Ambiguity in Stochastic Optimization Problems with Nonlinear Dependence on a Probability Measure via Wasserstein Metric

Vlasta Kaňková [1]

Abstract. Many economic and financial applications lead to deterministic optimization problems depending on a probability measure. It happens very often (in applications) that these problems have to be solved on the data base. Point estimates of an optimal value and estimates of an optimal solutions set can be obtained by this approach. A consistency, a rate of convergence and normal properties, of these estimates, have been discussed (many times) not only under assumptions of independent data corresponding to the distributions with light tails, but also for weak dependent data and the distributions with heavy tails. However, it is also possible to estimate (on the data base) a confidence intervals and bounds for the optimal value and the optimal solutions. To analyze this approach we focus on a special case of static problems depending nonlinearly on the probability measure. Stability results based on the Wasserstein metric and the Valander approach will be employed for the above mentioned analysis.

Keywords: Stochastic optimization problems, static problems, empirical measure, point estimates, interval estimates, nonlinear dependence

JEL classification: C44
AMS classification: 90C15

1 Introduction

To introduce a primary "classical" stochastic static one-objective optimization problem, let $(\Omega, \mathcal{S}, P)$ be a probability space; $\xi \, (:= \xi(\omega) = (\xi_1(\omega), \ldots, \xi_s(\omega))$ an s–dimensional random vector defined on $(\Omega, \mathcal{S}, P)$; $F(:= F_\xi(z), z \in R^s)$ the distribution function of ξ; P_F, Z_F the probability measure and a support corresponding to F; $X_F \subset X \subset R^n$ a nonempty set generally depending on F; $X \subset R^n$ a nonempty "deterministic" set; E_F an operator of mathematical expectation corresponding to the distribution function F. If $g_0(x, z)$ is a real-valued function defined on $R^n \times R^s$, then a primary classical problem of the stochastic optimization can be (in a rather general setting) introduced in the form:

Find

$$\varphi(F, X) = \inf\{\mathsf{E}_F g_0(x, \xi) | x \in X\}. \tag{1}$$

The "deterministic" constraints set X, in the problem (1), has been (from the beginning of the stochastic optimization) often replaced by the set X_F depending on the probability measure (see, e.g., [1] or [12]). Simultaneously, it has been soon recognized that these problems have to be often solved (in applications) on the data base; the probability measure P_F has to be often replaced by an empirical measure P_{F^N}. Consequently, instead of the original Problem (1) the following Empirical Problem often has to be solved:

Find

$$\varphi(F^N, X_{F^N}) = \inf\{\mathsf{E}_{F^N} g_0(x, \xi) | x \in X_{F^N}\}. \tag{2}$$

A great effort has been paid to investigate a relationship between an optimal value and an optimal solution of the problem (1) with $X := X_F$, generally, and they point estimates obtained by the problem (2). Works dealing with confidence intervals have begun to appear about year 2000. We can recall here the paper [6],

[1]The Institute of Information Theory and Automation
Academy of Sciences of the Czech Republic
Pod Vodárenskou věží 4, 182 08 Praha 8, Czech Republic, kankova@utia.cas.cz

where the Kolmogorov distance has been employed to study an ambiguity. The Kantorovich distance has been employed to get ambiguity results in [11], the ambiguity in chance constrained problems has been investigated in [3]. The optimization problems with a nonlinear dependence on the probability measure have begun more to appear only in the last time (see, e.g., [2], [4], [7] or [8]).

