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On the Isolated Calmness Property of Implicitly Defined Multifunctions

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Dedicated to Roger J-B Wets on the occasion of his 85th birthday.

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The paper deals with an extension of the available theory of SCD (subspace containing derivatives) mappings to mappings between spaces of different dimensions. This extension enables us to derive workable sufficient conditions for the isolated calmness of implicitly defined multifunctions *around* given reference points. This stability property differs substantially from isolated calmness *at* a point and, possibly in conjunction with the Aubin property, offers a new useful stability concept. The application area includes a broad class of parameterized generalized equations, where the respective conditions ensure a rather strong type of Lipschitzian behavior of their solution maps.

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1. Introduction

Analysis of Lipschitzian stability of set-valued mappings is one of the most important parts of modern variational analysis. Above all, the notions of the Aubin and the calmness property play a central role both in parameter-dependent equilibria (especially in presence of unknown parameters) and in qualification conditions of generalized differential calculus. But also the so-called isolated calmness and (the existence of) single-valued Lipschitzian localization have a great importance, e.g., in connection with Newton-type methods for nonsmooth problems.

There are various pointwise characterizations of the above mentioned stability notions in terms of generalized derivatives as, e.g., the Mordukhovich or the Levy-Rockafellar criteria.

Recently, in connection with the so-called SCD (subspace containing derivatives) mappings and the associated SCD semismooth* Newton method in [6], the authors derived for such mappings a characterization of the strong metric subregularity on a neighborhood which amounts ([3, Theorem 3I.2]) to the isolated calmness on a neighborhood of their inverses.

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Both these properties differ from their counterparts *at a point* rather substantially and one obtains thus a useful amendment to the available arsenal of regularity and stability properties. In [6], one finds both a characterization of the strong metric subregularity on a neighborhood for general mappings and special mappings having SCD and semismooth* properties. These characterizations of strong metric subregularity, however, were presented in [6] only for mappings between spaces of the same dimension. So, in order to derive workable criteria for the isolated calmness around a reference point for, say, a class of implicitly given multifunctions, the basic framework has to be extended. This extension, along with the corresponding stability results, is the aim of the present paper.

The plan of the paper is as follows. In the next section one finds a necessary background from variational analysis which is used throughout the whole paper. Section 3 contains the basic elements of the theory of SCD mappings between spaces of different dimensions. In this development one uses the corresponding part of [6] as a template. In Section 4 several calculus rules are derived which are needed in the proofs of the stability results presented in Section 5. The main statement (Theorem 5.1) provides us with two types of conditions ensuring that the implicit multifunction, defined via the inclusion

$$0 \in H(x, y)$$

possesses the *isolated calmness property on a neighborhood* of the given *reference point*. One of these conditions is based on the so-called *outer limiting graphical derivative* and works for general mappings, whereas the other one is tailored to semismooth* SCD mappings and is available in a primal and a dual form. To illustrate the nature of these conditions, we use a class of parameterized *generalized equations* (GEs). In case of variational inequalities with polyhedral constraint sets, we work out these conditions to an efficient form expressed in terms of faces of the critical cone to the constraint set. For semismooth* SCD mappings, it appears that the specialized condition is easier to verify than the general one. In addition, we present in Section 5 another condition expressed in terms of the limiting (Mordukhovich) coderivative which ensures that the respective implicit mapping has both the Aubin and the isolated calmness property around the reference point.

The following notation is employed. Given a linear subspace $L \subseteq \mathbb{R}^n$, $L^\perp$ denotes its *orthogonal complement* and, for a closed cone K with vertex at the origin, K° signifies its (negative) *polar*. Given a multifunction F, $\operatorname{gph} F := \{(x, y) \,|\, y \in F(x)\}$ stands for its *graph*. For an element $u \in \mathbb{R}^n$, $\|u\|$ denotes its *Euclidean norm* and $\mathcal{B}_\delta(u)$ denotes the *closed ball* around u with radius δ. The *closed unit ball* in $\mathbb{R}^n$ is denoted by $\mathcal{B}_{\mathbb{R}^n}$. In a product space we use the norm $\|(u, v)\| := \sqrt{\|u\|^2 + \|v\|^2}$.

Given an $m \times n$ matrix A, we employ the *operator norm* $\|A\|$ with respect to the Euclidean norm and we denote the *range* of A by $\operatorname{rge} A$. Given a set $\Omega \subset \mathbb{R}^s$, we define the *distance* of a point x to Ω by $d_\Omega(x) := \operatorname{dist}(x, \Omega) := \inf\{\, \|y - x\| \,|\, y \in \Omega \,\}$ and the *indicator function* is denoted by δ_Ω. Finally, $x \xrightarrow{\Omega} \bar{x}$ denotes convergence within the set Ω. When a mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at x, we denote by $\nabla F(x)$ its *Jacobian*.

2. Background from variational analysis

Throughout the whole paper, we will frequently use the following basic notions of modern variational analysis. All the sets under consideration are supposed to be *locally closed* around the points in question without further mentioning.

Definition 2.1. Let A be a set in $\mathbb{R}^s$ and let $\bar{x} \in A$. Then

(i) The *tangent (contingent, Bouligand) cone* to A at $\bar{x}$ is given by

$$T_A(\bar{x}) := \operatorname*{Lim\,sup}_{t \downarrow 0} \frac{A - \bar{x}}{t},$$

the *paratingent cone* to A at $\bar{x}$ is given by

$$T_A^P(\bar{x}) := \operatorname*{Lim\,sup}_{\substack{x \xrightarrow{A} \bar{x} \\ t \downarrow 0}} \frac{A - x}{t}$$

and the *outer limiting tangent cone* to A at $\bar{x}$ is defined as

$$T_A^\sharp(\bar{x}) := \operatorname*{Lim\,sup}_{x \xrightarrow{A} \bar{x}} T_A(x) = \operatorname*{Lim\,sup}_{x \xrightarrow{A} \bar{x}} \Big(\operatorname*{Lim\,sup}_{t \downarrow 0} \frac{A - x}{t} \Big). \tag{1}$$

(ii) The set
$$\widehat{N}_A(\bar{x}) := \big(T_A(\bar{x})\big)^\circ$$
is the *regular (Fréchet) normal cone* to A at $\bar{x}$, and

$$N_A(\bar{x}) := \operatorname*{Lim\,sup}_{x \xrightarrow{A} \bar{x}} \widehat{N}_A(x)$$

is the *limiting (Mordukhovich) normal cone* to A at $\bar{x}$.

In this definition "Lim sup" stands for the Painlevé-Kuratowski *outer (upper) set limit*, see, e.g.,[1]. The outer limiting tangent cone $T_A^\sharp(\bar{x})$ was very recently defined in [6] and it is always contained in the paratingent cone $T_A^P(\bar{x})$. All the other objects from variational geometry are well-known and can be found in standard textbooks, see, e.g., [15].

If A is convex, then $\widehat{N}_A(\bar{x}) = N_A(\bar{x})$ amounts to the classical normal cone in the sense of convex analysis and we will write $N_A(\bar{x})$.

The above listed cones enable us to describe the local behavior of set-valued maps via various generalized derivatives. All the set-valued mappings under consideration are supposed to have *locally closed graph* around the points in question.

Definition 2.2. Consider a multifunction $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and let $(\bar{x}, \bar{y}) \in \operatorname{gph} F$.

(i) The multifunction $DF(\bar{x}, \bar{y}) : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ given by $\operatorname{gph} DF(\bar{x}, \bar{y}) = T_{\operatorname{gph} F}(\bar{x}, \bar{y})$ is called the *graphical derivative* of F at $(\bar{x}, \bar{y})$.

(ii) The *outer limiting graphical derivative* of F at $(\bar{x}, \bar{y})$ is the multifunction $D^\sharp F(\bar{x}, \bar{y}) : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ given by

$$\operatorname{gph} D^\sharp F(\bar{x}, \bar{y}) = T^\sharp_{\operatorname{gph} F}(\bar{x}, \bar{y}).$$

(iii) The multifunction $D_* F(\bar{x}, \bar{y}) : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ given by $\operatorname{gph} D_* F(\bar{x}, \bar{y}) = T^P_{\operatorname{gph} F}(\bar{x}, \bar{y})$ is called the *strict (paratingent) derivative* of F at $(\bar{x}, \bar{y})$.

(iv) The multifunction $\widehat{D}^*F(\bar{x},\bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ defined by

$$\operatorname{gph} \widehat{D}^*F(\bar{x},\bar{y}) = \{(y^*,x^*) \,|\, (x^*,-y^*) \in \widehat{N}_{\operatorname{gph} F}(\bar{x},\bar{y})\}$$

is called the *regular (Fréchet) coderivative* of F at $(\bar{x},\bar{y})$.

(v) The multifunction $D^*F(\bar{x},\bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, defined by

$$\operatorname{gph} D^*F(\bar{x},\bar{y}) = \{(y^*,x^*) \,|\, (x^*,-y^*) \in N_{\operatorname{gph} F}(\bar{x},\bar{y})\}$$

is called the *limiting (Mordukhovich) coderivative* of F at $(\bar{x},\bar{y})$.

The outer limiting graphical derivative has been introduced by the authors in [6].

If F is single-valued, we can omit the second argument and write $DF(x)$, $\widehat{D}^*F(x),\ldots$ instead of $DF\big(x,F(x)\big)$, $\widehat{D}^*F\big(x,F(x)\big),\ldots$. However, be aware that when considering limiting objects at x where F is not continuous, it is not enough to consider only sequences $x_k \to x$ but we must work with sequences $\big(x_k,F(x_k)\big) \to \big(x,F(x)\big)$.

Definition 2.3. Let $U \subset \mathbb{R}^n$ be open and consider a mapping $F : U \to \mathbb{R}^m$. The *B-Jacobian* of F at $x \in U$ is defined as

$$\overline{\nabla} F(x) := \left\{ A \,\middle|\, \begin{array}{l} \exists x_k \to x : F \text{ is Fréchet differen-} \\ \text{tiable at } x_k \text{ and } A = \lim\limits_{k\to\infty} \nabla F(x_k) \end{array} \right\}. \tag{2}$$

Recall that the Clarke Generalized Jacobian is given by $\operatorname{co} \overline{\nabla} F(x)$, i.e., the convex hull of the B-Jacobian.

There exists the following relation between the B-Jacobian and the limiting coderivative of a single-valued mapping F, which states that every element from the B-Jacobian defines a certain subspace contained in the graph of the coderivative.

Proposition 2.4. ([6, Proposition 2.4]) *Let $U \subset \mathbb{R}^n$ be open and let $F : U \to \mathbb{R}^m$ be a mapping. Let F be continuous at $x \in U$ and let $A \in \overline{\nabla} F(x)$. Then*

$$(y^*, A^T y^*) \in \operatorname{gph} D^*F(x) \;\forall y^* \in \mathbb{R}^m.$$

If the mapping $F : U \to \mathbb{R}^m$ is Lipschitz continuous, then by Rademacher's Theorem F is differentiable almost everywhere in U and $\|\nabla F(x)\|$ is bounded there by the Lipschitz constant of F. Thus $\overline{\nabla} F(\bar{x}) \neq \emptyset$ for Lipschitz continuous mappings F.

