
&\lesssim_{L_A,B_A,R,k_e,\mathcal{B}} \|\mathfrak{z}_n(r)\|^2_{\mathcal{H}(B_{2R})} \lesssim l(r,\mathfrak{z}_n(r)).
\end{aligned} \tag{5.21}$$

Similarly by using the interpolation inequality (4.13) and Lemma 5.11, based on the computation of (4.15), we get

$$\|\partial_x \bar{f}_n(r)\|^2_{L^2(B_{2R})} \lesssim_{L_A, B_A, R, k_e, \mathcal{B}} l(r, \mathfrak{z}_n(r)),$$

where the constant of inequality is independent of n but depends on the properties of φ and its first two derivatives, consequently, we have, for some $C_{\bar{f}} > 0$,

$$\int_0^t \|\bar{f}_n(r)\|^2_{H^1(B_{2R})}\, dr \le C_{\bar{f}} \int_0^t l(r, \mathfrak{z}_n(r))\, dr, \quad \forall t \in [0, R]. \tag{5.22}$$

Now we move to the crucial estimate of integral involving $\bar{g}_n$. It is the part where we follow the idea of [25, Proposition 3.4] and [30, Proposition 4.4]. Let m be a natural number, whose value will be set later. Define the following partition of $[0, R]$,

$$\left\{0, \frac{1 \cdot R}{2^m}, \frac{2 \cdot R}{2^m}, \cdots, \frac{2^m \cdot R}{2^m}\right\},$$

and set

$$r_m := \frac{(k+1)\cdot R}{2^m} \text{ and } t_{k+1} := \frac{(k+1)\cdot R}{2^m} \text{ if } r \in \left[\frac{k \cdot R}{2^m}, \frac{(k+1)\cdot R}{2^m}\right).$$

Now observe that, for every $t \in [0, R]$,

$$\begin{aligned}
&\int_0^t \langle \bar{\mathfrak{v}}_n(r), \bar{g}_n(r)\rangle_{H^1(B_{2R})}\, dr \\
&= \int_0^t \langle \bar{\mathfrak{v}}_n(r), \varphi(Y(u_n(r)) - Y(u_h(r)))\dot{h}_n(r)\rangle_{H^1(B_{2R})}\, dr \\
&\quad + \int_0^t \langle \bar{\mathfrak{v}}_n(r) - \bar{\mathfrak{v}}_n(r_m), \varphi Y(u_h(r))(\dot{h}_n(r) - \dot{h}(r))\rangle_{H^1(B_{2R})}\, dr \\
&\quad + \int_0^t \langle \bar{\mathfrak{v}}_n(r_m), \varphi(Y(u_h(r)) - Y(u_h(r_m)))(\dot{h}_n(r) - \dot{h}(r))\rangle_{H^1(B_{2R})}\, dr \\
&\quad + \int_0^t \langle \bar{\mathfrak{v}}_n(r_m), \varphi Y(u_h(r_m))(\dot{h}_n(r) - \dot{h}(r))\rangle_{H^1(B_{2R})}\, dr \\
&=: G_1^{n,m}(t) + G_2^{n,m}(t) + G_3^{n,m}(t) + G_4^{n,m}(t).
\end{aligned} \tag{5.23}$$

For $G_1^{n,m}$, Lemmata 3.4, 5.4 and 5.11 followed by (5.17) imply

$$
\begin{aligned}
|G_1^{n,m}(t)| &\lesssim_\varphi \int_0^t \|\bar{\mathfrak{v}}_n(r)\|^2_{H^1(B_{2R})}\,dr + \int_0^t \|Y(u_n(r)) - Y(u_h(r))\|^2_{H^1(B_{2R})} \|\dot{h}_n(r)\|^2_{H_\mu}\,dr \\
&\lesssim_R \int_0^t \|\bar{\mathfrak{v}}_n(r)\|^2_{H^1(B_{2R})}\,dr \\
&\quad + \int_0^t \|u_n(r) - u_h(r)\|^2_{H^1(B_{2R})} \left(1 + \|u_n(r)\|^2_{H^1(B_{2R})} + \|u_h(r)\|^2_{H^1(B_{2R})}\right) \|\dot{h}_n(r)\|^2_{H_\mu}\,dr \\
&\lesssim_{\mathcal{B}} \int_0^t (1 + l(r, \mathfrak{z}_n(r))) \left(1 + \|\dot{h}_n(r)\|^2_{H_\mu}\right) dr, \qquad \forall t \in [0, R].
\end{aligned} \tag{5.24}
$$

To estimate $G_2^{n,m}(t)$ we invoke $\langle h, k\rangle_{H^1(B_{2R})} \leq \|h\|_{L^2(B_{2R})}\|k\|_{H^2(2R))}$ followed by the Hölder inequality and Lemmata 3.4, 5.4, and 5.15 to get, for every $t \in [0, R]$,

$$
\begin{aligned}
|G_2^{n,m}(t)| &\lesssim_{R,\varphi} \int_0^t \|\mathfrak{v}_n(r) - \mathfrak{v}_n(r_m)\|_{L^2(B_{2R})} \|Y(u_h(r))\|_{H^2(B_{2R})} \|\dot{h}_n(r) - \dot{h}(r)\|_{H_\mu}\,dr \\
&\lesssim_R \left(\int_0^t \|\mathfrak{v}_n(r) - \mathfrak{v}_n(r_m)\|^2_{L^2(B_{2R})}\,dr\right)^{\frac{1}{2}} \\
&\quad \times \left(\int_0^t \left[1 + \|u_h(r)\|^4_{H^2(B_{2R})}\right] \|\dot{h}_n(r) - \dot{h}(r)\|^2_{H_\mu}\,dr\right)^{\frac{1}{2}} \\
&\lesssim \sqrt{M_\mu} \left(\int_0^t |r - r_m|\,dr\right)^{\frac{1}{2}} \sup_{r \in [0, \frac{\mathcal{T}}{2}]} \left[1 + \|u_h(r)\|^4_{H^2(B_{\mathcal{T}-r})}\right] \\
&\lesssim \frac{R\sqrt{M_\mu}}{2^{m/2}} \sup_{r \in [0, \frac{\mathcal{T}}{2}]} \left[1 + (l(r, z_h(r)))^2\right] \leq \frac{R\sqrt{M_\mu}}{2^{m/2}}\,(1 + \mathcal{B}^2),
\end{aligned}
$$

where in the last and the second last step we have used, respectively, Lemma 5.11 for $\mathcal{T}$ instead of T and

$$
\left(\int_0^t |r - r_m|\,dr\right)^{\frac{1}{2}} \leq \left(\int_0^R |r - r_m|\,dr\right)^{\frac{1}{2}} = \left(\sum_{k=1}^{2^m} \int_{t_{k-1}}^{t_k} \left|r - \frac{kR}{2^m}\right| dr\right)^{\frac{1}{2}} \leq \frac{R}{2^{m/2}}.
$$

Moreover, in the third last step we have also applied the following: since $R = T$ and $\dot{h}_n \to \dot{h}$ weakly in $L^2(0, T; H_\mu)$, the sequence $\dot{h}_n - \dot{h}$ is bounded in $L^2(0, T; H_\mu)$ i.e. $\exists M_\mu > 0$ such that

$$\int_0^T \|\dot{h}_n(r) - \dot{h}(r)\|^2_{H_\mu}\, dr \le M_\mu, \qquad \text{for all } n. \tag{5.25}$$

Before moving to $G_3^{n,m}(t)$ note that, since $2R = \frac{\mathcal{T}}{2}$, due to Remark 5.12, for every $s, t \in [0, \frac{\mathcal{T}}{2}]$,

$$\|u_h(t) - u_h(s)\|_{H^1(B_{2R})} \le \int_s^t \|v_h(r)\|_{H^1(B_{2R})}\, dr \lesssim \sqrt{\mathcal{B}}|t-s|.$$

Consequently, by the Hölder inequality followed by Lemmata 3.4, 5.15, and 5.4 we obtain

$$\begin{aligned}
|G_3^{n,m}(t)| &\lesssim_\varphi \left(\int_0^t \left[\|v_n(r_m)\|^2_{H^1(B_{2R})} + \|v_h(r_m)\|^2_{H^1(B_{2R})}\right] dr\right)^{\frac{1}{2}} \\
&\qquad \times \left(\int_0^t \|Y(u_h(r)) - Y(u_h(r_m))\|^2_{H^1(B_{2R})} \|\dot{h}_n(r) - \dot{h}(r))\|^2_{H_\mu}\, dr\right)^{\frac{1}{2}} \\
&\lesssim_{T,\mathcal{B}} \left(\int_0^t \|u_h(r) - u_h(r_m)\|^2_{H^1(B_{2R})}\left[1 + \|u_h(r)\|^2_{H^1(B_{2R})} + \|u_h(r_m)\|^2_{H^1(B_{2R})}\right]\right. \\
&\qquad \left. \times \|\dot{h}_n(r) - \dot{h}(r)\|^2_{H_\mu}\, dr\right)^{\frac{1}{2}} \\
&\lesssim_{T,\mathcal{B}} \left(\int_0^t |r - r_m|\, \|\dot{h}_n(r) - \dot{h}(r)\|^2_{H_\mu}\, dr\right)^{\frac{1}{2}} \\
&\le \left(\sum_{k=1}^{2^m} \int_{t_{k-1}}^{t_k} \left|r - \frac{kR}{2^m}\right| \|\dot{h}_n(r) - \dot{h}(r)\|^2_{H_\mu}\, dr\right)^{\frac{1}{2}} \\
&\le \sqrt{\frac{R}{2^m}} \left(\int_0^t \|\dot{h}_n(r) - \dot{h}(r)\|^2_{H_\mu}\, dr\right)^{\frac{1}{2}} \le \sqrt{R\frac{M_\mu}{2^m}}, \qquad t \in [0, R].
\end{aligned} \tag{5.26}$$

Finally we start estimating $G_4^{n,m}(t)$ by noting that for every $t \in [0, R]$,

$$\text{there exists} \quad k_t \le 2^m \quad \text{such that} \quad t \in \left[\frac{(k_t - 1)\cdot R}{2^m}, \frac{k_t \cdot R}{2^m}\right).$$

