Non-Singular Fixed-Time Tracking Control of Uncertain Nonlinear Pure-Feedback Systems With Practical State Constraints

Chaoqun Guo, Jiangping Hu, *Senior Member, IEEE*, Yanzhi Wu, and Sergej Čelikovský, *Senior Member, IEEE*

***Abstract*— In this paper, a fixed-time tracking control problem is investigated for an uncertain high-order nonlinear pure-feedback systems with practical state constraints. To this end, a new nonlinear transformation function with lower change rate at the state constraint boundary is first proposed, which can not only handle both constrained and unconstrained states in a unified way, but also reduce the control magnitude at the constraint boundary. With the help of the proposed transformation function, the original system is transformed to a new system without state constraints. Then, a non-singular fixed-time adaptive tracking controller is designed by applying an adding a power integrator technique and an adaptive neural network method. It is shown that the practical fixed-time stability can be guaranteed for the closed-loop system under the proposed tracking controller. Finally, two numerical examples are presented to demonstrate the proposed fixed-time tracking control strategy.**

***Index Terms*— Fixed-time tracking control, nonlinear pure-feedback system, state constraint, nonlinear transformation function, adding a power integrator technique.**

I. Introduction

In some practical systems, system states are subject to certain constraints due to the physical limitations or security requirements [1], [2]. For example, velocity and displacement of unmanned vehicle system suffer constraints of traffic rules and road conditions. The violation of the state constraints may cause the deterioration of the system performance and even the safety accidents. Therefore, it is necessary to ensure the satisfaction of state constraints while achieving stability control. In recent years, this significant issue has attracted extensive attention and research.

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In the past decade, the methods of set invariance control [3], model predictive control [4] and barrier Lyapunov function (BLF) [5] have been developed to deal with the constrained systems. Among them, the barrier Lyapunov function (BLF) method has been widely used. Reference [6] employed the BLF to deal with static symmetric full state constraints. Reference [7] developed the integral BLF to handle static asymmetric full state constraints. For time-varying full state constraints, an adaptive neural network control scheme was established by using a tangent BLF in reference [8]. However, the above BLF-based controls produced additional feasibility conditions, which need extra complex offline calculations and even have no solution sometimes. To avoid this defect, a nonlinear transformation function (NTF) method was developed in reference [9], which can transform the original constrained system to a new one without constraints. Recently, the NTF method has been extensively used. For example, in reference [10], an adaptive neural dynamic control of pure-feedback nonlinear systems with full state constraints was designed by introducing another NTF. In reference [11], an NTF based tracking controller was designed for uncertain non-strict feedback systems with full state constraints.

Nevertheless, in practical systems, it is quite common that only partial states, not full, are subject to constraints. The above-mentioned controls were developed for the systems with full state constraints, which will fail for the systems with unconstrained states. To tackle this issue, reference [12] studied the partial state constraints-based control by dividing the full state into two parts and requiring that the first m states of the system were constrained states and the last $n-m$ states were free states. However, the constrained state sequence was not arbitrary in reference [12]. Although the sequence of partial constrained states was arbitrary in references [13] and [14], a set of artificial constraints need to be imposed on the free states. Once the free state sequence changes, it is necessary to reimpose artificial constraints and redesign the controller. Therefore, a unified control way that can simultaneously deal with the constrained and unconstrained sates without changing the control structure is significantly to be researched.

Up to now, only a few literatures have begun to tackle the constrained and unconstrained states in a unified way. Reference [15] developed a unified NTF to achieve tracking control for non-strict-feedback nonlinear systems with or without state constraints. Reference [16] proposed another unified NTF for high-order strict-feedback systems and applied the proposed unified NTF to the tracking control of robotic systems [17]. However, the cascade connections were affine in the references [15], [16], [17] and therefore easier to handle. In this paper, the so-called pure-feedback systems having nonaffine connections between cascades will be considered. Pure-feedback systems were introduced in [18] and thoroughly studied along with their more specific version - the so-called triangular systems [19], [20], [21]. Up to now, very few studies have been presented for the control of pure-feedback systems with constraints in a unified way. Although references [22] and [23] studied unified tracking controls for pure-feedback systems with both constrained and unconstrained states by developing the unified NTFs, but only asymptotic convergence, not fixed-time convergence, were achieved.

As is well known, the fixed-time convergence has faster convergence rate and better disturbance rejection [24], [25], [26], [27]. To achieve fixed-time convergence, the homogeneity method [28] and terminal sliding mode method [29] have been proposed for linear systems and second-order systems, respectively. For high-order systems, the modified backstepping method with fractional feedback terms was proposed and widely applied to the fixed-time tracking of nonlinear systems with state constraints [30], [31], [32], [33]. However, the singularity problem may occur in fixed-time control design due to the derivatives of the fractional power terms. To overcome the singularity problem, reference [34] proposed the switching functions and applied it into the finite-time control. Moreover, reference [35] developed the adding a power integrator technique to avoid the singularities in fixed-time control. However, for uncertain nonlinear pure-feedback systems with partial or full state constraints, the fixed-time tracking without singularities in a unified way has not been studied.

Motivated by above discussion, we are devoted to investigating the non-singular fixed-time tracking of uncertain nonlinear pure-feedback systems with partial or full state constraints. The main difficulties and challenges are: 1) To remove the singularities in fixed-time control, the introduction of the adding a power integrator technique will increase the complexity and difficulty of the traditional backstepping design. Especially for constrained pure-feedback systems with nonaffine cascade characteristic, designing a non-singular fixed-time tracking controller is more complex and challenging. 2) How to design a new unified NTF that can not only result in a smaller control effort to pull state back from the constraint boundary, but also be applicable to the multiple practical constraint situations given below is another challenge. Specifically, the considered constraint situations include: ① All states are constrained; ② All states are not constrained; ③ Some states are constrained and other states are not constrained. The main contributions are summarized as follows:

1) A new unified NTF is proposed to handle the constrained and unconstrained states uniformly. Since NTF converts the state x into a variable ξ, the variable ξ tends to infinity when state x tends to the constraint boundary. This leads to a large control effort to ensure the boundedness of the ξ. However, compared with references [16] and [23], the NTF proposed in this paper makes the ξ tend to infinity more slowly for the same process that x tends to the constraint boundary, which can lead to a smaller control effort than that of [16] and [23].

2) The adaptive neural network technique is applied to deal with the uncertain nonlinearities of the pure-feedback systems and the adding a power integrator technique is used to overcome the singularities in fixed-time control process.

3) The non-singular fixed-time tracking controller for uncertain nonlinear pure-feedback systems with or without state constraints is first developed in a unified way in this paper, which not only realizes the practical fixed-time tracking, but also ensures that the constrained states satisfy the corresponding constraints by a unified control structure. Compared with asymptotic tracking of the pure-feedback systems with or without state constraints [22], [23], this paper achieves the fixed-time tracking. Moreover, in contrast to references [16] and [17], there is no singularities in this paper.

The remainder of this paper is organized below. The preliminaries and problem formulation are presented in Section II. Section III gives a unified nonlinear transformation function and an adaptive fixed-time control strategy. At the same time, practical fixed-time convergence is analyzed for the closed-loop system. In Section IV, numerical examples are given to validate the proposed fixed-time control strategy. Finally, conclusions are presented in Section V.

II. Preliminaries and Problem Formulation

A. Preliminaries

Definition 1: Consider the system $\dot{x} = f(t, x), x(0) = x_0$, where $x \in \mathbb{R}^n$, the nonlinear function $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous with respect to t and Lipschitz continuous with respect to x, and $f(t, 0) \equiv 0$. The equilibrium of the system is said to be practically fixed-time stable on $\mathbb{R}^n$, if it is stable and $\forall\epsilon > 0$ there exists a settling time $T(\epsilon) > 0$ such that $\|x(t, x_0)\| \le \epsilon$, $\forall x_0 \in \mathbb{R}^n$, $\forall t \ge T(\epsilon)$.

Lemma 1 ([17]): Consider the system $\dot{x} = f(t, x)$, where $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous with respect to t and Lipschitz continuous with respect to x, and $f(t, 0) \equiv 0$. The equilibrium of the system is practically fixed-time stable if $\forall\delta > 0$ there exist a positive definite function $V_\delta(t, x)$ and parameters $k_1 > 0$, $k_2 > 0$, $0 < \gamma < 1$, $\beta > 1$, and $0 < \theta < 1$ such that

$$\dot{V}_\delta(t, x) \le -k_1 V_\delta(t, x)^\gamma - k_2 V_\delta(t, x)^\beta + \delta.$$

Furthermore, there exists a settling time T such that

$$V_\delta(t, x) \le \min\Big\{\Big(\frac{\delta}{k_1\theta}\Big)^{\frac{1}{\gamma}}, \Big(\frac{\delta}{k_2\theta}\Big)^{\frac{1}{\beta}}\Big\},$$

when $t \ge T$, and the upper bound of the settling time T is given by:

$$T \le \frac{1}{k_1(1-\theta)(1-\gamma)} + \frac{1}{k_2(1-\theta)(\beta-1)}.$$

Lemma 2 ([36]): (Mean value theorem) If function $f(x)$ is continuous on a closed interval $[a, b]$, and is differentiable on an open interval (a, b), then there exists a ϵ $(a < \epsilon < b)$ such that

$$f(b) - f(a) = f'(\epsilon)(b - a) \tag{1}$$

holds.

Lemma 3 ([37]): The following three inequalities hold.

① For arbitrary constants $x_1 > 0$, $x_1 \geq x_2$, and $p > 1$, it holds:

$$(x_1 - x_2)^p \geq x_2^p - x_1^p.$$

② For arbitrary constants $p > 0$, $x_1 \geq 0$, and $x_2 > 0$, it holds:

$$x_1^p (x_2 - x_1) \leq \frac{1}{1+p}(x_2^{1+p} - x_1^{1+p}).$$

③ For arbitrary constants $x_i \in \mathbb{R}$ and $p > 0$, it holds:

$$(\sum_{i=1}^{n} |x_i|)^p \leq \max\{n^{p-1}, 1\} \sum_{i=1}^{n} |x_i|^p.$$

Lemma 4 ([35]): For a real-valued function $\gamma(x, y) > 0$ and positive constants c, d, the following inequality holds:

$$|x|^c |y|^d \leq \frac{c}{c+d}\gamma(x, y)|x|^{c+d} + \frac{d}{c+d}\gamma^{-\frac{c}{d}}(x, y)|y|^{c+d}.$$

Lemma 5 ([35]): For $x \in \mathbb{R}$, $y \in \mathbb{R}$, and scalar $p \leq 1$, it holds:

$$|x^p - y^p| \leq 2^{1-p}|x - y|^p.$$

According to the radial basis function neural network (RBFNN) technique [38], the unknown continuous function $F(x) \in \mathbb{R}$ can be approximated by a linearly parameterized model as follows:

$$F(x) = W^T S(x) + \varepsilon(x), \quad x \in \mathbb{R}^n, \tag{2}$$

where $W \in \mathbb{R}^N$ is the weight vector of a radial basis function neural networks, $S(x) = [S_1(x), \ldots, S_N(x)]^T \in \mathbb{R}^N$ is the basis function vector. Particularly, the basis function is generally given by

$$S_i(x) = \exp\left[-\frac{(x - \tau_i)^T (x - \tau_i)}{\psi_i^2}\right], \quad i = 1, \ldots, N, \tag{3}$$

where $\psi_i \in \mathbb{R}$, $\tau_i \in \mathbb{R}^n$ are the width and the center of the basis function, respectively. $\varepsilon \in \mathbb{R}$ is the estimation error. In this paper, the unknown weight vector W and estimation error ε satisfy the following assumption:

Assumption 1: In the linearly parameterized model (2), assume that $\|W\| \leq \bar{W}$, $|\varepsilon| \leq \bar{\varepsilon}$, where $\bar{W}$ and $\bar{\varepsilon}$ are unknown positive constants. Furthermore, define $w = \max\{\bar{W}, \bar{\varepsilon}\}$.

