KYBERNETIKA — VOLUME *59* (2023), NUMBER 3, PAGES 461–483

CONSENSUS OF MULTI-AGENT SYSTEMS AND STABILIZATION OF LARGE-SCALE SYSTEMS WITH TIME DELAYS AND NONLINEARITIES - A COMPARISON OF BOTH PROBLEMS

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The problem of stabilization of large-scale systems and the consensus problem of multi-agent systems are related, similar tools for their solution are used. Therefore, they are occasionally confused. Although both problems show similar features, one can also observe important differences. A comparison of both problems is presented in this paper. In both cases, attention is paid to the explanation of the effects of the time delays. The most important fact is that, if the time delays are heterogeneous, full synchronization of the multi-agent systems cannot be achieved; however, stabilization of the large-scale network is reachable. In the case of nonlinear systems, we show that the stabilization of a large-scale nonlinear system is possible under more restrictive assumptions compared to the synchronization of a nonlinear multi-agent system.

Keywords: large-scale interconnected systems, multi-agent systems, time delays, nonlinearity

Classification: 93A14, 93B36

1. INTRODUCTION

The stabilization of a large-scale interconnected system and the synchronization of a multi-agent system are two important problems of the recent control theory. Both problems share several similar features; however, many differences between these two types of problems exist. This paper focuses on highlighting the commonalities and differences between these problems.

The large-scale interconnected system is a dynamical system which can be decomposed into a set of subsystems that mutually interact through interconnections. Frequently, these interconnections are physically present in the system. Examples of large-scale interconnected systems include, but are not limited to, power networks, parallel chemical reactors with common precooling, flexible structures, etc. [2]. Since large-scale interconnected systems usually cannot be controlled or stabilized in a centralized manner (such control would be overly complicated, prone to failures, etc.), one seeks a decentralized control law (see, e. g. [14]). Here, the goal is to find a control law identical for all subsystems so that, for any given subsystem, the proposed controller makes use

DOI: 10.14736/kyb-2023-3-0461

of the state information of this particular subsystem only. In other words, the control action is independent of the states of other subsystems. This means the control law must be sufficiently robust to mitigate the effects of the other subsystems interfering through interconnections; see [1] or [8], among others. Recently, the formulation of the algorithms for finding decentralized control is conducted with the help of linear matrix inequalities (LMI).

In this paper, it is assumed that the subsystems are identical. This assumption, although it may seem to be overly restrictive, is valid in many real-world applications [2, 14], among others. The goal pursued in this paper is to find a control law that is identical for all subsystems. Moreover, the complexity of the control design problem (measured, e. g., in the number of variables in LMIs to be solved) should be independent of the number of subsystems.

Controlling large-scale systems is usually dependent on the use of communication networks for signal transmission, whether from the sensors to the controllers or from the controllers to the actuators, hence, a cost-effective implementation of the control scheme can be achieved. Nevertheless, this also brings certain difficulties: time delays caused by the transmission of the signal through the network or due to the occurring packet dropouts are inevitable. The control law must be capable of stabilizing the system under fast-varying time delays (the derivative of the time delay equals 1), see e. g. [3, 4] or [5]. Here, the Razumikhin functional is used to find the desired control algorithm. Application of the Razumikhin functional yields methods useful for systems with the aforementioned fast varying delays; unfortunately, these results tend to be rather conservative. Therefore, the recently developed descriptor approach (based on the Lyapunov–Krasovskii functional, but, despite this fact, still applicable to systems exhibiting fast-varying delays, as shown in [9]) is useful. Its application to large-scale systems yields good results, [21]. Moreover, let us mention that the sampled control of large-scale systems is presented in [20] or the quantized control studied in [22], among other sources.

The second problem considered in this paper - the synchronization problem of multi-agent systems - sees also many applications in practice: control of platoons of vehicles, control of swarms of autonomous drones, to name a few; it has also been intensively studied in the recent past. One can distinguish two basic problems: the leader following problem and the consensus problem. For a detailed description, the reader is referred to [12] or [16]. In this paper, only the consensus problem is considered. The characteristic feature of the multi-agent synchronization problem is the restricted communication between agents. The agents are able to communicate only with the neighboring agents, the number of these agents is typically much smaller than the total number of agents.