Note that on such interval $r_m = \frac{k_t \cdot R}{2^m}$. Then by Lemma 5.11 we have

$$
\begin{aligned}
|G_4^{n,m}(t)| &\leq \Bigg| \sum_{k=1}^{k_t-1} \int_{t_{k-1}}^{t_k} \left\langle \bar{\mathfrak{v}}_n\left(\frac{k\cdot R}{2^m}\right), \varphi Y\left(u_h\left(\frac{k\cdot R}{2^m}\right)\right)(\dot{h}_n(r)-\dot{h}(r))\right\rangle_{H^1(B_{2R})} dr \\
&+ \int_{t_{k_t-1}}^{t} \left\langle \bar{\mathfrak{v}}_n\left(\frac{(k_t-1)\cdot R}{2^m}\right), \varphi Y\left(u_h\left(\frac{(k_t-1)\cdot R}{2^m}\right)\right)(\dot{h}_n(r)-\dot{h}(r))\right\rangle_{H^1(B_{2R})} dr \Bigg| \\
&\leq \sum_{k=1}^{2^m} \left| \left\langle \bar{\mathfrak{v}}_n\left(\frac{k\cdot R}{2^m}\right), \varphi Y\left(u_h\left(\frac{k\cdot R}{2^m}\right)\right) \int_{t_{k-1}}^{t_k} (\dot{h}_n(r)-\dot{h}(r))\,dr \right\rangle_{H^1(B_{2R})} \right| \\
&+ \sup_{1\leq k\leq 2^m} \sup_{t_k\leq t\leq t_{k-1}} \left| \left\langle \bar{\mathfrak{v}}_n\left(\frac{(k-1)\cdot R}{2^m}\right), \varphi Y\left(u_h\left(\frac{(k-1)\cdot R}{2^m}\right)\right) \int_{t_{k-1}}^{t} (\dot{h}_n(r)-\dot{h}(r))\,dr \right\rangle_{H^1(B_{2R})} \right| \\
&\leq \sum_{k=1}^{2^m} \left\| \bar{\mathfrak{v}}_n\left(\frac{k\cdot R}{2^m}\right) \right\|_{H^1(B_{2R})} \left\| \varphi Y\left(u_h\left(\frac{k\cdot R}{2^m}\right)\right) \int_{t_{k-1}}^{t_k} (\dot{h}_n(r)-\dot{h}(r))\,dr \right\|_{H^1(B_{2R})} \\
&+ \sup_{1\leq k\leq 2^m} \sup_{t_k\leq t\leq t_{k-1}} \left\| \bar{\mathfrak{v}}_n\left(\frac{(k-1)\cdot R}{2^m}\right) \right\|_{H^1(B_{2R})} \\
&\times \left\| \varphi Y\left(u_h\left(\frac{(k-1)\cdot R}{2^m}\right)\right) \int_{t_{k-1}}^{t} (\dot{h}_n(r)-\dot{h}(r))\,dr \right\|_{H^1(B_{2R})} \\
&\lesssim_{\varphi,\mathcal{B}} \sum_{k=1}^{2^m} \left\| Y\left(u_h\left(\frac{k\cdot R}{2^m}\right)\right) \int_{t_{k-1}}^{t_k} (\dot{h}_n(r)-\dot{h}(r))\,dr \right\|_{H^1(B_{2R})} \\
&+ \sup_{1\leq k\leq 2^m} \sup_{t_k\leq t\leq t_{k-1}} \left\| Y\left(u_h\left(\frac{(k-1)\cdot R}{2^m}\right)\right) \int_{t_{k-1}}^{t} (\dot{h}_n(r)-\dot{h}(r))\,dr \right\|_{H^1(B_{2R})} \\
&=: G_4^{n,m,1} + G_4^{n,m,2},
\end{aligned} \tag{5.27}
$$

where the right hand side does not depend on t. By invoking Lemmata 3.4, 5.4, the Hölder inequality, Lemma 5.11 and (5.25) we estimate $G_4^{n,m,2}$ as

$$
\begin{aligned}
G_4^{n,m,2} &\lesssim_R \sup_{1\leq k\leq 2^m} \sup_{t_k\leq t\leq t_{k-1}} \left\| Y\left(u_h\left(\frac{(k-1)\cdot R}{2^m}\right)\right) \right\|_{H^1(B_{2R})} \left(\int_{t_{k-1}}^{t} \|\dot{h}_n(r)-\dot{h}(r)\|_{H_\mu}\,dr \right) \\
&\leq \sup_{1\leq k\leq 2^m} \sup_{t_k\leq t\leq t_{k-1}} \left[1 + \left\| u_h\left(\frac{(k-1)\cdot R}{2^m}\right) \right\|_{H^1(B_{2R})} \right] \left(\int_{t_{k-1}}^{t} \|\dot{h}_n(r)-\dot{h}(r)\|_{H_\mu}\,dr \right)
\end{aligned} \tag{5.28}
$$

$$\lesssim_{\mathcal{B}} \sup_{1\le k\le 2^m} \left(\frac{R}{2^m}\right)^{\frac12} \left(\int_{t_{k-1}}^{t_k} \|\dot h_n(r)-\dot h(r)\|^2_{H_\mu}\,dr\right)^{\frac12} \le \sqrt{R\frac{M_\mu}{2^m}}.$$

For $G_4^{n,m,1}$ recall that, by Lemma 3.4, for every $\phi\in H^1(B(x,r))$ the multiplication operator

$$Y(\phi): K \ni k \mapsto Y(\phi)\cdot k \in H^1(B(x,r)),$$

is γ-radonifying and hence compact. Hence by Lemma 5.14 we infer that for every k,

$$\left\| Y\left(u_h\left(\frac{k\cdot R}{2^m}\right)\right)\int_{t_{k-1}}^{t_k}(\dot h_n(r)-\dot h(r))\,dr\right\|_{H^1(B_{2R})} \to 0 \text{ as } n\to 0. \tag{5.29}$$

Hence each term of the sum in $G_4^{n,m,1}$ converges to 0 as $n\to\infty$. Consequently, by substituting the computation between (5.24) and (5.27) into (5.23) we obtain

$$\begin{aligned}\int_0^t \langle \bar{\mathfrak{v}}_n(r),\bar g_n(r)\rangle_{H^1(B_{2R})}\,dr &\lesssim_{R,L_A,B_A,\varphi,\mathcal{B}} \int_0^t (1+l(r,\mathfrak{z}_n(r)))\left(1+\|\dot h_n(r)\|^2_{H_\mu}\right)dr\\ &+\sqrt{R\frac{M_\mu}{2^m}}+\sum_{k=1}^{2^m}\left\| Y\left(u_h\left(\frac{k\cdot R}{2^m}\right)\right)\int_{t_{k-1}}^{t_k}(\dot h_n(r)-\dot h(r))\,dr\right\|_{H^1(B_{2R})}.\end{aligned}$$

Therefore, with (5.22) and (5.17), from (5.20) we have

$$\begin{aligned} l(t,\mathfrak{z}_n(t)) &\lesssim \int_0^t (1+l(r,\mathfrak{z}_n(r)))\left(1+\|\dot h_n(r)\|^2_{H_\mu}\right)dr+\sqrt{R\frac{M_\mu}{2^m}}\\ &+\sum_{k=1}^{2^m}\left\| Y\left(u_h\left(\frac{k\cdot R}{2^m}\right)\right)\int_{t_{k-1}}^{t_k}(\dot h_n(r)-\dot h(r))\,dr\right\|_{H^1(B_{2R})},\quad t\in[0,R],\end{aligned}$$

and by the Gronwall Lemma, with the observation that all the terms in right hand side except the first are independent of t, and $h_n\in S_{\mathcal{M}}$ further we get

$$\sup_{t\in[0,R]} l(t,\mathfrak{z}_n(t)) \lesssim e^{T+\mathcal{M}}\left\{\sqrt{R\frac{M_\mu}{2^m}}+\sum_{k=1}^{2^m}\left\| Y\left(u_h\left(\frac{k\cdot R}{2^m}\right)\right)\int_{t_{k-1}}^{t_k}(\dot h_n(r)-\dot h(r))\,dr\right\|_{H^1(B_{2R})}\right\}. \tag{5.30}$$

Hence, for given any $\alpha>0$ we can choose m such that

$$\sqrt{R\frac{M_\mu}{2^m}}<\alpha,\quad \text{for every } n\in\mathbb{N}.$$

Thus, for such chosen m, due to (5.29) by taking $n \to \infty$ in (5.30) we conclude that, for every $\alpha > 0$,

$$0 < \limsup_{n\to\infty} \sup_{t\in[0,R]} l(t, \mathfrak{z}_n(t)) < \alpha. \tag{5.31}$$

Therefore, due to (5.17) we conclude the proof of assertion (5.19). □

Hence, the Proposition 5.13 follows. □

Now we come back to the proof of **Statement 1**. The previous proposition shows that every sequence in $K_{\mathcal{M}}$ has a convergent subsequence. Hence $K_{\mathcal{M}}$ is sequentially relatively compact subset of $\mathcal{X}_T$. Let $\{z_n\}_{n\in\mathbb{N}} \subset K_{\mathcal{M}}$ which converges to $z \in \mathcal{X}_T$. But Proposition 5.13 shows that there exists a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ which converges to some element z_h of $K_{\mathcal{M}}$ in the strong topology of $\mathcal{X}_T$. Hence $z = z_h$ and $K_{\mathcal{M}}$ is a closed subset of $\mathcal{X}_T$. This completes the proof of **Statement 1**.

Below we state a basic result that we have used in the proof of Proposition 5.13. A statement of this type can be found in [25], see the proof of Proposition 3.4.

