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On the coincidence of the pan-integral and the Choquet integral

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Abstract

We introduce the concept of weak (M)-property of a monotone measure and prove that this condition is not only sufficient, but also necessary for the coincidence of the pan-integral and the Choquet integral on monotone measure spaces. The previous results we obtained are substantially improved. An open problem concerning the weak (M)-property is raised.

Keywords: Monotone measure; (M)-property; Weak (M)-property; Choquet integral; Pan-integral

1. Introduction

In [6,9,11] we studied the relationship between the pan-integral and the Choquet integral (both these integrals belong to the class of decomposition integrals introduced by Even and Lehrer [1]. The counterpart of the decomposition integrals, superdecomposition integrals, was introduced in [7]. Interestingly enough, the Choquet integral is both decomposition and superdecomposition integral, see [4].) and obtained some interesting results. On this topic, we recall our several works, as follows: in [6] we introduced the concept of (M)-property of monotone measures (its original idea was proposed by Mesiar). We considered finite monotone measures on finite spaces and showed that if the pan-integral and the Choquet integral are coincident, then the considered monotone measure has (M)-property. Under the same assumption, in [9] we obtained that the (M)-property of monotone measure is also sufficient for the coincidence of the pan-integral and the Choquet integral, while by using the characteristics of *minimal atom of monotone measure* (which was introduced by Ouyang et al. [8]) presented another equivalence condition (see Theorem 4.6 in [9]). Following these researches, in [11] it was shown that the (M)-property is still sufficient for the coincidence of the pan-integral and the Choquet integral on general monotone measure spaces (not required that the space is finite set while the considered monotone measure is finite). But, for general cases, we were not able to find any necessary and sufficient condition for the coincidence of these two types of integrals.

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In this paper, we continue our research on this topic. We introduce the concept of *weak (M)-property* of a monotone measure and study some of its properties, and discuss the relationship between this new concept and the (M)-property. Then we show that the weak (M)-property is a necessary and sufficient condition for the coincidence of the pan-integral and the Choquet integral on general monotone measure spaces. The aforementioned results are substantially improved.

2. Preliminaries

Let X be a nonempty set and $\mathcal{A}$ be a σ-algebra of subsets of X. Let $\mathbb{F}^+$ denote the set of all finite nonnegative measurable functions defined on measurable space $(X, \mathcal{A})$, and χ_C denote the characteristic function of $C \in \mathcal{A}$. Unless otherwise stated, all the considered subsets are supposed to belong to $\mathcal{A}$.

By a *monotone measure* μ on $(X, \mathcal{A})$ we mean that μ is a set function $\mu : \mathcal{A} \to [0, +\infty]$ and satisfies $\mu(\emptyset) = 0$ and $\mu(R) \leq \mu(S)$ whenever $R \subset S$, where "$\subset$" is used in a non-strict sense.

We use $\mathfrak{M}$ to denote the set of all monotone measures defined on $(X, \mathcal{A})$. When $\mu \in \mathfrak{M}$, the triple $(X, \mathcal{A}, \mu)$ is called a monotone measure space [13,15].

Let $\mu \in \mathfrak{M}$, μ is called (i) *finite*, if $\mu(X) < \infty$; (ii) *superadditive*, if for any Q, S with $Q \cap S = \emptyset$, $\mu(Q \cup S) \geq \mu(Q) + \mu(S)$ holds.

We recall the Choquet and pan- integrals. Let $\mu \in \mathfrak{M}$ be fixed and $g \in \mathbb{F}^+$.

The Choquet integral of g with respect to μ is defined by

$$\mathbf{Ch}_\mu(g) = \sup\left\{ \sum_{i=1}^{s} \lambda_i \mu(C_i) : \sum_{i=1}^{s} \lambda_i \chi_{C_i} \leq g, \right.$$
$$\left. C_1 \subset C_2 \subset \cdots \subset C_s, \lambda_i > 0, s \in \mathbb{N} \right\}.$$

The pan-integral of g with respect to μ is defined by

$$\mathbf{Pan}_\mu(g) = \sup\left\{ \sum_{i=1}^{t} \lambda_i \mu(P_i) : \sum_{i=1}^{t} \lambda_i \chi_{P_i} \leq g, \right.$$
$$\left. P_i \cap P_j = \emptyset \ (i \neq j), \lambda_i > 0, t \in \mathbb{N} \right\}.$$

