Estimation of expected shortfall in linear model

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1 Introduction

Consider the linear regression model

$$\mathbf{Y}_n = \mathbf{X}_n\boldsymbol{\beta} + \mathbf{Z}_n$$

with with observations $\mathbf{Y}_n = (Y_1, \ldots, Y_n)^\top$ and variables $\mathbf{Z}_n = (Z_1, \ldots, Z_n)^\top$, i.i.d. and unobservable, distributed according unknown distribution function F. The designed matrix $\mathbf{X}$ of order $n \times (p+1)$ is known and $x_{i0} = 1$ for $i = 1, \ldots, n$ (i.e., β_0 is an intercept). All inference on $\mathbf{Z}$ and on F is possible only by means of $\mathbf{Y}$, either after estimating the unknown parameter $\boldsymbol{\beta} = (\beta_0, \beta_1, \ldots, \beta_p)^\top$, or by using a procedure invariant to $\boldsymbol{\beta}$. Our problem is to estimate the possible loss of an asset or a portfolio Z in a given period and with a particular confidence level α by means of the expected shortfall equal to $\mathsf{CVaR}_\alpha(\mathsf{Z}) = (1-\alpha)^{-1}\int_\alpha^1 \mathsf{F}^{-1}(\mathsf{t})\mathsf{dt}$.
It has been shown by Bassett et al. (2004) that

$$\mathsf{CVaR}_\alpha(Z) = \frac{1}{1-\alpha}\min_{\xi\in\mathbb{R}} \rho_\alpha(Z-\xi) + \mathbf{E}Z$$

where $\rho_\alpha(z) = z\,(\alpha - I[z<0])\,, \quad z \in \mathbb{R}$ is the quantile criterion function such that the solution of the minimization $\min_{\xi\in\mathbb{R}} \rho_\alpha(X-\xi)$ is the α-quantile of Z. Hence, if independent observations $Z_1, Z_2, \ldots, Z_n$ of Z were available, the estimate of $\mathsf{CVaR}_\alpha(Z)$ could be obtained from the empirical quantile function based on the order statistics $Z_{n:1} \le X_{n:2} \le \ldots \le Z_{n:n}$. It would have the form:

$$\widehat{\mathsf{CVaR}}_\alpha(X) = \frac{1}{\lfloor n(1-\alpha)\rfloor}\sum_{i=\lfloor n(1-\alpha)\rfloor}^{n} \mathrm{Z}_{n:i}.$$

However, because only the observations of Y are available, we should look for an alternative solution of explicit estimating of $\mathsf{CVaR}_\alpha(\mathsf{Z})$. A possible estimate can be based on the regression quantile of the model or on its functional, e.g. on its intercept component, on the average regression quantile or on the two-step regression quantile with an R-estimate of its slope componens.

References

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