Lemma 5.14. *Let X, Y be separable Hilbert spaces and let $C : X \to Y$ be a compact operator. Then the operator $K : L^2(0, T; X) \to C([0, T]; Y)$ defined as*

$$Kg(t) = C \int_0^t g(s)\, ds\,,$$

where the integral $\int_0^t g(s)\, ds$ is meant in the Bochner sense, is compact. In particular, if $g_n \to g$ weakly in $L^2(0, T; X)$ then Kg_n converges to Kg strongly in $C([0, T]; Y)$.

Proof of Lemma 5.14. Clearly the operator K is bounded. Let $B_{L^2_T X}$ stand for the centered unit ball in $L^2(0, T; X)$. In order to prove compactness of K, in view of the Arzelà-Ascoli Theorem, see [62, Lemma 1] (and, for a very general formulation, [33, Theorem 8.2.10]), we only need to show that the following two conditions hold.

(1) for every fixed $t \in [0, T]$ the set

$$\left\{Kg(t) : g \in B_{L^2_T X}\right\} \subset Y \quad \text{is relatively compact in } Y;$$

(2) the set of function

$$\left\{Kg : g \in B_{L^2_T X}\right\} \subset C([0, T]; Y)$$

is uniformly equi-continuous.

To prove (1) we note first that for $t \in [0, T]$ fixed

$$\left| \int_0^t g(s)\, ds \right|_X \le \sqrt{T},\ g \in B_{L^2_T X}\,.$$

Since $C : X \to Y$ is compact, the set

$$\left\{ C \int_0^t g(s)\, ds : g \in B_{L^2_T X} \right\},$$

being an image of a bounded set in X, is relatively compact in Y.

To prove (2) it is enough to note that for any $g \in B_{L^2_T X}$ and $s, t \in [0, T]$

$$|Kg(t) - Kg(s)| \le \|C\| \int_s^t |g(r)|\, dr \le \|C\| \sqrt{|t-s|}.$$

Thus the proof of Lemma 5.14 is complete. □

The following Lemma is about the Lipschitz property of the difference of solutions that we have used in proving Proposition 5.13.

Lemma 5.15. *Let $R > 0$, $I = [-a, a]$ and $h_n, h \in S_{\mathcal{M}}$. There exists a positive constant $C := C(R, \mathcal{B}, M, a)$ such that for $t, s \in [0, R]$ the following holds*

$$\sup_{x \in I} \|\mathfrak{v}_n(t) - \mathfrak{v}_n(s)\|_{L^2(B(x,2R))} \lesssim C\, |t-s|^{\frac{1}{2}}, \tag{5.32}$$

where $\mathfrak{v}_n$ is defined just after (5.14).

Proof of Lemma 5.15. Due to triangle inequality it is sufficient to show

$$\sup_{x \in I} \|v_h(t) - v_h(s)\|_{L^2(B(x,2R))} \lesssim C|t-s|^{\frac{1}{2}}, \quad t, s \in [0, R].$$

From the proof of existence part in Theorem 4.1 we have, for $t, s \in [0, R]$,

$$\|v_h(t) - v_h(s)\|_{L^2(B(x,2R))} \le \int_s^t \|\partial_{xx} u_h(r)\|_{L^2(B(x,2R))}\, dr + \int_s^t \left[\|f_h(r)\|_{L^2 B(x,2R))} + \|g_h(r)\|_{L^2(B(x,2R))} \right] dr, \tag{5.33}$$

where

$$f_h(r) := A_{u_h(r)}(v_h(r), v_h(r)) - A_{u_h(r)}(\partial_x u_h(r), \partial_x u_h(r)), \text{ and } g_h(r) := Y(u_h(r))\dot{h}(r).$$

But, since $h \in S_{\mathcal{M}}$, the Hölder inequality followed by Lemmata 3.4 and 5.4 yield

$$\sup_{x\in I}\int_s^t \|g_h(r)\|_{L^2(B(x,2R))}\,dr \le |t-s|^{\frac12}\left(\int_s^t \sup_{x\in I}\|Y(u_h(r))\|^2_{L^2(B(x,2R))}\|\dot h(r)\|^2_{H_\mu}\,dr\right)^{\frac12}$$
$$\lesssim_{R,\mathcal{B},M} |t-s|^{\frac12}, \qquad \text{for} \quad t,s\in[0,R],$$

where we also applied 5.11 with $2R$ instead of T and, based on (5.21), we also have

$$\sup_{x\in I}\int_s^t \|f_h(r)\|_{L^2(B(x,2R))}\,dr \le |t-s|^{\frac12}\left(\int_s^t \sup_{x\in I}\|A_{u_h(r)}(v_h(r),v_h(r))\|^2_{L^2(B(x,2R))}\,dr\right)^{\frac12}$$
$$+|t-s|^{\frac12}\left(\int_s^t \sup_{x\in I}\|A_{u_h(s)}(\partial_x u_h(r),\partial_x u_h(r))\|^2_{L^2(B(x,2R))}\,dr\right)^{\frac12}$$
$$\lesssim |t-s|^{\frac12}\left(\int_s^t \sup_{x\in I}\|u_h(r)\|^2_{L^2(B(x,2R))}\{\|v_h(s)\|^4_{L^2(B(x,2R))}+\|\partial_x u_h(s)\|^4_{L^2(B(x,2R))}\}\,ds\right)^{\frac12}$$
$$\lesssim |t-s|\,\mathcal{B}^{\frac32} \qquad \text{for} \quad t,s\in[0,R].$$

Finally, by the Hölder inequality and Lemma 5.11, we obtain, for $t,s\in[0,R]$,

$$\sup_{x\in I}\int_s^t \|\partial_{xx}u_h(s)\|_{L^2(B(x,2R))}\,dr \le \left(\int_s^t 1\,dr\right)^{\frac12}\left(\int_s^t \sup_{x\in I}\|u_h(r)\|^2_{H^2(B(x,2R))}\,dr\right)^{\frac12}$$
$$\lesssim \sqrt{\mathcal{B}}|t-s|.$$

Therefore, by collecting the estimates in (5.33) we get the required inequality (5.32) and we are done with the proof of Lemma 5.15. $\square$

5.3. *Proof of Statement 2*

Recall that $\mathcal{M}>0$ is given and a sequence $\{h_n\}_{n\in\mathbb{N}}\subset\mathscr{S}_{\mathcal{M}}$ is also given which converges in law to $h\in\mathscr{S}_{\mathcal{M}}$ as $\varepsilon_n\to 0$. It will be useful to introduce the following notation for the processes

$$Z_n := (U_n,V_n) = J^{\varepsilon_n}\left(W+\frac{1}{\sqrt{\varepsilon_n}}h_n\right), \quad z_n := (u_n,v_n) = J^0(h_n).$$

Let us fix any $x\in\mathbb{R}$. Then set N a natural number such that

$$N > \|(u_0,v_0)\|_{\mathcal{H}(B(x,T))}.$$

For each $n\in\mathbb{N}$ we define an $\mathfrak{F}_t$-stopping time

$$\tau_n(\omega) := \inf\{t > 0 : \|Z_n(t,\omega)\|_{\mathcal{H}(B(x,T-t))} \geq N\} \wedge T, \quad \omega \in \Omega. \tag{5.34}$$

Recall that for $z = (u, v) \in \mathcal{H}_{loc}$, we set

$$\mathbf{e}(t, T; x, z) = \frac{1}{2}\left\{\|u\|^2_{H^2(B(x,T-t))} + \|v\|^2_{H^1(B(x,T-t))}\right\} = \frac{1}{2}\|z\|^2_{\mathcal{H}(B(x,T-t))}, \quad t \in [0, T].$$

In this framework we prove the following key result.

Proposition 5.16. *Let us define $\mathcal{Z}_n := Z_n - z_n$. For τ_n defined in (5.34) we have*

$$\lim_{n\to\infty} \sup_{x\in[-a,a]} \mathbb{E}\left[\sup_{t\in[0,\frac{T}{2}]} \mathbf{e}(t \wedge \tau_n, T; x, \mathcal{Z}_n(t \wedge \tau_n))\right] = 0.$$

Proof of Proposition 5.16. Let us fix any $n \in \mathbb{N}$. To avoid complexity of notation we use an abuse of notation and write all the norms without reference of the center of the ball x and we will write $\mathbf{e}(t, z)$ in place of $\mathbf{e}(t, T; x, z)$ unless any conflict arises. First note that under our notation $Z_n = (U_n, V_n)$ and $z_n = (u_n, v_n)$, respectively, are the unique global strong solutions to the Cauchy problem

$$\begin{cases} \partial_{tt}U_n = \partial_{xx}U_n + A_{U_n}(\partial_t U_n, \partial_t U_n) - A_{U_n}(\partial_x U_n, \partial_x U_n) + Y(U_n)\dot{h}_n, \\ \qquad\qquad + \sqrt{\varepsilon_n}Y(U_n)\dot{W}, \\ (U_n(0), \partial_t U_n(0)) = (u_0, v_0), \quad \text{where } V_n := \partial_t U_n, \end{cases}$$

and

$$\begin{cases} \partial_{tt}u_n = \partial_{xx}u_n + A_{u_n}(\partial_t u_n, \partial_t u_n) - A_{u_n}(\partial_x u_n, \partial_x u_n) + Y(u_n)\dot{h}_n, \\ (u_n(0), \partial_t u_n(0)) = (u_0, v_0), \quad \text{where } v_n := \partial_t u_n. \end{cases}$$

Hence $\mathcal{Z}_n$ solves uniquely the Cauchy problem, with null initial data,

$$\begin{aligned} \partial_{tt}\mathcal{U}_n = \partial_{xx}\mathcal{U}_n &- A_{U_n}(\partial_x U_n, \partial_x U_n) + A_{u_n}(\partial_x u_n, \partial_x u_n) + A_{U_n}(\partial_t U_n, \partial_t U_n) \\ &- A_{u_n}(\partial_t u_n, \partial_t u_n) + Y(U_n)\dot{h}_n - Y(u_n)\dot{h}_n + \sqrt{\varepsilon_n}Y(U_n)\dot{W}, \end{aligned}$$

where $\mathcal{V}_n := \partial_t\mathcal{U}_n$. This is equivalent to say, for all $t \in [0, \frac{T}{2}]$,

$$\mathcal{Z}_n(t) = \int_0^t S_{t-s}\begin{pmatrix} 0 \\ f_n(s) \end{pmatrix} ds + \int_0^t S_{t-s}\begin{pmatrix} 0 \\ g_n(s) \end{pmatrix} dW(s). \tag{5.35}$$