If one attempts to solve the synchronization problem of multi-agent systems with the application of communication networks, the same challenges as described in the case of the networked control of large-scale systems have to be overcome. Many papers deal with multi-agent systems with homogeneous delays (delays of all agents are equal). This assumption simplifies the analysis considerably, even though it is rather unnatural; see [10] or [30] for details. Paper [18] uses Lyapunov–Krasovskii functionals for the synchronization problem of nonlinear multi-agent systems, they are analogous to the functionals used in this paper. The main focus of [23, 28] is the consensus problem of nonlinear multi-agent systems with input delay. It was shown that, even if full synchronization is not achievable, a bound on the synchronization error can be derived.

The presence of heterogeneous time delays in the multi-agent systems makes the synchronization problem more challenging. As shown in [24] for the case of symmetric graph topology or in [26] (where general interconnecting topologies are considered), heterogeneous delays may cause a steady synchronization error. This error does not, for time increasing to infinity, converge to zero. Fortunately, one can derive an estimate of this error, see also [13, 15, 32]. Let us also note that [17, 33], and [31] investigate synchronization of multi-agent systems such that time delays are different in every communication channel.

The large-scale interconnected systems control and the synchronization of multi-agent systems were compared in [7], however, only for systems without delays and nonlinearities. In this paper, a similar comparison is presented for nonlinear problems and systems with delays. Analogies between both problems, such as methods for computation of the control in both cases, are described. It will be demonstrated that the asymptotic stabilization of the large-scale system can be achieved with heterogeneous time delays in the system. On the other hand, this is not always the case of the full synchronization of the multi-agent system. In this case, we derive a bound on the synchronization error.

Purpose of this paper

- to provide a comparison of the properties of the control of large-scale interconnected systems subject to delayed control signals with multi-agent systems, again with time delays. Special attention is paid to the effects caused by uncertainties in the systems as well as to the effects of heterogeneous time delays,
- to present algorithms derived with the descriptor approach for both the large-scale system stabilization and for the multi-agent consensus synchronization; these algorithms lead to effective and not overly conservative design methods,
- to conduct an analogous comparison for nonlinear multi-agent and large-scale systems.

Notation

1. The LMI $P > 0$ means matrix P is a square symmetric positive definite matrix.
2. The elements below the diagonal are not written explicitly, they are replaced by an asterisk for symmetric matrices: $\left(\begin{smallmatrix} a & b \\ b^T & c \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & b \\ * & c \end{smallmatrix}\right)$.
3. If no confusion can arise, the time argument t is often omitted for brevity, and the time delay is written using subscript: $x = x(t)$, $x(t-\tau) = x_\tau(t) = x_\tau$. However, if the time argument is different from t, it is written in full.
4. The m-dimensional identity matrix is denoted by the symbol I_m.
5. The symbol $\|.\|$ stands for the Euclidean norm (even for matrices).
6. The Kronecker product is denoted by the symbol $\otimes$.
7. For a square matrix A we define $\mathcal{H}(A) = A + A^T$.

2. DEFINITION OF LARGE-SCALE INTERCONNECTED SYSTEMS AND MULTI-AGENT SYSTEMS

2.1. Large-scale interconnected system

Throughout this and the following sections, the subsystems composing the large-scale system are assumed to be identical and linear, and their number is denoted by N. Let n, m, p be integers; consider matrices $A, \tilde{A} \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $G \in \mathbb{R}^{n\times p}$. The ith subsystem, resp. the ith agent can be defined using these matrices:

$$\dot{x}_i = Ax_i + Bu_i + \mathcal{I}_i + Gw_i, \quad x(0) = x_0. \tag{1}$$

The state is denoted by the symbol $x_i(t) \in \mathbb{R}^n$, the control is represented by the symbol $u_i \in \mathbb{R}^m$; the symbol $w_i \in \mathbb{R}^p$ stands for the external disturbance. Finally, the symbol $\mathcal{I}_i$ denotes the interconnection term: $\mathcal{I}_i = \sum_{j=1}^N l_{ij}\tilde{A}x_j$. To be specific, $l_{ij} = 1$ if there is a direct connection from subsystem j to subsystem i, in another case, we set $l_{ij} = 0$. Matrix $L = (l_{ij})$ is called the *interconnection matrix* in the case of large-scale interconnected systems.