B. Problem Formulation

Consider an uncertain high-order nonlinear pure-feedback system as follows:

$$\begin{cases} \dot{x}_i = f_i(\bar{x}_i, x_{i+1}), i = 1, \ldots, n-1, \\ \dot{x}_n = f_n(\bar{x}_n, u), \\ y = x_1, \end{cases} \tag{4}$$

where $x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$ is the state; $\bar{x}_i$ denotes $[x_1, \ldots, x_i]^T \in \mathbb{R}^i$; $y \in \mathbb{R}$ is the output; $u \in \mathbb{R}$ is the control input; $f_i(\cdot)$ $(i = 1, \ldots, n)$ are unknown nonlinear functions. The system states need to satisfy the following constraints:

$$-h_{i1}(t) < x_i(t) < h_{i2}(t), \quad i = 1, \ldots, n, \tag{5}$$

and we allow that $h_{i1}(t)$ and $h_{i2}(t)$ can include the following practical situations:

1) $h_{i1}(t)$ and $h_{i2}(t)$, $i = 1, \ldots, n$, are bounded time-varying strictly positive smooth functions. This means that all states are constrained all the time.

2) $h_{i1}(t) = h_{i2}(t) \equiv +\infty$, $i = 1, \ldots, n$, which means that there is no constraint on the whole system.

3) $h_{i1}(t) = h_{i2}(t) \equiv +\infty$ for partial $i \in \{1, \ldots, n\}$, and the other $h_{i1}(t)$, $h_{i2}(t)$ are bounded time-varying strictly positive smooth functions. This denotes that the researched objective is a system with partial state constraints.

4) $0 < h_{ij}(t) \leq +\infty$ for $i \in \{1, \ldots, n\}$, $j = 1, 2$, which implies that the state x_i may be constrained for a period of time and unconstrained over certain period.

Assumption 2: The initial states satisfy $-h_{i1}(0) < x_i(0) < h_{i2}(0)$, $i = 1, \ldots, n$.

Assumption 3: $f_i(\bar{x}_i, x_{i+1})$ $(i = 1, \ldots, n-1)$ and $f_n(\bar{x}_n, u)$ are continuously differentiable with respect to all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$.

According to Assumption 3 and Lemma 2, there exists a constant $\alpha \in (0, 1)$ such that the following equation holds:

$$f_n(\bar{x}_n, u) = f_n(\bar{x}_n, 0) + g_n(\bar{x}_n, \alpha u)u, \tag{6}$$

where $g_n(\bar{x}_n, \alpha u) = \frac{\partial f_n(\bar{x}_n, \nu)}{\partial \nu}|_{\nu=\alpha u}$.

Assumption 4: We assume that $\underline{g}_n \leq g_n(\cdot) \leq \overline{g}_n$, where $\underline{g}_n$, $\overline{g}_n$ are unknown positive constants.

For the nonlinear system (4), the reference output is given by $y_d(t)$, which is bounded and smooth, and satisfies the constraint (5), i.e., $-h_{11}(t) < y_d(t) < h_{12}(t)$.

The objective of this paper is to design a fixed-time controller u for the system (4) to achieve practical fixed-time output tracking, that is,

$$|y(t) - y_d(t)| \leq \zeta, \quad \forall t \geq T, \tag{7}$$

where ζ and T are some positive constants, and at the same time, the controller u guarantees that the state constraints (5) are not violated all the time.

III. Main Results

In this section, firstly, a unified nonlinear transformation function is constructed to handle the cases with and without state constraints. Thus the system (4) can be transformed to a new unconstrained system by using the nonlinear transformation function. Secondly, a non-singular practical fixed-time tracking controller is constructed by introducing the adding a power integrator technique and the adaptive neural network technique. Thirdly, the convergence analysis is given to show that the practical fixed-time output tracking of the closed-loop system is achieved and the state constraints are always satisfied simultaneously.

A. Unified Nonlinear Transformation Function

In order to uniformly handle more practical systems that meets all the constraint situations listed in (5), a unified nonlinear transformation function is proposed as follows:

$$\xi_i(t) = \frac{h_{i1}(t) + h_{i2}(t)}{4} \ln \frac{h_{i1}(t) + x_i(t)}{h_{i2}(t) - x_i(t)}. \tag{8}$$

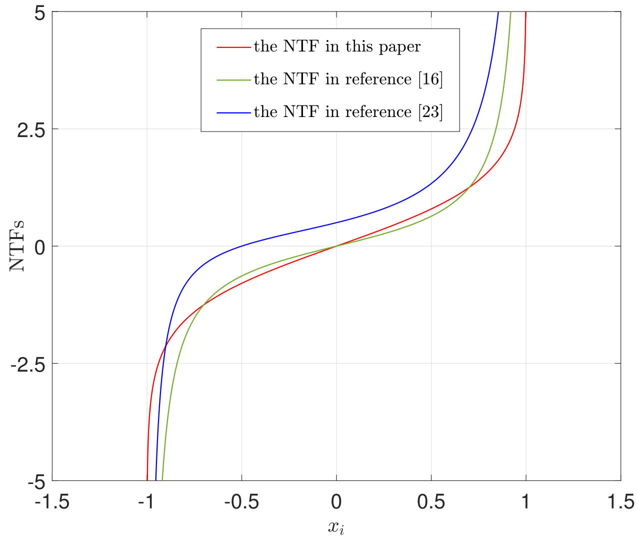

Fig. 1. The nonlinear transformation functions (NTFs).

The nonlinear transformation (8) has the following properties:

$$\begin{cases}\text{(i)} \lim\limits_{x_i(t)\to -h_{i1}(t)} \xi_i(t) = -\infty;\\ \text{(ii)} \lim\limits_{x_i(t)\to h_{i2}(t)} \xi_i(t) = +\infty;\\ \text{(iii)} \lim\limits_{h_{i1}(t)=h_{i2}(t)\to +\infty} \xi_i(t) = x_i(t).\end{cases} \tag{9}$$

Remark 1: According to the properties (i) and (ii), one can obtain that if $\xi_i(t)$ is bounded, then the condition $-h_{i1}(t) < x_i(t) < h_{i2}(t)$ holds for any $-h_{i1}(0) < x_i(0) < h_{i2}(0)$. Therefore, in order to ensure that the constraints are not violated, we just need to ensure that $\xi_i(t)$ is bounded. For the property (iii), $h_{i1}(t) = h_{i2}(t) = +\infty$ means that there is no state constraint and thus $\xi_i(t) = x_i(t)$. Consequently, the proposed transformation (8) can deal with the cases with and without state constraints in a unified manner.

Remark 2: The transformation function (8) can be applied to the systems with both constrained and unconstrained states. Moreover, the transformation function (8) is also applicable to the systems that start from an unconstrained initial state to a constrained state space. For example, when an unmanned vehicle drives into a narrow tunnel starting from an open area, the state constraint is continuously changing from infinity to a bounded region. In this case, the constraint functions can be selected as $h_{ij}(t) = e^{\frac{1}{t-t_0}} + k_{ij}$ $(j = 1, 2)$, where t_0 is the initial time instant, k_{ij} is a positive constant. Thus the state constraint decreases monotonically from infinity to $1 + k_{ij}$.

Remark 3: It is noted that when the state x_i tends to the boundary of constraint, the transformed variable $\xi_i(t)$ tends to $-\infty$ or $+\infty$, which often needs a large control effort to make the $\xi_i(t)$ be bounded. However, compared with the unified NTFs in [16] and [23], the change of ξ_i tending to $-\infty$ or $+\infty$ in this paper is slower than that in [16] and [23], respectively. When the state constraint is set as $|x_i| \le 1$, the evolution of the transformed variable $\xi_i(t)$ is shown in Fig. 1. More details can also be found in Table I. From Table I, it can be further found that the change rates of the transformed variable ξ under different x intervals near the boundary 1 are smaller than those in references [16] and [23].

TABLE I
THE CHANGE RATE OF ξ IN DIFFERENT x INTERVALS

References	Change rate of ξ in different x intervals		
	$[0.85, 0.9]$	$[0.9, 0.95]$	$[0.95, 0.99]$
This paper	6.24	10.38	23.5
[16]	27.34	81.4	648.82
[23]	50.06	150.04	1200

Under the unified nonlinear transformation function (8), differentiating $\xi = [\xi_1, \ldots, \xi_n]^T$ yields a new system as follows:

$$\begin{cases}\dot{\xi}_i = F_i(\bar{x}_{i+1}, \xi_{i+1}) + \xi_{i+1}, \quad i = 1, \ldots, n-1,\\ \dot{\xi}_n = F_n(\bar{x}_n) + \varphi_n g_n u,\end{cases} \tag{10}$$

where

$$\begin{aligned} &F_i(\bar{x}_{i+1}, \xi_{i+1})\\ &\quad = \varphi_i f_i(\bar{x}_i, x_{i+1}) + \phi_i - \xi_{i+1}, \quad i = 1, \ldots, n-1,\\ &F_n(\bar{x}_n) = \varphi_n f_n(\bar{x}_n) + \phi_n,\\ &\varphi_i = \frac{\partial \xi_i}{\partial x_i} = \frac{\big(h_{i1}(t) + h_{i2}(t)\big)^2}{4\big(h_{i1}(t) + x_i(t)\big)\big(h_{i2}(t) - x_i(t)\big)},\\ &\phi_i = \frac{h_{i1}(t) + h_{i2}(t)}{4}\Big(\frac{\dot{h}_{i1}(t)}{h_{i1}(t) + x_i(t)} - \frac{\dot{h}_{i2}(t)}{h_{i2}(t) - x_i(t)}\Big)\\ &\qquad + \frac{\dot{h}_{i1}(t) + \dot{h}_{i2}(t)}{4} \ln \frac{h_{i1}(t) + x_i(t)}{h_{i2}(t) - x_i(t)}, \end{aligned}$$

for $i = 1, \ldots, n$, and $f_n(\bar{x}_n) = f_n(\bar{x}_n, 0)$, $g_n = g_n(\bar{x}_n, \alpha u)$ are defined in (6).

Obviously, the original system (4) with the state constraint (5) is transformed to an unconstrained system (10). The constraints on state x can be guaranteed by ensuring the boundedness of the variable ξ. In sequel, based on the transformed system (10), we need to design a fixed-time controller to not only ensure that the state constraints are not violated, but also realize the practical fixed-time output tracking.

According to the unified nonlinear transformation function (8), a nonlinear transformation is given for the reference output y_d as follows:

$$\xi_d(t) = \frac{h_{i1}(t) + h_{i2}(t)}{4} \ln \frac{h_{i1}(t) + y_d(t)}{h_{i2}(t) - y_d(t)}. \tag{11}$$

B. Controller Design and Convergence Analysis

Define

$$\begin{cases} e_1 = \xi_1 - \xi_d,\\ e_k = \xi_k^{1/q_k} - \xi_k^{*1/q_k}, \quad k = 2, \ldots, n,\end{cases} \tag{12}$$

where $q_k = 1 - (k-1)\tau$, $\tau = \frac{p}{q} \in (0, \frac{1}{n})$, and p is a positive even constant, q is a positive odd constant. It can be verified that $q_n < q_{n-1} < \ldots < q_2 < 1$. ξ_k^* is the virtual controller to be designed at the step k of the following Algorithm 1. where $\psi_k(\cdot) = n-(k-1)+\epsilon_k e_k^{\kappa}+v_{k-1}+G_{k-1}+l_k\sqrt{1+\hat{\theta}_k^2} > 0$, the parameters ϵ_k, $k = 1, \ldots, n$ are positive constants,

Algorithm 1 Power integrator based control design.