For fixed $\mu \in \mathfrak{M}$, we say that the pan-integral coincides with the Choquet integral, denoted by $\mathbf{Pan}_\mu \equiv \mathbf{Ch}_\mu$, if for all $g \in \mathbb{F}^+$,

$$\mathbf{Pan}_\mu(g) = \mathbf{Ch}_\mu(g).$$

In general, for some $\mu \in \mathfrak{M}$ these two kinds of integrals are incomparable (see [6,9]). Concerning the relationship between these two kinds of integrals, we have the following result (see [6,15]).

Proposition 2.1. *Let $(X, \mathcal{A}, \mu)$ be a monotone measure space. Then for all $g \in \mathbb{F}^+$, $\mathbf{Pan}_\mu(g) \leq \mathbf{Ch}_\mu(g)$ if and only if μ is superadditive.*

3. (M)-property and weak (M)-property

In [6] we introduced the concept of (M)-property of monotone measure and used it to study the relationship between the pan-integral and the Choquet integral, see also [9,11]. We present this concept.

Definition 3.1. *[6] Let $\mu \in \mathfrak{M}$. If for any $U \subset V$, there is $T \subset U$ such that*

$$\mu(T) = \mu(U) \quad \text{and} \quad \mu(V) = \mu(T) + \mu(V \setminus T),$$

then we say that μ has (M)-property.

In [6,9,11], we showed that the (M)-property is a sufficient condition for $\mathbf{Pan}_\mu \equiv \mathbf{Ch}_\mu$, and when X is a finite space and $\mu \in \mathfrak{M}$ is finite, then the (M)-property is also necessary for $\mathbf{Pan}_\mu \equiv \mathbf{Ch}_\mu$.

To further discuss the equivalence of the pan-integral and the Choquet integral, we propose a new concept, as follows.

Definition 3.2. *Let $\mu \in \mathfrak{M}$. We say that μ has* weak (M)-property, *if for any $U \subset V$, $\mu(V) < \infty$ and any $\epsilon > 0$, there is $T \subset U$ such that*

$$\mu(T) > \mu(U) - \epsilon$$

and

$$\mu(T) + \mu(V \setminus T) \le \mu(V) < \mu(T) + \mu(V \setminus T) + \epsilon.$$

Comparing Definitions 3.1 and 3.2, the following result is obvious.

Proposition 3.3. *If μ has (M)-property, then it has weak (M)-property.*

Note 3.4. The additivity of μ implies its (M)-property, thus an additive measure also has weak (M)-property. In general case, weak (M)-property does not imply (M)-property even if X is finite as shown in the following example.

Example 3.5. Let X be a finite set and $\mathcal{A} = 2^X$. Define the monotone measure $\mu : 2^X \to [0, +\infty]$ by $\mu(X) = \infty$ and $\mu(E) = |E|$ if $E \neq X$, where $|E|$ denotes the cardinality of E. We can easily check that μ has weak (M)-property. However, μ has no (M)-property. Let $U \subset X$, $U \neq \emptyset$ and $U \neq X$, then for any $T \subset U$, if $\mu(T) = \mu(U)$, then $T \neq \emptyset$ and $T \neq X$. Thus $\mu(X) > \mu(T) + \mu(X \setminus T)$.

For finite monotone measures on finite spaces, we have the following result:

Proposition 3.6. *Suppose that X is a finite set, and $\mu \in \mathfrak{M}$ is finite. If μ has weak (M)-property then it has (M)-property, and hence weak (M)-property and (M)-property are equivalent.*

Proof. Assume that μ has weak (M)-property. Let $U, V \in \mathcal{A}$, $U \subset V$ be given. Due to the finiteness of X, U has only finite subsets. Let

$$a = \min\Big\{ \max\big\{\mu(U) - \mu(T),\ \mu(V) - \mu(T) - \mu(V \setminus T)\big\} :\ T \subset U \Big\}.$$

If $a > 0$ then there is no T such that both $\mu(T) > \mu(U) - \epsilon$ and $\mu(V) < \mu(T) + \mu(V \setminus T) + \epsilon$ for $\epsilon = \frac{a}{2}$. Thus $a = 0$ and which implies there exists at least one subset T such that $\mu(T) = \mu(U)$ and $\mu(V) = \mu(T) + \mu(V \setminus T)$. □

The following result shows the superadditivity is a necessary condition of weak (M)-property.