To recall optimization problems with a nonlinear dependence on the probability measure, let $\bar{g}_0(:= \bar{g}_0(x, z, y))$ be a real–valued function defined on $R^n \times R^s \times R^{m_1}$; $h(:= h(x, z)) = (h_1(x, z), \ldots, h_{m_1}(x, z))$ be an m_1–dimensional vector function defined on $R^n \times R^s$. A stochastic static one-objective optimization problem with the nonlinear dependence on the probability measure can be introduced in the form:

Find

$$\bar{\varphi}(F, X_F) = \inf\{\mathsf{E}_F \bar{g}_0(x, \xi, \mathsf{E}_F h(x, \xi)) | x \in X_F\}, \tag{3}$$

where a nonlinear dependence can appear also in the constraints set X_F. We consider the types of X_F :

$$\begin{aligned} a. \quad & X_F := X, \\ b. \quad & X_F := \{x \in X : \mathsf{E}_F \bar{g}_i(x, \xi, \mathsf{E}_F h(x, \xi)) \le 0, \; i = 1, \ldots, m\}, \\ & \text{where} \quad \bar{g}_i(x, z, y), \; i = 1, \ldots, m \quad \text{are defined on} \quad R^n \times R^s \times R^{m_1}. \end{aligned} \tag{4}$$

Of course, we consider that all mathematical expectations in (1), (2), (3), (4) exist and they are finite.

To define second order stochastic dominance constraints set let $Y(\xi)$, $g(x, \xi)$ be for every $x \in X$ random variables with distribution function $F_{Y(\xi)}$, $F_{g(x, \xi)}$. Let, moreover, for every $x \in X$ there exist finite $\mathsf{E}_F g(x, \xi)$, $\mathsf{E}_F Y(\xi)$ and

$$F^2_{g(x, \xi)}(u) = \int_{-\infty}^{u} F_{g(x, \xi)}(v) dv, \quad F^2_{Y(\xi)}(u) = \int_{-\infty}^{u} F_{Y(\xi)}(v) dv, \quad u \in R^1.$$

Rather general second order stochastic dominance constraints set X_F can be defined by

c. $$X_F = \{x \in X : F^2_{g(x, \xi)}(u) \le F^2_{Y(\xi)}(u) \quad \text{for every} \quad u \in R^1\},$$

or equaivalently by

$$X_F = \{x \in X : \mathsf{E}_F(u - g(x, \xi))^+ \le \mathsf{E}_F(u - Y(\xi))^+ \quad \text{for every} \quad u \in R^1\}. \tag{5}$$

The proof of the last equivalence can be found in [10].

Very often it is necessary (instead of the underlying problem (3) also here) to solve empirical problem

Find

$$\bar{\varphi}(F^N, X_{F^N}) = \inf\{\mathsf{E}_{F^N} \bar{g}_0(x, \xi, \mathsf{E}_{F^N} h(x, \xi)) | x \in X_{F^N}\}. \tag{6}$$

2 Some Definitions, Assumptions and Auxiliary Assertion

Our analysis of the ambiguity is based on the Wasserstein metric and $\mathcal{L}_1$ distance in R^s . To this end, let $\mathcal{P}(R^s)$ denote the set of all (Borel) probability measures on R^s and let the system $\mathcal{M}^1_1(R^s)$ be defined by the relation:

$$\mathcal{M}^1_1(R^s) = \{\nu \in \mathcal{P}(R^s) : \int_{R^s} \|z\|_1 d\nu(z) < \infty\}, \quad \|\cdot\|_1 \quad \text{denotes} \quad \mathcal{L}_1 \quad \text{norm in} \quad R^s.$$

2.1 Definitions and Assumptions

First, we define a system of the assumptions:

A.1 1. $g(x, z)$, $Y(z)$ are for $x \in X$ Lipschitz functions of $z \in R^s$ with the Lipschitz constant L_g (corresponding to the $\mathcal{L}_1$ norm) not depending on x,

A.2
1. $\{\xi^i\}_{i=1}^{\infty}$ is an independent random sequence corresponding to F,
2. F^N is an empirical distribution function determined by $\{\xi^i\}_{i=1}^{N}$, $N = 1, 2, \ldots$,

B.1 $P_F \in \mathcal{M}_1^1(R^s)$, there exist $\varepsilon > 0$ and an ε-neighbourhood $X(\varepsilon)$ of X such that