Here

$$\begin{aligned} f_n(s) := &-A_{U_n(s)}(\partial_x U_n(s), \partial_x U_n(s)) + A_{u_n(s)}(\partial_x u_n(s), \partial_x u_n(s)) + A_{U_n(s)}(V_n(s), V_n(s)) \\ &- A_{u_n(s)}(v_n(s), v_n(s)) + Y(U_n(s))\dot{h}_n(s) - Y(u_n(s))\dot{h}_n(s), \end{aligned}$$

and

$$g_n(s) := \sqrt{\varepsilon_n} Y(U_n(s)).$$

Invoking Proposition C.1, with that by taking $k = 1, L = I$, implies for every $t \in [0, \frac{T}{2}]$ and $x \in [-a, a]$,

$$\mathbf{e}(t, T; x, \mathcal{Z}_n(t)) \leq \int_0^t \mathbb{V}(r, \mathcal{Z}_n(r))\, dr + \int_0^t \langle \mathcal{V}_n(r), g_n(r) dW(r) \rangle_{L^2(B_{T-r})}$$
$$+ \int_0^t \langle \partial_x \mathcal{V}_n(r), \partial_x [g_n(r) dW(r)] \rangle_{L^2(B_{T-r})}, \tag{5.36}$$

with

$$\mathbb{V}(r, \mathcal{Z}_n(r)) = \langle \mathcal{U}_n(r), \mathcal{V}_n(r) \rangle_{L^2(B_{T-r})} + \langle \mathcal{V}_n(r), f_n(r) \rangle_{L^2(B_{T-r})} + \langle \partial_x \mathcal{V}_n(r), \partial_x f_n(r) \rangle_{L^2(B_{T-r})}$$
$$+ \frac{1}{2} \sum_{j=1}^{\infty} \| g_n(r) e_j \|^2_{L^2(B_{T-r})} + \frac{1}{2} \sum_{j=1}^{\infty} \| \partial_x [g_n(r) e_j] \|^2_{L^2(B_{T-r})},$$

for a given sequence $\{e_j\}_{j \in \mathbb{N}}$ of orthonormal basis of H_μ.

Observe that, for any $\tau \in [0, T]$, by the Cauchy-Schwartz inequality

$$\sup_{0 \leq t \leq \tau} \int_0^{t \wedge \tau_n} \mathbb{V}(r, \mathcal{Z}_n(r))\, dr \leq 2 \int_0^{\tau \wedge \tau_n} \mathbf{e}(r, \mathcal{Z}_n(r))\, dr \tag{5.37}$$
$$+ \frac{1}{2} \int_0^{\tau \wedge \tau_n} \left(\| f_n(r) \|^2_{H^1(B_{T-r})} + \| g_n(r) \cdot \|^2_{\mathscr{L}_2(H_\mu, H^1(B_{T-r}))} \right) dr,$$

where $g_n(r)\cdot$ denotes the multiplication operator in $\mathscr{L}_2(H_\mu, H^1(B_{T-r}))$, see Lemma 3.4.

Next, we define the process

$$\mathcal{Y}(t) := \int_0^t \langle \mathcal{V}_n(r), g_n(r) dW(r) \rangle_{H^1(B_{T-r})}. \tag{5.38}$$

By taking $\int_0^t \xi(r)\, dW(r)$ with

$$\xi(r) : H_\mu \ni k \mapsto \langle \mathcal{V}_n(r), g_n(r)(k) \rangle_{H^1(B_{T-r})} \in \mathbb{R},$$

a Hilbert-Schmidt operator, note that

$$\mathcal{Q}(t) := \int_0^t \xi(r) \circ \xi(r)^\star \, dr,$$

is quadratic variation of $\mathbb{R}$-valued martingale $\mathcal{Y}$. Thus

$$\begin{aligned}\mathcal{Q}(t) &\leq \int_0^t \|\xi(r)\|_{\mathscr{L}_2(H_\mu,\mathbb{R})} \|\xi(r)^\star\|_{\mathscr{L}_2(\mathbb{R},H_\mu)} \, dr = \int_0^t \|\xi(r)\|^2_{\mathscr{L}_2(H_\mu,\mathbb{R})} \, dr \\ &= \int_0^t \sum_{j=1}^\infty |\xi(r)(e_j)|^2 \, dr = \int_0^t \sum_{j=1}^\infty |\langle \mathcal{V}_n(r), g_n(r)(e_j)\rangle_{H^1(B_{T-r})}|^2 \, dr, \quad t \in [0, \frac{T}{2}].\end{aligned} \tag{5.39}$$

On the other hand by the Cauchy-Schwartz inequality

$$\sum_{j=1}^\infty |\langle \mathcal{V}_n(r), g_n(r)(e_j)\rangle_{H^1(B_{T-r})}|^2 \leq \|\mathcal{V}_n(r)\|^2_{H^1(B_{T-r})} \|g_n(r) \cdot \|^2_{\mathscr{L}_2(H_\mu, H^1(B_{T-r}))}.$$

Therefore,

$$\mathcal{Q}(t) \leq \int_0^t \|\mathcal{V}_n(r)\|^2_{H^1(B_{T-r})} \|g_n(r) \cdot \|^2_{\mathscr{L}_2(H_\mu, H^1(B_{T-r}))} \, dr, \quad t \in [0, \frac{T}{2}]. \tag{5.40}$$

Invoking the Davis inequality with (5.40) followed by the Young inequality gives

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq t\leq\tau} |\mathcal{Y}(t\wedge\tau_n)|\right] &\leq 3\mathbb{E}\left[\sqrt{\mathcal{Q}(\tau\wedge\tau_n)}\right] \\ &\leq 3\mathbb{E}\left[\sup_{0\leq t\leq\tau\wedge\tau_n} \|\mathcal{V}_n(t\wedge\tau_n)\|_{H^1(B_{T-t})} \left\{\int_0^{\tau\wedge\tau_n} \|g_n(r)\cdot\|^2_{\mathscr{L}_2(H_\mu,H^1(B_{T-r}))} \, dr\right\}^{\frac{1}{2}}\right] \\ &\leq 3\mathbb{E}\left[\varepsilon \sup_{0\leq t\leq\tau\wedge\tau_n} \|\mathcal{V}_n(t)\|^2_{H^1(B_{T-t})} + \frac{1}{4\varepsilon}\int_0^{\tau\wedge\tau_n} \|g_n(r)\cdot\|^2_{\mathscr{L}_2(H_\mu,H^1(B_{T-r}))} \, dr\right] \\ &\leq 6\varepsilon\, \mathbb{E}\left[\sup_{0\leq t\leq\tau\wedge\tau_n} \mathbf{e}(t, \mathcal{Z}_n(t))\right] + \frac{3}{4\varepsilon}\mathbb{E}\left[\int_0^{\tau\wedge\tau_n} \|g_n(r)\cdot\|^2_{\mathscr{L}_2(H_\mu,H^1(B_{T-r}))} \, dr\right].\end{aligned} \tag{5.41}$$

By choosing ε such that $6\varepsilon = \frac{1}{2}$ and taking $\sup_{0\leq s\leq t}$ followed by expectation $\mathbb{E}$ on the both sides of (5.36) after evaluating it at $\tau \wedge \tau_n$ we obtain

$$\mathbb{E}\left[\sup_{0\le s\le t\wedge\tau_n}\mathbf{e}(s,\mathcal{Z}_n(s))\right]\le\mathbb{E}\left[\sup_{0\le s\le t}\int_0^{s\wedge\tau_n}\mathbb{V}(r,\mathcal{Z}_n(r))\,dr\right]+\mathbb{E}\left[\sup_{0\le s\le t}\mathcal{Y}(s\wedge\tau_n)\right].$$

Consequently, using (5.37) and (5.41) we infer that

$$\mathbb{E}\left[\sup_{0\le s\le t\wedge\tau_n}\mathbf{e}(s,\mathcal{Z}_n(s))\right]\le 4\mathbb{E}\left[\int_0^{t\wedge\tau_n}\mathbf{e}(r,\mathcal{Z}_n(r))\,dr\right]+\mathbb{E}\left[\int_0^{t\wedge\tau_n}\|f_n(r)\|^2_{H^1(B_{T-r})}\,dr\right]$$
$$+19\mathbb{E}\left[\int_0^{t\wedge\tau_n}\|g_n(r)\cdot\|^2_{\mathcal{L}_2(H_\mu,H^1(B_{T-r}))}\,dr\right]. \tag{5.42}$$

Now since the Hilbert-Schmidt operator $g_n(r)\cdot$ is defined as

$$H_\mu\ni k\mapsto g_n(r)\cdot k\in H^1(B_{T-r}),$$

Lemmata 3.4 and 5.4 give,

$$\sup_{x\in[-a,a]}\mathbb{E}\left[\int_0^{t\wedge\tau_n}\|g_n(r)\cdot\|^2_{\mathcal{L}_2(H_\mu,H^1(B_{T-r}))}\,dr\right]\lesssim_T\mathbb{E}\left[\int_0^{t\wedge\tau_n}\|\sqrt{\varepsilon_n}Y(U_n(r))\|^2_{H^1(B_{T-r})}\,dr\right]$$
$$\le C^2_{Y,T}\varepsilon_n\,\mathbb{E}\left[\int_0^{t\wedge\tau_n}\left(1+\|Z_n(r)\|^2_{\mathcal{H}_{T-r}}\right)dr\right]\lesssim_T\varepsilon_n\,(1+N^2). \tag{5.43}$$