Let us now recall the following regularity notions.

Definition 2.5. Consider a mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and let $(\bar{x},\bar{y}) \in \operatorname{gph} F$. Then

(i) F is said to be *metrically subregular at* $(\bar{x},\bar{y})$ if there exists $\kappa \geq 0$ along with some neighborhood U of $\bar{x}$ such that

$$\operatorname{dist}(x, F^{-1}(\bar{y})) \leq \kappa \operatorname{dist}(\bar{y}, F(x)) \;\forall x \in U. \tag{3}$$

(ii) F is said to be *strongly metrically subregular at* $(\bar{x},\bar{y})$ if there is $\kappa \geq 0$ together with some neighborhood U of $\bar{x}$ such that

$$\|x - \bar{x}\| \leq \kappa \operatorname{dist}(\bar{y}, F(x)) \;\forall x \in U. \tag{4}$$

(iii) F is said to be *metrically regular around* $(\bar{x}, \bar{y})$ if there is $\kappa \geq 0$ together with neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$\operatorname{dist}(x, F^{-1}(y)) \leq \kappa \operatorname{dist}(y, F(x)) \ \forall (x, y) \in U \times V. \tag{5}$$

Note that condition (4) implies that $F^{-1}(\bar{y}) \cap U = \{\bar{x}\}$.

Related with these regularity properties are the following Lipschitzian properties.

Definition 2.6. Let $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ be a mapping and let $(\bar{y}, \bar{x}) \in \operatorname{gph} S$. Then

(i) S is *calm* at $(\bar{y}, \bar{x})$ if there exists $\kappa \geq 0$ along with a neighborhood U of $\bar{x}$ such that

$$S(y) \cap U \subset S(\bar{y}) + \kappa \|y - \bar{y}\| \mathcal{B}_{\mathbb{R}^n} \ \forall y \in \mathbb{R}^m.$$

(ii) S has the *isolated calmness* property at $(\bar{y}, \bar{x})$ if there exists $\kappa \geq 0$ along with a neighborhood U of $\bar{x}$ such that

$$S(y) \cap U \subset \{\bar{x}\} + \kappa \|y - \bar{y}\| \mathcal{B}_{\mathbb{R}^n} \ \forall y \in \mathbb{R}^m. \tag{6}$$

(iii) S has the *Aubin* property around $(\bar{y}, \bar{x})$ if there is some constant $\kappa \geq 0$ along with neighborhoods V of $\bar{y}$ and U of $\bar{x}$ such that

$$S(y) \cap U \subset S(y') + \kappa \|y - y'\| \mathcal{B}_{\mathbb{R}^n} \ \forall y, y' \in V.$$

Now the condition (6) defining isolated calmness ensures that $S(\bar{y}) \cap U = \{\bar{x}\}$.

It is well-known, see, e.g., [3], that the property of (strong) metric subregularity for F at $(\bar{x}, \bar{y})$ with constant κ is equivalent with the property of *(isolated) calmness* for F^{-1} at $(\bar{y}, \bar{x})$ with constant κ. Further, F is metrically regular around $(\bar{x}, \bar{y})$ with constant κ if and only if F^{-1} has the *Aubin property* around $(\bar{y}, \bar{x})$ with constant κ.

The properties of metric regularity and strong metric subregularity are stable under Lipschitzian and calm perturbations, respectively, cf. [3]. Further note that the property of metric regularity holds around all points belonging to the graph of F sufficiently close to the reference point, whereas the property of (strong) metric subregularity is guaranteed to hold only at the reference point. This leads to the following definition.

Definition 2.7. (i) We say that the mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is *(strongly) metrically subregular around* $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if there is $\kappa \geq 0$ and a neighborhood W of $(\bar{x}, \bar{y})$ such that F is (strongly) metrically subregular with constant κ at every point $(x, y) \in \operatorname{gph} F \cap W$.

In this case we will also speak about *(strong) metric subregularity on a neighborhood.*

(ii) We say that the mapping $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is called *(isolatedly) calm around* $(\bar{y}, \bar{x}) \in \operatorname{gph} S$ if there is some constant $\kappa \geq 0$ along with some neighborhood W of $(\bar{y}, \bar{x})$ such that S is isolatedly calm with constant κ at every point $(y, x) \in \operatorname{gph} S \cap W$.

In this case we will also speak about *(isolated) calmness on a neighborhood.*

The notion of (strong) metric subregularity on a neighborhood was introduced in [6, Definition 2.8]. Due to the relation between (strong) metric subregularity of F and (isolated) calmness of F^{-1} we immediately obtain the following result.

Lemma 2.8. *Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a mapping and let $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. Then F is (strongly) metrically subregular around $(\bar{x}, \bar{y})$ if and only if F^{-1} is isolatedly calm around $(\bar{y}, \bar{x})$.*

Note that every polyhedral multifunction, i.e., a mapping whose graph is the union of finitely many convex polyhedral sets, is both metrically subregular and calm around every point of its graph by Robinson's result [13]. In this paper, we will restrict our investigations to the properties of strong metric subregularity and isolated calmness on a neighborhood. Let us first have a closer look on Definition 2.7.

The mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is strongly metrically subregular around $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if and only if there is some $\kappa \geq 0$ together with some neighborhood W of $(\bar{x}, \bar{y})$ such that for every $(x, y) \in \operatorname{gph} F \cap W$ there is some neighborhood U_{xy} of x with

$$\operatorname{dist}(x', F^{-1}(y)) \leq \kappa \operatorname{dist}(y, F(x')) \ \forall x' \in U_{xy}.$$

Note that the neighborhoods U_{xy} depends both on x amd y and can be arbitrarily small.

Similarly, the mapping $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is isolatedly calm around $(\bar{y}, \bar{x}) \in \operatorname{gph} S$ if and only if there is some $\kappa \geq 0$ together with some neighborhood W of $(\bar{y}, \bar{x})$ such that for every $(y, x) \in \operatorname{gph} S \cap W$ there is some neighborhood U_{yx} of x with

$$S(y') \cap U_{yx} \subset \{x\} + \kappa \|y' - y\| \mathcal{B}_{\mathbb{R}^n} \ \forall y' \in \mathbb{R}^m.$$

In this paper we will use the following point-based characterizations of the above regularity properties.

Theorem 2.9. *Consider a mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and let $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. Then*

(i) (Levy-Rockafellar criterion, see, e.g., [3, Theorem 4.1]) *F is strongly metrically subregular at $(\bar{x}, \bar{y})$ if and only if*

$$0 \in DF(\bar{x}, \bar{y})(u) \ \Rightarrow u = 0. \tag{7}$$

(ii) (Mordukhovich criterion, see, e.g., [11, Theorem 3.3]) *F is metrically regular around $(\bar{x}, \bar{y})$ if and only if*

$$0 \in D^*F(\bar{x}, \bar{y})(y^*) \ \Rightarrow \ y^* = 0. \tag{8}$$

(iii) *F is strongly metrically subregular around $(\bar{x}, \bar{y})$ if and only if*

$$0 \in D^\sharp F(\bar{x}, \bar{y})(u) \ \Rightarrow u = 0. \tag{9}$$

The characterization (iii) of strong metric subregularity on a neighborhood was shown in [6, Theorem 6.1] for the special case $m = n$. But a close inspection of the proof of [6, Theorem 6.1] shows that it can be used without any modification to show the general case as well.

Next we introduce the semismooth* sets and mappings.

Definition 2.10. (i) A set $A \subseteq \mathbb{R}^s$ is called *semismooth** at a point $\bar{x} \in A$ if for every $\epsilon > 0$ there is some $\delta > 0$ such that

$$|\langle x^*, x - \bar{x} \rangle| \leq \epsilon \|x - \bar{x}\| \|x^*\|$$

holds for all $x \in A \cap \mathcal{B}_\delta(\bar{x})$ and all $x^* \in \widehat{N}_A(x)$.

(ii) A set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is called *semismooth** at a point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, if $\operatorname{gph} F$ is semismooth* at $(\bar{x}, \bar{y})$, i.e., for every $\epsilon > 0$ there is some $\delta > 0$ such that

$$|\langle x^*, x - \bar{x}\rangle - \langle y^*, y - \bar{y}\rangle| \le \epsilon \|(x, y) - (\bar{x}, \bar{y})\| \|(x^*, y^*)\|$$

holds for all $(x, y) \in \operatorname{gph} F \cap \mathcal{B}_\delta(\bar{x}, \bar{y})$ and all $(y^*, x^*) \in \operatorname{gph} \widehat{D}^* F(x, y)$.

Note that the above definitions of semismooth* sets and multifunctions are not the same as the ones introduced in [5], but by [5, Proposition 3.2, Corollary 3.3] they are equivalent.

The class of semismooth* sets and mappings is rather broad.

Proposition 2.11. (i) *Any closed convex set $A \subset \mathbb{R}^s$ is semismooth* at each $\bar{x} \in A$.*

(ii) *Assume that we are given closed sets $A_i \subset \mathbb{R}^s$, $i = 1, \dots p$, and a point $\bar{x} \in A := \bigcup_{i=1}^p A_i$. If the sets A_i, $i \in \bar{I} := \{j \,|\, \bar{x} \in A_j\}$, are semismooth* at $\bar{x}$, then so is the set A.*

(iii) *Every closed subanalytic set A is semismooth* at each $\bar{x} \in A$.*

The first two statements of this proposition can be found in [5, Propsition 3.4, Proposition 3.5], whereas the last statement follows from [9, Theorem 2].

We now state a sufficient condition for the semismooth* property of sets with constraint structure.