Assumption 2.1. For all $i = 1, \ldots, N$ holds $l_{ii} = 0$.

This assumption guarantees that no subsystem is connected "with itself" (the interconnection of subsystems has no loops). This is a natural condition, as the interconnections are intended only for the description of interference between different subsystems; hence the existence of such connections is not meaningful in this setting.

Assumption 2.2. Matrix L is symmetric.

This assumption is not necessary (the subsequent analysis can be conducted under the assumption of the directed communication topology), but will simplify the presentation considerably. It is noteworthy that this assumption is satisfied in many real-world systems.

2.2. Multi-agent system

A multi-agent system is composed of autonomous systems (agents); the control of an agent is computed using information from its neighboring agents only. In the sequel, we assume the number of agents is N; moreover, the agents are supposed to be identical.

$$\dot{x}_i = Ax_i + Bu_i + Gw_i, \quad x(0) = x_0. \tag{2}$$

The dimensions of all involved vectors and matrices, as well as their meaning, remain the same as in the case of the large-scale system. One can see that the main difference from the large-scale systems is the absence of the term describing the physical interconnections.

Let $\bar{x} = \frac{1}{N}\sum_{i=1}^N x_i$. The goal of the consensus synchronization problem of a multi-agent system is to achieve the consensus defined as:

$$\lim_{t\to\infty} \sum_{i=1}^N \|x_i - \bar{x}\| = 0. \tag{3}$$

This means that all agents have to follow the same trajectory in the limit, despite different initial conditions. To reach this goal, knowledge of the interconnection topology is vital for the controller design. This knowledge is concentrated in the so-called *interconnection matrix* $E \in \mathbb{R}^{N\times N}$ defined as follows: if the agent j has access to the state of the agent j, then set $\epsilon_{ij} = 1$; otherwise, set $\epsilon_{ij} = 0$. Then, matrix E is defined as $E = (\epsilon_{ij})$. Again, we assume no agent is connected to itself: $\epsilon_{ii} = 0$.

The control of the ith agent is computed using the state of the ith agent as well as from the values of the agents that send information to the ith agent.

Note also that there are no terms $\mathcal{I}_i$ in the multi-agent systems. This is because the agents are not physically interconnected, the connection between them is established through the control signal.

The Laplacian matrix $\bar{L}$ is defined as $\bar{L} = (\bar{l}_{ij})$, $\bar{l}_{ij} = -\epsilon_{ij}$ for $i \neq j$ and $\bar{l}_{ii} = \sum_{j=1}^{N} \epsilon_{ij}$.

Assumption 2.3. Matrix $\bar{L}$ is symmetric.

Assumption 2.2 (for the interconnected systems) or Assumption 2.3 (for multi-agent systems) imply the existence of a real diagonal matrix D and an orthogonal matrix T that satisfy (in the case of the large-scale system)

$$L = T^T DT, \tag{4}$$

or, for multi-agent systems

$$\bar{L} = T^T DT. \tag{5}$$

Furthermore, without loss of generality, we assume that $D = \mathrm{diag}(d_1, \dots, d_N)$ and $d_1 \leq \cdots \leq d_N$.

Remark 2.4. In the consensus problem of multi-agent systems, we have $d_1 = 0$. This case is related to the average dynamics; this eigenvalue is not important for the synchronization of the multi-agent system. On the other hand, $d_2 > 0$. For a more detailed discussion on this topic, see, for instance, [6].

Remark 2.5. Let us note that the analysis can be conducted for more general interconnection topologies (see, e. g., [26]), Assumptions 2.2 and 2.3 being superfluous. However, this leads to some technical complications. Thus, to simplify the presentation, these assumptions are supposed to be valid. The most important conclusions of this paper remain valid even for systems with these more general interconnection topologies.