Here we observe that the constant in inequality (5.43) does not depend on a due to Lemma 3.4. To estimate the terms involving f_n we have

$$\begin{aligned}\|f_n(r)\|^2_{H^1(B_{T-r})}&\lesssim\|A_{U_n(r)}(\partial_xU_n(r),\partial_xU_n(r))-A_{u_n(r)}(\partial_xu_n(r),\partial_xu_n(r))\|^2_{H^1(B_{T-r})}\\&\quad+\|A_{U_n(r)}(V_n(r),V_n(r))-A_{u_n(r)}(v_n(r),v_n(r))\|^2_{H^1(B_{T-r})}\\&\quad+\|Y(U_n(r))\dot h_n(r)-Y(u_n(r))\dot h_n(r)\|^2_{H^1(B_{T-r})}\\&=:f_n^1+f_n^2+f_n^3.\end{aligned} \tag{5.44}$$

By doing the computation based on Lemmata 4.7 and 5.4 we obtain

$$\begin{aligned}f_n^1&\lesssim\|A_{U_n(r)}(\partial_xU_n(r),\partial_xU_n(r))-A_{u_n(r)}(\partial_xU_n(r),\partial_xU_n(r))\|^2_{H^1(B_{T-r})}\\&\quad+\|A_{u_n(r)}(\partial_xU_n(r),\partial_xU_n(r))-A_{u_n(r)}(\partial_xu_n(r),\partial_xU_n(r))\|^2_{H^1(B_{T-r})}\\&\quad+\|A_{u_n(r)}(\partial_xu_n(r),\partial_xU_n(r))-A_{u_n(r)}(\partial_xu_n(r),\partial_xu_n(r))\|^2_{H^1(B_{T-r})}\\&\lesssim_{T,x}\|U_n(r)-u_n(r)\|^2_{H^2(B_{T-r})}\left(1+\|\partial_xU_n(r)\|^2_{H^1(B_{T-r})}+\|\partial_xU_n(r)\|^2_{H^1(B_{T-r})}\right)\times\end{aligned}$$

$$
\begin{aligned}
&\times \left(1 + \|u_n(r)\|^2_{H^2(B_{T-r})}\right) \\
&+ \|u_n(r)\|^2_{H^2(B_{T-r})} \|\partial_x[U_n(r) - u_n(r)]\|^2_{H^1(B_{T-r})} \|\partial_x[u_n(r)]\|^2_{H^1(B_{T-r})} \\
\lesssim\ & \|\mathcal{Z}_n(r)\|^2_{\mathcal{H}_{T-r}} \left[\left(1 + \|Z_n(r)\|^2_{\mathcal{H}_{T-r}}\right)\left(1 + \|z_n(r)\|^2_{\mathcal{H}_{T-r}}\right) + \|z_n(r)\|^4_{\mathcal{H}_{T-r}}\right],
\end{aligned} \tag{5.45}
$$

and, by similar calculations,

$$
f_n^2 \lesssim_{T,x} \|\mathcal{Z}_n(r)\|^2_{\mathcal{H}_{T-r}} \left[\left(1 + \|Z_n(r)\|^2_{\mathcal{H}_{T-r}}\right)\left(1 + \|z_n(r)\|^2_{\mathcal{H}_{T-r}}\right) + \|z_n(r)\|^4_{\mathcal{H}_{T-r}}\right]. \tag{5.46}
$$

Furthermore, Lemmata 5.4 and 3.4 implies

$$
\begin{aligned}
f_n^3 &\lesssim_{T,x} \|U_n(r) - u_n(r)\|^2_{H^1(B_{T-r})} \left[1 + \|U_n(r)|^2_{H^1(B_{T-r})} + \|u_n(r)|^2_{H^1(B_{T-r})}\right] \|\dot{h}_n(r)\|^2_{H_\mu} \\
&\lesssim \|\mathcal{Z}_n(r)\|^2_{\mathcal{H}_{T-r}} \left(1 + \|Z_n(r)\|^2_{\mathcal{H}_{T-r}} + \|z_n(r)\|^2_{\mathcal{H}_{T-r}}\right) \|\dot{h}_n(r)\|^2_{H_\mu}.
\end{aligned} \tag{5.47}
$$

Hence by substituting (5.45)–(5.47) in (5.44) we get

$$
\begin{aligned}
\|f_n(r)\|^2_{H^1(B_{T-r})} \lesssim_{T,x} & \|\mathcal{Z}_n(r)\|^2_{\mathcal{H}_{T-r}} \left[\left(1 + \|Z_n(r)\|^2_{\mathcal{H}_{T-r}}\right)\left(1 + \|z_n(r)\|^2_{\mathcal{H}_{T-r}}\right) + \|z_n(r)\|^4_{\mathcal{H}_{T-r}}\right] \\
&+ \|\mathcal{Z}_n(r)\|^2_{\mathcal{H}_{T-r}} \left(1 + \|Z_n(r)\|^2_{\mathcal{H}_{T-r}} + \|z_n(r)\|^2_{\mathcal{H}_{T-r}}\right) \|\dot{h}_n(r)\|^2_{H_\mu},
\end{aligned}
$$

consequently, the definition of τ_n and Lemma 5.11 suggest

$$
\begin{aligned}
\mathbb{E}\left[\int_0^{t\wedge\tau_n} \|f_n(r)\|^2_{H^1(B_{T-r})}\, dr\right] &\lesssim \mathbb{E}\left[\int_0^{t\wedge\tau_n} \left\{\|\mathcal{Z}_n(r)\|^2_{\mathcal{H}_{T-r}} \left[\left(1 + N^2\right)\left(1 + \mathcal{B}^2\right) + \mathcal{B}^4\right]\right.\right. \\
&\qquad \left.\left. + \|\mathcal{Z}_n(r)\|^2_{\mathcal{H}_{T-r}} \left(1 + N^2 + \mathcal{B}^2\right)\left(1 + \mathcal{B}^2\right) \|\dot{h}_n(r)\|^2_{H_\mu}\right\} dr\right] \\
&\lesssim \mathbb{E}\left[\int_0^{t\wedge\tau_n} \mathbf{e}(r, T; x, \mathcal{Z}_n(r))\, C_{N,\mathcal{B}} \left(1 + \|\dot{h}_n(r)\|^2_{H_\mu}\right) dr\right],
\end{aligned} \tag{5.48}
$$

for some constant $C_{N,\mathcal{B}} > 0$ depends on $N, \mathcal{B}$, where $\mathcal{B}$ is a function of x which is bounded on compact sets. Then substitution of (5.43) and (5.48) in (5.42) implies, here we write dependency of $\mathbf{e}$ on x and T explicitly,

$$
\begin{aligned}
\mathbb{E}\left[\sup_{0\le s\le t\wedge\tau_n} \mathbf{e}(s, T; x, \mathcal{Z}_n(s))\right] &\lesssim_{T,x} \varepsilon_n\ (1 + N^2) \\
&+ C_{N,\mathcal{B}} \mathbb{E}\left[\int_0^{t\wedge\tau_n} [\sup_{0\le s\le r\wedge\tau_n} \mathbf{e}(s, T; x, \mathcal{Z}_n(s))] \left(1 + \|\dot{h}_n(r)\|^2_{H_\mu}\right) dr\right].
\end{aligned}
$$

Therefore, invoking the stochastic Gronwall Lemma, see [30, Lemma 3.9], gives,

$$\sup_{x\in[-a,a]} \mathbb{E}\left[\sup_{0\le s\le t\wedge\tau_n} \mathbf{e}(s,T;x,\mathcal{Z}_n(s))\right] \lesssim_{T,a} \varepsilon_n\,(1+N^2)\exp\left[C_{N,\mathcal{B}}(T+\mathcal{M})\right]. \quad (5.49)$$

Since $\varepsilon_n \to 0$ as $n\to\infty$ and

$$\mathbb{E}\left[\sup_{0\le s\le t\wedge\tau_n} \mathbf{e}(s,T;x,\mathcal{Z}_n(s))\right] = \mathbb{E}\left[\sup_{0\le s\le t} \mathbf{e}(s\wedge\tau_n,T;x,\mathcal{Z}_n(s\wedge\tau_n))\right],$$

inequality (5.49) gives $\lim_{n\to\infty}\sup_{x\in[-a,a]}\mathbb{E}\left[\sup_{0\le t\le T}\mathbf{e}(t\wedge\tau_n,T;x,\mathcal{Z}_n(t\wedge\tau_n))\right]=0$. Hence we are done with the proof of Proposition 5.16. □

To proceed further we also need the following stochastic analogue of Lemma 5.11.

Lemma 5.17. *There exists a constant $\mathscr{B} := \mathscr{B}(N,T,\mathcal{M}) > 0$ such that*

$$\limsup_{n\to\infty}\sup_{x\in[-a,a]} \mathbb{E}\left[\sup_{t\in[0,\frac{T}{2}]} \mathbf{e}(t\wedge\tau_n,T;x,Z_n(t\wedge\tau_n))\right] \le \mathscr{B}.$$

Proof of Lemma 5.17. Let us fix sequence $\{e_j\}_{j\in\mathbb{N}}$ of orthonormal basis of H_μ. Let us also fix any $n\in\mathbb{N}$. With the notation of this subsection, Proposition C.1, with $k=1$, $L=I$, implies for every $t\in[0,\frac{T}{2}]$ and $x\in[-a,a]$,

$$\mathbf{e}(t,T;x,Z_n(t)) \le \int_0^t \mathbb{V}(r,x,Z_n(r))\,dr + \int_0^t \langle V_n(r), g_n(r)dW(r)\rangle_{H^1(B(x,T-r))},$$

with

$$\begin{aligned}\mathbb{V}(r,x,Z_n(r)) &:= \langle U_n(r),V_n(r)\rangle_{L^2(B(x,T-r))} + \langle V_n(r), f_n(r)\rangle_{H^1(B(x,T-r))}\\ &\quad + \frac{1}{2}\sum_{j=1}^{\infty}\|g_n(r)e_j\|^2_{H^1(B(x,T-r))},\end{aligned}$$

where for simplification we avoid writing the dependency of l.h.s. on T explicitly, and

$$\begin{aligned} f_n(r) &:= A_{U_n(r)}(V_n(r),V_n(r)) - A_{U_n(r)}(\partial_x U_n(r),\partial_x U_n(r)) + Y(U_n(r))\dot{h}_n(r),\\ g_n(r) &:= \sqrt{\varepsilon_n}\,Y(U_n(r)).\end{aligned}$$