Proposition 2.12. *Let $A = \{x \in \mathbb{R}^s \,|\, \Phi(x) \in D\}$, where $\Phi : \mathbb{R}^s \to \mathbb{R}^p$ is continuously differentiable and $D \subset \mathbb{R}^p$ is a closed set. Given $\bar{x} \in A$, assume that the mapping $x \mapsto F(x) := \Phi(x) - D$ is metrically subregular at $(\bar{x}, 0)$ and assume that D is semismooth* at $\Phi(\bar{x})$. Then A is semismooth* at $\bar{x}$.*

Proof. By metric subregularity of F there exists a real $\kappa > 0$ together with some open neighborhood U such that (3) holds. It follows that for every $x \in A \cap U$ the mapping F is metrically subregular with constant κ at $(x, 0)$ and thus, by [7, Theorem 3] there holds

$$N_A(x) \subset \{\nabla\Phi(x)^T y^* \,|\, y^* \in N_D\big(\Phi(x)\big) \cap \kappa \|x^*\| \mathcal{B}_{\mathbb{R}^p}\},\ x \in A \cap U.$$

Since D is semismooth* at $\Phi(\bar{x})$, by [5, Proposition 3.2] there is some radius $\rho > 0$ such that

$$|\langle y^*, d - \Phi(\bar{x})\rangle| \le \frac{\epsilon}{2L\kappa} \|d - \Phi(\bar{x})\| \|y^*\| \ \forall d \in D \cap \mathcal{B}_\rho\big(\Phi(\bar{x})\big)\ \forall y^* \in N_D(d),$$

where L denotes the Lipschitz constant of Φ on some ball $B_\delta(\bar{x}) \subset U$. Next choose $0 < \bar{\delta} < \min\{\delta, \rho/L\}$ such that

$$\|\Phi(\bar{x}) - \Phi(x) - \nabla\Phi(x)(\bar{x} - x)\| \le \frac{\epsilon}{2\kappa} \|x - \bar{x}\|,\ x \in \mathcal{B}_{\bar{\delta}}(\bar{x}),$$

and consider $x \in A \cap \mathcal{B}_{\bar{\delta}}(\bar{x})$ and $x^* \in N_A(x)$ together with $y^* \in N_D\big(\Phi(x)\big)$ satisfying $\|y^*\| \le \kappa \|x^*\|$ and $x^* = \nabla\Phi(x)^T y^*$. Then

$$
\begin{aligned}
|\langle x^*, x-\bar{x}\rangle| &= |\langle y^*, \nabla\Phi(x)(x-\bar{x})\rangle| \\
&\leq |\langle y^*, \Phi(x)-\Phi(\bar{x})\rangle| + |\langle y^*, \Phi(\bar{x})-\Phi(x)-\nabla\Phi(x)(x-\bar{x})\rangle| \\
&\leq \frac{\epsilon}{2L\kappa}\|\Phi(x)-\Phi(\bar{x})\|\|y^*\| + \|y^*\|\|\Phi(\bar{x})-\Phi(x)-\nabla\Phi(x)(\bar{x}-x)\| \\
&\leq \frac{\epsilon}{2L\kappa}L\|x-\bar{x}\|\kappa\|x^*\| + \frac{\epsilon}{2\kappa}\kappa\|x^*\|\|x-\bar{x}\| = \epsilon\|x-\bar{x}\|\|x^*\|,
\end{aligned}
$$

verifying that A is semismooth* at $\bar{x}$. □

In case of single-valued Lipschitzian mappings the semismooth* property is equivalent with the semismooth property introduced by Gowda [8], which is weaker than the one in [12].

3. Preliminaries

This section is composed from two parts. The first one, Section 3.1, contains a generalization of the basic facts about the SCD mappings from [6, Section 3] to multifunctions between different finite-dimensional spaces. Section 3.2 is then devoted to the important notion of SCD regularity, playing a crucial role in the subsequent development.

3.1. SCD mappings

Let us denote by $\mathbb{Z}_{nm}$ the metric space of all n-dimensional subspaces of $\mathbb{R}^{n+m}$ equipped with the metric

$$d_{\mathbb{Z}_{nm}}(L_1, L_2) := \|P_1 - P_2\|, \tag{10}$$

where P_i is the symmetric $(n+m)\times(n+m)$ matrix representing the orthogonal projection onto $L_i, i = 1, 2$. Throughout the whole paper we make use of the following relationships.

Lemma 3.1. (i) *Let A_k be a sequence of $(n+m)\times(n+l)$ full-column-rank matrices converging to a full-column-rank matrix A and let $L_k \in \mathbb{Z}_{nl}$ be a sequence of subspaces converging to $L \in \mathbb{Z}_{nl}$. Then $\lim_{k\to\infty} d_{\mathbb{Z}_{nm}}(A_k L_k, AL) = 0$.*

(ii) *The metric space $\mathbb{Z}_{nm}$ is (sequentially) compact.*

The above statements can be proved in the same way as their counterparts in [6, Lemma 3.1(iii),(iv)] and therefore the proofs are omitted.

To be consistent with the notation in [6] we will write $\mathbb{Z}_n$ instead of $\mathbb{Z}_{nn}$.

With each $L \in \mathbb{Z}_{nm}$ one can associate its *adjoint* subspace L^* defined by

$$L^* := \{(-v^*, u^*) \in \mathbb{R}^m \times \mathbb{R}^n \mid (u^*, v^*) \in L^\perp\}. \tag{11}$$

Since $\dim L^\perp = m$, it follows that $L^* \in \mathbb{Z}_{mn}$ (i.e., its dimension is m). It is easy to see that

$$L^* = S_{nm}L^\perp, \text{ where } S_{nm} = \begin{pmatrix} 0 & -I_m \\ I_n & 0 \end{pmatrix}, \tag{12}$$

yielding $(L^*)^\perp = \{z \mid S_{nm}^T z \in (L^\perp)^\perp = L\}$.

Hence we obtain

$$\begin{aligned}(L^*)^* &= \{(-u,v)\in\mathbb{R}^n\times\mathbb{R}^m \,\big|\, (v,u)\in(L^*)^\perp\}\\ &= \{(-u,v)\in\mathbb{R}^n\times\mathbb{R}^m \,\big|\, S_{nm}^T(v,u)=(u,-v)\in L\} = -L = L. \qquad (13)\end{aligned}$$

Further, if we denote by P_{L^*}, $P_{L^\perp}$ and P_L the symmetric $(n+m)\times(n+m)$ matrices representing the orthogonal projections onto L^*, $L^\perp$ and L, respectively, then we have $P_{L^\perp}=I_{n+m}-P_L$ and, since S_{nm} is orthogonal,

$$P_{L^*}=S_{nm}P_{L^\perp}S_{nm}^T=I_{n+m}-S_{nm}P_LS_{nm}^T.$$

We conclude that for any two subspaces $L_1,L_2\in\mathbb{Z}_{nm}$ there holds

$$\begin{aligned}d_{\mathbb{Z}_{mn}}(L_1^*,L_2^*) &= \|I_{n+m}-S_{nm}P_{L_1}S_{nm}^T-(I_{n+m}-S_{nm}P_{L_2}S_{nm}^T)\|\\ &= \|P_{L_1}-P_{L_2}\| = d_{\mathbb{Z}_{nm}}(L_1,L_2)\end{aligned}$$

and thus the mapping $L\to L^*$ is an isometry between $\mathbb{Z}_{nm}$ and $\mathbb{Z}_{mn}$. In what follows, the symbol L^* signifies both the adjoint subspace to some $L\in\mathbb{Z}_{nm}$ as well as an arbitrary subspace from $\mathbb{Z}_{mn}$. This double role, however, cannot lead to a confusion. Consider now a mapping $F:\mathbb{R}^n\rightrightarrows\mathbb{R}^m$.

Definition 3.2. We say that F is *graphically smooth of dimension* n at $(\bar x,\bar y)$ if $T_{\operatorname{gph} F}(\bar x,\bar y)\in\mathbb{Z}_{nm}$. By $\mathcal{O}_F$ we denote the subset of $\operatorname{gph} F$, where F is graphically smooth of dimension n.

Clearly, for $(x,y)\in\mathcal{O}_F$ and $L=T_{\operatorname{gph} F}(x,y)=\operatorname{gph} DF(x,y)$ we have the relations $L^\perp=\hat N_{\operatorname{gph} F}(x,y)$ and $L^*=\operatorname{gph}\widehat D^*F(x,y)$.

As a next step we introduce the four derivative-like mappings $\widehat{\mathcal{S}}F:\mathbb{R}^n\times\mathbb{R}^m\rightrightarrows\mathbb{Z}_{nm}$, $\widehat{\mathcal{S}^*}F:\mathbb{R}^n\times\mathbb{R}^m\rightrightarrows\mathbb{Z}_{mn}$, $\mathcal{S}F:\mathbb{R}^n\times\mathbb{R}^m\rightrightarrows\mathbb{Z}_{nm}$ and $\mathcal{S}^*F:\mathbb{R}^n\times\mathbb{R}^m\rightrightarrows\mathbb{Z}_{mn}$ defined by

$$\widehat{\mathcal{S}}F(x,y):=\begin{cases}\operatorname{gph} DF(x,y) & \text{if } (x,y)\in\mathcal{O}_F\\ \emptyset & \text{otherwise,}\end{cases}$$

$$\widehat{\mathcal{S}^*}F(x,y):=\begin{cases}\operatorname{gph}\widehat D^*F(x,y) & \text{if } (x,y)\in\mathcal{O}_F\\ \emptyset & \text{otherwise,}\end{cases}$$

$$\mathcal{S}F(x,y):=\operatorname*{Lim\,sup}_{(u,v)\xrightarrow{\operatorname{gph} F}(x,y)}\widehat{\mathcal{S}}F(u,v)=\left\{L\in\mathbb{Z}_{nm}\,\middle|\,\begin{array}{l}\exists(x_k,y_k)\xrightarrow{\mathcal{O}_F}(x,y)\text{ such that}\\ \lim d_{\mathbb{Z}_{nm}}\big(L,\operatorname{gph} DF(x_k,y_k)\big)=0\end{array}\right\},$$

and

$$\mathcal{S}^*F(x,y):=\operatorname*{Lim\,sup}_{(u,v)\xrightarrow{\operatorname{gph} F}(x,y)}\widehat{\mathcal{S}^*}F(u,v)=\left\{L^*\in\mathbb{Z}_{mn}\,\middle|\,\begin{array}{l}\exists(x_k,y_k)\xrightarrow{\mathcal{O}_F}(x,y)\text{ such that}\\ \lim d_{\mathbb{Z}_{mn}}\big(L^*,\operatorname{gph}\widehat D^*F(x_k,y_k)\big)=0\end{array}\right\}.$$

Both $\mathcal{S}F$ and $\mathcal{S}^*F$ constitute generalized derivatives of F whose elements, by virtue of the above definitions, are subspaces of the graphs of the outer limiting graphical derivative and the limiting coderivative:

$$L\subset\operatorname{gph} D^\sharp F(x,y)\subset\operatorname{gph} D_*F(x,y)\ \forall L\in\mathcal{S}F(x,y), \qquad (14)$$

$$L^*\subset\operatorname{gph} D^*F(x,y)\ \forall L^*\in\mathcal{S}^*F(x,y). \qquad (15)$$

In what follows $\mathcal{S}F$ will be called *SC (subspace containing) limiting graphical derivative* and $\mathcal{S}^*F$ will be termed *SC limiting coderivative at* (x, y).

Due to the isometry $L \to L^*$ we obtain a useful relationship between $\mathcal{S}F(\bar{x}, \bar{y})$ and $\mathcal{S}^*F(\bar{x}, \bar{y})$. It holds, namely, that

$$\mathcal{S}^*F(\bar{x}, \bar{y}) = \{L^* \mid L \in \mathcal{S}F(\bar{x}, \bar{y})\} \quad \text{and} \quad \mathcal{S}F(\bar{x}, \bar{y}) = \{L \mid L^* \in \mathcal{S}^*F(\bar{x}, \bar{y})\}, \tag{16}$$

which enables us together with (12) a simple conversion of the statements in terms of $L \in \mathcal{S}F(\bar{x}, \bar{y})$ to statements in terms of $L^* \in \mathcal{S}^*F(\bar{x}, \bar{y})$ and vice versa.

On the basis of $\mathcal{S}^*F(\bar{x}, \bar{y})$ we may now introduce the following notion playing a crucial role in the sequel.