Proposition 3.7. *If μ has weak (M)-property, then it is superadditive.*

Proof. Let $Q, S \in \mathcal{A}$ be disjoint. Suppose that $\mu(Q \cup S) < \infty$ without loss of generality. By the weak (M)-property, for any n there exists $O_n \subset Q$ with

$$\mu(O_n) > \mu(Q) - \frac{1}{n}$$

such that

$$\mu(O_n) + \mu((Q \cup S) \setminus O_n) \le \mu(Q \cup S) < \mu(O_n) + \mu((Q \cup S) \setminus O_n) + \frac{1}{n}.$$

Thus

$$\mu(Q \cup S) \ge \mu(O_n) + \mu((Q \cup S) \setminus O_n) > \mu(Q) - \frac{1}{n} + \mu(S).$$

Letting $n \to \infty$ we have $\mu(Q \cup S) \ge \mu(Q) + \mu(S)$. This shows that μ is superadditive. □

In [8], we introduced the concept of *minimal atoms*. Recall that a measurable set A is said to be a minimal atom of the monotone measure μ if $\mu(A) > 0$ and for each $O \subsetneqq A$ we have $\mu(O) = 0$. Note that for monotone measures defined on a finite space, each set with positive measure contains at least one minimal atom [8].

The following result indicates that weak (M)-property implies the additivity for minimal atoms.

Proposition 3.8. *Suppose that μ has weak (M)-property. Let $A \subset E$ with $\mu(E) < \infty$, and A be a minimal atom of μ, then*

$$\mu(E) = \mu(A) + \mu(E \setminus A).$$

In particular, if A_1, A_2 are minimal atoms of μ, $\mu(A_1 \cup A_2) < \infty$ and $A_1 \cap A_2 = \emptyset$, then

$$\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2).$$

Proof. Since A is a minimal atom of μ, the only subset of A which has positive measure is A itself. Thus $\mu(A) + \mu(E \setminus A) \leq \mu(E) \leq \mu(A) + \mu(E \setminus A) + \epsilon$ for any $\epsilon > 0$, which then implies that $\mu(E) = \mu(A) + \mu(E \setminus A)$. □

4. Coincidence of the pan- and Choquet integrals

Now we present our main result.

Theorem 4.1. *Suppose that $(X, \mathcal{A}, \mu)$ is a monotone measure space. Then the following statements (i) and (ii) are equivalent:*

(i) μ has weak (M)-property;

(ii) $\boldsymbol{Ch}_\mu \equiv \boldsymbol{Pan}_\mu$, i.e., for each $f \in \mathbb{F}^+$,

$$\boldsymbol{Ch}_\mu(f) = \boldsymbol{Pan}_\mu(f).$$

Proof. $(ii) \Rightarrow (i)$: Let $A \subset B$ with $\mu(B) < \infty$ be given. Then $\mathbf{Pan}_\mu(\chi_A + \chi_B) = \mathbf{Ch}_\mu(\chi_A + \chi_B) = \mu(A) + \mu(B)$. For any $\epsilon > 0$, there are mutual disjoint sets E_i and nonnegative numbers λ_i, $i = 1, 2, \cdots, n$ such that $\sum_{i=1}^{n} \lambda_i \chi_{E_i} \leq \chi_A + \chi_B$ and $\sum_{i=1}^{n} \lambda_i \mu(E_i) > \mu(A) + \mu(B) - \epsilon$. Obviously, $\lambda_i \leq 2$ for $E_i \subset A$ and $\lambda_i \leq 1$ otherwise. Denote $C = \bigcup_{E_i \subset A} E_i$, then $C \subset A$ and $2\chi_C + \chi_{B \setminus C} \leq \chi_A + \chi_B$. Since $\mathbf{Ch}_\mu \equiv \mathbf{Pan}_\mu$, it holds that $\mathbf{Pan}_\mu(g) = \mathbf{Ch}_\mu(g)$ for all g, which can also be seen as that $\mathbf{Pan}_\mu(g) \leq \mathbf{Ch}_\mu(g)$ for all g. By Proposition 2.1, we have that μ is superadditive, therefore $\mu(C) + \mu(B \setminus C) \leq \mu(B)$ and