2.3. Time delays in the large-scale and multi-agent systems

Let $x = (x_1^T, \dots, x_N^T)^T$, $u = (u_1^T, \dots, u_N^T)^T$. In the subsequent text, it is supposed that the delayed states are used to compute the control input. This is realistic - the delays are caused by information transmission throughout the communication network. The delays need not be equal for all subsystems or agents; however, they are assumed to be uniformly bounded. The existence of the upper bound on the delays (and the availability of this quantity to the control designer) is a common requirement imposed on systems with time delays; the analysis would be impossible to conduct without knowledge of the maximal time delay.

Assumption 2.6. Let $\bar{\tau} > 0$ be a constant denoting the maximal delay which can occur in the entire system. Then the delay of the ith agent, resp. subsystem is a measurable function $\tau_i : [0, \infty) \to [0, \bar{\tau}]$. This constant is known and available for the control design.

Then we define vector $\tilde{x}$ by

$$\tilde{x} = (x^T_{1,\tau_1}, \dots, x^T_{N,\tau_N})^T. \tag{6}$$

This vector contains the state values used to compute the control inputs u_i; this procedure is detailed in the subsequent sections in detail.

The problem of stabilization of the large-scale system can be formulated as follows: find matrix $K \in \mathbb{R}^{m\times n}$ so that, if

$$u_i = K\tilde{x}_i = Kx_{i,\tau_i}. \tag{7}$$

for all $i = 1, \dots, N$, the large-scale interconnected system is asymptotically stabilized. The important feature is that all control gain matrices are equal for all subsystems. As will be shown, this requirement allows us to simplify the controller design.

In the multi-agent system, the control signal of the ith agent ($i = 1, \dots, N$) equals to

$$u_i = \sum_{i=1}^{N} \epsilon_{ji} K(\tilde{x}_j - \tilde{x}_i) = \sum_{i=1}^{N} \epsilon_{ji} K(x_{j,\tau_j} - x_{i,\tau_i}) \tag{8}$$

Matrix K is again equal for all agents.

2.4. Compacted formulation of large-scale and multi-agent systems

To facilitate the notation, the set of the differential equations describing the N subsystems of a large-scale system or agents in the multi-agent system can be written in a compact form using the Kronecker product. To be specific, the dynamics of the overall large-scale system can be written as

$$\dot{x} = (I_N \otimes A + L \otimes \tilde{A})x + (I_N \otimes BK)\tilde{x} + (I_N \otimes G)w \tag{9}$$

while the dynamics of the multi-agent system is described as

$$\dot{x} = (I_N \otimes A)x + (\bar{L} \otimes BK)\tilde{x} + (I_N \otimes G)w. \tag{10}$$

As one can see, the interconnection term is different.

Remark 2.7. The first important difference between both problems is that the dynamics (9) is directly used for the design of the stabilizing control of the large-scale system, however, the so-called *disagreement vector* e defined as $e_i = x_i - \bar{x}$, $e = (e^T_1, \dots, e^T_N)$, must be introduced to find the synchronizing control of a multi-agent system. This vector has no analogy in the theory of large-scale systems. It is necessary to introduce this vector since in the problem of the multi-agent system synchronization, the absolute value of the states of the agents is not important; the differences from the average dynamics matter. Note that (as shown, for example, in [25]), with $\mathbf{1} = (1, \dots, 1)^T \in \mathbb{R}^N$, one has

$$\bar{L}\mathbf{1} = 0. \tag{11}$$

One can see that the average dynamics is governed by the autonomous equation (with $\bar{w} = \frac{1}{N}\sum_{i=1}^{N} w_i$)

$$\dot{\bar{x}} = A\bar{x} + G\bar{w}. \tag{12}$$

Eq. (12) combined with (10) yields

$$\dot{e} = (I_N \otimes A)e + (\bar{L} \otimes BK)\tilde{x} + (I_N \otimes G)(w - \mathbf{1} \otimes \bar{w}). \tag{13}$$

This is the most important equation for the design of a synchronizing control for a multi-agent system. Thus, Eq. (13) can be regarded as the counterpart of Eq. (9) that is used to design the stabilizing control of large-scale systems.