Definition 3.3. A mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is said to have the *SCD property* at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, provided $\mathcal{S}^*F(\bar{x}, \bar{y}) \neq \emptyset$. F is termed an *SCD mapping* if it has the SCD property at all points of $\operatorname{gph} F$.

By virtue of (16), the SCD property at $(\bar{x}, \bar{y})$ is obviously equivalent with the condition $\mathcal{S}F(\bar{x}, \bar{y}) \neq \emptyset$.

Since we consider convergence in the compact metric space $\mathbb{Z}_{nm}$, by using similar arguments as in the proof of [6, Lemma 3.6], we readily obtain the following result.

Lemma 3.4. *A mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ has the SCD property at $(x, y) \in \operatorname{gph} F$ if and only if $(x, y) \in \operatorname{cl} \mathcal{O}_F$. Further, F is an SCD mapping if and only if we have $\operatorname{cl} \mathcal{O}_F = \operatorname{cl} \operatorname{gph} F$, i.e., F is graphically smooth of dimension n at the points of a dense subset of its graph.*

The derivatives $\mathcal{S}F$ and $\mathcal{S}^*F$ can be considered as a generalization of the B-Jacobian to multifunctions. In case of single-valued continuous mappings one has the following relationship.

Lemma 3.5. *Let $U \subset \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}^m$ be continuous. Then for every $x \in U$ there holds*

$$\mathcal{S}f(x) := \mathcal{S}\big(x, f(x)\big) \supseteq \{\text{rge}\,(I, A) \mid A \in \overline{\nabla} f(x)\}, \tag{17}$$

$$\mathcal{S}^*f(x) := \mathcal{S}^*\big(x, f(x)\big) \supseteq \{\text{rge}\,(I, A^T) \mid A \in \overline{\nabla} f(x)\}. \tag{18}$$

If f is Lipschitz continuous near x, these inclusions hold with equality and f has the SCD property around x.

Proof. We can carry over the proof of [6, Lemma 3.11] with marginal modifications. □

3.2. SCD regularity

For $m = n$ we recall the following weakening of metric regularity tailored to SCD mappings.

Definition 3.6. ([6, Definition 4.1]) A mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is called *SCD regular* around $(\bar{x}, \bar{y})$, provided it has the SCD property on a neighborhood of $(\bar{x}, \bar{y})$ and for all $L^* \in \mathcal{S}^*F(\bar{x}, \bar{y})$ one has the implication

$$(v^*, 0) \in L^* \;\Rightarrow\; v^* = 0. \tag{19}$$

It is easy to see, cf. [6, Lemma 4.5], that implication (19) is equivalent with the requirement that

$$(u,0)\in L \;\Rightarrow\; u=0 \quad \text{for all} \quad L\in \mathcal{S}F(\bar{x},\bar{y}). \tag{20}$$

Further we observe that SCD regularity persists on a neighborhood of $(\bar{x},\bar{y})$, cf. [6, Proposition 4.8], and, taking into account (15) and the Mordukhovich criterion, condition (19) is implied by the (classical) metric regularity of F around $(\bar{x},\bar{y})$.

The main vehicle in our stability analysis of SCD mappings in Section 5 are the following statements taken over from [6, Theorem 6.2, Corollary 6.4].

Theorem 3.7. *Assume that $F:\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a SCD regular mapping around a point $(\bar{x},\bar{y})\in \operatorname{gph} F$. Then there is a neighborhood $\mathcal{U}$ of $(\bar{x},\bar{y})$ such that F is strongly metrically subregular at each point of $\operatorname{gph} F\cap\mathcal{U}$ at which F is semismooth*.*

Corollary 3.8. *Assume that $F:\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is semismooth* and has the SCD property around $(\bar{x},\bar{y})\in \operatorname{gph} F$. Then F is strongly metrically subregular around $(\bar{x},\bar{y})$ if and only if F is SCD regular around $(\bar{x},\bar{y})$.*

Conversely, thanks to Theorem 2.9(iii) and (14), strong metric subregularity around $(\bar{x},\bar{y})$ implies the SCD regularity at $(\bar{x},\bar{y})$ even in absence of the semismooth* property. Since by virtue of [3, Theorem 3H.3] F is strongly metrically subregular at $(\bar{x},\bar{y})$ if and only if F^{-1} is isolatedly calm at $(\bar{y},\bar{x})$, Corollary 3.8 thus provides us with a workable characterization of isolated calmness of inverses to SCD mappings having the semismooth* property.

Under the semismooth* and the SCD property, let us now compare Corollary 3.8 with the characterization of strong metric subregularity on a neighborhood provided by Theorem 2.9(iii). To this aim we write down relation (9) equivalently in the form

$$(u,0)\in \operatorname{gph} D^\sharp F(\bar{x},\bar{y}) \;\Rightarrow\; u=0. \tag{21}$$

By taking into account (14) and (20), we see that we need not to check (21) for the whole graph of $D^\sharp F(\bar{x},\bar{y})$, but only for the part which is given by the subspaces contained in $\mathcal{S}F(\bar{x},\bar{y})$. It seems that for the analysis of strong metric subregularity and isolated calmness on a neighborhood of semismooth* SCD mappings the outer limiting graphical derivative is much too large and contains useless parts. Moreover, it seems that the outer limiting graphical derivative is much harder to compute than the SC limiting graphical derivative.

Because of the mentioned relationship between the metric regularity and SCD regularity and Theorem 3.7 we arrive finally at the following corollary.

Corollary 3.9. *Assume that an SCD mapping $F:\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is metrically regular and semismooth* around $(\bar{x},\bar{y})$. Then F^{-1} not only has the Aubin property around $(\bar{y},\bar{x})$, but it is also isolatedly calm around $(\bar{y},\bar{x})$.*

4. Calculus

In this section we present some calculus rules for SCD mappings which can be useful in various situations.

Consider a mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ defined by

$$\operatorname{gph} F = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \,|\, \Phi(x, y) \in \operatorname{gph} Q\}, \tag{22}$$

where $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^l \times \mathbb{R}^m$ is a continuously differentiable function and $Q : \mathbb{R}^l \rightrightarrows \mathbb{R}^m$ is a closed-graph mapping.

Theorem 4.1. *Assume that $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, Q has the SCD property at $\Phi(\bar{x}, \bar{y})$ and the $(l + m) \times (n + m)$ matrix $\nabla\Phi(\bar{x}, \bar{y})$ has full row rank $l + m$. Then F has the SCD property at $(\bar{x}, \bar{y})$,*

$$\mathcal{S}F(\bar{x}, \bar{y}) = \{L \in \mathbb{Z}_{nm} \,|\, \nabla\Phi(\bar{x}, \bar{y})L \in \mathcal{S}Q\big(\Phi(\bar{x}, \bar{y})\big)\} \tag{23}$$

and

$$\mathcal{S}^*F(\bar{x}, \bar{y}) = \{L^* \in \mathbb{Z}_{mn} \,|\, L^* = S_{nm}\nabla\Phi(\bar{x}, \bar{y})^T S_{lm}^T M^* \text{ with } M^* \in \mathcal{S}^*Q\big(\Phi(\bar{x}, \bar{y})\big)\}, \tag{24}$$

where the matrices S_{nm} and S_{lm} are given by (12).

Proof. Since $\nabla\Phi(\bar{x}, \bar{y})$ is surjective, the mapping Φ is metrically regular around $\big((\bar{x}, \bar{y}), \Phi(\bar{x}, \bar{y})\big)$, cf. [15, Example 9.44]. Moreover, there is an open neighborhood $\mathcal{W}$ of $(\bar{x}, \bar{y})$ such that $\nabla\Phi(x, y)$ is surjective for all $(x, y) \in \mathcal{W}$ and $\widetilde{\mathcal{W}} = \Phi(\mathcal{W})$ is open. By virtue of [15, Exercise 6.7] it holds that

$$T_{\operatorname{gph} F}(x, y) = \{w \in \mathbb{R}^n \times \mathbb{R}^m \,|\, \nabla\Phi(x, y)w \in T_{\operatorname{gph} Q}\big(\Phi(x, y)\big)\} \tag{25}$$

for all $(x, y) \in \operatorname{gph} F \cap \mathcal{W}$. We now claim that

$$\mathcal{O}_Q \cap \widetilde{\mathcal{W}} = \{\Phi(x, y) \,|\, (x, y) \in \mathcal{O}_F \cap \mathcal{W}\}. \tag{26}$$

Indeed, consider $(x, y) \in \mathcal{O}_F \cap \mathcal{W}$ and take two tangents $q_1, q_2 \in T_{\operatorname{gph} Q}\big(\Phi(x, y)\big)$. Since $\nabla\Phi(x, y)$ is surjective, there exist w_i, $i = 1, 2$, with $\nabla\Phi(x, y)w_i = q_i$ implying $w_i \in T_{\operatorname{gph} F}(x, y)$ by (25). Since $T_{\operatorname{gph} F}(x, y)$ is a subspace, we have

$$\alpha_1 w_1 + \alpha_2 w_2 \in T_{\operatorname{gph} F}(x, y) \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}$$

and consequently $\nabla\Phi(x, y)(\alpha_1 w_1 + \alpha_2 w_2) = \alpha_1 q_1 + \alpha_2 q_2 \in T_{\operatorname{gph} Q}\big(\Phi(x, y)\big)$.

Hence $T_{\operatorname{gph} Q}\big(\Phi(x, y)\big)$ is a subspace. From $(x, y) \in \mathcal{O}_F$ we deduce that the dimension of the subspace $T_{\operatorname{gph} F}(x, y)$ is n. On the other hand, by (25) together with the surjectivity of $\nabla\Phi(x, y)$, the dimension of $T_{\operatorname{gph} F}(x, y)$ equals to the dimension of the subspace $T_{\operatorname{gph} Q}\big(\Phi(x, y)\big)$ plus $(n + m) - (k + m)$, the dimension of the nullspace of $\nabla\Phi(x, y)$.

Hence, the dimension of $T_{\operatorname{gph} Q}\big(\Phi(x, y)\big)$ is k and $\Phi(x, y) \in \mathcal{O}_Q \cap \widetilde{\mathcal{W}}$ is verified.

Next, consider $z \in \mathcal{O}_Q \cap \widetilde{\mathcal{W}}$. Then we can find $(x, y) \in \mathcal{W}$ such that $z = \Phi(x, y)$ and using similar arguments as above, we can show that $T_{\operatorname{gph} F}(x, y)$ is a subspace of dimension n implying $(x, y) \in (x, y) \in \mathcal{O}_F \cap \mathcal{W}$. Hence our claim (26) holds true.

Since Q has the SCD property at $\Phi(\bar{x}, \bar{y})$, we have $\mathcal{S}Q\big(\Phi(\bar{x}, \bar{y})\big) \neq \emptyset$.

Consider $M \in \mathcal{S}Q\big(\Phi(\bar{x}, \bar{y})\big)$ together with a sequence $z_k \xrightarrow{\mathcal{O}_Q} \Phi(\bar{x}, \bar{y})$ such that we have $M_k := T_{\operatorname{gph} Q}(z_k) \xrightarrow{\mathbb{Z}_{lm}} M$. For every k sufficiently large we can find $(x_k, y_k) \in \mathcal{W}$ with $z_k = \Phi(x_k, y_k)$ and, due to the metric regularity of Φ, $(x_k, y_k) \to (\bar{x}, \bar{y})$.