Next, we set

$$\psi_n(t,x) := \mathbb{E}\left[\sup_{0\le s\le t} \mathbf{e}(s\wedge\tau_n,T;x,Z_n(s\wedge\tau_n))\right], \quad t\in[0,T].$$

Now, we intent to follow the procedure of Proposition 5.16. By the Cauchy-Schwartz inequality, for $\tau \in [0, \frac{T}{2}]$ and $x \in [-a, a]$, we have

$$\sup_{0\le t\le\tau} \int_0^{t\wedge\tau_n} \mathbb{V}(r, x, Z_n(r))\,dr \le 2\int_0^{\tau\wedge\tau_n} \mathbf{e}(r, T; x, Z_n(r))\,dr + \frac{1}{2}\int_0^{\tau\wedge\tau_n} \left(\|f_n(r)\|^2_{H^1(B(x,T-r))} + \|g_n(r)\cdot\|^2_{\mathscr{L}_2(H_\mu, H^1(B(x,T-r)))}\right) dr.$$

Since the g_n here is same as in Proposition 5.16, the computation of (5.38)–(5.43) fits here too and we have

$$\mathbb{E}\left[\sup_{0\le s\le t\wedge\tau_n} \mathbf{e}(s, T; x, Z_n(s))\right] \lesssim_T \mathbb{E}\left[\int_0^{t\wedge\tau_n} \mathbf{e}(r, T; x, Z_n(r))\,dr\right] + \mathbb{E}\left[\int_0^{t\wedge\tau_n} \|f_n(r)\|^2_{H^1(B(x,T-r))}\,dr\right] + \varepsilon_n(1+N^2). \tag{5.50}$$

Invoking Lemmata 3.4 and 5.4 implies, to save space we write B_{T-r} instead of $B(x, T-r)$,

$$\begin{aligned}
\|f_n(r)\|^2_{H^1(B_{T-r})} &\lesssim \|A_{U_n(r)}(\partial_x U_n(r), \partial_x U_n(r))\|^2_{H^1(B_{T-r})} + \|A_{U_n(r)}(V_n(r), V_n(r))\|^2_{H^1(B_{T-r})} \\
&\quad + \|Y(U_n(r))\dot{h}_n(r)\|^2_{H^1(B_{T-r})} \\
&\lesssim_{T,x} \left(1 + \|U_n(r)\|^2_{H^1(B_{T-r})}\right)\left[1 + \|\partial_x U_n(r)\|^2_{H^1(B_{T-r})} + \|V_n(r)\|^2_{H^1(B_{T-r})} + \|\dot{h}_n(r)\|^2_{H_\mu}\right] \\
&\lesssim \left(1 + \|Z_n(r)\|^2_{\mathcal{H}_{T-r}}\right)\left[1 + \|Z_n(r)\|^2_{\mathcal{H}_{T-r}} + \|\dot{h}_n(r)\|^2_{H_\mu}\right].
\end{aligned}$$

So from (5.50) and the definition (5.34) we get

$$\begin{aligned}
\sup_{x\in[-a,a]} \mathbb{E}\left[\sup_{0\le s\le t\wedge\tau_n} \mathbf{e}(s, T; x, Z_n(s))\right] &\lesssim_{T,a} N^2\mathbb{E}\left[t\wedge\tau_n\right] + \varepsilon_n(1+N^2) \\
&\quad + (1+N^2)\mathbb{E}\left[\int_0^{t\wedge\tau_n} \left(1 + N^2 + \dot{h}_n(r)\|^2_{H_\mu}\right) dr\right] \\
&\lesssim_T N^2 T + (1+N^2)T + \mathcal{M} + \varepsilon_n(1+N^2).
\end{aligned}$$

Since $\lim_{n\to\infty} \varepsilon_n = 0$, taking $\limsup_{n\to\infty}$ on both the sides we get the required bound, and hence, the Lemma 5.17. $\square$

Lemma 5.18. *The sequence of $\mathcal{X}_T$-valued process $\{\mathcal{Z}_n\}_{n\in\mathbb{N}}$ converges in probability to* 0.

Proof of Lemma 5.18. We aim to show that for every $x \in \mathbb{R}$ and $R, \delta, \alpha > 0$ there exists a natural number n_0 such that

$$\mathbb{P}\left[\sup_{t\in[0,T]} \|\mathcal{Z}_n(t)\|_{\mathcal{H}_{B(x,R)}} > \delta\right] < \alpha \quad \text{for all} \quad n \geq n_0. \tag{5.51}$$

Let us choose and fix $x \in \mathbb{R}, \delta > 0, \alpha > 0$. In first step, we prove (5.51) for the case when R is set to be T. Let us also set $\mathcal{T} = 2T$. Then, since $\|\cdot\|_{\mathcal{H}_{B(x,r)}}$ is increasing in r and for $t \in [0,T]$ we have $\mathcal{T} - t \geq T = R$, and

$$\mathbb{P}\left[\sup_{t\in[0,T]} \|\mathcal{Z}_n(t)\|_{\mathcal{H}_{B(x,R)}} > \delta\right] \leq \mathbb{P}\left[\sup_{t\in[0,T]} \|\mathcal{Z}_n(t)\|_{\mathcal{H}_{B(x,\mathcal{T}-t)}} > \delta\right]. \tag{5.52}$$

Further note that, since $0 \leq \mathbf{e}(t, \mathcal{T}; x, \mathcal{Z}_n(t,\omega)) = \frac{1}{2}\|\mathcal{Z}_n(t,\omega)\|^2_{\mathcal{H}_{B(x,\mathcal{T}-t)}}$, due to (5.52) instead of showing (5.51), in the setting $R = T$, it is enough to show that there exists $n_0 \in \mathbb{N}$ such that

$$\mathbb{P}\left[\sup_{t\in[0,T]} \mathbf{e}(t, \mathcal{T}; x, \mathcal{Z}_n(t,\omega)) > \delta^2/2\right] < \alpha \quad \text{for all} \quad n \geq n_0. \tag{5.53}$$

But, since x is fix in the argument now, there exists $a > 0$ such that $x \in [-a, a]$ and the following holds

$$\mathbb{P}\left[\sup_{t\in[0,T]} \mathbf{e}(t, \mathcal{T}; x, \mathcal{Z}_n(t)) > \delta^2/2\right] \leq \sup_{x\in[-a,a]} \mathbb{P}\left[\sup_{t\in[0,T]} \mathbf{e}(t, \mathcal{T}; x, \mathcal{Z}_n(t)) > \delta^2/2\right].$$

Consequently instead of (5.53) it is sufficient to show that the existence of $n_0 \in \mathbb{N}$ such that

$$\sup_{x\in[-a,a]} \mathbb{P}\left[\sup_{t\in[0,T]} \mathbf{e}(t, \mathcal{T}; x, \mathcal{Z}_n(t,\omega)) > \delta^2/2\right] < \alpha \quad \text{for all} \quad n \geq n_0. \tag{5.54}$$

To prove (5.54), let us define a sequence $\{\kappa_n\}_{n\in\mathbb{N}}$ of stopping time via replacing T by $\mathcal{T}$ in (5.34). Now choose $N > \|(u_0, v_0)\|_{\mathcal{H}_{a+T}}$ and $n_0 \in \mathbb{N}$ such that, based on Lemma 5.17 for $\mathcal{T}$ instead of T,

$$\frac{2}{N^2} \sup_{n\in\mathbb{N}} \sup_{x\in[-a,a]} \mathbb{E}\left[\sup_{t\in[0,T]} \mathbf{e}(t\wedge\kappa_n, \mathcal{T}; x, Z_n(t\wedge\kappa_n))\right] < \frac{\alpha}{2} \text{ for all } n \geq n_0, \tag{5.55}$$

and, due to Proposition 5.16 for $\mathcal{T}$ instead of T,

$$\sup_{x\in[-a,a]} \mathbb{E}\left[\sup_{t\in[0,T]} \mathbf{e}(t\wedge\kappa_n, \mathcal{T}; x, \mathcal{Z}_n(t\wedge\kappa_n))\right] < \frac{\delta^2\alpha}{4} \text{ for all } n \geq n_0. \tag{5.56}$$

Thus the Markov inequality followed by using of (5.55) and (5.56), for $n \geq n_0$, gives

$$\sup_{x\in[-a,a]} \mathbb{P}\left[\sup_{t\in[0,T]} \mathbf{e}(t,\mathcal{T};x,\mathcal{Z}_n(t)) > \delta^2/2\right]$$
$$= \sup_{x\in[-a,a]} \mathbb{P}\left[\sup_{t\in[0,T]} \mathbf{e}(t,\mathcal{T};x,\mathcal{Z}_n(t)) > \delta^2/2 \text{ and } \kappa_n = \mathcal{T}\right]$$
$$+ \sup_{x\in[-a,a]} \mathbb{P}\left[\sup_{t\in[0,T]} \mathbf{e}(t,\mathcal{T};x,\mathcal{Z}_n(t)) > \delta^2/2 \text{ and } \mathbf{e}(t,\mathcal{T};x,Z_n(t)) \geq \frac{N^2}{2}\right]$$
$$\leq \frac{2}{\delta^2} \sup_{x\in[-a,a]} \mathbb{E}\left[\sup_{t\in[0,T]} \mathbf{e}(t,\mathcal{T};x,\mathcal{Z}_n(t))\right]$$
$$+ \frac{2}{N^2} \sup_{x\in[-a,a]} \mathbb{E}\left[\sup_{t\in[0,T]} \mathbf{e}(t,\mathcal{T};x,Z_n(t))\right] < \alpha. \tag{5.57}$$