Step 1: Consider

$$e_1 = \xi_1 - \xi_d.$$

Construct the Lyapunov function as

$$V_1 = \tfrac{1}{2}e_1^2 + \tfrac{1}{2}\tilde{\theta}_1^2, \tag{15}$$

then, according to the derivative of V_1, design the virtual controller as

$$\xi_2^* = -e_1^{q_2}\psi_1(e_1, \hat{\theta}_1). \tag{18}$$

Step k: $(2 \le k \le n-1)$ Consider

$$e_k = \xi_k^{1/q_k} - \xi_k^{*1/q_k}.$$

Construct the Lyapunov function with a power integrator as

$$V_k = V_{k-1} + \int_{\xi_k^*}^{\xi_k}(s^{1/q_k} - \xi_k^{*1/q_k})^{2-q_k}ds + \tfrac{1}{2}\tilde{\theta}_k^2, \tag{35}$$

then, according to the derivative of V_k, design the virtual controller as

$$\xi_{k+1}^* = -e_k^{q_{k+1}}\psi_k(e_1, \xi_2, \ldots, \xi_k, \hat{\theta}_1, \ldots, \hat{\theta}_k).$$

Step n: Consider

$$e_n = \xi_n^{1/q_n} - \xi_n^{*1/q_n}.$$

Construct the Lyapunov function with a power integrator as

$$V_n = V_{n-1} + \int_{\xi_n^*}^{\xi_n}(s^{1/q_n} - \xi_n^{*1/q_n})^{2-q_n}ds + \tfrac{1}{2}\tilde{\theta}_n^2, \tag{46}$$

and design the controller u as

$$u = -\frac{1}{\varphi_n \underline{g}_n}e_n^{q_n+1}\Big(1 + \epsilon_n e_n^{\kappa} + v_{n-1} + G_{n-1} + l_n\sqrt{1+\hat{\theta}_n^2}\Big), \tag{47}$$

$\kappa > 1$ is an even constant. $l_k = \rho_{k-1} + \sum_{l=1}^{k-1} H_{k-1,l}$, and $\rho_{k-1} = \frac{2-q_{k-1}}{d}$, $v_{k-1} = 2^{1-q_k}\frac{q_k}{d}(\frac{1}{2}\frac{d}{2-q_{k-1}}\frac{1}{2^{1-q_k}})^{-(2-q_{k-1})/q_k} > 0$, $H_{k-1,l}(\cdot) = (2-q_k)2^{2-q_k}\frac{1}{d}\mu_l^d\hat{G}_{k-1,l}^d > 0$. $\hat{G}_{k-1,l} > 0$ and $G_{k-1} > 0$ are $\mathcal{C}^1$ functions. The positive constant $\underline{g}_n$ is the lower bound of g_n. In addition, the parameters $\tilde{\theta}_k$ and $\hat{\theta}_k$ are defined in the detailed controller design process which is given in Appendix A.

Remark 4: It is worth noting that in the references [17], [30], [31], [32], and [33], the fixed-time controllers suffered from a singularity problem. Specifically, there is often a state related term with fractional power less than 1 like x^l $(l < 1)$ in the virtual controller α_i. Thus the next virtual controller α_{i+1} contains the derivative of α_i, which leads to a singularity at $x = 0$. To overcome this drawback, this paper designs the controller (47) by introducing the adding a power integrator into the Lyapunov functions V_i $(i = 2, \ldots, n)$, and can avoid the singularities caused by the derivatives of the virtual controllers in the traditional backstepping design process.

Under the controller u in Algorithm 1, the objectives in Theorem 1 are achieved.

Theorem 1: Consider the uncertain high-order nonlinear pure-feedback system (4) with the state constraints given by (5). Under Assumptions 1-4, the proposed controller (47) can achieve the following objectives:

(1) All the states in closed-loop system (4) are bounded.

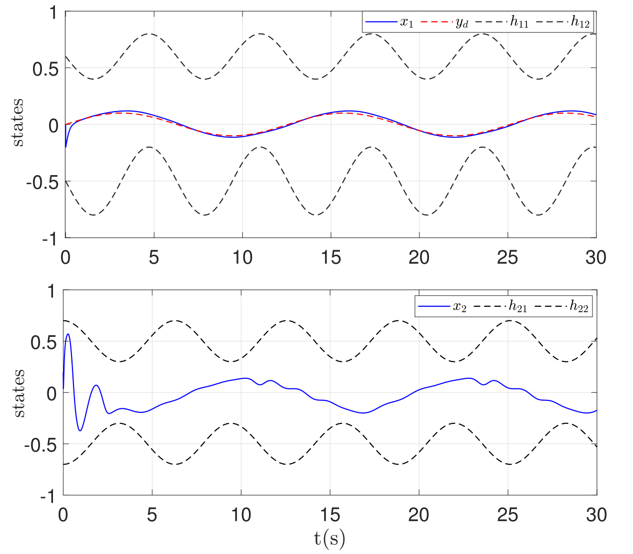

Fig. 2. The trajectories of the states x_1, x_2, and the reference output y_d under the case that x_1, x_2 are all constrained.

(2) The practical fixed-time output tracking is achieved, that is, $|z| = |y - y_d| \le \zeta = \sqrt{2\mathcal{R}}$ when $t \ge T$. The settling time is estimated as

$$T \le \frac{2}{\Gamma_1(1-\theta)(2-d)} + \frac{2}{\Gamma_2(1-\theta)(d+\kappa-2)},$$

where $\mathcal{R}$ is defined in (53) in Appendix B, $\Gamma_1 = \frac{\varpi_1}{\varpi^{d/2}}$, and $\Gamma_2 = \frac{\varpi_2}{\varpi^{(d+\kappa)/2}}$, $\varpi_1 = \min_{k=1,\ldots,n}\{1, \frac{\sigma_{k,1}}{d}\}$, $\varpi_2 = \min_{k=1,\ldots,n}\{1, \frac{\sigma_{k,2}}{d+\kappa}\}$, $\varpi = \max\{\frac{1}{2}, 2^{1-q_k}\}$, θ is a constant satisfies $0 < \theta < 1$, $\sigma_{k,1}$ and $\sigma_{k,2}$ are given positive constants, $\kappa >$ is an even constant.

(3) The state constraints are satisfied all the time.

Proof: The proof is shown in Appendix B.

IV. Simulation Examples

This section contains two examples. Example 1 shows the relevant control results of numerical system under the three cases respectively as: 1. all states are constrained; 2. partial state is constrained; 3. all states are unconstrained, respectively. Example 2 shows the control results of Brusselator model.

Example 1: Consider the following pure-feedback nonlinear system:

$$\begin{cases} \dot{x}_1 = x_1 + x_2 + x_2^2, \\ \dot{x}_2 = x_1x_2 + u + 0.1\sin(u), \\ y = x_1. \end{cases} \tag{13}$$

The reference output is given by $y_d = 0.1\sin(0.5t)$. The initial state is set as $x(0) = [-0.2, 0.2]^T$ and the parameters in the controller (47) are selected as follows: $q_2 = \frac{197}{199}$, $q_3 = \frac{195}{199}$, $\tau = \frac{2}{199}$, $\kappa = \frac{4}{3}$, $\epsilon_1 = 0.01$, $\epsilon_2 = 0.01$, $\sigma_{1,1} = 0.01$, $\sigma_{1,2} = 0.01$, $\sigma_{2,1} = 0.01$, $\sigma_{2,2} = 0.01$, $\hat{\theta}_1(0) = \hat{\theta}_2(0) = 0$, $\underline{g}_n = 0.9$, $\psi_i = 3$, $\tau_i = 0$.

Case 1: All states are constrained.

The state constraint functions are given by $h_{11}(t) = 0.5 + 0.3\sin(t)$, $h_{12}(t) = 0.6 - 0.2\sin(t)$, $h_{21}(t) = 0.5 + 0.2\cos(t)$, and $h_{22}(t) = 0.5 + 0.2\cos(t)$. Then, Fig. 2 shows the

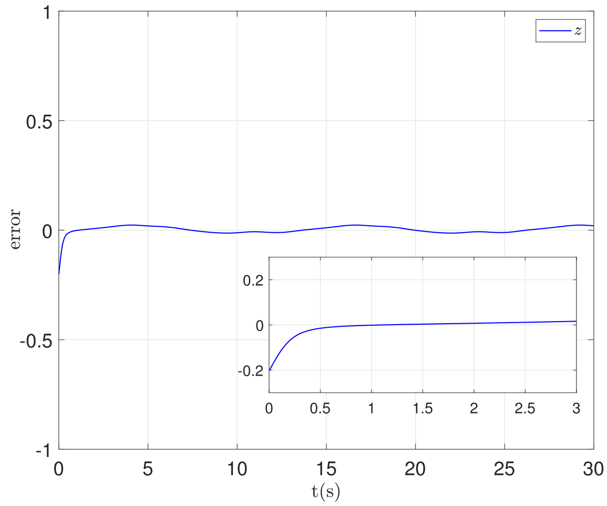

Fig. 3. The trajectory of the output tracking error z under the case that x_1, x_2 are all constrained.

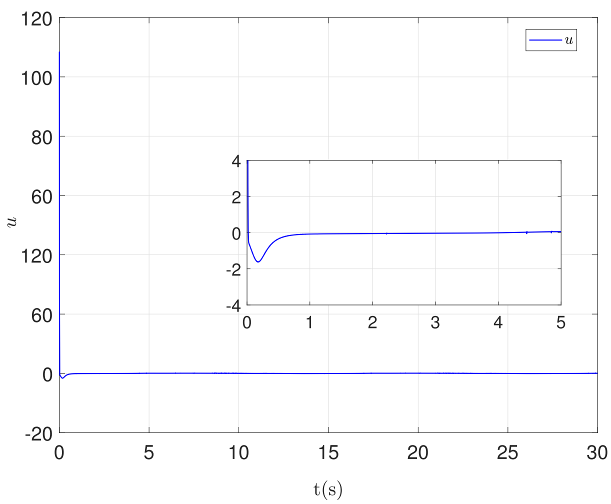

Fig. 4. The trajectory of the control input u under the case that x_1, x_2 are all constrained.

trajectories of the desired output y_d and the states x_1, x_2 under the proposed controller (47). In Fig. 2, we can see that the state x_1 can track the reference output y_d at about $T = 2s$. Moreover, the states x_1 and x_2 are always within the given constraints all the time. Fig. 3 shows the output tracking error and demonstrates that the tracking error is bounded in fixed time, that is, $|z| \leq 0.05$ for $t \geq 2s$. The control input u is shown in Fig. 4.

Case 2: Partial state is constrained.