Further, $M_k^\perp$ converges in $\mathbb{Z}_{ml}$ to $M^\perp$. Let $L_k := T_{\operatorname{gph} F}(x_k, y_k) = \nabla\Phi(x_k, y_k)^{-1} M_k$, where $\nabla\Phi(x_k, y_k)^{-1}$ denotes the inverse of the linear mapping induced by $\nabla\Phi(x_k, y_k)$. By our claim (26) we have that $L_k \in \mathbb{Z}_{nm}$ and, since $L_k^\perp = \nabla\Phi(x_k, y_k)^T M_k^\perp$ by [14, Corollary 16.3.2], $L_k^* = S_{nm}\nabla\Phi(x_k, y_k)^T M_k^\perp$ converges to

$$L^* := S_{nm}\nabla\Phi(\bar{x}, \bar{y})^T M^\perp = S_{nm}\nabla\Phi(\bar{x}, \bar{y})^T S_{lm}^T M^*$$

by Lemma 3.1(i). On the other hand, since $\nabla\Phi(\bar{x}, \bar{y})^T M^\perp = \big(\nabla\Phi(\bar{x}, \bar{y})^{-1} M\big)^\perp$, we obtain $L = \nabla\Phi(\bar{x}, \bar{y})^{-1} M$. These arguments show the inclusion "$\supset$" in (23) and (24).

In order to show the reverse inclusion, consider $L \in \mathcal{S}F(\bar{x}, \bar{y})$ together with sequences

$$(x_k, y_k) \xrightarrow{\mathcal{O}_F} (\bar{x}, \bar{y}) \quad \text{and} \quad L_k := T_{\operatorname{gph} F}(x_k, y_k) \xrightarrow{\mathbb{Z}_{nm}} L.$$

Following (25) and (26) together with the surjectivity of $\nabla\Phi(x_k, y_k)$, we obtain that $M_k := \nabla\Phi(x_k, y_k) L_k = T_{\operatorname{gph} Q}\big(\Phi(x_k, y_k)\big) \in \mathbb{Z}_{lm}$. The metric space $\mathbb{Z}_{lm}$ is compact and thus, after possibly passing to a subsequence, we may assume that M_k converges in $\mathbb{Z}_{lm}$ to some $M \in \mathcal{S}Q\big(\Phi(\bar{x}, \bar{y})\big)$. Utilizing the same arguments as before, we obtain that the sequence

$$L_k^* = S_{nm}\nabla\Phi(x_k, y_k)^T S_{lm}^T M_k^*$$

converges to $L^* = S_{nm}\nabla\Phi(\bar{x}, \bar{y})^T S_{lm}^T M^*$ and $L = \nabla\Phi(\bar{x}, \bar{y})^{-1} M$. This completes the proof. □

As a first consequence of this theorem we derive that graphically Lipschitzian mappings have the SCD property.

Definition 4.2. (cf.[15, Definition 9.66]) A mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is *graphically Lipschitzian of dimension d* at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if there is an open neighborhood W of $(\bar{x}, \bar{y})$ and a one-to-one mapping Φ from W onto an open subset of $\mathbb{R}^{n+m}$ with Φ and Φ^{-1} continuously differentiable, such that $\Phi(\operatorname{gph} F \cap W)$ is the graph of a Lipschitz continuous mapping $f : U \to \mathbb{R}^{n+m-d}$, where U is an open set in $\mathbb{R}^d$.

Many mappings $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, important in applications, are graphically Lischitzian of dimension n. As an example we mention the subdifferential mapping of prox-regular and subdifferentially continuous functions $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, cf. [15, Proposition 13.46].

Corollary 4.3. *Assume that $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is graphically Lipschitzian of dimension n at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. Then F has the SCD property at $(\bar{x}, \bar{y})$.*

Proof. Let Φ, W, U and f be defined as in Definition 4.2 and observe that we have $\operatorname{gph} F \cap W = \{(x, y) \mid \Phi(x, y) \in \operatorname{gph} Q\}$, where

$$Q(u) := \begin{cases} \{f(u)\} & \text{if } u \in U, \\ \emptyset & \text{else.} \end{cases}$$

By Lemma 3.5, Q has the SCD property at $\big(\bar{u}, f(\bar{u})\big) := \Phi(\bar{x}, \bar{y})$ and the statement follows from Theorem 4.1. □

Let us now provide a calculus rule for the outer limiting tangent cone.

Proposition 4.4. *Let $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ be continuously differentiable, let $A \subset \mathbb{R}^m$ be a closed set and consider $C := \{x \in \mathbb{R}^n \mid \Phi(x) \in A\}$.*
Then for any $\bar{x} \in C$ there holds

$$T^\sharp_C(\bar{x}) \subset \{u \mid \nabla\Phi(\bar{x})u \in T^\sharp_A\big(\Phi(\bar{x})\big)\}. \tag{27}$$

If $\nabla\Phi(\bar{x})$ has full row rank m then this inclusion holds with equality.

Proof. By [15, Theorem 6.31], for any $x \in C$ there holds the inclusion

$$T_C(x) \subset \{u \mid \nabla\Phi(x)u \in T_A\big(\Phi(x)\big)\}. \tag{28}$$

Consider $u \in T^\sharp_C(\bar{x})$ together with sequences $(x_k, u_k) \overset{\text{gph}\, T_C}{\longrightarrow} (\bar{x}, u)$. Then

$$(\Phi(x_k), \nabla\Phi(x_k)u_k) \to (\Phi(\bar{x}), \nabla\Phi(\bar{x})u)$$

and $\nabla\Phi(x_k)u_k \in T_A\big(\Phi(x_k)\big)$ verifying $\nabla\Phi(\bar{x})u \in T^\sharp_A\big(\Phi(\bar{x})\big)$. This proves (27). Now assume that $\nabla\Phi(\bar{x})$ has full row rank. Then Φ is metrically regular with some constant κ around $\big(\bar{x}, \Phi(\bar{x})\big)$, see, e.g., [15, Example 9.44]. In addition, we can find a neighborhood U of $\bar{x}$ such that $\nabla\Phi(x)$ has full row rank for every $x \in U$ and we conclude from [15, Exercise 6.7] that inclusion (28) holds with equality for every $x \in U$.
Consider $v \in T^\sharp_A\big(\Phi(\bar{x})\big)$ together with sequences $(y_k, v_k) \overset{\text{gph}\, T_A}{\longrightarrow} (\Phi(\bar{x}), v)$.
By metric regularity of Φ, for every k sufficiently large we can find $x_k \in \Phi^{-1}(y_k)$ with $\|x_k - \bar{x}\| \le \kappa\|y_k - \Phi(\bar{x})\|$ so that $x_k \to \bar{x}$ and $x_k \in U$. Consider $u \in \mathbb{R}^n$ with $\nabla\Phi(\bar{x})u = v$. For the pseudo-inverse $\nabla\Phi(\bar{x})^\dagger := \nabla\Phi(\bar{x})^T\big(\nabla\Phi(\bar{x})\nabla\Phi(\bar{x})^T\big)^{-1}$ there holds

$$u = \nabla\Phi(\bar{x})^\dagger v + \big(I - \nabla\Phi(\bar{x})^\dagger\nabla\Phi(\bar{x})\big)u.$$

Since the pseudo-inverses $\nabla\Phi(x_k)^\dagger$ converge to $\nabla\Phi(\bar{x})^\dagger$, we conclude that the sequence

$$u_k := \nabla\Phi(x_k)^\dagger v_k + \big(I - \nabla\Phi(x_k)^\dagger\nabla\Phi(x_k)\big)u$$

converges to u. Further, since $\nabla\Phi(x_k)u_k = v_k \in T_A\big(\Phi(x_k)\big)$, we have $u_k \in T_C(x_k)$ and $u \in T^\sharp_C(\bar{x})$ follows. This justifies the inclusion $T^\sharp_C(\bar{x}) \supset \{u \mid \nabla\Phi(\bar{x})u \in T^\sharp_A\big(\Phi(\bar{x})\big)\}$ and the proof of the proposition is complete. □

The next calculus rule is essential for the main stability result presented in Section 5. Let us consider the situation when $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^l \times \mathbb{R}^k$ is given via

$$F(x) := \begin{pmatrix} G(x) \\ H(x) \end{pmatrix} \tag{29}$$

where $G : \mathbb{R}^n \to \mathbb{R}^l$ is a C^1 function and $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^k$ has a closed graph.

Proposition 4.5. *Consider $(\bar{x}, \bar{z}) \in \operatorname{gph} H$. Then for the mapping F given by (29) one has:*

(i) $T^\sharp_{\operatorname{gph} F}\big(\bar{x}, (G(\bar{x}), \bar{z})\big) = \big\{\big(u, (\nabla G(\bar{x})u, w)\big) \bigm| (u, w) \in T^\sharp_{\operatorname{gph} H}(\bar{x}, \bar{z})\big\}. \quad (30)$

(ii) *If H is semismooth* at $(\bar{x}, \bar{z})$, then F is semismooth* at $\big(\bar{x}, (G(\bar{x}), \bar{z})\big)$.*

(iii) *Assume that* H *has the SCD property at* $(\bar x,\bar z)$. *Then* F *has the SCD property at* $\big(\bar x,(G(\bar x),\bar z)\big)$ *and one has that*

$$\mathcal{S}F\big(\bar x,(G(\bar x),\bar z)\big)=\Big\{\{\big(u,(\nabla G(\bar x)u,w)\big)\,|\,(u,w)\in M\}\,\Big|\,M\in\mathcal{S}H(\bar x,\bar z)\Big\}, \tag{31}$$

$$\mathcal{S}^*F\big(\bar x,(G(\bar x),\bar z)\big)= \tag{32}$$
$$=\Big\{\{\big((q^*,w^*),\nabla G(\bar x)^Tq^*+u^*\big)\,|\,q^*\in\mathbb{R}^l,\ (w^*,u^*)\in M^*\}\,\Big|\,M^*\in\mathcal{S}^*H(\bar x,\bar z)\Big\}.$$

Proof. Let $\tilde H:\mathbb{R}^n\rightrightarrows\mathbb{R}^l\times\mathbb{R}^k$ be given by $\tilde H(x)=\{0\}\times H(x)$. Then

$$\operatorname{gph}F=\{(x,p,z)\,|\,\Phi(x,p,z)\in\operatorname{gph}\tilde H\},$$

where $\Phi:\mathbb{R}^n\times\mathbb{R}^l\times\mathbb{R}^k\to\mathbb{R}^n\times\mathbb{R}^l\times\mathbb{R}^k$ is given by $\Phi(x,p,z)=(x,p-G(x),z)^T$. Note that for every triple (x,p,z) the following Jacobian is nonsingular:

$$\nabla\Phi(x,p,z)=\begin{pmatrix} I_n & 0 & 0\\ -\nabla G(x) & I_l & 0\\ 0 & 0 & I_k\end{pmatrix}.$$

Ad (i): Obviously we have $T^\sharp_{\operatorname{gph}\tilde H}\big(\bar x,(0,\bar z)\big)=\{\big(u,(0,w)\big)\,|\,(u,w)\in T^\sharp_{\operatorname{gph}H}(\bar x,\bar z)\}$. Thus we obtain from Proposition 4.4, yielding (30):

$$\begin{aligned}T^\sharp_{\operatorname{gph}F}\big(\bar x,(G(\bar x),\bar z)\big)&=\Big\{\big(u,(q,w)\big)\,\Big|\,\nabla\Phi(\bar x,G(\bar x),\bar z)(u,q,w)\in T^\sharp_{\operatorname{gph}\tilde H}(\bar x,0,\bar z)\Big\}\\ &=\Big\{\big(u,(q,w)\big)\,|\,(u,w)\in T^\sharp_{\operatorname{gph}H}(\bar x,\bar z),\ q-\nabla G(\bar x)u=0\}.\end{aligned}$$

Ad (ii): Since H is semismooth* at $(\bar x,\bar z)$, $\tilde H$ is semismooth* at $\big(\bar x,(0,\bar z)\big)$. Surjectivity of $\nabla\Phi(\bar x,G(\bar x),\bar z)$ ensures that the mapping $\Phi(\cdot)-\operatorname{gph}\tilde H$ is metrically regular around $\big(\bar x,(G(\bar x),\bar z)\big)$, cf. [15, Example 9.44] and therefore metrically subregular as well. Now the claimed statement follows from Proposition 2.12.