Now we move to prove (5.51) when R is not set to T. Since the closure of $B(x,R)$ is compact and $B(x,R) \subset \cup_{y\in B(x,R)} B(y,T)$, we can find finitely many center $\{x_i\}_{i=1}^m$ such that $B(x,R) \subset \cup_{i=1}^m B(x_i,T)$. Moreover, since $B(x,R)$ is bounded, there exists $a>0$ such that $B(x,R) \in [-a,a]$. In particular, $x_i \in [-a,a]$ for all $i=1,\dots,m$. Then since $\|\mathcal{Z}_n(t,\omega)\|_{\mathcal{H}_{B(x,R)}} \leq \sum_{i=1}^m \|\mathcal{Z}_n(t,\omega)\|_{\mathcal{H}_{B(x_i,T)}}$, we have

$$\sup_{x\in[-a,a]} \mathbb{P}\left[\sup_{t\in[0,T]} \|\mathcal{Z}_n(t)\|_{\mathcal{H}_{B(x,R)}} > \delta\right] \leq \sup_{x\in[-a,a]} \mathbb{P}\left[\sup_{t\in[0,T]} \sum_{i=1}^m \|\mathcal{Z}_n(t)\|_{\mathcal{H}_{B(x_i,T)}} > \delta\right]$$
$$\leq \sum_{i=1}^m \sup_{x\in[-a,a]} \mathbb{P}\left[\sup_{t\in[0,T]} \|\mathcal{Z}_n(t)\|_{\mathcal{H}_{B(x,T)}} > \delta\right] \leq m \sup_{x\in[-a,a]} \mathbb{P}\left[\sup_{t\in[0,T]} \mathbf{e}(t,\mathcal{T};x,\mathcal{Z}_n(t)) > \delta^2/2\right].$$

Now by taking α as α/m in (5.57), of course with new a, we get that there exists an $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$\sup_{x\in[-a,a]} \mathbb{P}\left[\omega\in\Omega : \sup_{t\in[0,T]} \|\mathcal{Z}_n(t,\omega)\|_{\mathcal{H}_{B(x,R)}} > \delta\right] < \alpha.$$

Hence the Lemma 5.18. □

Now we come back to the proof of **Statement 2**. Recall that $S_{\mathcal{M}}$ is a separable metric space. Since, by the assumptions, the sequence $\{\mathscr{L}(h_n)\}_{n\in\mathbb{N}}$ of laws on $S_{\mathcal{M}}$ converges weakly to the law $\mathscr{L}(h)$, the Skorokhod representation theorem, see for example [44, Theorem 3.30], there exists a probability space $(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{\mathbb{P}})$, and on this probability space, one can construct processes $(\tilde{h}_n,\tilde{h},\tilde{W})$ such that the joint distribution of $(\tilde{h}_n,\tilde{W})$ is same as that of (h_n,W), the distribution of $\tilde{h}$ coincide with that of h, and $\tilde{h}_n \xrightarrow[n\to\infty]{} \tilde{h}$, $\tilde{\mathbb{P}}$-a.s. pointwise on $\tilde{\Omega}$, in the weak topology of S_M. By Proposition 5.13 this implies that

$$J^0\circ\tilde{h}_n \to J^0\circ\tilde{h} \quad \text{in} \quad \mathcal{X}_T \;\; \tilde{\mathbb{P}}\text{-a.s. pointwise on } \tilde{\Omega}.$$

Next, we claim that

$$\mathscr{L}(z_n) = \mathscr{L}(\tilde{z}_n), \qquad \text{for all } n$$

where

$$z_n := J^0 \circ h : \Omega \to \mathcal{X}_T \quad \text{and} \quad \tilde{z}_n := J^0 \circ \tilde{h}_n : \tilde{\Omega} \to \mathcal{X}_T.$$

To avoid complexity, we will write $J^0(h)$ for $J^0 \circ h$. Let B be an arbitrary Borel subset of $\mathcal{X}_T$. Thus, since from Proposition 5.13 $J^0 : S_{\mathcal{M}} \to \mathcal{X}_T$ is Borel, $(J^0)^{-1}(B)$ is Borel in $S_{\mathcal{M}}$. So we have

$$\mathscr{L}(z_n)(B) = \mathbb{P}\left[J^0(h_n)(\omega) \in B\right] = \mathbb{P}\left[h_n^{-1}\left((J^0)^{-1}(B)\right)\right] = \mathscr{L}(h_n)\left((J^0)^{-1}(B)\right).$$

But, since $\mathscr{L}(h_n) = \mathscr{L}(\tilde{h}_n)$ on $\mathcal{X}_T$, this implies $\mathscr{L}(z_n)(B) = \mathscr{L}(\tilde{z}_n)(B)$. Hence the claim and by a similar argument we also have $\mathscr{L}(z_h) = \mathscr{L}(z_{\tilde{h}})$.

Before moving forward, note that from Lemma 5.18, the sequence of $\mathcal{X}_T$-valued random variables, defined from Ω, $J^{\varepsilon_n}(h_n) - J^0(h_n)$ converges in measure $\mathbb{P}$ to 0. Consequently, because $\mathscr{L}(h_n) = \mathscr{L}(\tilde{h}_n)$ and $J^{\varepsilon_n} - J^0$ is measurable, we infer that $J^{\varepsilon_n}(\tilde{h}_n) - J^0(\tilde{h}_n) \xrightarrow{\tilde{\mathbb{P}}} 0$ as $n \to \infty$. Hence, we can choose a subsequence $\{J^{\varepsilon_n}(\tilde{h}_n) - J^0(\tilde{h}_n)\}_{n \in \mathbb{N}}$, indexed again by n, of $\mathcal{X}_T$-valued random variables converges to 0, $\tilde{\mathbb{P}}$-almost surely.

Now we claim to have the proof of **Statement 2**. Indeed, for any globally Lipschitz continuous and bounded function $\psi : \mathcal{X}_T \to \mathbb{R}$, see [31, Theorem 11.3.3], we have

$$\begin{aligned}
&\left| \int_{\mathcal{X}_T} \psi(x)\, d\mathscr{L}(J^{\varepsilon_n}(h_n)) - \int_{\mathcal{X}_T} \psi(x)\, d\mathscr{L}(J^0(h)) \right| \\
&= \left| \int_{\mathcal{X}_T} \psi(x)\, d\mathscr{L}(J^{\varepsilon_n}(\tilde{h}_n)) - \int_{\mathcal{X}_T} \psi(x)\, d\mathscr{L}(J^0(\tilde{h})) \right| \\
&\leq \left| \int_{\tilde{\Omega}} \left\{ \psi\left(J^{\varepsilon_n}(\tilde{h}_n)\right) - \psi\left(J^0(\tilde{h}_n)\right) \right\} d\tilde{\mathbb{P}} \right| \\
&\quad + \left| \int_{\tilde{\Omega}} \psi\left(J^0(\tilde{h}_n)\right) d\tilde{\mathbb{P}} - \int_{\tilde{\Omega}} \psi\left(J^0(\tilde{h})\right) d\tilde{\mathbb{P}} \right|.
\end{aligned}$$

Since $J^0(\tilde{h}_n) \xrightarrow[n \to \infty]{} J^0(\tilde{h})$, $\mathbb{P}$-a.s. and ψ is bounded and continuous, we deduce that the 2nd term in right hand side above converges to 0 as $n \to \infty$. Moreover we claim that the 1st term also goes to 0. Indeed, it follows from the dominated convergence theorem because the term is bounded by

$$L_{\psi} \int_{\tilde{\Omega}} |J^{\varepsilon_n}(\tilde{h}_n) - J^0(\tilde{h}_n)| \, d\tilde{\mathbb{P}},$$

where L_{ψ} is Lipschitz constant of ψ, and the sequence $\{J^{\varepsilon_n}(\tilde{h}_n) - J^0(\tilde{h}_n)\}_{n\in\mathbb{N}}$ converges to 0, $\tilde{\mathbb{P}}$-a.s.

Therefore, **Statement 2** holds true and we complete the proof of Theorem 5.7.

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Appendix A. Intrinsic and extrinsic formulation

Here we recall the intrinsic and extrinsic formulation of SGWE from [15] and state, without proof, the equivalence result between them. Consider the following SGWE Cauchy problem

$$\begin{cases} \mathbf{D}_t \partial_t u = \mathbf{D}_x \partial_x u + Y_u(\partial_t u, \partial_x u)\, \dot{W}, \\ u(0,\cdot) = u_0, \\ \partial_t u(t,\cdot)_{|t=0} = v_0 \end{cases} \tag{A.1}$$

Assume that u_0, v_0 are $\mathfrak{F}_0$-measurable random variables with values in $H^2_{\text{loc}}(\mathbb{R}, M)$ and $H^1_{\text{loc}}(\mathbb{R}, TM)$ respectively such that $u_0(x,\omega) \in M$ and $v_0(x,\omega) \in T_{u_0(x,\omega)}M$ hold for every $\omega \in \Omega$ and $x \in \mathbb{R}$.

Definition A.1 *([15, Definition 2.3])*. A process $u : \mathbb{R}_+ \times \mathbb{R} \times \Omega \to M$ is called an intrinsic solution of problem (A.1) provided the following six conditions are satisfied:

(i) $u(t,x,\cdot)$ is $\mathfrak{F}_t$-measurable for every $x \in \mathbb{R}$ and every $t \geq 0$,
(ii) $u(\cdot,\cdot,\omega)$ belongs to $\mathcal{C}^1(\mathbb{R}_+ \times \mathbb{R}, M)$ for every $\omega \in \Omega$,
(iii) $\mathbb{R}^+ \ni t \mapsto u(t,\cdot,\omega) \in H^2_{\text{loc}}(\mathbb{R}, M)$ is continuous for every $\omega \in \Omega$,
(iv) $\mathbb{R}^+ \ni t \mapsto \partial_t u(t,\cdot,\omega) \in H^1_{\text{loc}}(\mathbb{R}; TM)$ is continuous for every $\omega \in \Omega$,
(v) $u(0,x,\omega) = u_0(x,\omega)$ and $\partial_t u(0,x,\omega) = v_0(x,\omega)$ holds for every $x \in \mathbb{R}$ almost surely,

(vi) and for every vector field X on M, and every $t \geq 0$ and $R > 0$

$$\begin{aligned}\langle \partial_t u(t), X(u(t))\rangle_{T_{u(t)}M} &= \langle v_0, X(u_0)\rangle_{T_{u(t)}M} + \int_0^t \langle \mathbf{D}_x \partial_x u(s), X(u(s))\rangle_{T_{u(s)}M}\, ds \\ &+ \int_0^t \langle \partial_t u(s), \nabla_{\partial_t u(s)} X\rangle_{T_{u(s)}M}\, ds \\ &+ \int_0^t \langle X(u(s)), Y_{u(s)}(\partial_t u(s), \partial_x u(s))\, dW(s)\rangle_{T_{u(s)}M},\end{aligned}$$

holds in $L^2(-R, R)$ almost surely.