1. $\bar{g}_0(x, z, y)$ is, for $x \in X(\varepsilon)$, $z \in R^s$, a Lipschitz function of $y \in Y(\varepsilon)$ with the Lipschitz constant L_y; $Y(\varepsilon) = \{y \in R^{m_1} : y = h(x, z) \quad \text{for some} \quad x \in X(\varepsilon), z \in R^s\}$, $\mathsf{E}_F h(x, \xi)$, $\mathsf{E}_{F^N} h(x, \xi) \in Y(\varepsilon)$, for $x \in X(\varepsilon)$, $N = 1, 2, \ldots$,
2. for every $x \in X(\varepsilon)$ and every $y \in Y(\varepsilon)$ there exist finite mathematical expectations $\mathsf{E}_F \bar{g}_0(x, \xi, y)$, $\mathsf{E}_{F^N} \bar{g}_0(x, \xi, y)$,
3. $h_j(x, z)$, $j = 1, \ldots, m_1$ are for every $x \in X(\varepsilon)$ Lipschitz functions of $z \in R^s$ with the Lipschitz constants L_h^i (corresponding to the $\mathcal{L}_1$ norm),
4. $\bar{g}_0(x, z, y)$ is for every $x \in X(\varepsilon)$, $y \in Y(\varepsilon)$ a Lipschitz function of $z \in R^s$ with the Lipschitz constant L_z (corresponding to the $\mathcal{L}_1$ norm),

B.2 $\mathsf{E}_F \bar{g}_0(x, \xi, \mathsf{E}_F h(x, \xi))$, $\mathsf{E}_{F^N} \bar{g}_0(x, \xi, \mathsf{E}_{F^N} h(x, \xi))$, $N = 1, \ldots$ are continuous functions on X,

C.1
- $\bar{g}_0(x, z, y)$ is for every $z \in Z_F$ and $y \in Y(\varepsilon)$ a Lipschitz function of $x \in X$ with the Lipschitz constant L_C not depending on $z \in Z_F$, $y \in Y(\varepsilon)$,
- $h_j(x, z)$, $j = 1, \ldots, m_1$ are for every $z \in Z_F$ Lipschitz functions on X with the Lipschitz constant L_C^h not depending on $z \in Z_F$.

Further, we recall two Definitions and simultaneously define for $\varepsilon \in R^1$ the sets $X_F^{b,\varepsilon}$, $X_F^{c,\varepsilon}$.

Definition 1. [13] Let $\|\cdot\| = \|\cdot\|_n$ denote the Euclidean norm in R^n. If X', $X'' \subset R^n$ are two non–empty sets, then the Hausdorff distance of these sets $\Delta[X', X''] := \Delta_n[X', X'']$ is defined by

$$\Delta_n[X', X''] = \max[\delta_n(X', X''), \delta_n(X'', X')], \quad \delta_n(X', X'') = \sup_{x' \in X'} \inf_{x'' \in X''} \|x' - x''\|.$$

Definition 2. [13] Let $\hat{h}(x)$ be a real–valued function defined on a nonempty convex set $\mathcal{K} \subset R^n$. $\hat{h}(x)$ is a strongly convex function on $\mathcal{K}$ with a parameter $\bar{\rho} > 0$ if

$$\hat{h}(\lambda x^1 + (1-\lambda)x^2) \le \lambda \hat{h}(x^1) + (1-\lambda)\hat{h}(x^2) - \lambda(1-\lambda)\bar{\rho}\|x^1 - x^2\|_n^2 \quad \text{for every} \quad x^1, x^2 \in \mathcal{K}, \lambda \in \langle 0, 1\rangle.$$

$$\begin{aligned} X_F^{b,\varepsilon} &= \{x \in X : \mathsf{E}_F \bar{g}_1(x, \xi, \mathsf{E}_F h(x, \xi)) \le \varepsilon\} \quad \text{in the case of the constraints set } b.) \text{ with } m = 1, \\ X_F^{c,\varepsilon} &= \{x \in X : \mathsf{E}_F(u - g(x, \xi))^+ - \mathsf{E}_F(u - Y(\xi))^+ \le \varepsilon \quad \text{for every} \quad u \in R^1\} \\ & \qquad \text{in the case of the constraints set } c.). \end{aligned} \tag{7}$$

Evidently $X_F = X_F^0 := X_F^{b,0}$ in the case b.) with $m = 1$; $X_F = X_F^0 := X_F^{c,0}$ in the case set c.).