$$\begin{aligned} 2\mu(C) + \mu(B \setminus C) &\geq 2 \sum_{E_i \subset C} \mu(E_i) + \sum_{E_i \setminus C \neq \emptyset} \mu(E_i) \\ &\geq \sum_{i=1}^{n} \lambda_i \mu(E_i) > \mu(A) + \mu(B) - \epsilon. \end{aligned}$$

Since $C \subset A$, $\mu(B) < 2\mu(C) + \mu(B \setminus C) - \mu(A) + \epsilon \leq \mu(C) + \mu(B \setminus C) + \epsilon$. Again by the superadditivity, we have $\mu(B) \geq \mu(C) + \mu(B \setminus C)$, thus $\mu(C) + \mu(B) \geq 2\mu(C) + \mu(B \setminus C) > \mu(A) + \mu(B) - \epsilon$, which implies $\mu(C) > \mu(A) - \epsilon$.

$(i) \Rightarrow (ii)$: Since μ has weak (M)-property, by Proposition 3.7, μ is superadditive. Therefore, by Theorem 10.7 in [15] (see also Lemma 3.1 in [6]), we have

$$\mathbf{Pan}_\mu(f) \leq \mathbf{Ch}_\mu(f)$$

for each $f \in \mathbb{F}^+$. To finish our proof, now we prove that for any $f \in \mathbb{F}^+$, it holds

$$\mathbf{Ch}_\mu(f) \leq \mathbf{Pan}_\mu(f).$$

We distinguish two cases to prove this inequality.

Case 1. $\mathbf{Ch}_\mu(f) < \infty$.

For any $\epsilon > 0$ there are $\lambda_i > 0$, $B_1 \subset B_2 \subset \cdots \subset B_n$ and $\sum_{i=1}^n \lambda_i \chi_{B_i} \le f$, such that $\sum_{i=1}^n \lambda_i \mu(B_i) > \mathbf{Ch}_\mu(f) - \epsilon$. By the weak (M)-property of μ, for $B_1 \subset B_2$ there is $B_1^{(1)} \subset B_1$ such that $\mu(B_1^{(1)}) > \mu(B_1) - \frac{\epsilon}{n^2(\sum\limits_{i=1}^n \lambda_i)}$ and

$$\mu(B_2) < \mu(B_1^{(1)}) + \mu(B_2 \setminus B_1^{(1)}) + \frac{\epsilon}{n^2(\sum\limits_{i=1}^n \lambda_i)}.$$

Similarly, for $B_1^{(i-1)} \subset B_{i+1}$ there exists $B_1^{(i)} \subset B_1^{(i-1)}$ such that $\mu(B_1^{(i)}) > \mu(B_1^{(i-1)}) - \frac{\epsilon}{n^2(\sum\limits_{i=1}^n \lambda_i)} > \mu(B_1) - \frac{i\epsilon}{n^2(\sum\limits_{i=1}^n \lambda_i)}$

and

$$\mu(B_{i+1}) < \mu(B_1^{(i)}) + \mu(B_{i+1} \setminus B_1^{(i)}) + \frac{\epsilon}{n^2(\sum\limits_{i=1}^n \lambda_i)}, \ i = 2, 3, \cdots, n-1.$$