3. PROBLEMS WITH DELAYED CONTROL - EQUAL DELAYS, NO EXTERNAL DISTURBANCES

In the absence of disturbance signals ($G = 0$, $w = 0$) and identical delays in all subsystems, resp. identical delays in all agents, the controller design methods for the stabilization of the interconnected large-scale system and for the consensus synchronization of the multi-agent system are more or less identical. The procedure can be briefly summarized as follows:

1. A suitable Lyapunov–Krasovskii functional V is proposed. It is defined using matrices $P_i \in \mathbb{R}^{n\times n}$, $P_i > 0$ (the number of matrices P_i depends on the specific choice of this functional) so that the functional V is formulated using matrices $I_N \otimes P_i$.
2. A set of LMIs that guarantees negative definiteness of the derivative of V (denoted as $\dot{V}$) is derived; the dimension of this set of LMI depends on N.
3. One can prove that the LMI given in the previous step is equivalent to another set of LMIs whose dimension is independent of N.
4. A control gain K is found by satisfying this set of LMIs.

The size of the controller design problem can be reduced; thanks to a set of LMIs defined in the fourth step, the size is independent of the number of subsystems or agents. This is a common feature for both large-scale interconnected systems as well as multi-agent systems. Moreover, the dimension reduction method in Step 4 is in both cases identical.

Let us describe the procedure in a more detailed way. Assume $\tau_1 = \cdots = \tau_N = \tau$. This assumption is rather unrealistic, nevertheless, it facilitates the analysis.

Using (11), the disagreement vector with delays e_τ is useful in reformulating the disagreement dynamics as

$$\dot{e} = (I_N \otimes A)e + (\bar{L} \otimes BK)e_\tau. \tag{14}$$

The dynamics of the large-scale system (9) is rewritten similarly:

$$\dot{x} = (I_N \otimes A)x + (L \otimes \tilde{A})x + (I_N \otimes BK)x_\tau. \tag{15}$$

For example, for the control design of the large-scale interconnected system, the Lyapunov–Krasovskii functional V (given in [9] by Eq. (3.101)) can be used. Assume $n \times n$-dimensional symmetric matrices $\bar{P}_1 > 0$, $\bar{P}_2 > 0$, $\bar{P}_3 > 0$ are given. Then we define the functional V as

$$\begin{aligned} V =& x^T(I_N \otimes \bar{P}_1)x + \int_{t-\bar{\tau}}^{t} x^T(s)(I_N \otimes \bar{P}_2)x(s)\,\mathrm{d}s \\ &+ \bar{\tau}\int_{-\bar{\tau}}^{0}\int_{t+\theta}^{t} x^T(s)(I_N \otimes \bar{P}_3)x(s)\,\mathrm{d}s\mathrm{d}\theta. \end{aligned}$$

To design the synchronizing control of a multi-agent system, one can merely replace x by e to derive the design method: to solve the synchronization problem for the multi-agent system, the Lyapunov–Krasovskii functional V' is defined as

$$\begin{aligned} V' =& e^T(I_N \otimes \bar{P}_1)e + \int_{t-\bar{\tau}}^{t} e^T(s)(I_N \otimes \bar{P}_2)e(s)\,\mathrm{d}s \\ &+ \bar{\tau}\int_{-\bar{\tau}}^{0}\int_{t+\theta}^{t} e^T(s)(I_N \otimes \bar{P}_3)e(s)\,\mathrm{d}s\mathrm{d}\theta. \end{aligned}$$

The following theorem guarantees the stability of the interconnected system (14) or synchronization of the multi-agent system (1).