2.2 Brief Survey of Former Results

Lemma 1. [9] Let $\mathcal{K} \subset R^n$ be a nonempty, convex, compact set, $x_0 = \arg\min_{x \in \mathcal{K}} \hat{h}(x)$, If

1. $\hat{h}(x)$ is a strongly convex continuous function on $\mathcal{K}$ with a parameter $\bar{\rho} > 0$, $\mathcal{K}^{\varepsilon} = \{x \in \mathcal{K} : \hat{h}(x) \le \varepsilon\}$, $\varepsilon > \hat{h}(x_0)$,
2. for $\varepsilon_1, \ \varepsilon_2 > \hat{h}(x_0)$, $\varepsilon_1 < \varepsilon_2$ it holds that
   - there exists $x_2 \in \mathcal{K}$ such that $\hat{h}(x_2) = \varepsilon_2$,
   - for every $x_2 \in \mathcal{K}^{\varepsilon_2}$, $\hat{h}(x_2) = \varepsilon_2$, there exists a projection $x_1 := x_1(x_2)$ on $\mathcal{K}^{\varepsilon_1}$, $\hat{h}(x_1) = \varepsilon_1$,
   - $\hat{h}(x(\lambda))$ is an increasing function of λ, $\lambda \in \langle 0, 1\rangle$, $x(\lambda) = \lambda x_2 + (1-\lambda)x_1$,

then $\qquad [\Delta_n[\mathcal{K}^{\varepsilon_1}, \mathcal{K}^{\varepsilon_2}]]^2 \le \frac{2}{\bar{\rho}}|\hat{h}(x_2) - \hat{h}(x_1)|.$

If we denote by $F_i(z_i)$, $i = 1, \ldots, s$ one dimensional marginal distributions corresponding to F and by $\mathcal{N}_0$ the set of all natural numbers, then we can recall the following assertion.

Theorem 1. [7] Let $P_F \in \mathcal{M}_1^1(R^s)$ and let, moreover, X be a nonempty compact set, Assumptions A.2, B.1, B.2 be fulfilled, then for every $N \in \mathcal{N}_0$ it holds

$$|\bar{\varphi}(F, X) - \bar{\varphi}(F^N, X)| \le D \sum_{i=1}^{s} \int_{-\infty}^{\infty} |F_i(z_i) - F_i^N(z_i)| dz_i, \quad \text{where} \quad 0 \le D \le L_y \sum_{j=1}^{m_1} L_h^i + L_z. \tag{8}$$

Further, we little modify the assertions of the paper [9].

Theorem 2. Let $P_F \in \mathcal{M}_1^1(R^s)$, X be nonempty, compact convex sets, X_F be given by the constraint set b.) with $m = 1$. Let, moreover, X_F, X_{F^N}, $N \in \mathcal{N}$, be nonempty compact sets. If

1. $\mathsf{E}_F \bar{g}_1(x, \xi, \mathsf{E}_F h(x, \xi))$ is a strongly convex, with a parameter $\bar{\rho} > 0$, function on X that fulfils the assumptions of Lemma 1 (setting $\mathsf{E}_F \bar{g}_1(x, \xi, \mathsf{E}_F h(x, \xi)) := \hat{h}(x)$, $X := \mathcal{K}$),
2. $x_0 = \arg\min_{x \in X} \mathsf{E}_F \bar{g}_1(x, \xi, \mathsf{E}_F h(x, \xi)), \quad \varepsilon_1 > \mathsf{E}_F \bar{g}_1(x_0, \xi, \mathsf{E}_F h(x_0, \xi))$,
3. $X_F := X_F^{\varepsilon_1} = \{x \in X : \mathsf{E}_F \bar{g}_1(x, \xi, \mathsf{E}_F h(x, \xi)) \le \varepsilon_1\}$,
4.
   - Assumption A.2 is fulfilled,
   - $\bar{g}_1$ fulfils Assumptions B.1, B.2, (setting $\bar{g}_1 := \bar{g}_0$), $\bar{g}_0$ fulfils Assumptions B.1, B.2, C.1,
5. there exists $\bar{\varepsilon}_0 > 0$, $\bar{\varepsilon}_0 := \bar{\varepsilon}_0(\varepsilon_1)$ such that $X_F^{\varepsilon_1 - \varepsilon_0}$ is for $0 < \varepsilon_0 < \bar{\varepsilon}_0$ a nonempty compact set,