Denote $A_1 = B_1^{(n-1)}$, $B_{i+1}^{(1)} = B_{i+1} \setminus B_1^{(i)}$, $i = 1, 2, \cdots, n-1$ and $l_1 = \sum_{i=1}^n \lambda_i$. Then,

$$\begin{aligned} l_1 \chi_{A_1} + \sum_{i=2}^n \lambda_i \chi_{B_i^{(1)}} &= \left(\sum_{i=1}^n \lambda_i\right) \chi_{A_1} + \sum_{i=2}^n \lambda_i \chi_{B_i^{(1)}} \\ &\le \lambda_1 \chi_{B_1} + \sum_{i=2}^n \lambda_i \left(\chi_{B_i^{(1)}} + \chi_{B_1^{(i-1)}}\right) \\ &= \sum_{i=1}^n \lambda_i \chi_{B_i} \le f \end{aligned}$$

and,

$$\begin{aligned} l_1 \mu(A_1) + \sum_{i=2}^n \lambda_i \mu(B_i^{(1)}) &= \left(\sum_{i=1}^n \lambda_i\right) \mu(A_1) + \sum_{i=2}^n \lambda_i \mu(B_i^{(1)}) \\ &= \lambda_1 \mu(A_1) + \sum_{i=2}^n \lambda_i \left(\mu(B_i^{(1)}) + \mu(A_1)\right) \\ &> \lambda_1 \left(\mu(B_1) - \frac{(n-1)\epsilon}{n^2(\sum\limits_{i=1}^n \lambda_i)}\right) \\ &\quad + \sum_{i=2}^n \lambda_i \left(\mu(B_i^{(1)}) + \mu(B_1^{(i-1)}) - \frac{(n-i)\epsilon}{n^2(\sum\limits_{i=1}^n \lambda_i)}\right) \\ &> \lambda_1 \left(\mu(B_1) - \frac{(n-1)\epsilon}{n^2(\sum\limits_{i=1}^n \lambda_i)}\right) + \sum_{i=2}^n \lambda_i \left(\mu(B_i) - \frac{(n-i+1)\epsilon}{n^2(\sum\limits_{i=1}^n \lambda_i)}\right) \\ &> \sum_{i=1}^n \lambda_i \mu(B_i) - \frac{\epsilon}{n}. \end{aligned}$$

As we have done above, for $B_2^{(1)} \subset B_3^{(1)} \subset \cdots \subset B_n^{(1)}$, we can find $\{B_2^{(i)}\}_{i=2}^{n-1}$ with $B_2^{(n-1)} \subset B_2^{(n-2)} \subset \cdots \subset B_2^{(2)} \subset B_2^{(1)}$ such that $\mu(B_2^{(i)}) > \mu(B_2^{(1)}) - \frac{(i-1)\epsilon}{n^2(\sum_{i=1}^n \lambda_i)}$ and

$$\mu(B_{i+1}^{(1)}) < \mu(B_2^{(i)}) + \mu(B_{i+1}^{(1)} \setminus B_2^{(i)}) + \frac{\epsilon}{n^2(\sum\limits_{i=1}^n \lambda_i)}.$$

Denote $A_2 = B_2^{(n-1)}$, $B_{i+1}^{(2)} = B_{i+1}^{(1)} \setminus B_2^{(i)}$, $i = 2, 3, \cdots, n-1$ and $l_2 = \sum_{i=2}^n \lambda_i$. Then we have,

$$\sum_{i=1}^2 l_i \chi_{A_i} + \sum_{i=3}^n \lambda_i \chi_{B_i^{(2)}} \le l_1 \chi_{A_1} + \sum_{i=2}^n \lambda_i \chi_{B_i^{(1)}} \le f,$$

and

$$\sum_{i=1}^2 l_i \mu(A_i) + \sum_{i=3}^n \lambda_i \mu(B_i^{(2)}) > l_1 \mu(A_1) + \sum_{i=2}^n \lambda_i \mu(B_i^{(1)}) - \frac{\epsilon}{n}$$
$$> \sum_{i=1}^n \lambda_i \mu(B_i) - \frac{2\epsilon}{n}.$$

Generally, for $B_i^{(i-1)} \subset B_{i+1}^{(i-1)} \subset \cdots \subset B_n^{(i-1)}$, $i = 2, 3, \cdots, n-1$, we can find $\{B_i^{(j)}\}_{j=i}^{n-1}$ with $B_i^{(n-1)} \subset \cdots \subset B_i^{(i)} \subset B_i^{(i-1)}$ such that $\mu(B_i^{(j)}) > \mu(B_i^{(i-1)}) - \frac{(j-i+1)\epsilon}{n^2(\sum_{i=1}^n \lambda_i)}$ and

$$\mu(B_{j+1}^{(i-1)}) = \mu(B_i^{(j)}) + \mu(B_{j+1}^{(i-1)} \setminus B_i^{(j)}) + \frac{\epsilon}{n^2(\sum\limits_{i=1}^n \lambda_i)}.$$