In order to verify the effectiveness of the controller (47) for the system (12) with both constrained and unconstrained states, we set that state x_1 is constrained by $h_{11}(t) = 0.5 + 0.3\sin(t)$ and $h_{12}(t) = 0.6 - 0.2\sin(t)$, while state x_2 has no constraint, that is $h_{21}(t) = h_{22}(t) \equiv +\infty$. Then, the dynamics of states x_1, x_2, and reference output y_d are shown in Fig. 5. From Fig. 5, it can be found that x_1 and x_2 are bounded while x_1 does not violate its constraint all the time. Compared with the case that all states are constrained, the x_2 in Fig. 5 exceeds the constraints given in case 1 due to there is no constraint on state x_2. Moreover, the output tracking error $|z| \leq 0.05$ for $t \geq 2s$, which is shown in Fig. 6. And the control input is shown in Fig. 7

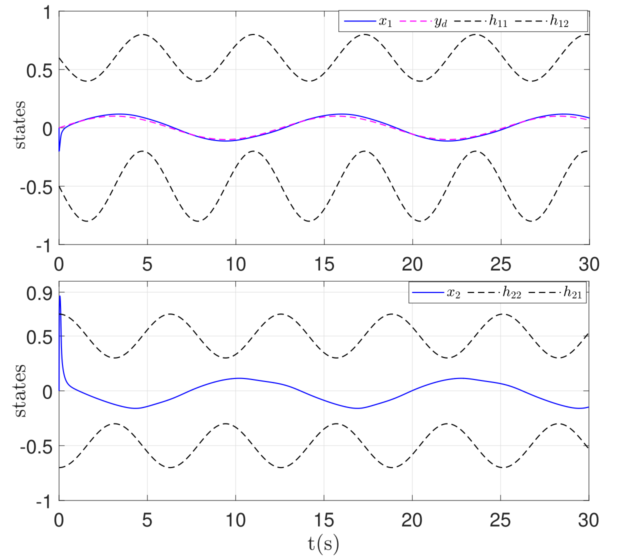

Fig. 5. The trajectories of the states x_1, x_2, and the reference output y_d under the case that x_1 is constrained, x_2 is unconstrained.

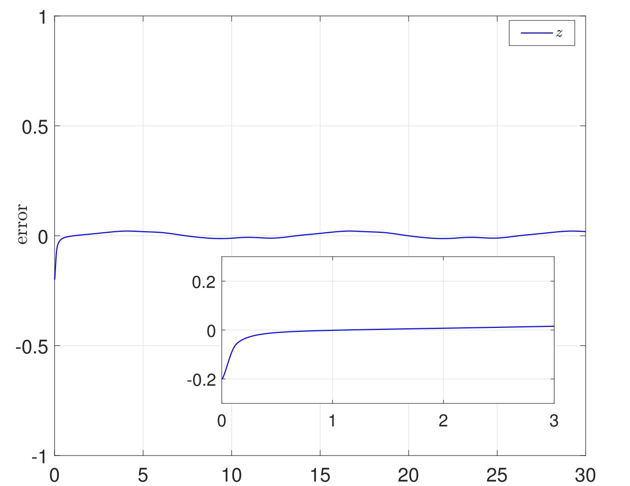

Fig. 6. The trajectory of the output tracking error z under the case that x_1 is constrained, x_2 is unconstrained.

Case 3: All states are not constrained.

Considering that there is no constraints on states, then the trajectories of states, tracking error, and control input are shown in Fig. 8, Fig. 9, and Fig. 10, respectively. According to Fig. 8 and Fig. 9, we obtain that state x_1 can track the reference output in a fixed-time. Moreover, from Fig. 8, the maximum value of the state x_2 is larger than that in case 2 due to no constraints on neither x_1 and x_2.

The above simulation results illustrate that the proposed fixed-time controller can handle the output tracking problem for the uncertain nonlinear pure-feedback system with and without state constraints while keeping one control structure.

Example 2: To further illustrate the applicability of the proposed controller, consider the following disturbed Brusselator model [39]:

$$\begin{cases} \dot{x}_1 = C - (D+1)x_1 + x_1^2 x_2 + d_1, \\ \dot{x}_2 = Dx_1 - x_1^2 x_2 + (2+\cos(x_1))u + d_2, \\ y = x_1, \end{cases} \tag{14}$$

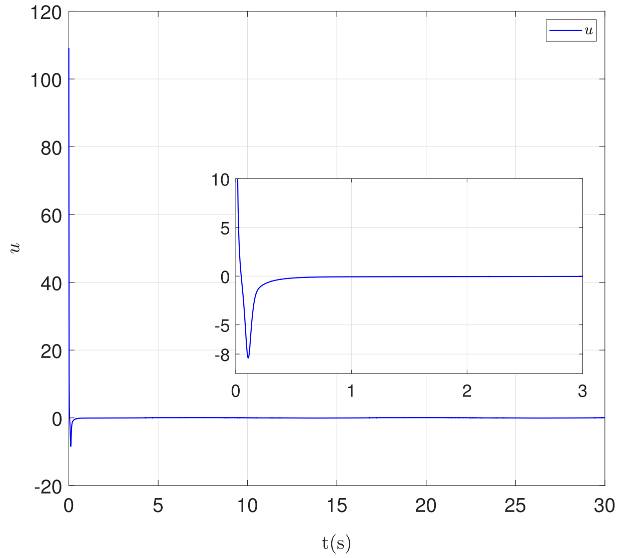

Fig. 7. The trajectory of the control input u under the case that x_1 is constrained, x_2 is unconstrained.

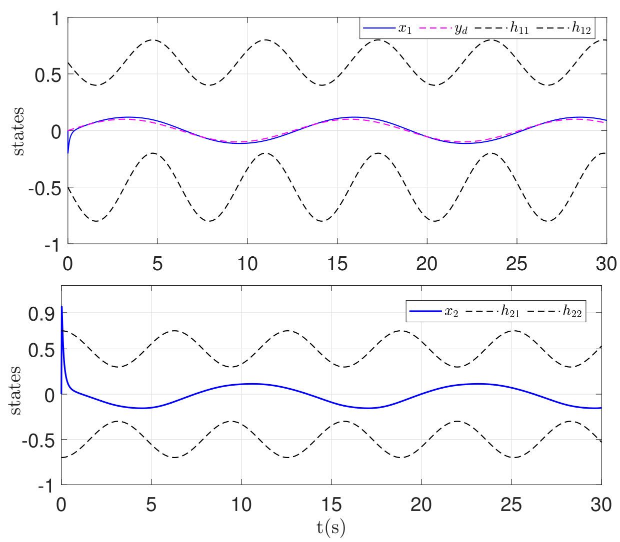

Fig. 8. The trajectories of the states x_1, x_2, and the reference output y_d under the case that x_1, x_2 are all unconstrained.

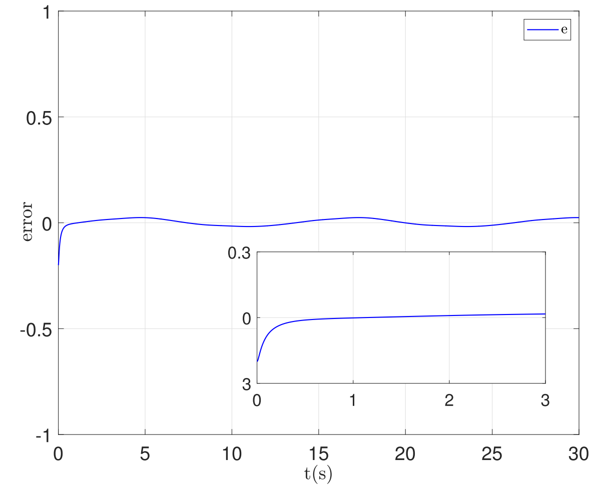

Fig. 9. The trajectory of the output tracking error z under the case that x_1, x_2 are all unconstrained.

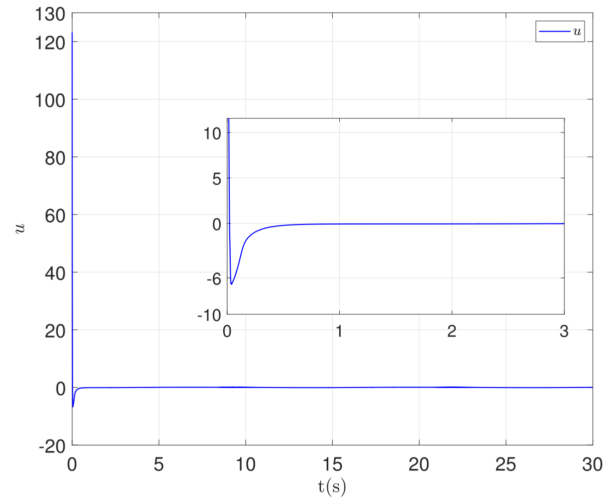

Fig. 10. The trajectory of the control input u under the case that x_1, x_2 are all unconstrained.

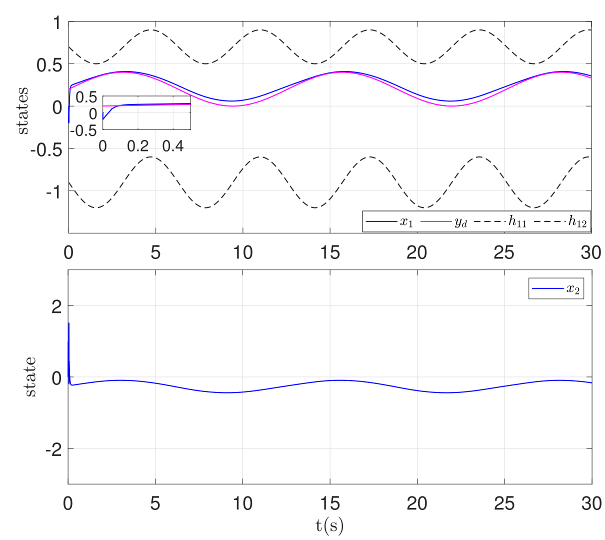

Fig. 11. The trajectories of the states x_1, x_2, and the reference output y_d of Brusselator system.

where the state variables x_1 and x_2 denote the concentrations of the chemical reaction intermediates, the positive numbers C and D are the parameters of the supply of "reservoir" chemicals, d_1 and d_2 denote the external disturbances. In real cases, the concentrations of some reaction intermediates are usually constrained within a certain range to ensure reaction effectiveness. Thus, we assume that the state x_1 is constrained by $h_{11}(t) = 0.5 + 0.3\sin(t)$ and $h_{12}(t) = 0.6 - 0.2\sin(t)$, and the state x_2 has no constraint. We choose $C = 1$, $D = 3$, $d_1 = 2\cos(x_1)x_2$, $d_2 = 2\sin(x_2)x_1$, and the initial states and the other parameters are same as those in Example 1.

Then the states x_1, x_2 and the reference output y_d are shown in Fig. 11. The tracking error and control input are shown in Fig. 12 and Fig. 13, respectively. From Fig. 11, we can see that the states are bounded and the constraint is not violated. In Fig. 12, it is shown that the tracking error $|z| \leq 0.05$ for $t \geq 2s$. Therefore, the proposed controller (47) not only achieves the practical fixed-time output tracking of the Brusselator system (14), but also ensures that the constrained state satisfy its constraint condition.

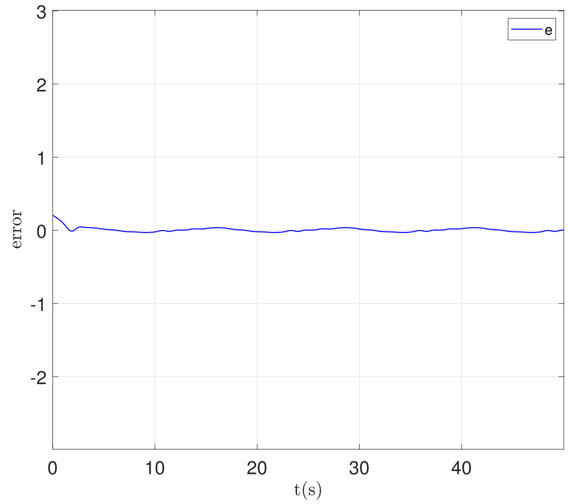

Fig. 12. The trajectory of the output tracking error z of Brusselator system.