Definition A.2 *([15, Definition 2.6]).* A process $u : \mathbb{R}_+ \times \mathbb{R} \times \Omega \to M$ is called an extrinsic solution of problem (A.1) if and only if the following six conditions are satisfied.

(a) $u(t, x, \cdot)$ is $\mathfrak{F}_t$-measurable for every $t \geq 0$ and $x \in \mathbb{R}$,
(b) $\mathbb{R}^+ \ni t \mapsto u(t, \cdot, \omega) \in H^2_{\text{loc}}(\mathbb{R}; \mathbb{R}^n)$ is continuous for every $\omega \in \Omega$,
(c) $\mathbb{R}^+ \ni t \mapsto u(t, \cdot, \omega) \in H^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^n)$ is continuously differentiable for every $\omega \in \Omega$,
(d) $u(t, x, \omega) \in M$ for every $x \in \mathbb{R}$ and every $\omega \in \Omega$,
(e) $u(0, x, \omega) = u_0(x, \omega)$ and $\partial_t u(0, x, \omega) = v_0(x, \omega)$ holds for every $x \in \mathbb{R}$ almost surely,
(f) and for every $t \geq 0$ and $R > 0$

$$\begin{aligned}\partial_t u(t) &= v_0 + \int_0^t \left[\partial_{xx} u(s) - A_{u(s)}(\partial_x u(s), \partial_x u(s)) + A_{u(s)}(\partial_t u(s), \partial_t u(s))\right] ds \\ &+ \int_0^r Y_{u(s)}(\partial_t u(s), \partial_x u(s))\, dW(s),\end{aligned}$$

holds in $L^2((-R, R); \mathbb{R}^n)$ almost surely.

The next result state the equivalence between the intrinsic solution and extrinsic solution to the problem (A.1).

Theorem A.3 *([15, Theorem 12.1]). Assume that u_0, v_0 are $\mathfrak{F}_0$-measurable random variables with values in $H^2_{loc}(\mathbb{R}, M)$ and $H^1_{loc}(\mathbb{R}, TM)$ respectively such that $u_0(x, \omega) \in M$ and $v_0(x, \omega) \in T_{u_0(x,\omega)}M$ hold for every $\omega \in \Omega$ and $x \in \mathbb{R}$. Suppose also that M is a compact submanifold of $\mathbb{R}^n$ as in Definition A.2. Then a process $u : \mathbb{R}_+ \times \mathbb{R} \times \Omega \to M$ is an intrinsic solution of problem (A.1) if and only if it is an extrinsic solution of the same problem.*

Appendix B. Existence and uniqueness result

In this part we recall a result about the existence of a uniqueness global solution, in strong sense, to problem (A.1). We ask the reader to refer [15, Theorem 11.1] for the proof.

Theorem B.1. *Fix $T > 0$ and $R > T$. For every $\mathfrak{F}_0$-measurable random variable u_0, v_0 with values in $H^2_{loc}(\mathbb{R}, M)$ and $H^1_{loc}(\mathbb{R}, TM)$, there exists a process $u : [0, T) \times \mathbb{R} \times \Omega \to M$, which we denote by $u = \{u(t), t < T\}$, such that the following hold:*

(1) $u(t, x, \cdot) : \Omega \to M$ *is $\mathfrak{F}_t$-measurable for every $t < T$ and $x \in \mathbb{R}$,*
(2) $[0, T) \ni t \mapsto u(t, \cdot, \omega) \in H^2((-R, R); \mathbb{R}^n)$ *is continuous for almost every $\omega \in \Omega$,*
(3) $[0, T) \ni t \mapsto u(t, \cdot, \omega) \in H^1((-R, R); \mathbb{R}^n)$ *is continuously differentiable for almost every $\omega \in \Omega$,*
(4) $u(t, x, \omega) \in M$, *for every $t < T, x \in \mathbb{R}$, $\mathbb{P}$-almost surely,*
(5) $u(0, x, \omega) = u_0(x, \omega)$ *and* $\partial_t u(0, x, \omega) = v_0(x, \omega)$ *holds, for every $x \in \mathbb{R}$, $\mathbb{P}$-almost surely,*
(6) *for every $t \geq 0$ and $R > 0$,*

$$\partial_t u(t) = v_0 + \int_0^t \left[\partial_{xx} u(s) - A_{u(s)}(\partial_x u(s), \partial_x u(s)) + A_{u(s)}(\partial_t u(s), \partial_t u(s))\right] ds$$
$$+ \int_0^t Y_{u(s)}(\partial_t u(s), \partial_x u(s))\, dW(s),$$

holds in $L^2((-R, R); \mathbb{R}^n)$, $\mathbb{P}$-almost surely.

Moreover, if there exists another process $U = \{U(t); t \geq 0\}$ satisfy the above properties, then $U(t, x, \omega) = u(t, x, \omega)$ for every $|x| < R - t$ and $t \in [0, T)$, $\mathbb{P}$-almost surely.

Appendix C. Energy inequality for stochastic wave equation

Recall the following slightly modified version of [15, Proposition 6.1] for a one (spatial) dimensional linear inhomogeneous stochastic wave equation. For $l \in \mathbb{N}$, we use the symbol $D^l h$ to denote the $\mathbb{R}^{n\times 1}$-vector $\left(\frac{d^l h^1}{dx^l}, \frac{d^l h^2}{dx^l}, \cdots, \frac{d^l h^n}{dx^l}\right)$.

Proposition C.1. *Assume that $T > 0$ and $k \in \mathbb{N}$. Let W be a cylindrical Wiener process on a Hilbert space K. Let f and g be progressively measurable processes with values, respectively, in $H^k_{loc}(\mathbb{R}; \mathbb{R}^n)$ and $\mathscr{L}_2(K, H^k_{loc}(\mathbb{R}; \mathbb{R}^n))$ such that, for every $R > 0$,*

$$\int_0^T \left\{\|f(s)\|_{H^k((-R,R);\mathbb{R}^n)} + \|g(s)\|^2_{\mathscr{L}_2(K, H^k((-R,R);\mathbb{R}^n))}\right\} ds < \infty,$$

$\mathbb{P}$-almost surely. Let z_0 be an $\mathscr{F}_0$-measurable random variable with values in

$$\mathcal{H}^k_{loc} := H^{k+1}_{loc}(\mathbb{R}; \mathbb{R}^n) \times H^k_{loc}(\mathbb{R}; \mathbb{R}^n).$$

Assume that an $\mathcal{H}^k_{loc}$-valued process $z = z(t)$, $t \in [0, T]$, satisfies

$$z(t) = S_t z_0 + \int_0^t S_{t-s} \begin{pmatrix} 0 \\ f(s) \end{pmatrix} ds + \int_0^t S_{t-s} \begin{pmatrix} 0 \\ g(s) \end{pmatrix} dW(s), \qquad 0 \le t \le T.$$

Given $x \in \mathbb{R}$, we define the energy function $\mathbf{e} : [0, T] \times \mathcal{H}^k_{loc} \to \mathbb{R}^+$ by, for $z = (u, v) \in \mathcal{H}^k_{loc}$,

$$\mathbf{e}(t, T; x, z) = \frac{1}{2} \left\{ \|u\|^2_{L^2(B(x,T-t))} + \sum_{l=0}^{k} \left[\|D^{l+1}u\|^2_{L^2(B(x,T-t))} + \|D^l v\|^2_{L^2(B(x,T-t))} \right] \right\}.$$

Assume that $L : [0, \infty) \to \mathbb{R}$ is a non-decreasing $\mathcal{C}^2$-smooth function and define the second energy function $\mathbf{E} : [0, T] \times \mathcal{H}^k_{loc} \to \mathbb{R}$, by

$$\mathbf{E}(t, z) = L(\mathbf{e}(t, T; x, z)),\ z = (u, v) \in \mathcal{H}^k_{loc}.$$

Let $\{e_j\}$ be an orthonormal basis of K. We define a function $V : [0, T] \times \mathcal{H}^k_{loc} \to \mathbb{R}$, by

$$\begin{aligned} V(t, z) = L'(\mathbf{e}(t, T; x, z)) & \left[\langle u, v\rangle_{L^2(B(x,T-t))} + \sum_{l=0}^{k} \langle D^l v, D^l f(t)\rangle_{L^2(B(x,T-t))} \right] \\ & + \frac{1}{2} L'(\mathbf{e}(t, T; x, z)) \sum_j \sum_{l=0}^{k} |D^l[g(t)e_j]|^2_{L^2(B(x,T-t))} + \\ & + \frac{1}{2} L''(\mathbf{e}(t, T; x, z)) \sum_j \left[\sum_{l=0}^{k} \langle D^l v, D^l[g(t)e_j]\rangle_{L^2(B(x,T-t))} \right]^2, \ (t, z) \in [0, T] \times \mathcal{H}^k_{loc}, \end{aligned}$$

where we suppress the dependency of the left hand side on T and x. Then $\mathbf{E}$ is continuous on $[0, T] \times \mathcal{H}^k_{loc}$, and for every $0 \le t \le T$,

$$\begin{aligned} \mathbf{E}(t, z(t)) \ \le\ & \mathbf{E}(0, z_0) + \int_0^t V(r, z(r)\, dr \\ & + \sum_{l=0}^{k} \int_0^t L'(\mathbf{e}(r, z(r))) \langle D^l v(r), D^l[g(r)\, dW(r)]\rangle_{L^2(B(x,T-r))}, \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

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