then for $N \in \mathcal{N}_0$, $\hat{C} = L_y \sum_{j=1}^{m_1} L_h^i + L_z$ fulfilling the relations

$$\hat{C}[\sum_{i=1}^{s} \int_{-\infty}^{\infty} |F_i(z_i) - F_i^N(z_i)| dz_i] \le \bar{\varepsilon}_0, \quad \sum_{i=1}^{s} \int_{-\infty}^{\infty} |F_i(z_i) - F_i^N(z_i)| dz_i \le 1, \tag{9}$$

the next assertion is valid

$$|\bar{\varphi}(F, X_F) - \bar{\varphi}(F^N, X_{F^N})| \le D \sum_{i=1}^{s} \int_{-\infty}^{\infty} |F_i(z_i) - F_i^N(z_i)| dz_i, \quad D = \hat{C} + 2[\max[L_C, m_1 L_y L_C^h][\frac{4}{\bar{\rho}}\hat{C}10]^{1/2}. \tag{10}$$

Theorem 3. Let $P_F \in \mathcal{M}_1^1(R^s)$, $X \subset R^n$ be a nonempty compact set, X_F correspond to the constraint set b.) with $m = 1$, X_F, X_{F^N}, $N = 1, 2, \ldots$ be nonempty compact sets, If

1.
   - Assumption A.2 is fulfilled,
   - $\bar{g}_1$ fulfils Assumptions B.1, B.2, (setting $\bar{g}_1 := \bar{g}_0$), $\bar{g}_0$ fulfils Assumptions B.1, B.2, C.1,
2. there exists $\bar{\varepsilon}_0 > 0$ such that X_F^{ε} (defined by the relation (7)) are nonempty compact sets for every $\varepsilon \in \langle -\bar{\varepsilon}_0, \bar{\varepsilon}_0 \rangle$ and, moreover, there exists a constant $\bar{C} > 0$ such that

$$\Delta_n[X_F^{\varepsilon}, X_F^{\varepsilon'}] \le \bar{C}|\varepsilon - \varepsilon'| \quad \text{for} \quad \varepsilon, \varepsilon' \in \langle -\bar{\varepsilon}_0, \bar{\varepsilon}_0 \rangle,$$

then for $N \in \mathcal{N}_0$ fulfilling inequality $\hat{C} \sum_{i=1}^{s} \int_{-\infty}^{\infty} |F_i(z_i) - F_i^N(z_i)| dz_i] \le \bar{\varepsilon}_0$ it holds that

$$|\bar{\varphi}(F, X_F) - \bar{\varphi}(F^N, X_{F^N})| \le D \sum_{i=1}^{s} \int_{-\infty}^{\infty} |F_i(z_i) - F_i^N(z_i)| dz_i, \quad D = \hat{C}[1 + 2\max[L_C, 2m_1 L_y L_C^h]10\bar{C}]. \tag{11}$$

If the assumption 2 holds for every $\bar{\varepsilon}_0 \in R^1$, then the inequality (11) holds also for all $N \in \mathcal{N}_0$.