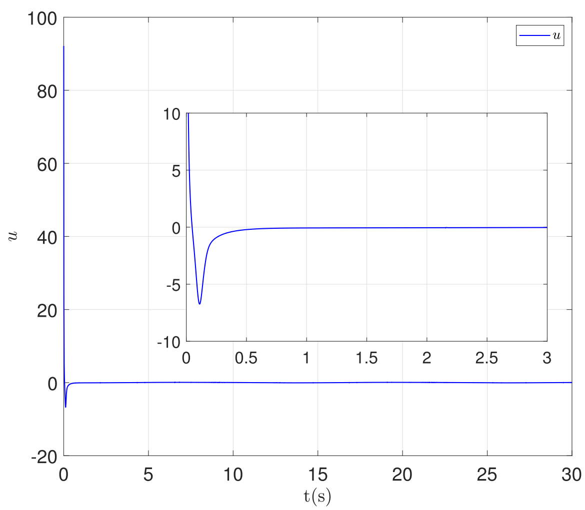

Fig. 13. The trajectory of the control input u of Brusselator system.

V. Conclusion

This paper has solved the fixed-time output tracking problem for uncertain high-order nonlinear pure-feedback systems with partial state constraints in a unified way. A new nonlinear transformation function has been proposed to deal with the constrained and unconstrained states. Based on the unified nonlinear transformation, the original partially constrained systems have been transformed to the systems without constraints. The adding a power integrator technique has been introduced and delicately combined with the adaptive neural network technique to facilitate the fixed-time controller design. Thus, an adaptive neural network based fixed-time tracking controller has been constructed for the uncertain high-order nonlinear pure-feedback systems, which needs only one structure to ensure not only practical fixed-time tracking but also state constraints. Moreover, the effectiveness of the proposed controller has been validated by numerical simulations.

Appendix A
Design Process of the Controller u

Step 1: Construct a Lyapunov function as

$$V_1 = \frac{1}{2}e_1^2 + \frac{1}{2}\tilde{\theta}_1^2, \tag{15}$$

where $\tilde{\theta}_1 = \theta_1 - \hat{\theta}_1$, $\hat{\theta}_1$ is the estimation of θ_1. θ_1 and $\hat{\theta}_1$ will be given later. According to

$$\begin{aligned} \dot{e}_1 &= F_1(\bar{x}_2, \xi_2) + \xi_2 - \dot{\xi}_d \\ &= \bar{F}_1(\bar{x}_2, \xi_2, \dot{\xi}_d) + \xi_2, \end{aligned}$$

where $\bar{F}_1(\bar{x}_2, \xi_2, \dot{\xi}_d) = F_1(\bar{x}_2, \xi_2) - \dot{\xi}_d$, then we can derive that

$$\begin{aligned} \dot{V}_1 &= e_1(\bar{F}_1(\bar{x}_2, \xi_2, \dot{\xi}_d) + \xi_{i2}) + \tilde{\theta}_1\dot{\tilde{\theta}}_1 \\ &= e_1\bar{F}_1(\bar{x}_2, \xi_2, \dot{\xi}_d) + e_1\xi_2^* + e_1(\xi_2 - \xi_2^*) + \tilde{\theta}_1\dot{\tilde{\theta}}_1. \end{aligned} \tag{16}$$

Using the neural network approximation (2), we have $\bar{F}_1(\bar{x}_2, \xi_2, \dot{\xi}_d) = W_1^T S(X_1) + \varepsilon_1(X_1)$, where $X_1 = [x_1, x_2, \xi_2, \dot{\xi}_d]^T$, $W_1 \in \mathbb{R}^M$, $S(X_1) \in \mathbb{R}^M$ and $\varepsilon_1(X_1)$ are weight vector, basis function vector, and estimation error, respectively. According to Assumption 1, $||W_1|| \leq \bar{W}_1$, $|\varepsilon_1| \leq \bar{\varepsilon}_1$, where $\bar{W}_1$ and $\bar{\varepsilon}_1$ are unknown positive constants. Define $w_1 = \max\{\bar{W}_1, \bar{\varepsilon}_1\}$, we obtain that $\bar{F}_1(\bar{x}_2, \xi_2, \dot{\xi}_d) \leq w_1\mu_1(X_1)$, where $\mu_1(X_1) = 1 + ||S(X_1)||$. Then, according to Lemma 4, we get

$$e_1\bar{F}_1(\bar{x}_2, \xi_2, \dot{\xi}_d) \leq |e_1|w_1\mu_1 \cdot 1^{q_2} \leq \rho_1\theta_1 e_1^d + c_1, \tag{17}$$

where $\rho_1 = \frac{1}{d}\mu_1^d$, $d = 1 + q_2$, $c_1 = \frac{q_2}{d}$ and $\theta_1 = w_1^d$.

Design the virtual controller ξ_2^* and the adaptive law of $\hat{\theta}_1$ as follows:

$$\begin{aligned} \xi_2^* &= -e_1^{q_2}(n + \epsilon_1 e_1^\kappa + l_1\sqrt{1 + \hat{\theta}_1^2}) \\ &= -e_1^{q_2}\psi_1(e_1, \hat{\theta}_1), \end{aligned} \tag{18}$$

$$\dot{\hat{\theta}}_1 = -\sigma_{1,1}\hat{\theta}_1^{q_2} - \sigma_{1,2}\hat{\theta}_1^{q_2+\kappa} + l_1 e_1^d, \tag{19}$$

where ϵ_1, $\sigma_{1,1}$, $\sigma_{1,2}$ are positive constants, $\kappa > 1$ is an even constant, $l_1 = \rho_1$, $\psi_1(e_1, \hat{\theta}_1) = n + \epsilon_1 e_1^\kappa + l_1\sqrt{1 + \hat{\theta}_1^2} > 0$ is a $\mathcal{C}^1$ function.

Substituting (17)-(19) into (16), we get

$$\begin{aligned} \dot{V}_1 \leq& -ne_1^d - \epsilon_1 e_1^{d+\kappa} + \sigma_{1,1}\tilde{\theta}_1\hat{\theta}_1^{q_2} + \sigma_{1,2}\tilde{\theta}_1\hat{\theta}_1^{q_2+\kappa} \\ &+ C_1 + e_1(\xi_2 - \xi_2^*), \end{aligned} \tag{20}$$

where $C_1 = c_1$.

By employing the second term of Lemma 3, we have

$$\begin{aligned} \sigma_{1,1}\tilde{\theta}_1\hat{\theta}_1^{q_2} &= \sigma_{1,1}\hat{\theta}_1^{q_2}(\theta_1 - \hat{\theta}_1) \\ &\leq \sigma_{1,1}\frac{1}{d}(\theta_1^d - [\theta_1 - \hat{\theta}_1]^d). \end{aligned}$$

According to the adaptive law (19), we can verify that $\hat{\theta}_1(t) \geq 0$ for any given initial value $\hat{\theta}_1(0) \geq 0$ if $\dot{\hat{\theta}}_1(t) \geq 0$. If $\dot{\hat{\theta}}_1(t) < 0$, then $\hat{\theta}_1(t)$ will decrease until $\hat{\theta}_1(t) = 0$ at a certain time t_d. Due to the fact that $l_1 e_1^d \geq 0$, it can be found that $\dot{\hat{\theta}}_1(t) \geq 0$ when $\hat{\theta}_1(t) = 0$. Thus, $\hat{\theta}_1(t) \geq 0$ after $t \geq t_d$. Therefore, if we choose an initial value $\hat{\theta}_1(0) \geq 0$, then we have $\hat{\theta}_1(t) \geq 0$, which means that $\theta_1(t) - \tilde{\theta}_1(t) = \hat{\theta}_1(t) \geq 0$. Next, according to $\theta_1 = w_1^d > 0$ and the first term of Lemma 3, the following inequality can be obtained:

$$\sigma_{1,1}\tilde{\theta}_1\hat{\theta}_1^{q_2} \leq \frac{2\sigma_{1,1}}{d}\theta_1^d - \frac{\sigma_{1,1}}{d}\tilde{\theta}_1^d. \tag{21}$$

Similarly, it can be obtained that:

$$\sigma_{1,2}\tilde{\theta}_1\hat{\theta}_1^{q_2+\kappa} \le \frac{2\sigma_{1,2}}{d+\kappa}\theta_1^{d+\kappa} - \frac{\sigma_{1,2}}{d+\kappa}\tilde{\theta}_1^{d+\kappa}. \tag{22}$$

With the help of inequalities (21), (22), the inequality (20) can be further derived as

$$\begin{aligned}\dot{V}_1 \le& -ne_1^d - \epsilon_1 e_1^{d+\kappa} - \frac{\sigma_{1,1}}{d}\tilde{\theta}_1^d - \frac{\sigma_{1,2}}{d+\kappa}\tilde{\theta}_1^{d+\kappa}\\ &+ e_1(\xi_2 - \xi_2^*) + C_1 + \Lambda_1,\end{aligned} \tag{23}$$

where $\Lambda_1 = \frac{2\sigma_{1,1}}{d}\theta_1^d + \frac{2\sigma_{1,2}}{d+\kappa}\theta_1^{d+\kappa}$.

Step 2: Take the following Lyapunov function:

$$V_2 = V_1 + \int_{\xi_2^*}^{\xi_2}(s^{1/q_2} - \xi_2^{*1/q_2})^{2-q_2}ds + \frac{1}{2}\tilde{\theta}_2^2, \tag{24}$$

where $\tilde{\theta}_2 = \theta_2 - \hat{\theta}_2$, $\hat{\theta}_2$ is the estimation of θ_2. In addition, θ_2, $\hat{\theta}_2$ will be given later. Differentiating V_2 yields:

$$\begin{aligned}\dot{V}_2 \le& -ne_1^d - \epsilon_1 e_1^{d+\kappa} - \frac{\sigma_{1,1}}{d}\tilde{\theta}_1^d\\ &- \frac{\sigma_{1,2}}{d+\kappa}\tilde{\theta}_1^{d+\kappa} + e_1(\xi_2 - \xi_2^*) + C_1 + \Lambda_1\\ &+ e_2^{2-q_2}(F_2(\bar{x}_3, \xi_3) + \xi_3) + \tilde{\theta}_2\dot{\tilde{\theta}}_2\\ &+ (2-q_2)\frac{d(-\xi_2^{*1/q_2})}{dt}\int_{\xi_2^*}^{\xi_2}(s^{1/q_2} - \xi_2^{*1/q_2})^{1-q_2}ds.\end{aligned} \tag{25}$$

Employing the neural network approximation (2), we have $F_2(X_2) = W_2^T S(X_2) + \varepsilon_2$, where $X_2 = [\bar{x}_3, \xi_3]^T$, $W_2 \in \mathbb{R}^M$, $S(X_2) \in \mathbb{R}^M$ and $\varepsilon_2(X_1)$ are weight vector, basis function vector, and estimation error, respectively. According to Assumption 1, $||W_2|| \le \bar{W}_2$, $|\varepsilon_2| \le \bar{\varepsilon}_2$, where $\bar{W}_2$ and $\bar{\varepsilon}_2$ are unknown positive constants. Define $w_2 = \max\{\bar{W}_2, \bar{\varepsilon}_2\}$, we obtain that $F_2(X_2) \le w_2\mu_2(X_2)$, where $\mu_2(X_2) = 1 + ||S(X_2)||$.

Using Lemma 4, it can be obtained that

$$\begin{aligned}e_2^{2-q_2}F_2(\bar{x}_3, \xi_3) &\le |e_2|^{2-q_2}w_2\mu_2 \cdot 1^{q_3}\\ &\le \rho_2 w_2^{d/(2-q_2)}e_2^d + m_2,\end{aligned} \tag{26}$$

where $\rho_2 = \frac{2-q_2}{d}\mu_2^{d/(2-q_2)}$, $m_2 = \frac{q_3}{d}$.