Assume there are $n \times n$-dimensional matrices $P > 0$, $R > 0$, $S > 0$, Q (non-singular) and M, a $m \times n$-dimensional matrix Y, as well as a constant $\varepsilon > 0$. In the case of a large-scale interconnected system, define using these matrices the following auxiliary matrices $\phi_{i,j}$ by

$$\begin{aligned} \phi_{11} =& \mathcal{H}\Big((I_N \otimes Q)(I_N \otimes A + L \otimes \tilde{A})\Big) + (I_N \otimes (S - R)), \\ \phi_{12} =& (I_N \otimes (P - Q)) + \varepsilon(I_N \otimes Q^T)(I_N \otimes A + L \otimes \tilde{A})^T, \\ \phi_{14} =& (I_N \otimes BY) + (I_N \otimes (R - M)), \\ \phi_{24} =& (I_N \otimes \varepsilon BY), \\ \phi_{13} =& (I_N \otimes M), \\ \phi_{22} =& (I_N \otimes (-\varepsilon(Q + Q^T) + \bar{\tau}^2 R)), \\ \phi_{33} =& -(I_N \otimes (R + S)), \\ \phi_{34} =& I_N \otimes (R - M^T), \\ \phi_{44} =& I_N \otimes (M + M^T - 2R), \end{aligned}$$

while in the case of synchronizing the multi-agent system, these matrices are changed as follows:

$$\begin{aligned} \phi_{11} =& \mathcal{H}\Big((I_N \otimes Q)(I_N \otimes A)\Big) + (I_N \otimes (S - R)), \\ \phi_{12} =& (I_N \otimes (P - Q)) + \varepsilon(I_N \otimes Q^T)(I_N \otimes A)^T, \\ \phi_{14} =& (\bar{L} \otimes BY) + (I_N \otimes (R - M)), \end{aligned}$$

$$\phi_{24} =(\bar{L} \otimes \varepsilon BY).$$

Then, let matrix Φ be given as

$$\Phi = \begin{pmatrix} \phi_{11} & \phi_{12} & \phi_{13} & \phi_{14} \\ * & \phi_{22} & 0 & \phi_{24} \\ * & * & \phi_{33} & \phi_{34} \\ * & * & * & \phi_{44} \end{pmatrix}.$$

Proposition 5.3 from [9] (reformulated to suit our problems) yields the following:

Proposition 3.1. Let matrices P, Q, R, S, Φ and scalar ε satisfy the above conditions. If the pair of LMIs

$$0 >\Phi, \tag{16}$$

$$0 >I_N \otimes \begin{pmatrix} -R & M \\ * & -R \end{pmatrix} \tag{17}$$

is satisfied, then the control gain given by $K = YQ^{-1}$ is such that the interconnected system (15) is stable and multi-agent system (1) is synchronized by the control signals u_i given by (8).

Remark 3.2. Matrices $\bar{P}_1, \bar{P}_3, \bar{P}_2$ used in the definition of functional V are closely related to matrices P, R, S from the preceding proposition: $P = Q^T \bar{P}_1 Q$, $S = Q^T \bar{P}_2 Q$ and $R = Q^T \bar{P}_3 Q$; moreover $\dot{V} < 0$ if $x \neq 0$.

The interconnections appear only in two terms: ϕ_{11} and ϕ_{22} for the large-scale interconnected system; the interconnections appear in ϕ_{14} and ϕ_{24} for the multi-agent system. Nevertheless, from this point on, the way how to deal with these LMIs is identical for both cases. To begin, define matrix $\mathbb{T} = (T \otimes I_n, T \otimes I_n, T \otimes I_n, T \otimes I_n)$ where T is defined in (4) or (5). Then, let

$$\Gamma = \mathbb{T}^{-1}\Phi\mathbb{T}.$$

The properties of the Kronecker product imply that matrices Γ and Φ are closely related: with exception of elements containing matrices L or $\bar{L}$, their elements are equal. To be more specific, the expression $I_N \otimes A + L \otimes \tilde{A}$ in the case of the large-scale interconnected system is replaced by $I_N \otimes A + D \otimes \tilde{A}$. Also, the expression $\bar{L} \otimes BY$ is replaced by $D \otimes BY$ in the case of synchronizing the multi-agent system. Let us introduce the following matrices and matrix-valued functions (depending on real parameters d', d''):

$$\begin{aligned} \lambda_{11}(d') &=\mathcal{H}(Q(A + d'\tilde{A})) + S - R, \\ \lambda_{12}(d') &=P - Q + \varepsilon Q^T (A + d'\tilde{A})^T, \\ \lambda_{14}(d'') &=d'' BY + R - M, \\ \lambda_{24}(d'') &=d'' \varepsilon BY, \\ \lambda_{13} &=M, \\ \lambda_{22} &= - \varepsilon(Q + Q^T) + \bar{\tau}^2 R, \end{aligned}$$