Ad (iii): It is easy to see that $\tilde H$ has the SCD property at $\big(\bar x,(0,\bar z)\big)$ with

$$\mathcal{S}\tilde H\big(\bar x,(0,\bar z)\big)=\Big\{\{\big(u,(0,w)\big)\,|\,(u,w)\in M\}\,\Big|\,M\in\mathcal{S}H(\bar x,\bar z)\Big\},$$
$$\mathcal{S}^*\tilde H\big(\bar x,(0,\bar z)\big)=\Big\{\{\big((q^*,w^*),u^*\big)\,|\,q^*\in\mathbb{R}^l,\ (w^*,u^*)\in M^*\}\,\Big|\,M^*\in\mathcal{S}^*H(\bar x,\bar z)\Big\}.$$

Next we can apply Theorem 4.1 to obtain

$$\mathcal{S}F\big(\bar x,(G(\bar x),\bar z)\big)=\{\nabla\Phi(\bar x,G(\bar x),\bar z)^{-1}\tilde M\,|\,\tilde M\in\mathcal{S}\tilde H\big(\bar x,(0,\bar z)\big)\},$$
$$\mathcal{S}^*F\big(\bar x,(G(\bar x),\bar z)\big)=\{S_{n(l+k)}\nabla\Phi(\bar x,G(\bar x),\bar z)^TS^T_{n(l+k)}\tilde M^*\,|\,\tilde M^*\in\mathcal{S}^*\tilde H\big(\bar x,(0,\bar z)\big)\}.$$

Straightforward calculations yield that

$$\nabla\Phi(\bar x,G(\bar x),\bar z)^{-1}=\begin{pmatrix} I_n & 0 & 0\\ \nabla G(\bar x) & I_l & 0\\ 0 & 0 & I_k\end{pmatrix},$$

$$S_{n(l+k)}\nabla\Phi(\bar x,G(\bar x),\bar z)^TS^T_{n(l+k)}=\begin{pmatrix} I_l & 0 & 0\\ 0 & I_k & 0\\ \nabla G(\bar x)^T & 0 & I_n\end{pmatrix}$$

and the formulas (31), (32) follow. □

5. Isolated calmness on a neighborhood of implicit multifunctions

We consider now a multifunction $H : \mathbb{R}^l \times \mathbb{R}^k \rightrightarrows \mathbb{R}^k$ with closed graph and a point $\big((\bar{x}, \bar{y}), \bar{z}\big) \in \operatorname{gph} H$. Then the relation

$$\operatorname{gph} \Sigma = H^{-1}(\bar{z}) \tag{33}$$

defines the so-called *implicit multifunction* $\Sigma : \mathbb{R}^l \rightrightarrows \mathbb{R}^k$. Our aim is now to ensure a certain stability property of Σ around $(\bar{x}, \bar{y})$ by imposing suitable assumptions on H around $(\bar{x}, \bar{y}, \bar{z})$. Usually one puts $\bar{z} = 0$ so that

$$\operatorname{gph} \Sigma = \{(x, y) \,|\, 0 \in H(x, y)\}. \tag{34}$$

It is easy to see that any stability property of Σ around $(\bar{x}, \bar{y})$ is inherited by the same stability property of the inverse to the "extended" mapping $F : \mathbb{R}^{l+k} \rightrightarrows \mathbb{R}^{l+k}$ given by

$$F(x, y) = \begin{pmatrix} x \\ H(x, y) \end{pmatrix} \tag{35}$$

around $\big((\bar{x}, \bar{y}), (\bar{x}, 0)\big)$. In fact, in this way, e.g., the classical Implicit Function Theorem or the Clarke Implicit Function Theorem have been proved. Alternatively, one can combine a suitable characterization of the examined property in terms of a generalized derivative with the available calculus, as shown, e.g., in [10, Section 4.3] or [4, Section 4] in case of the Aubin property. In our approach we will use the mapping (35) along with Theorem 2.9(iii) and Corollary 3.8.

Theorem 5.1. *Consider the inclusion $0 \in H(x, y)$ and a point $\big((\bar{x}, \bar{y}), 0\big) \in \operatorname{gph} H$. Then any of the following two conditions ensures the isolated calmness property of the respective implicit solution map Σ around $(\bar{x}, \bar{y})$.*

(i)
$$0 \in D^\sharp H\big((\bar{x}, \bar{y}), 0\big)(0, v) \;\Rightarrow\; v = 0. \tag{36}$$

(ii) *The mapping H has both the SCD property and the semismooth* property around $\big((\bar{x}, \bar{y}), 0\big)$ and either the implication*

$$\big((0, v), 0\big) \in L \;\Rightarrow\; v = 0 \tag{37}$$

holds for all $L \in \mathcal{S}H\big((\bar{x}, \bar{y}), 0\big)$, or, equivalently, the implication

$$\big(w^*, (u^*, 0)\big) \in L^* \;\Rightarrow\; w^* = 0,\; u^* = 0 \tag{38}$$

holds for all $L^* \in \mathcal{S}^* H\big((\bar{x}, \bar{y}), 0\big)$.

Proof. In the first case we conclude from Proposition 4.5(i) that the mapping F given by (35) fulfills

$$D^\sharp F\big((\bar{x}, \bar{y}), (\bar{x}, 0)\big)(u, v) = \{(u, w) \,|\, w \in D^\sharp H\big((\bar{x}, \bar{y}), 0\big)(u, v)\}.$$

Thus it follows from Theorem 2.9(iii) that condition (36) is equivalent with strong metric subregularity of F around $\big((\bar{x}, \bar{y}), (\bar{x}, 0)\big)$ and the claimed isolated calmness of Σ around $(\bar{x}, \bar{y})$ follows.

In the second case, note that by Proposition 4.5(ii), (iii) the mapping F has the SCD property around $\big((\bar{x}, \bar{y}), (\bar{x}, 0)\big)$ and is semismooth* around $\big((\bar{x}, \bar{y}), (\bar{x}, 0)\big)$.

Further we have

$$\mathcal{S}F\big((\bar{x},\bar{y}),(\bar{x},0)\big) = \Big\{\{\big((u,v),(u,w)\big) \,|\, \big((u,v),w\big) \in L\} \,\Big|\, L \in \mathcal{S}H\big((\bar{x},\bar{y}),0\big)\Big\},$$

$$\mathcal{S}^*F\big((\bar{x},\bar{y}),(\bar{x},0)\big) =$$

$$= \Big\{\{\big((q^*,w^*),(q^*+u^*,v^*)\big) \,|\, q^* \in \mathbb{R}^l,\ \big(w^*,(u^*,v^*)\big) \in L^*\} \,\Big|\, L^* \in \mathcal{S}^*H\big((\bar{x},\bar{y}),0\big)\Big\}.$$

Implications (20) and (19) now yield conditions (37), (38) which, by Corollary 3.8, are equivalent with the strong metric subregularity of F around $\big((\bar{x},\bar{y}),(\bar{x},0)\big)$. The proof is complete. □

Theorem 5.1 can well be applied to parameterized GEs. To this aim consider the case when $H : \mathbb{R}^l \times \mathbb{R}^k \rightrightarrows \mathbb{R}^k$ is given via

$$H(x,y) := f(x,y) + Q(x,y), \tag{39}$$

where $x \in \mathbb{R}^l$ is the *perturbation parameter*, $y \in \mathbb{R}^k$ is the *decision variable*, $f : \mathbb{R}^l \times \mathbb{R}^k \to \mathbb{R}^k$ is continuously differentiable and $Q : \mathbb{R}^l \times \mathbb{R}^k \rightrightarrows \mathbb{R}^k$ has a closed graph.

Proposition 5.2. *Consider the reference point $(\bar{x},\bar{y}) \in H^{-1}(0)$ and assume that one of the following conditions hold true:*

(i) $$0 \in \nabla_y f(x,y)v + D^\sharp Q\big((\bar{x},\bar{y}),-f(\bar{x},\bar{y})\big)(0,v) \ \Rightarrow\ v = 0. \tag{40}$$

(ii) *Q has the SCD property around $\big((\bar{x},\bar{y}),-f(\bar{x},\bar{y})\big)$ and is semismooth* on a neighborhood of $\big((\bar{x},\bar{y}),-f(\bar{x},\bar{y})\big)$ and either one of the implications holds true:*

$$\big((0,v),-\nabla_y f(\bar{x},\bar{y})v\big) \in M \ \Rightarrow\ v = 0 \ \textit{ for all } \ M \in \mathcal{S}Q\big((\bar{x},\bar{y}),-f(\bar{x},\bar{y})\big) \tag{41}$$

and, for all $M^* \in \mathcal{S}^*Q\big((\bar{x},\bar{y}),-f(\bar{x},\bar{y})\big)$

$$\big(w^*,(u^*,-\nabla_y f(\bar{x},\bar{y})^T w^*)\big) \in M^* \ \Rightarrow\ w^* = 0,\ u^* = 0. \tag{42}$$

Then the respective solution mapping $\Sigma : \mathbb{R}^l \rightrightarrows \mathbb{R}^k$ is isolatedly calm around $(\bar{x},\bar{y})$.