Using Lemma 5, we get

$$\begin{aligned}e_1(\xi_2 - \xi_2^*) &\le |e_1|\left|(\xi_2^{1/q_2})^{q_2} - (\xi_2^{*1/q_2})^{q_2}\right|\\ &\le 2^{1-q_2}|e_1|\left|\xi_2^{1/q_2} - \xi_2^{*1/q_2}\right|^{q_2}\\ &= 2^{1-q_2}|e_1||e_2|^{q_2}\\ &\le \frac{1}{2}e_1^d + \upsilon_1 e_2^d,\end{aligned} \tag{27}$$

where $\upsilon_1 = 2^{1-q_2}\frac{q_2}{d}(\frac{1}{2}2^{q_2-1}d)^{-1/q_2} > 0$.

Since $e_2 = \xi_2^{1/q_2} - \xi_2^{*1/q_2}$ and $\xi_2^* = -e_1^{q_2}\psi_1(\cdot)$, then $|\xi_2| \le (|e_2| + |\xi_2^*|^{1/q_2})^{q_2} \le |e_2|^{q_2} + |\xi_2| = |e_2|^{q_2} + |e_1|^{q_2}\psi_1$, which will be applied to the following second inequality.

$$\begin{aligned}&\left|\frac{d(-\xi_2^{*1/q_2})}{dt}\right|\\ &\le \left(\psi_1^{1/q_2} + \frac{1}{q_2}\psi_1^{\frac{1}{q_2}-1}\epsilon_1\kappa e_1^\kappa\right)(|\bar{F}_1| + |\xi_2|)\\ &\quad + \frac{1}{q_2}\psi_1^{\frac{1}{q_2}-1}e_1 l_1\frac{\hat{\theta}_1}{\sqrt{1+\hat{\theta}_1^2}}\big(-\sigma_{1,1}\hat{\theta}_1^{q_2}\\ &\quad - \sigma_{1,2}\hat{\theta}_1^{q_2+\kappa} + l_1 e_1^d\big)\\ &\le (|e_1|^{q_2} + |e_2|^{q_2})(1+\psi_1)\left(\psi_1^{\frac{1}{q_2}} + \frac{1}{q_2}\psi_1^{\frac{1}{q_2}-1}\epsilon_1\kappa e_1^\kappa\right)\\ &\quad + w_1\mu_1\left(\psi_1^{\frac{1}{q_2}} + \frac{1}{q_2}\psi_1^{\frac{1}{q_2}-1}\epsilon_1\kappa e_1^\kappa\right)\\ &\quad + \frac{1}{q_2}\psi_1^{\frac{1}{q_2}-1}\sqrt{1+e_1^2}\,l_1\left(\sigma_{1,1}\frac{\hat{\theta}_1^d}{\sqrt{1+\hat{\theta}_1^2}} + \sigma_{1,2}\frac{\hat{\theta}_1^{d+\kappa}}{\sqrt{1+\hat{\theta}_1^2}}\right.\\ &\quad \left. + l_1 e_1^d\frac{1}{2}\sqrt{1+\hat{\theta}_1^2}\right)\\ &\le (|e_1^{q_2}| + |e_2|^{q_2})\tilde{G}_1(e_1, \hat{\theta}_1) + w_1\mu_1\hat{G}_1(e_1, \hat{\theta}_1)\\ &\quad + \check{G}_1(e_1, \hat{\theta}_1),\end{aligned} \tag{28}$$

where $\tilde{G}_1(e_1, \hat{\theta}_1) > 0$, $\hat{G}_1(e_1, \hat{\theta}_1) > 0$, and $\check{G}_1(e_1, \hat{\theta}_1) > 0$ are $\mathcal{C}^1$ functions, and

$$\tilde{G}_1(e_1, \hat{\theta}_1) = (1+\psi_1)(\psi_1^{\frac{1}{q_2}} + \frac{1}{q_2}\psi_1^{\frac{1}{q_2}-1}\epsilon_1\kappa e_1^\kappa),$$

$$\begin{aligned}\check{G}_1(e_1, \hat{\theta}_1) =& \frac{1}{q_2}\psi_1^{\frac{1}{q_2}-1}\sqrt{1+e_1^2}\,l_1\left(\sigma_{1,1}\frac{\hat{\theta}_1^d}{\sqrt{1+\hat{\theta}_1^2}}\right.\\ &\left. + \sigma_{1,2}\frac{\hat{\theta}_1^{d+\kappa}}{\sqrt{1+\hat{\theta}_1^2}} + l_1 e_1^d\frac{1}{2}\sqrt{1+\hat{\theta}_1^2}\right).\end{aligned}$$

$$\hat{G}_1(e_1, \hat{\theta}_1) = \psi_1^{\frac{1}{q_2}} + \frac{1}{q_2}\psi_1^{\frac{1}{q_2}-1}\epsilon_1\kappa e_1^\kappa.$$

In addition, we can derive that

$$\begin{aligned}&\left|(2-q_2)\int_{\xi_2^*}^{\xi_2}(s^{1/q_2} - \xi_2^{*1/q_2})^{1-q_2}ds\right|\\ &\quad \le (2-q_2)|e_2|^{1-q_2}|\xi_2 - \xi_2^*|\\ &\quad \le (2-q_2)|e_2|^{1-q_2}\left|(\xi_2^{1/q_2})^{q_2} - (\xi_2^{*1/q_2})^{q_2}\right|\\ &\quad \le (2-q_2)|e_2|^{1-q_2}2^{1-q_2}|e_2|^{q_2}\\ &\quad \le (2-q_2)2^{1-q_2}|e_2|.\end{aligned} \tag{29}$$

According to (28) and (29) and Lemma 4, we obtain that

$$\begin{aligned}&\left|(2-q_2)\frac{d(-\xi_2^{*1/q_2})}{dt}\int_{\xi_2^*}^{\xi_2}(s^{1/q_2} - \xi_2^{*1/q_2})^{1-q_2}ds\right|\\ &\le (2-q_2)2^{1-q_2}|e_2|\Big[(|e_1|^{q_2} + |e_2|^{q_2})\tilde{G}_1\\ &\quad + w_1\mu_1\hat{G}_1 + \check{G}_1\Big]\\ &\le \frac{1}{2}e_1^d + G_1 e_2^d + (2-q_2)2^{1-q_2}\frac{1}{d}\mu_1^d w_1^d\hat{G}_1^d e_2^d + g_2,\end{aligned} \tag{30}$$

where $G_1 = (2-q_2)2^{1-q_2}[\frac{1}{d}(\frac{1}{2}\frac{d}{q_2}\frac{1}{2^{1-q_2}(2-q_2)})^{-q_2}\tilde{G}_1^d + \tilde{G}_1 + \frac{1}{d}\check{G}_1^d] > 0$ is a $\mathcal{C}^1$ function, $g_2 = (2-q_2)2^{1-q_2}\frac{2q_2}{d} > 0$ is a constant.

Substituting (26), (27), (30) into (25) yields

$$\begin{aligned}\dot{V}_2 \le& -(n-1)e_1^d - \epsilon_1 e_1^{d+\kappa} - \frac{\sigma_{1,1}}{d}\tilde{\theta}_1^d\\ &- \frac{\sigma_{1,2}}{d+\kappa}\tilde{\theta}_1^{d+\kappa} + \Lambda_1 + (\upsilon_1 + G_1)e_2^d\end{aligned}$$

$$
\begin{aligned}
&+ l_2\theta_2 e_2^d + C_2 + \tilde{\theta}_2\dot{\hat{\theta}}_2 \\
&+ e_2^{2-q_2}\xi_3^* + e_2^{2-q_2}(\xi_3 - \xi_3^*),
\end{aligned} \tag{31}
$$

where $\theta_2 = \max\{w_2^{d/(2-q_2)}, w_1^d\} > 0$, $l_2 = \rho_2 + (2-q_2)2^{1-q_2}\frac{1}{d}\mu_1^d\hat{G}_1^d > 0$, $C_2 = C_1 + m_2 + g_2$ is a positive constant.

Next, we design the virtual controller ξ_3^* as follows:

$$
\begin{aligned}
\xi_3^* &= -e_2^{q_3}\Big(n - 1 + \epsilon_2 e_2^\kappa + (v_1 + G_1) + l_2\sqrt{1 + \hat{\theta}_2^2}\Big) \\
&= -e_2^{q_3}\psi_2(e_1, \xi_2, \hat{\theta}_1, \hat{\theta}_2),
\end{aligned} \tag{32}
$$

and the adaptive law is given as follows:

$$
\dot{\hat{\theta}}_2 = -\sigma_{2,1}\hat{\theta}_2^{q_2} - \sigma_{2,2}\hat{\theta}_2^{q_2+\kappa} + l_2 e_2^d, \tag{33}
$$

where $\psi_2(\cdot) = n - 1 + \epsilon_2 e_2^\kappa + (v_1 + G_1) + l_2\sqrt{1 + \hat{\theta}_2^2}\big) > 0$ is a $\mathcal{C}^1$ function, $\sigma_{2,1}$, $\sigma_{2,2}$ are positive constants.

Substituting (32) and (33) into (31), and according to the similar analysis of (21), (22), we get

$$
\begin{aligned}
\dot{V}_2 \le &-(n-1)(e_1^d + e_2^d) - (\epsilon_1 e_1^{d+\kappa} + \epsilon_2 e_2^{d+\kappa}) \\
&- \frac{\sigma_{1,1}}{d}\tilde{\theta}_1^d - \frac{\sigma_{1,2}}{d+\kappa}\tilde{\theta}_1^{d+\kappa} + C_2 + \Lambda_2 \\
&- \frac{\sigma_{2,1}}{d}\tilde{\theta}_2^d - \frac{\sigma_{2,2}}{d+\kappa}\tilde{\theta}_2^{d+\kappa} + e_2^{2-q_2}(\xi_3 - \xi_3^*),
\end{aligned} \tag{34}
$$

where $\Lambda_2 = \Lambda_1 + (\frac{2\sigma_{2,1}}{d}\theta_2^d + \frac{2\sigma_{2,2}}{d+\kappa}\theta_2^{d+\kappa})$.