$$\begin{aligned}\lambda_{33} =& -(R+S),\\ \lambda_{34} =& R - M^T,\\ \lambda_{44} =& M + M^T - 2R,\end{aligned}$$

$$\Lambda(d', d'') = \begin{pmatrix} \lambda_{11}(d') & \lambda_{12}(d') & \lambda_{13} & \lambda_{14}(d'') \\ * & \lambda_{22} & 0 & \lambda_{24}(d'') \\ * & * & \lambda_{33} & \lambda_{34} \\ * & * & * & \lambda_{44} \end{pmatrix}$$

The procedure described above yields that there exists a permutation matrix $\Pi \in \mathbb{R}^{4nN \times 4nN}$ such that the following is true for the large-scale interconnected system

$$\Pi^T \Gamma \Pi = \operatorname{diag}\Big(\Lambda(d_1, 1), \ldots, \Lambda(d_N, 1)\Big) \tag{18}$$

while one has a slightly different relation for the multi-agent system:

$$\Pi^T \Gamma \Pi = \operatorname{diag}\Big(\Lambda(0, 0), \Lambda(0, d_2), \ldots, \Lambda(0, d_N)\Big) \tag{19}$$

(here, note that $d_1 = 0$; as explained above, this eigenvalue is irrelevant to the control design as it corresponds to the average dynamics). In both cases, the dependence on d, d' is convex. Thus it suffices to verify only $\Lambda(d_1, 1) < 0$ and $\Lambda(d_N, 1) < 0$ (for the large-scale interconnected system) or $\Lambda(0, d_2) < 0$ and $\Lambda(0, d_N) < 0$ (for the multi-agent system; here, one cannot assume $\Lambda(0, 0) < 0$ since this case corresponds to the average dynamics that is not affected by the synchronizing control). As a result, one gets

Theorem 3.3. Let there exist $n \times n$-dimensional matrices M, $P > 0$, $R > 0$, $S > 0$, Q non-singular, a $m \times n$-dimensional matrix Y and a scalar $\varepsilon > 0$ satisfying

$$0 > \begin{pmatrix} -R & M \\ * & -R \end{pmatrix}. \tag{20}$$

Then

1. If $0 > \Lambda(d_1, 1),\ 0 > \Lambda(d_N, 1)$ then the control (7) with the control gain $K = YQ^{-1}$ asymptotically stabilizes the large-scale interconnected system (14).
2. If $0 > \Lambda(0, d_2),\ 0 > \Lambda(0, d_N)$ then the multi-agent system (14) achieves consensus by the control (8) with $K = YQ^{-1}$. That means, $\lim_{t\to\infty} \|e(t)\| = 0$.

Remark 3.4. The case of equal time delays is simple, the procedures to obtain the stabilizing or synchronizing control law are more or less identical. It is noteworthy that the presence of the parameter ε is somewhat problematic as this parameter has to be defined a-priori. It cannot be obtained as part of the LMIs solution because it appears in a multiple with other variables. However, as [9] points out, this approach tolerates a fairly wide range of this parameter.

In the case of equal delays, the large-scale system (9) **is asymptotically stabilized and the full consensus synchronization of the multi-agent system** (10) **is achieved.**

4. PROBLEMS WITH DELAYED CONTROL WITH PERTURBATIONS AND WITH HETEROGENEOUS DELAYS

In this section, we investigate large-scale systems and multi-agent systems with external disturbances first. As will be seen, the state of the large-scale system cannot converge to 0; analogously, one cannot expect the full synchronization of the multi-agent system. However, the errors caused by these disturbances can be estimated using the methods of the H_∞ control.

First, let us recall the definition of the H_∞-stability.