Denote $A_i = B_i^{(n-1)}$, $B_{j+1}^{(i)} = B_{j+1}^{(i-1)} \setminus B_i^{(j)}$, $j = i, \cdots, n-1$ and $l_i = \sum_{j=i}^n \lambda_j$. Then

$$\sum_{j=1}^i l_j \chi_{A_j} + \sum_{j=i+1}^n \lambda_j \chi_{B_j^{(i)}} \le f,$$

and

$$\sum_{j=1}^i l_j \mu(A_j) + \sum_{j=i+1}^n \lambda_j \mu(B_j^{(i)}) > \sum_{i=1}^n \lambda_i \mu(B_i) - \frac{i\epsilon}{n}.$$

If we take $A_n = B_n^{(n-1)}$ and $l_n = \lambda_n$, we find a sequence of mutual disjoint sets $\{A_i\}_{i=1}^n$ and a sequence of nonnegative numbers $\{l_i\}_{i=1}^n$ such that

$$\sum_{i=1}^n l_i \chi_{A_i} \le \sum_{i=1}^n \lambda_i \chi_{B_i} \le f,$$

and

$$\mathbf{Pan}_\mu(f) \ge \sum_{i=1}^n l_i \mu(A_i) > \sum_{i=1}^n \lambda_i \mu(B_i) - \epsilon > \mathbf{Ch}_\mu(f) - 2\epsilon.$$

By the arbitrariness of ϵ, we have $\mathbf{Pan}_\mu(f) \ge \mathbf{Ch}_\mu(f)$ and thus $\mathbf{Pan}_\mu(f) = \mathbf{Ch}_\mu(f)$ follows.

Case 2. $\mathbf{Ch}_\mu(f) = \infty$.

For any $M > 0$ there are $\lambda_i > 0$, $B_1 \subset B_2 \subset \cdots \subset B_n$ and $\sum_{i=1}^n \lambda_i \chi_{B_i} \le f$, such that $\sum_{i=1}^n \lambda_i \mu(B_i) > M$. If $\mu(B_n) = \infty$ then $\mathbf{Pan}_\mu(f) \ge \lambda_n \mu(B_n) = \infty = \mathbf{Ch}_\mu(f)$. If $\mu(B_n) < \infty$, by using the technique in Case 1, we can find a sequence of mutual disjoint sets $\{A_i\}_{i=1}^n$ and a sequence of nonnegative numbers $\{l_i\}_{i=1}^n$ such that

$$\sum_{i=1}^{n} l_i \chi_{A_i} \le \sum_{i=1}^{n} \lambda_i \chi_{B_i} \le f,$$

and

$$\mathbf{Pan}_\mu(f) \ge \sum_{i=1}^{n} l_i \mu(A_i) > \frac{M}{2}.$$

Since M is arbitrary, we conclude $\mathbf{Pan}_\mu(f) = \mathbf{Ch}_\mu(f) = \infty$. □

The following corollary is a direct result of Proposition 3.3 and Theorem 4.1:

Corollary 4.2. *(Ouyang et al. Theorem 4.1, [11]) Suppose that $(X, \mathcal{A}, \mu)$ is a monotone measure space. If μ has (M)-property, then*

$$\mathbf{Pan}_\mu(g) = \mathbf{Ch}_\mu(g)$$

holds for all $g \in \mathbb{F}^+$.

Note that the converse of Corollary 4.2 is not true even if X is finite.

Example 4.3. We consider μ shown in Example 3.5. It holds $\mathbf{Pan}_\mu \equiv \mathbf{Ch}_\mu$. In fact, for each $g \in \mathbb{F}^+$, $\mathbf{Pan}_\mu(g) = \mathbf{Ch}_\mu(g) = \infty$ if $\min_{x\in X} g(x) > 0$ and $\mathbf{Pan}_\mu(g) = \mathbf{Ch}_\mu(g) = \sum_{x\in X} g(x)$ otherwise. But μ has no (M)-property as shown in Example 3.5.