Step k: $(3 \le k \le n-1)$ Assume that for the Lyapunov function

$$
V_k = V_{k-1} + \int_{\xi_k^*}^{\xi_k}(s^{1/q_k} - \xi_k^{*1/q_k})^{2-q_k}ds + \frac{1}{2}\tilde{\theta}_k^2, \tag{35}
$$

there exist virtual controllers and adaptive laws $\xi_{k+1}^* = -e_k^{q_{k+1}}\psi_k(\cdot)$, $\dot{\hat{\theta}}_k = -\sigma_{k,1}\hat{\theta}_k^{q_2} - \sigma_{k,2}\hat{\theta}_k^{q_2+\kappa} + l_k e_k^d$, where $k = 3, \ldots, n-1$, $\psi_k(e_1, \xi_2, \ldots, \xi_k, \hat{\theta}_1, \ldots, \hat{\theta}_k) > 0$ is a $\mathcal{C}^1$ function, such that

$$
\begin{aligned}
\dot{V}_k \le &-(n-k+1)(e_1^d + e_2^d + \ldots + e_k^d) \\
&- (\epsilon_1 e_1^{d+\kappa} + \epsilon_2 e_2^{d+\kappa} + \ldots + \epsilon_k e_k^{d+\kappa}) \\
&- (\frac{\sigma_{1,1}}{d}\tilde{\theta}_1^d + \frac{\sigma_{2,1}}{d}\tilde{\theta}_2^d + \ldots + \frac{\sigma_{k,1}}{d}\tilde{\theta}_k^d) \\
&- (\frac{\sigma_{1,2}}{d+\kappa}\tilde{\theta}_1^{d+\kappa} + \frac{\sigma_{2,2}}{d+\kappa}\tilde{\theta}_2^{d+\kappa} + \ldots + \frac{\sigma_{k,2}}{d+\kappa}\tilde{\theta}_k^{d+\kappa}) \\
&+ C_k + \Lambda_k + e_k^{2-q_k}(\xi_{k+1} - \xi_{k+1}^*).
\end{aligned}
$$

Then, we prove that there is a similar conclusion at step $k+1$. Construct the following Lyapunov function:

$$
V_{k+1} = V_k + \int_{\xi_{k+1}^*}^{\xi_{k+1}}(s^{1/q_{k+1}} - \xi_{k+1}^{*1/q_{k+1}})^{2-q_{k+1}}ds + \frac{1}{2}\tilde{\theta}_{k+1}^2. \tag{36}
$$

Then, we have

$$
\begin{aligned}
\dot{V}_{k+1} \le &\dot{V}_k + e_{k+1}^{2-q_{k+1}}F_{k+1} + e_{k+1}^{2-q_{k+1}}\xi_{k+2}^* \\
&+ e_{k+1}^{2-q_{k+1}}(\xi_{k+2} - \xi_{k+2}^*) + (2-q_{k+1})\frac{d(-\xi_{k+1}^{*1/q_{k+1}})}{dt} \\
&\cdot \int_{\xi_{k+1}^*}^{\xi_{k+1}}(s^{1/q_{k+1}} - \xi_{k+1}^{*1/q_{k+1}})^{1-q_{k+1}}ds \\
&+ \tilde{\theta}_{k+1}\dot{\hat{\theta}}_{k+1}.
\end{aligned}
$$

For F_{k+1}, by employing neural network approximation (2), it can be estimated as $F_{k+1} = W_{k+1}^T S(X_{k+1}) + \varepsilon_{k+1}$, where $X_{k+1} = [\bar{x}_{k+2}, \xi_{k+2}]^T$, $||W_{k+1}|| \le \bar{W}_{k+1}$, $|\varepsilon_{k+1}| \le \bar{\varepsilon}_{k+1}$, and $w_{k+1} = \max\{\bar{W}_{k+1}, \varepsilon_{k+1}\}$. Then, according to Lemma 4, we obtain that

$$
\begin{aligned}
e_{k+1}^{2-q_{k+1}}F_{k+1} &\le |e_{k+1}|^{2-q_{k+1}}|F_{k+1}| \cdot 1^{q_{k+2}} \\
&\le \rho_{k+1}w_{k+1}^d\mu_{k+1}^d e_{k+1}^d + m_{k+1},
\end{aligned} \tag{37}
$$

where $\rho_{k+1} = \frac{2-q_{k+1}}{d}$, $m_{k+1} = \frac{q_{k+2}}{d}$.

In addition, according to Lemma 4 and Lemma 5, we obtain

$$
\begin{aligned}
e_k^{2-q_k}(\xi_{k+1} - \xi_{k+1}^*) &= |e_k|^{2-q_k}\big|(\xi_{k+1}^{1/q_{k+1}})^{q_{k+1}} - (\xi_{k+1}^{*1/q_{k+1}})^{q_{k+1}}\big| \\
&\le \frac{1}{2}e_k^d + v_k e_{k+1}^d,
\end{aligned} \tag{38}
$$

where $v_k = 2^{1-q_{k+1}}\frac{q_{k+1}}{d}(\frac{1}{2}\frac{d}{2-q_k}\frac{1}{2^{1-q_{k+1}}})^{-(2-q_k)/q_{k+1}} > 0$ is a constant.

Similar to the derivation of (28), we have

$$
\begin{aligned}
&\left|\frac{d(-\xi_{k+1}^{*1/q_{k+1}})}{dt}\right| \\
&\le \left|\frac{\partial\big(-\xi_{k+1}^{*1/q_{k+1}}\big)}{\partial e_1}\right|\big(|\bar{F}_1| + |\xi_2|\big) \\
&\quad + \left|\frac{\partial\big(-\xi_{k+1}^{*1/q_{k+1}}\big)}{\partial \xi_2}\right|\big(|F_2| + |\xi_3|\big) + \ldots \\
&\quad + \left|\frac{\partial\big(-\xi_{k+1}^{*1/q_{k+1}}\big)}{\partial \xi_k}\right|\big(|F_k| + |\xi_{k+1}|\big) \\
&\quad + \left|\frac{\partial\big(-\xi_{k+1}^{*1/q_{k+1}}\big)}{\partial \hat{\theta}_1}\big(-\sigma_{1,1}\hat{\theta}_1^{q_2} - \sigma_{1,2}\hat{\theta}_1^{q_2+\kappa} + l_1 e_1^d\big)\right| \\
&\quad + \ldots + \left|\frac{\partial\big(-\xi_{k+1}^{*1/q_{k+1}}\big)}{\partial \hat{\theta}_k}\big(-\sigma_{k,1}\hat{\theta}_k^{q_2} - \sigma_{k,2}\hat{\theta}_k^{q_2+\kappa} + l_k e_k^d\big)\right|.
\end{aligned}
$$

Since $e_k = \xi_k^{1/q_k} - \xi_k^{*1/q_k}$, then $|\xi_k| \le (|e_k| + |\xi_k^*|^{1/q_k})^{q_k} \le |e_k|^{q_k} + |\xi_k^*| = |e_k|^{q_k} + |e_{k-1}|^{q_k}\psi_{k-1}$. Further, according to the Proposition B.5 in reference [35], we can obtain that

$$
\begin{aligned}
\left|\frac{d(-\xi_{k+1}^{*1/q_{k+1}})}{dt}\right| &\le \big(|e_1|^{q_2} + |e_2|^{q_2} + \ldots + |e_{k+1}|^{q_2}\big)\tilde{G}_k(\bar{e}_k, \bar{\hat{\theta}}_k) \\
&\quad + \sum_{l=1}^{k} w_l\mu_l\hat{G}_{k,l}(\bar{e}_k, \bar{\hat{\theta}}_k) + \check{G}_k(\bar{e}_k, \bar{\hat{\theta}}_k),
\end{aligned} \tag{39}
$$

where $\bar{e}_k = [e_1, \ldots, e_k]^T$, $\bar{\hat{\theta}}_k = [\hat{\theta}_1, \ldots, \hat{\theta}_k]^T$, $\tilde{G}_k(\cdot) > 0$, $\hat{G}_{k,l}(\cdot) > 0$, $\check{G}_k(\cdot) > 0$ are $\mathcal{C}^1$ functions.

Moreover, the following inequality can also be derived:

$$
\begin{aligned}
(2-q_{k+1})&\int_{\xi_{k+1}^*}^{\xi_{k+1}}(s^{1/q_{k+1}} - \xi_{k+1}^{*1/q_{k+1}})^{1-q_{k+1}}ds \\
&\le (2-q_{k+1})2^{1-q_{k+1}}|e_{k+1}|.
\end{aligned} \tag{40}
$$

Then, according to Lemma 4, we have

$$
(2-q_{k+1})\frac{d(-\xi_{k+1}^{*1/q_{k+1}})}{dt}\int_{\xi_{k+1}^*}^{\xi_{k+1}}(s^{1/q_{k+1}} - \xi_{k+1}^{*1/q_{k+1}})^{1-q_{k+1}}ds
$$

$$\leq \sum_{l=1}^{k} \frac{1}{2} e_l^d + G_k(\cdot) e_{k+1}^d + \sum_{l=1}^{k} w_l^d H_{k,l}(\cdot) e_{k+1}^d + g_{k+1}, \quad (41)$$

where $G_k(\cdot) = (2-q_{k+1})2^{1-q_{k+1}}\big[\frac{1}{d}\big(\frac{1}{2}\frac{d}{q_2}\frac{2^{q_{k+1}-1}}{2-q_{k+1}}\big)^{-q_2} k\tilde{G}_k^d + \tilde{G}_k + \frac{1}{d}\check{G}_k^d\big] > 0$ and $H_{k,l}(\cdot) = (2-q_{k+1})2^{2-q_{k+1}}\frac{1}{d}\mu_l^d \hat{G}_{k,l}^d > 0$ are $\mathcal{C}^1$ functions, $g_{k+1} = (2-q_{k+1})2^{2-q_{k+1}}\frac{(k+1)q_2}{d}$ is a positive constant.

Combining the inequalities (37), (38) and (41), we obtain

$$\begin{aligned}\dot{V}_{k+1} \leq& -(n-k)(e_1^d + e_2^d + \ldots + e_k^d) \\ &- (\epsilon_1 e_1^{d+\kappa} + \epsilon_2 e_2^{d+\kappa} + \ldots + \epsilon_k e_k^{d+\kappa}) \\ &+ \big(v_k + G_k\big) e_{k+1}^d \\ &+ \big(\rho_k + \sum_{l=1}^{k} H_{k,l}\big)\theta_{k+1} e_{k+1}^d \\ &+ e_{k+1}^{2-q_{k+1}}(\xi_{k+2} - \xi_{k+2}^*) + e_{k+1}^{2-q_{k+1}}\xi_{k+2}^* \\ &+ \tilde{\theta}_{k+1}\dot{\hat{\theta}}_{k+1} + C_{k+1} + \Lambda_k, \end{aligned} \quad (42)$$

where $\theta_{k+1} = \max\{w_{k+1}^{d/(2-q_{k+1})}, w_k^d, \ldots, w_1^d\}$, $C_{k+1} = C_k + m_{k+1} + g_{k+1}$. Design ξ_{k+2}^* as follows:

$$\begin{aligned}\xi_{k+2}^* =& -e_{k+1}^{q_{k+2}}\Big(n - k + \epsilon_{k+1} e_{k+1}^{\kappa} \\ &+ v_k + G_k + l_{k+1}\sqrt{1+\hat{\theta}_{k+1}^2}\Big) \\ =& -e_{k+1}^{q_{k+2}}\psi_{k+1}(\cdot), \end{aligned} \quad (43)$$

and

$$\dot{\hat{\theta}}_{k+1} = -\sigma_{k+1,1}\hat{\theta}_{k+1}^{q_2} - \sigma_{k+1,2}\hat{\theta}_{k+1}^{q_2+\kappa} + l_{k+1} e_{k+1}^d, \quad (44)$$

where the $\mathcal{C}^1$ function $\psi_{k+1}(\cdot) = n-k+\epsilon_{k+1}e_{k+1}^{\kappa} + v_k + G_k + l_{k+1}\sqrt{1+\hat{\theta}_{k+1}^2} > 0$, and $\sigma_{k+1,1}$, $\sigma_{k+1,2}$ are positive constants, $l_{k+1} = \rho_k + \sum_{l=1}^{k} H_{k,l}$.

Substituting (43) and (44) into (42), we can derive that

$$\begin{aligned}\dot{V}_{k+1} \leq& -(n-k)(e_1^d + e_2^d + \ldots + e_k^d) \\ &- (\epsilon_1 e_1^{d+\kappa} + \epsilon_2 e_2^{d+\kappa} + \ldots + \epsilon_k e_k^{d+\kappa}) \\ &- (\frac{\sigma_{1,1}}{d}\tilde{\theta}_1^d + \frac{\sigma_{2,1}}{d}\tilde{\theta}_2^d + \ldots + \frac{\sigma_{k+1,1}}{d}\tilde{\theta}_{k+1}^d) \\ &- (\frac{\sigma_{1,2}}{d+\kappa}\tilde{\theta}_1^{d+\kappa} + \ldots + \frac{\sigma_{k+1,2}}{d+\kappa}\tilde{\theta}_{k+1}^{d+\kappa}) \\ &+ C_{k+1} + \Lambda_{k+1} + e_{k+1}^{2-q_{k+1}}(\xi_{k+2} - \xi_{k+2}^*), \end{aligned} \quad (45)$$

where $\Lambda_{k+1} = \Lambda_k + (\frac{2\sigma_{k+1,1}}{d}\theta_{k+1}^d + \frac{2\sigma_{k+1,2}}{d+\kappa}\theta_{k+1}^{d+\kappa})$.