Theorem 4. Let $P_F \in \mathcal{M}_1^1(R^s)$, $X \subset R^n$ be a nonempty compact set, X_F correspond to the constraints set c.), Assumption A.1, A.2 be fulfilled and let X_F, X_{F^N}, $N = 1, 2 \ldots$ be nonempty compact sets. If

1. $\bar{g}_0$ fulfils Assumptions B.1, B.2, C.1,
2. there exists $\bar{\varepsilon}_0 > 0$ such that X_F^{ε} (defined by the relation (7)) are nonempty compact sets for every $\varepsilon \in \langle -\bar{\varepsilon}_0, \bar{\varepsilon}_0 \rangle$ and, moreover, there exists a constant $\bar{C} > 0$ such that

$$\Delta_n[X_F^{\varepsilon}, X_F^{\varepsilon'}] \le \bar{C}|\varepsilon - \varepsilon'| \quad \text{for} \quad \varepsilon, \varepsilon' \in \langle -\bar{\varepsilon}_0, \bar{\varepsilon}_0 \rangle,$$

then for $N \in \mathcal{N}_0$ fulfilling inequality $2L_g \sum_{i=1}^{s} \int_{-\infty}^{\infty} |F_i(z_i) - F_i^N(z_i)|dz_i] \le \bar{\varepsilon}_0$ it holds that

$$|\bar{\varphi}(F, X_F) - \bar{\varphi}(F^N, X_{F^N})| \le D \sum_{i=1}^{s} \int_{-\infty}^{\infty} |F_i(z_i) - F_i^N(z_i)|dz_i, \quad D = \hat{C}[1 + 2\max[L_C, m_1 L_y L_C^h]20\bar{C}L_g \tag{12}$$

If the assumption 2 holds for every $\bar{\varepsilon}_0 \in R^1$, then the inequality (12) holds also for all $N \in \mathcal{N}_0$ [9].

Further, we recall Kolmogorov‘s limit Theorem. To this end we consider $s = 1$.

Proposition 1. [5] Let $s = 1$. If the probability measure corresponding to $F(z)$ is absolutely continuous with respect to the Lebesgue measure on R^1, Assumption A.2 is fulfilled, then

$$\begin{aligned} \lim_{N\to\infty} P\{\omega : (N)^{1/2} \sup_z |F(z) - F^N(z)| \le t\} \quad &= \sum_{j=-\infty}^{\infty} (-1)^j e^{-2j^2t^2} \quad \text{for} \quad t > 0, \\ &= 0 \qquad\qquad\qquad \text{for} \quad t \le 0. \end{aligned} \tag{13}$$

Evidently, in this case $K^N(t) = P\{\omega : \sup_z |F(z) - F^N(z)| \le t\}$ for $t \in R^1$, $N = 1, 2 \ldots$, is the distribution function of the random value $\sup_z |F(z) - F^N(z)|$, its quantils for $N \ge 100$ can be approximated employing the relation (13) (see, e.g., [5]).

3 Ambiguity Analysis

To analyze ambiguity, first, we introduce the following assumption:

C.2
- P_{F_i}, $i = 1, \ldots, s$ are absolutely continuous w.r.t. Lebesgue measure on R^1;
- $Z_{F_i} = \langle a_i, b_i \rangle$, $\langle a_i, b_i \rangle \subset \langle a, b \rangle$ for some $a_i, b_i, a, b \in R^1$, $a_i \le b_i$, $a \le b$, $i = 1, \ldots, s$,

and we define $k_N(\alpha)$ for $\alpha \in (0, 1)$, $N \in \mathcal{N}$ by

$$P\{\omega \in \Omega : \sup_{z_i} |F_i(z_i) - F_i^N(z_i)| \ge k_N^i(\alpha)\} \le \alpha, i = 1, 2, \ldots, s, \qquad k_N(\alpha) = \sup_i k_N^i(\alpha).$$