Proof. Clearly we have $\operatorname{gph} H = \{\big((x,y),z\big) \in \mathbb{R}^l \times \mathbb{R}^k \times \mathbb{R}^k \,|\, \Phi\big((x,y),z\big) \in \operatorname{gph} Q\}$ with $\Phi\big((x,y),z\big) = \big((x,y),z - f(x,y)\big)^T$ so that we can apply Proposition 4.4 and Theorem 4.1 to obtain

$$T^\sharp_{\operatorname{gph} H}\big((\bar{x},\bar{y}),0\big) = \nabla\Phi\big((\bar{x},\bar{y}),0\big)^{-1} T^\sharp_{\operatorname{gph} Q}\big(\Phi\big((\bar{x},\bar{y}),0\big)\big),$$

$$\mathcal{S}H\big((\bar{x},\bar{y}),0\big) = \nabla\Phi\big((\bar{x},\bar{y}),0\big)^{-1} \mathcal{S}Q\big(\Phi\big((\bar{x},\bar{y}),0\big)\big),$$

$$\mathcal{S}^*H\big((\bar{x},\bar{y}),0\big) = \left\{ L^* \in \mathbb{Z}_{k(l+k)} \,\middle|\, \begin{array}{l} L^* = S_{(l+k)k}\nabla\Phi\big((\bar{x},\bar{y}),0\big)^T S^T_{(l+k)k} M^* \\ \text{with } \ M^* \in \mathcal{S}^*Q\big(\Phi\big((\bar{x},\bar{y}),0\big)\big) \end{array} \right\}.$$

Straightforward calculations yield

$$\nabla\Phi\big((\bar{x},\bar{y}),0\big)^{-1} = \begin{pmatrix} I_{l+k} & 0 \\ \nabla f(\bar{x},\bar{y}) & I_k \end{pmatrix},$$

and
$$S_{(l+k)k}\nabla\Phi\big((\bar{x},\bar{y}),0\big)^T S_{(l+k)k}^T = \begin{pmatrix} I_k & 0 \\ \nabla f(\bar{x},\bar{y})^T & I_{l+k} \end{pmatrix}$$

and we arrive at the formulas

$$\begin{aligned} &D^\sharp H\big((\bar{x},\bar{y}),0\big)(u,v) = \\ &\qquad = \{\nabla_x f(\bar{x},\bar{y})u + \nabla_y f(\bar{x},\bar{y})v + w \,\big|\, w \in D^\sharp Q\big((\bar{x},\bar{y}),-f(\bar{x},\bar{y})\big)(u,v)\}, \end{aligned}$$

$$\begin{aligned} \mathcal{S}H\big((\bar{x},\bar{y}),0\big) = \Big\{&\{\big((u,v),\nabla_x f(\bar{x},\bar{y})u + \nabla_y f(\bar{x},\bar{y})v + w\big) \\ &\quad \mid \big((u,v),w\big) \in M\} \,\Big|\, M \in \mathcal{S}Q\big((\bar{x},\bar{y}),-f(\bar{x},\bar{y})\big)\Big\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{S}^*H\big((\bar{x},\bar{y}),0\big) = \Big\{&\big\{\big(w^*,(\nabla_x f(\bar{x},\bar{y})^T w^* + u^*, \nabla_y f(\bar{x},\bar{y})^T w^* + v^*)\big) \\ &\quad \mid (w^*,(u^*,v^*)) \in M^*\big\} \,\Big|\, M^* \in \mathcal{S}^*Q\big((\bar{x},\bar{y}),-f(\bar{x},\bar{y})\big)\Big\}. \end{aligned}$$

Conditions (36), (37), (38) thus read

$$\left.\begin{array}{l} u = 0,\ \nabla_x f(\bar{x},\bar{y})u + \nabla_y f(\bar{x},\bar{y})v + w = 0 \\ w \in D^\sharp Q\big((\bar{x},\bar{y}),-f(\bar{x},\bar{y})\big)(u,v) \end{array}\right\} \Rightarrow v = 0,$$

$$\left.\begin{array}{l} u = 0,\ \nabla_x f(\bar{x},\bar{y})u + \nabla_y f(\bar{x},\bar{y})v + w = 0 \\ \big((u,v),w\big) \in M \end{array}\right\} \Rightarrow v = 0$$

for all $M \in \mathcal{S}Q\big((\bar{x},\bar{y}),-f(\bar{x},\bar{y})\big)$,

$$\left.\begin{array}{l} \nabla_y f(\bar{x},\bar{y})^T w^* + v^* = 0 \\ \big(w^*,(u^*,v^*)\big) \in M^* \end{array}\right\} \Rightarrow w^* = 0, \nabla_x f(\bar{x},\bar{y})^T w^* + u^* = 0$$

for all $M^* \in \mathcal{S}^*Q\big((\bar{x},\bar{y}),-f(\bar{x},\bar{y})\big)$, which are equivalent to (40), (41) and (42), respectively. This completes the proof. □

Recall that for SCD mappings Q having the semismooth* property any of the three conditions (40), (41) and (42) is equivalent to the strong metric subregularity on a neighborhood of the mapping F given by (35) and thus the conditions (40), (41) and (42) are equivalent. Whereas (42) is a dual formulation of (41), conditions (40) and (41) might look quite different. Let us shed some light on this issue by the following application of Proposition 5.2 to parameterized variational inequalities with polyhedral constraint sets.

Consider the GE
$$0 \in H(x,y) := f(x,y) + N_D\big(g(x,y)\big), \tag{43}$$
where $f,g : \mathbb{R}^l \times \mathbb{R}^k \to \mathbb{R}^k$ are continuously differentiable and $D \subset \mathbb{R}^k$ is a convex polyhedral set, and let $0 \in H(\bar{x},\bar{y})$. In what follows we denote by

$$\mathcal{K}_D(d,d^*) := T_D(d) \cap [d^*]^\perp,\ (d,d^*) \in \operatorname{gph} N_D$$

the critical cone to D at d for d^*.

Our further development makes use of the following statements.

Proposition 5.3. *Let $D \subset \mathbb{R}^k$ be a convex polyhedral set. Then the normal cone mapping $N_D(\cdot)$ is an SCD mapping, which is semismooth* at every point of its graph. Further, for every point $(d,d^*) \in \operatorname{gph} N_D$ there holds*

$$\mathcal{S}N_D(d,d^*) = \mathcal{S}^*N_D(d,d^*) = \{(\mathbb{F}-\mathbb{F}) \times (\mathbb{F}-\mathbb{F})^\perp \,\big|\, \mathbb{F} \text{ is face of } \mathcal{K}_D(d,d^*)\}, \quad (44)$$

and the outer limiting tangent cone $T^\sharp_{\operatorname{gph} N_D}(d,d^*)$ is the union of all sets $\operatorname{gph} N_{\mathbb{F}_1-\mathbb{F}_2}$, where $\mathbb{F}_1, \mathbb{F}_2$ are closed faces of $\mathcal{K}_D(d,d^*)$ with $\mathbb{F}_2 \subset \mathbb{F}_1$.

Proof. Since the normal cone mapping N_D is the subdifferential mapping of the convex lsc function δ_D, it is an SCD mapping by [6, Corollary 3.28]. Further, since $\operatorname{gph} N_D$ is the union of finitely many convex polyhedral sets, N_D is semismooth* at every point of its graph by Proposition 2.11. Formula (44) can be found in [6, Example 3.29] and there remains to show the representation of $T^\sharp_{\operatorname{gph} N_D}(d,d^*)$.

For any $(d',d'^*) \in \operatorname{gph} N_D$ we have $T_{\operatorname{gph} N_D} = \operatorname{gph} N_{\mathcal{K}_D(d',d'^*)}$, cf. [3, Lemma 2E4]. Further, by the Critical Superface Lemma [3, Lemma 4H.2], for every sufficiently small neighborhood W of (d,d^*) the collection of all critical cones $\mathcal{K}_D(d',d'^*)$, $(d',d'^*) \in \operatorname{gph} N_D \cap W$ coincides with the collection of the so-called critical superfaces $\mathbb{F}_1 - \mathbb{F}_2$, where $\mathbb{F}_1, \mathbb{F}_2$ are faces of the critical cone $\mathcal{K}_D(d,d^*)$ with $\mathbb{F}_2 \subset \mathbb{F}_1$.
Now consider a quadruple $\big((d,d^*),(e,e^*)\big)$ satisfying $(e,e^*) \in T^\sharp_{\operatorname{gph} N_D}(d,d^*)$ together with sequences $\big((d_k,d_k^*),(e_k,e_k^*)\big) \to \big((d,d^*),(e,e^*)\big)$ with $(e_k,e_k^*) \in T_{\operatorname{gph} N_D}(d_k,d_k^*)$. Since the convex polyhedral set D has only finitely many faces, after possibly passing to a subsequence we can assume that there are two faces $\mathbb{F}_1, \mathbb{F}_2$ of $\mathcal{K}_D(d,d^*)$ with $\mathbb{F}_2 \subset \mathbb{F}_1$ such that $\mathcal{K}_D(d_k,d_k^*) = \mathbb{F}_1 - \mathbb{F}_2$ for all k. Thus,

$$(e_k,e_k^*) \in T_{\operatorname{gph} N_D}(d_k,d_k^*) = \operatorname{gph} N_{\mathcal{K}_D(d_k,d_k^*)} = \operatorname{gph} N_{\mathbb{F}_1-\mathbb{F}_2}$$

for all k and $(e,e^*) \in \operatorname{gph} N_{\mathbb{F}_1-\mathbb{F}_2}$ follows. Conversely, let $\mathbb{F}_2 \subset \mathbb{F}_1$ be two faces of $\mathcal{K}_D(d,d^*)$ and let $(e,e^*) \in \operatorname{gph} N_{\mathbb{F}_1-\mathbb{F}_2}$. Then there exists some sequence

$$(d_k,d_k^*) \overset{\operatorname{gph} N_D}{\longrightarrow} (d,d^*) \ \text{ with } \ \mathcal{K}_D(d_k,d_k^*) = \mathbb{F}_1 - \mathbb{F}_2 \ \forall k$$

so that $(e,e^*) \in \operatorname{gph} N_{\mathbb{F}_1-\mathbb{F}_2} = T_{\operatorname{gph} N_D}(d_k,d_k^*) \ \forall k$ implying $(e,e^*) \in T^\sharp_{\operatorname{gph} N_D}(d,d^*)$. The statement has been established. □

Proposition 5.4. *In the setting of (43), assume that $g(\bar x,\bar y) \in D$ and the Jacobian $\nabla g(\bar x,\bar y)$ has full row rank k. Then the mapping $Q(x,y) : \mathbb{R}^l \times \mathbb{R}^k \rightrightarrows \mathbb{R}^k$ given by $Q(x,y) = N_D\big(g(x,y)\big)$ has the SCD property around $\big((\bar x,\bar y),d^*\big)$ and is semismooth* around $\big((\bar x,\bar y),d^*\big)$ for every $d^* \in N_D\big(g(\bar x,\bar y)\big)$. Further one has*

$$\mathcal{S}Q\big((\bar x,\bar y),d^*\big) = \left\{\{\big((u,v),e^*\big) \,\big|\, \begin{pmatrix}\nabla g(\bar x,\bar y)(u,v)\\ e^*\end{pmatrix} \in (\mathbb{F}-\mathbb{F}) \times (\mathbb{F}-\mathbb{F})^\perp\} \,\Big|\, \mathbb{F} \text{ is face of } \mathcal{K}_D(g(\bar x,\bar y),d^*)\right\}$$

and

$$T^{\sharp}_{\operatorname{gph} Q}\big((\bar x,\bar y),d^*\big) = \left\{ \big((u,v),e^*\big) \;\middle|\; \begin{matrix} \begin{pmatrix}\nabla g(\bar x,\bar y)(u,v)\\ e^*\end{pmatrix} \in \operatorname{gph} N_{\mathbb{F}_1-\mathbb{F}_2} \text{ for faces} \\ \mathbb{F}_1,\mathbb{F}_2 \text{ of } \mathcal{K}_D(g(\bar x,\bar y),d^*) \text{ with } \mathbb{F}_2\subset\mathbb{F}_1 \end{matrix} \right\}. \tag{45}$$