Step n: Construct the Lyapunov function V_n as follows:

$$V_n = V_{n-1} + \int_{\xi_n^*}^{\xi_n} (s^{1/q_n} - \xi_n^{*1/q_n})^{2-q_n} ds + \frac{1}{2}\tilde{\theta}_n^2, \quad (46)$$

where $\tilde{\theta}_n = \theta_n - \hat{\theta}_n$, $\hat{\theta}_n$ is the estimation of θ_n, $\xi_n^* = \varphi_n g_n u$. Set $k = n-1$, then $V_n = V_{k+1}$. Thus, $\theta_n = \theta_{k+1}$, which is given in (42). Based on the analysis of (42), we have

$$\begin{aligned}\dot{V}_n \leq& -(e_1^d + \ldots + e_{n-1}^d) - (\epsilon_1 e_1^{d+\kappa} + \ldots + \epsilon_{n-1} e_{n-1}^{d+\kappa}) \\ &- (\frac{\sigma_{1,1}}{d}\tilde{\theta}_1^d + \frac{\sigma_{2,1}}{d}\tilde{\theta}_2^d + \ldots + \frac{\sigma_{n-1,1}}{d}\tilde{\theta}_{n-1}^d) \\ &- (\frac{\sigma_{1,2}}{d+\kappa}\tilde{\theta}_1^{d+\kappa} + \frac{\sigma_{2,2}}{d+\kappa}\tilde{\theta}_2^{d+\kappa} + \ldots + \frac{\sigma_{n-1,2}}{d+\kappa}\tilde{\theta}_{n-1}^{d+\kappa}) \\ &+ (v_{n-1} + G_{n-1})e_n^d + e_n^{2-q_n}(\xi_n^*) \\ &+ (\rho_{n-1} + \sum_{l=1}^{n-1} H_{n-1,l})\theta_n e_n^d + \tilde{\theta}_n\dot{\hat{\theta}}_n \\ &+ C_n + \Lambda_{n-1}. \end{aligned}$$

According to (42) and (43), we design the controller u and the adaptive law $\dot{\hat{\theta}}_n$ as follows:

$$u = -\frac{1}{\varphi_n \underline{g}_n} e_n^{q_{n+1}}\Big(1 + \epsilon_n e_n^{\kappa} + v_{n-1} + G_{n-1} + l_n\sqrt{1+\hat{\theta}_n^2}\Big), \quad (47)$$

$$\dot{\hat{\theta}}_n = -\sigma_{n,1}\hat{\theta}_n^{q_2} - \sigma_{n,2}\hat{\theta}_n^{q_2+\kappa} + l_n e_n^d, \quad (48)$$

where $\kappa > 1$ is an even constant, $\sigma_{n,1}$, $\sigma_{n,2}$ are positive constants, $l_n = \rho_{n-1} + \sum_{l=1}^{n-1} H_{n-1,l}$, and $\rho_{n-1} = \frac{2-q_{n-1}}{d}$, $v_{n-1} = 2^{1-q_n}\frac{q_n}{d}(\frac{1}{2}\frac{d}{2-q_{n-1}}\frac{1}{2^{1-q_n}})^{-(2-q_{n-1})/q_n} > 0$, $H_{n-1,l}(\cdot) = (2-q_n)2^{2-q_n}\frac{1}{d}\mu_l^d\hat{G}_{n-1,l}^d > 0$. $\hat{G}_{n-1,l} > 0$ and $G_{n-1} > 0$ are $\mathcal{C}^1$ functions. The positive constant $\underline{g}_n$ is the lower bound of g_n.

According to (47) and (48), we can obtain

$$\begin{aligned}\dot{V}_n \leq& -(e_1^d + \ldots + e_n^d) - (\epsilon_1 e_1^{d+\kappa} + \ldots + \epsilon_n e_n^{d+\kappa}) \\ &- (\frac{\sigma_{1,1}}{d}\tilde{\theta}_1^d + \ldots + \frac{\sigma_{n,1}}{d}\tilde{\theta}_n^d) \\ &- (\frac{\sigma_{1,2}}{d+\kappa}\tilde{\theta}_1^{d+\kappa} + \frac{\sigma_{n,2}}{d+\kappa}\tilde{\theta}_n^{d+\kappa}) + C_n + \Lambda_n, \end{aligned} \quad (49)$$

where $\Lambda_n = \Lambda_{n-1} + (\frac{2\sigma_{n,1}}{d}\theta_n^d + \frac{2\sigma_{n,2}}{d+\kappa}\theta_n^{d+\kappa})$.

Appendix B
Proof of the Theorem 1

Proof: According to Lemma 5, V_n can be analyzed as follows:

$$\begin{aligned}V_n \leq& \frac{1}{2}e_1^2 + \sum_{k=2}^{n} 2^{1-q_k} e_k^2 + \sum_{k=1}^{n}\frac{1}{2}\tilde{\theta}_k^2 \\ \leq& \varpi\Big(\sum_{k=1}^{n} e_k^2 + \sum_{k=1}^{n}\tilde{\theta}_k^2\Big), \end{aligned} \quad (50)$$

where $\varpi = \max\{\frac{1}{2}, 2^{1-q_k}\}$.

We have derived that

$$\begin{aligned}\dot{V}_n \leq& -(e_1^d + \ldots + e_n^d) - (\epsilon_1 e_1^{d+\kappa} + \ldots + \epsilon_n e_n^{d+\kappa}) \\ &- (\frac{\sigma_{1,1}}{d}\tilde{\theta}_1^d + \ldots + \frac{\sigma_{n,1}}{d}\tilde{\theta}_n^d) \\ &- (\frac{\sigma_{1,2}}{d+\kappa}\tilde{\theta}_1^{d+\kappa} + \frac{\sigma_{n,2}}{d+\kappa}\tilde{\theta}_n^{d+\kappa}) + \Omega \\ =& -\sum_{k=1}^{n} e_k^d - \sum_{k=1}^{n}\epsilon_k e_k^{d+\kappa} - \sum_{k=1}^{n}\frac{\sigma_{k,1}}{d}\tilde{\theta}_k^d \\ &- \sum_{k=1}^{n}\frac{\sigma_{k,2}}{d+\kappa}\tilde{\theta}_k^{d+\kappa} + \Omega, \end{aligned} \quad (51)$$

where $\Omega = C_n + \Lambda_n$. Then, according to the third term of Lemma 3, we can continue to obtain that

$$\dot{V}_n \leq -\varpi_1\sum_{k=1}^{n}(e_k^d + \tilde{\theta}_k^d) - \varpi_2\sum_{k=1}^{n}(e_k^{d+\kappa} + \tilde{\theta}_k^{d+\kappa}) + \Omega$$

$$\leq -\frac{\varpi_1}{\varpi^{d/2}}\left(\varpi\sum_{k=1}^{n}(e_k^2+\tilde{\theta}_k^2)\right)^{\frac{d}{2}}+\Omega$$
$$-\frac{\varpi_2}{\varpi^{(d+\kappa)/2}}2n^{1-\frac{d+\kappa}{2}}\left(\varpi\sum_{k=1}^{n}(e_k^2+\tilde{\theta}_k^2)\right)^{\frac{d+\kappa}{2}}$$
$$= -\Gamma_1 V_n^{\frac{d}{2}} - \Gamma_2 V_n^{\frac{d+\kappa}{2}} + \Omega, \tag{52}$$

where $\varpi_1 = \min_{k=1,\ldots,n}\{1, \frac{\sigma_{k,1}}{d}\}$, $\varpi_2 = \min_{k=1,\ldots,n}\{1, \frac{\sigma_{k,2}}{d+\kappa}\}$, $\Gamma_1 = \frac{\varpi_1}{\varpi^{d/2}}$, and $\Gamma_2 = \frac{\varpi_2}{\varpi^{(d+\kappa)/2}}$.

Then, using Lemma 1, we obtain that there exists a time T, such that

$$V_n \leq \mathcal{R} := \min\left\{\left(\frac{\Omega}{\Gamma_1\theta}\right)^{\frac{2}{d}}, \left(\frac{\Omega}{\Gamma_2\theta}\right)^{\frac{2}{d+\kappa}}\right\}, \tag{53}$$

when $t \geq T$, where

$$T \leq \frac{2}{\Gamma_1(1-\theta)(2-d)} + \frac{2}{\Gamma_2(1-\theta)(d+\kappa-2)},$$

and $0 < \theta < 1$ is a constant. This implies that the closed-loop system (12) is practically fixed-time stable. Moreover, because $V_n \in \mathcal{L}_\infty$, thus $e_k \in \mathcal{L}_\infty$, $\tilde{\theta}_k \in \mathcal{L}_\infty$, $k = 1, \ldots, n$. Then, we can derive that ξ_k^* is bounded. Thus $\xi_k = e_k + \xi_k^*$, $k = 2, \ldots, n$ are bounded, and $\xi_1 = e_1 + \xi_d$ is bounded. Therefore, all states of system (10) are bounded.

Due to $V_n \leq \mathcal{R}$ when $t \geq T$, thus, we have $|e_1| \leq \sqrt{2\mathcal{R}}$ when $t \geq T$. Define the output tracking error $z = y - y_d$. Recalling the transformation functions (8) and (11) and applying the mean value theorem, we get that there exists a constant $\hat{\xi}$ such that

$$\begin{aligned}
|z| &= \left|\frac{h_{11}(t)+h_{12}(t)}{2}\tanh\left(\frac{2\xi_1}{h_{11}(t)+h_{12}(t)}\right)\right.\\
&\quad \left. - \frac{h_{11}(t)+h_{12}(t)}{2}\tanh\left(\frac{2\xi_d}{h_{11}(t)+h_{12}(t)}\right)\right|\\
&= \frac{h_{11}(t)+h_{12}(t)}{2}\left|\frac{1}{\cosh^2(\hat{\xi})}\times\frac{2(\xi_1-\xi_d)}{h_{11}(t)+h_{12}(t)}\right|\\
&\leq \frac{h_{11}(t)+h_{12}(t)}{2}\times\frac{2}{h_{11}(t)+h_{12}(t)}|\xi_1-\xi_d|\\
&= |\xi_1-\xi_d|\\
&= e_1,
\end{aligned}$$

where $\hat{\xi} \in \left(\frac{2\xi_1}{h_{11}(t)+h_{12}(t)}, \frac{2\xi_d}{h_{11}(t)+h_{12}(t)}\right)$. Therefore, $|z| = |y - y_d| \leq |e_1| \leq \sqrt{2\mathcal{R}}$ when $t \geq T$, which means that the practical fixed-time output tracking objective (7) is achieved, that is,

$$|y(t) - y_d(t)| \leq \zeta = \sqrt{2\mathcal{R}}, \quad \forall t \geq T.$$

Moreover, since it has been obtained that ξ_k, $k = 1, \ldots, n$ are bounded, thus $-h_{k1}(t) < x_k < h_{k2}(t)$ is satisfied for $-h_{k1}(0) < x_k(0) < h_{k2}(0)$. Therefore, all the state constraints are not violated all the time.

This completes the proof.

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