Consequently if $1 - s\alpha > 0$, then according to C.2 and Theorems of Section 2 (we can obtain successively (for the corresponding constant D) that

$$\begin{aligned} &P\{\omega \in \Omega : \bigcup_{i=1}^{s} [\sup_{z_i} |F_i(z_i) - F_i^N(z_i)| \ge k_N(\alpha)]\} && \le && s\alpha, && N \in \mathcal{N}_0, \\ &P\{\omega \in \Omega : \bigcap_{i=1}^{s} [\sup_{z_i} |F_i(z_i) - F_i^N(z_i)| < k_N(\alpha)]\} && > && 1 - s\alpha, && N \in \mathcal{N}_0, \\ &P\{\omega : D \sum_{i=1}^{s} \int_{a_i}^{b_i} |F_i(z_i) - F_i^N(z_i)|dz_i < dsDk_N(\alpha)\} && > && 1 - s\alpha, \quad d = b - a, && N \in \mathcal{N}_0, \\ &.P\{\omega : |\bar{\varphi}(F, X) - \bar{\varphi}(F^N, X_{F^N})| < dsDk_N(\alpha)\} && > && 1 - s\alpha, && N \in \mathcal{N}_0. \end{aligned}$$

Further, if we denote by $\mathcal{F}$ a system of all $s-$ dimensional distribution functions F with one dimensional marginal distribution functions F_i, $i = 1, \ldots, s$ fulfilling the assumption C.2, then

$$F \in \mathcal{F} \Rightarrow P\{\omega : |\bar{\varphi}(F, X) - \bar{\varphi}(F^N, X_{F^N})| < dsDk_N(\alpha)\} > 1 - s\alpha, \quad N \in \mathcal{N}_0. \tag{14}$$

Theorem 5. Let $X \subset R^n$ be a nonempty compact set, assumption C.2 be fulfilled, $\alpha \in (0, 1)$, $1 - s\alpha > 0$, $F \in \mathcal{F}$, $N \in \mathcal{N}_0$ fulfil the corresponding Theorem of Section 2. If

1. the assumptions of Theorem 1 are fulfilled, $D = L_y \sum_{j=1}^{m_j} L_h^i + L_z$, then

$$F \in \mathcal{F} \Rightarrow P\{\omega : |\bar{\varphi}(F, X) - \bar{\varphi}(F^N, X_{F^N})| < dDsk_N(\alpha)\} > 1 - s\alpha, \ N = 1, 2, \ldots, \tag{15}$$

2. the assumptions of Theorem 2 are fulfilled, $D = \hat{C}[1 + 2[\max[L_C, m_1 L_y L_C^h][\frac{2}{\bar{\rho}}10]^{1/2}]$, then

$$F \in \mathcal{F} \Rightarrow P\{\omega : |\bar{\varphi}(F, X_F) - \bar{\varphi}(F^N, X_{F^N})| < dDsk_N(\alpha) > 1 - s\alpha\} \tag{16}$$

3. the assumptions of Theorem 3 are fulfilled, $D = \hat{C} + 2\max[L_C, m_1 L_y L_C^h]10\hat{C}\bar{C}$,

$$F \in \mathcal{F} \Rightarrow P\{\omega : |\bar{\varphi}(F, X_F) - \bar{\varphi}(F^N, X_{F^N})| < dDsk_N(\alpha)\} > 1 - s\alpha), \tag{17}$$

4. the assumptions of Theorem 4 are fulfilled, $D = \hat{C} + \max[L_C, m_1 L_y L_C^h]\bar{C}20L_g$. then

$$F \in \mathcal{F} \Rightarrow P\{\omega : |\bar{\varphi}(F, X_F) - \bar{\varphi}(F^N, X_{F^N})| < dDsk_N(\alpha)\}\} > 1 - s\alpha, \tag{18}$$

Consequently, under the assumptions of Theorem 5, we get.

$$P\{\omega : \bar{\varphi}(F^N, \ X_{F^N}) - dDsk_N(\alpha) \ < \bar{\varphi}(X, X_F) \ < \bar{\varphi}(F^N, \ X_{F^N}) + dDsk_N(\alpha)\} > 1 - s\alpha \tag{19}$$

with constant D determined by the corresponding results given in Theorem 5.

4 Conclusion

The contribution is focused on a special type of the stochastic optimization problems in which dependence on the probability measure is not linear. This type of problems corresponds to real-life situations rather often and it has been investigated in [8], [9]. This contribution tries to employ there achieved results to investigate ambiguity properties.

Acknowledgement

This work was supported by the Czech Science Foundation under grant 18-02739S.

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