Combining Propositions 3.6, 3.8 and Theorem 4.1, we go back to our previous result:

Corollary 4.4. *(Ouyang et al. Theorem 4.6, [9]) Let X be a finite space and $\mu \in \mathfrak{M}$ be finite. Then the following (i), (ii) and (iii) are equivalent:*
(i) μ has (M)-property;
(ii) For all $g \in \mathbb{F}^+$, it holds

$$\boldsymbol{Pan}_\mu(g) = \boldsymbol{Ch}_\mu(g);$$

(iii) For any $S \in \mathcal{A}$ and $\mu(S) > 0$, and any minimal atom A contained in S, we have

$$\mu(S) = \mu(A) + \mu(S \setminus A).$$

Example 4.5. Let $X = [0, 1]$, $\mathcal{A} = 2^X$, m_* be Lebesgue inner measure on 2^X. Let $\mathcal{L}_{[0,1]}$ denote the set of all Lebesgue measurable subsets in $[0, 1]$, m be Lebesgue measure on $\mathbb{R}^1$. Then m_* is a monotone measure on 2^X. We show that the inner measure m_* has weak (M)-property (on 2^X).

Let $U, V \in 2^X$ and $U \subset V$. From the definition of inner measure, for any $\epsilon > 0$, there is $D \in \mathcal{L}_{[0,1]}$ such that $D \subset V$ and

$$m(D) > m_*(V) - \epsilon.$$

Similarly, we consider the set U, there exists $T \in \mathcal{L}_{[0,1]}$ such that $T \subset A$ and

$$m_*(T) = m(T) > m_*(U) - \epsilon.$$

Thus,

$$\begin{aligned} m_*(V) &\ge m_*(T) + m_*(V \setminus TC) \\ &\ge m(T) + m(D \setminus T) = m(D \cup T) \\ &\ge m(D) > m_*(V) - \epsilon. \end{aligned}$$

That is,

$$m_*(T) + m_*(V \setminus T) \le m_*(V) < m_*(T) + m_*(V \setminus T) + \epsilon.$$

This shows that m_* has weak (M)-property.

By Theorem 4.1, for all $g \in \mathbb{F}^+$, we have

$$\mathbf{Pan}_{m_*}(g) = \mathbf{Ch}_{m_*}(g).$$

They are both equal to the Lebesgue integral whenever $g \in \mathbb{F}^+$ is Lebesgue measurable function.

Remark 4.6. Propositions 3.3 and 3.6 show that (M)-property of μ implies weak (M)-property, and if X is finite set and if $\mu \in \mathfrak{M}$ is finite, then weak (M)-property and (M)-property are equivalent. However, for a general X (not necessarily finite set) and μ is finite, we don't know whether weak (M)-property implies (M)-property. This induces the following question: when the monotone measure μ is finite, whether (M)-property of μ is a necessary and sufficient condition for $\mathbf{Pan}_\mu \equiv \mathbf{Ch}_\mu$. We leave this question as an open problem.

5. Conclusion

We have investigated the equivalence of the pan-integral and the Choquet integral and it is solved in full generality: a necessary and sufficient condition that the pan-integral coincides with the Choquet integral on monotone measure spaces is that the considered monotone measure has weak (M)-property (Theorem 4.1). The results obtained in this paper can be treated in a dual way for the equivalence of the pan-integral from above and the Choquet integral [5,14].

As is well known, in addition to the pan integral and the Choquet integral, the concave integral introduced by Lehrer [2] is also a kind of important nonlinear integral. In [3] it is shown that the concave integral coincides with the Choquet integral if and only if the considered monotone measure is supermodular, see Proposition 2 in [3]. Thus, the equivalence conditions of the pan-integral and the Choquet integral, and of the Choquet integral and the concave integral have been clearly described. For the equivalence of the concave integral and the pan integral, up to now we only obtained partial results: the subadditivity of monotone measure is a sufficient condition for which these two kinds of integrals coincide, but not necessary, see Theorem 9 in [10]. For more researches, see also [8,9,12]. In further study we will try to find necessary and sufficient conditions that the concave integral is equivalent to the pan integral on monotone measure spaces.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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