Proof. Obviously

$$\operatorname{gph} Q = \{\big((x,y),d^*\big) \mid \Phi(x,y,d^*) := \begin{pmatrix} g(x,y)\\ d^*\end{pmatrix} \in \operatorname{gph} N_D\}.$$

The full-rank assumption imposed on $\nabla g(\bar x,\bar y)$ ensures that $\nabla g(x,y)$ has full row rank for all (x,y) belonging to some neighborhood U of $(\bar x,\bar y)$ and it follows that $\nabla\Phi(x,y,d^*)$ has full row rank for all $(x,y,d^*)\in U\times\mathbb{R}^k$. Hence, by Theorem 4.1, for any $(x,y,d^*)\in \operatorname{gph} Q\cap U\times\mathbb{R}^k$ the mapping Q has the SCD property at $\big((x,y),d^*\big)$ and, together with (44),

$$\mathcal{S}Q\big((x,y),d^*\big) = \Big\{\{\big((u,v),e^*\big) \mid \begin{pmatrix}\nabla g(x,y)(u,v)\\ e^*\end{pmatrix} \in (\mathbb{F}-\mathbb{F})\times(\mathbb{F}-\mathbb{F})^{\perp}\} \;\Big|\; \mathbb{F} \text{ is face of } \mathcal{K}_D(g(x,y),d^*)\Big\}.$$

Further, the mapping $(x,y,d^*)\rightrightarrows \Phi(x,y,d^*)-\operatorname{gph} N_D$ is metrically regular around the point $\big((x,y,d^*),(0,0)\big)$ by [15, Example 9.44] and consequently also metrically subregular. This allows us to invoke Proposition 2.12 in order to guarantee the semismooth* property of Q at $\big((x,y),d^*\big)$. Finally, formula (45) follows from Proposition 4.4 and Lemma 5.3. □

Taking into account our above considerations about the equivalence of (40), (41) and the strong metric subregularity of F on a neighborhood for semismooth* SCD mappings Q, we arrive at the following result.

Proposition 5.5. *In the setting of* (43)*, assume that* $\big((\bar x,\bar y),0\big)\in\operatorname{gph} H$ *and that the Jacobian* $\nabla g(\bar x,\bar y)$ *has full row rank* k*. Then the following statements are equivalent.*

(i) *The mapping* $F:\mathbb{R}^l\times\mathbb{R}^k\rightrightarrows\mathbb{R}^l\times\mathbb{R}^k$ *given by*

$$F(x,y) = \begin{pmatrix} x\\ f(x,y)+N_D\big(g(x,y)\big)\end{pmatrix}$$

is strongly metrically subregular around $\big((\bar x,\bar y),(\bar x,0)\big)$.

(ii) *The implication*

$$\left.\begin{aligned} \nabla_y g(\bar x,\bar y)v &\in \mathbb{F}-\mathbb{F}\\ -\nabla_y f(\bar x,\bar y)v &\in (\mathbb{F}-\mathbb{F})^{\perp}\end{aligned}\right\} \;\Rightarrow\; v=0 \tag{46}$$

holds for every face $\mathbb{F}$ *of the critical cone* $\mathcal{K}_D\big(g(\bar x,\bar y),-f(\bar x,\bar y)\big)$.

(iii) The implication

$$\left.\begin{array}{l}\nabla_y g(\bar{x},\bar{y})v \in \mathbb{F}_1 - \mathbb{F}_2\\ -\nabla_y f(\bar{x},\bar{y})v \in (\mathbb{F}_1 - \mathbb{F}_2)^\circ\\ \langle \nabla_y g(\bar{x},\bar{y})v, -\nabla_y f(\bar{x},\bar{y})v\rangle = 0\end{array}\right\} \Rightarrow\ v = 0 \tag{47}$$

holds for every pair $\mathbb{F}_1, \mathbb{F}_2$ of faces of the critical cone $\mathcal{K}_D\big(g(\bar{x},\bar{y}), -f(\bar{x},\bar{y})\big)$ with $\mathbb{F}_2 \subset \mathbb{F}_1$.

This result is quite surprising since the implications in (ii) are only a proper subset of those in (iii) with $\mathbb{F}_1 = \mathbb{F}_2$ and it is by no means evident why the remaining implications in (iii) with $\mathbb{F}_2 \neq \mathbb{F}_1$ are superfluous. These considerations demonstrate that for testing the strong metric subregularity on a neighborhood of semismooth* SCD mappings the approach via (46) may require less effort in comparison with (47) based on the outer limiting graphical derivative.

Concerning the isolated calmness of the solution map Σ related to (43), we arrive at the following result.

Proposition 5.6. *In the setting of* (43)*, let $0 \in H(\bar{x},\bar{y})$. If the implication* (46) *holds for every face $\mathbb{F}$ of the critical cone $\mathcal{K}_D\big(g(\bar{x},\bar{y}), -f(\bar{x},\bar{y})\big)$, then the respective solution mapping $\Sigma : \mathbb{R}^l \rightrightarrows \mathbb{R}^k$ is isolatedly calm around $(\bar{x},\bar{y})$.*

Proof. If $\nabla g(\bar{x},\bar{y})$ has full row rank, the assertion follows from Proposition 5.5.

If the Jacobian $\nabla g(\bar{x},\bar{y})$ does not possess full row rank, we simply consider the generalized equation

$$0 \in \tilde{H}\big((x,p),y\big) = \tilde{f}\big((x,p),y\big) + N_D\big(\tilde{g}\big((x,p),y\big)\big)$$

where $\tilde{f}, \tilde{g} : \mathbb{R}^l \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ are given by

$$\tilde{f}\big((x,p),y\big) = f(x,y),\ \ \tilde{g}\big((x,p),y\big) = g(x,y) - p.$$

Since the Jacobian $\nabla \tilde{g}\big((x,p),y\big)$ has full row rank and $\nabla_y \tilde{g}\big((x,p),y\big) = \nabla_y g(x,y)$, $\nabla_y \tilde{f}\big((x,p),y\big) = \nabla_y f(x,y)$ for all (x,p,y), we can conclude that the respective solution mapping $\tilde{\Sigma} : \mathbb{R}^l \times \mathbb{R}^k \to \mathbb{R}^l$ is isolatedly calm around $\big((\bar{x},0),\bar{y}\big)$ and, together with the observation that $\Sigma(x) = \tilde{\Sigma}(x,0)\ \forall x \in \mathbb{R}^l$, the isolated calmness of Σ around $(\bar{x},\bar{y})$ follows. □

Let us illustrate the above conditions via a simple academic example.

Example 5.7. Let $l = k = 1$ and consider the parameterized GE (43), where $f(x,y) = -y$, $g(x,y) = y - x$ and $D = \mathbb{R}_+$. With $(\bar{x},\bar{y}) = (0,0)$ we observe that all the assumptions of Proposition 5.6 are fulfilled,

$$\mathcal{K}_D\big(g(\bar{x},\bar{y}), -f(\bar{x},\bar{y})\big) = T_D\big(g(\bar{x},\bar{y})\big) = \mathbb{R}_+,$$

and one has to consider the faces $\mathbb{F}_1 = \mathbb{R}_+$, $\mathbb{F}_2 = \{0\}$. We have thus to check the validity of the implications

$$\begin{pmatrix} v \\ v \end{pmatrix} \in (\mathbb{F}_1 - \mathbb{F}_1) \times (\mathbb{F}_1 - \mathbb{F}_1)^\perp = \mathbb{R} \times \{0\} \ \Rightarrow \ v = 0$$

$$\begin{pmatrix} v \\ v \end{pmatrix} \in (\mathbb{F}_2 - \mathbb{F}_2) \times (\mathbb{F}_2 - \mathbb{F}_2)^\perp = \{0\} \times \mathbb{R} \ \Rightarrow \ v = 0,$$

which are evidently fulfilled. Consequently the respective solution mapping Σ is isolatedly calm around $(0, 0)$. This conclusion is correct because, as one can easily compute,

$$\Sigma(x) = \begin{cases} \{0\} \cup \{x\} & \text{if } x \le 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

Note that Σ does not have the Aubin property around $(0, 0)$. □

Finally, let us compare condition (38) with a standard criterion for the Aubin property of Σ around the reference point. On the basis of the theory from [10, Chapter 4] one obtains the following result.

Proposition 5.8. *Consider the inclusion* $0 \in H(x, y)$ *and assume that the implication*

$$(u^*, 0) \in D^*H\big((\bar{x}, \bar{y}), 0\big)(w^*) \Rightarrow w^* = 0, \ u^* = 0 \tag{48}$$

holds true. Then Σ *has the Aubin property around* $(\bar{x}, \bar{y})$.

Note that the condition (48) ensures both a qualification condition needed to compute the coderivative of Σ and the satisfaction of the Mordukhovich criterion $D^*\Sigma(\bar{x}, \bar{y})(0) = \{0\}$. Since $L^* \subset \operatorname{gph} D^*H(\bar{x}, \bar{y}, 0)\ \forall L^* \in \mathcal{S}H\big((\bar{x}, \bar{y}), 0\big)$, it follows that for H being SCD and semismooth* around $(\bar{x}, \bar{y})$ condition (48) implies not only the Aubin property but also the isolated calmness of Σ around $(\bar{x}, \bar{y})$. This is an important fact emphasizing the importance of the SCD and semismooth* property in stability issues. Observe that the conjunction of the Aubin and the isolated calmness property represents a new useful stability notion, where the isolated calmness specifies the nature of Lipschitzian behavior and the Aubin property ensures the non-emptiness of a localization.

Of course, to ensure in the setting of Proposition 5.5 the conjunction of the Aubin property and the isolated calmness property around a point, we can use also condition (46) along with some specific non-restrictive criterion for the Aubin property, see, e.g. [4, Theorem 4.4].

6. Conclusion

As explained in Corolary 4.3, graphically Lipschitzian mappings of dimension n are SCD mappings for which both SC limiting derivatives can be computed. For GEs with such multi-valued parts thus the respective conditions (41) and (42) can be used in a large number of parameterized GEs corresponding, e.g., to variational inequalities of the 2nd kind, hemivariational inequalities or implicit complementarity problems.

In [6] one finds also a relationship between SCD and strong metric regularity. This indicates that in some cases the SC limiting derivatives could be used also to ensure that an implicitly defined mapping has a single-valued and Lipschitzian localization around the reference point. This task we postpone to a future research.

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