Definition 4.1. The dynamical system

$$\dot{\xi} = \mathcal{A}\xi + \mathcal{G}w \tag{21}$$

is H_∞-stable, if the following conditions hold:

1. if $w = 0$ on $[0, \infty)$, system (21) is asymptotically stable;
2. there exists a constant $\gamma > 0$ so that, if the initial conditions are zero, inequality $\int_0^T \|x(s)\|^2 \, \mathrm{d}s \le \gamma \int_0^T \|w(s)\|^2 \, \mathrm{d}s$ is satisfied for all $T \ge 0$.

As noted, e. g., in [19], this definition can be generalized to time-delay systems.

Let us consider the case of identical delay for both large-scale as well as multi-agent systems in the presence of disturbances expressed by the vector w.

Consider first the large-scale interconnected system with external disturbances (1) with $G \neq 0$.

According to the Proposition 5.3 in [9], when applied to the case of a large-scale interconnected system composed of subsystems (1), the H_∞ stability of this large-scale interconnected system holds if the following LMI

$$\left(\begin{array}{c|cc} \multirow{4}{*}{Φ} & I_N \otimes G & I_N \otimes Q^T \\ & \varepsilon I_N \otimes G & 0 \\ & 0 & 0 \\ & 0 & 0 \\ \hline \multirow{2}{*}{$*$} & -\gamma^2 I_{pN} & 0 \\ & * & -I_{nN} \end{array}\right) < 0 \tag{22}$$

is satisfied in combination with (17). Then, one can conduct dimension reduction as in the previous case. Let us define the (matrix-valued) function Λ' by the following formula:

$$\Lambda'(d', d'') = \left(\begin{array}{c|cc} \multirow{4}{*}{$\Lambda(d', d'')$} & G & Q^T \\ & \varepsilon G & 0 \\ & 0 & 0 \\ & 0 & 0 \\ \hline \multirow{2}{*}{$*$} & -\gamma^2 I_p & 0 \\ & * & -I_n \end{array}\right). \tag{23}$$

The MIMO version of Proposition 5.3 in [9] is required to prove H_∞-stability of the large-scale interconnected system. Nevertheless, a close inspection of its proof shows that this generalization is straightforward, hence its proof is omitted here. Due to this Proposition, LMI (22) together with (17) imply H_∞-stability of (9). This means, the relation $\int_0^T \|x(s)\|^2\,\mathrm{d}s \le \gamma \int_0^T \|w(s)\|^2\,\mathrm{d}s$ holds for all $T > 0$ if the initial conditions are zero; moreover, in absence of disturbances, the large-scale system (9) is asymptotically stabilized. The fact that the inputs are delayed with different delays does not play a role here. Note also that validity of (22) and (17) is implied by $\Lambda'(d_1, 1) < 0$, $\Lambda'(d_N, 1) < 0$ in combination with (20). Hence, one arrives at

Proposition 4.2. Consider system (9) where the control loop time delays satisfy Assumption 2.6. Let $\Lambda'(d_1, 1) < 0$, $\Lambda'(d_N, 1) < 0$ and (20) hold. Then there exists a constant $\gamma > 0$ so that $\int_0^T \|x(s)\|^2\,\mathrm{d}s \le \gamma \int_0^T \|w(s)\|^2\,\mathrm{d}s$ if all initial conditions are zero. Moreover, if $w = 0$ for all $t \ge 0$, then system (9) is asymptotically stabilized.

If the delays in the multi-agent system (10) are equal, then, using the similar reasoning one obtains

Proposition 4.3. Consider multi-agent system (10). Let the delays in all agents be equal and $e(t) = 0$ for $t \in [-\bar{\tau}, 0]$. Let also $\Lambda'(d_2, 0) < 0$, $\Lambda'(d_N, 0) < 0$ and (20) hold. Then there exists a constant $\gamma > 0$ so that $\int_0^T \|e(s)\|^2\,\mathrm{d}s \le \gamma \int_0^T \|w(s) - \mathbf{1} \otimes \bar{w}(s)\|^2\,\mathrm{d}s$. Moreover, if the disturbance is equal for all agents, the synchronization error converges asymptotically to zero (this case reduces to the problem solved in the previous section).

The situation becomes more complicated in the problem of synchronization of multi-agent systems with heterogeneous delays. From (8) follows that the control of one particular agent needs not only knowledge of its own state but also knowledge of the states of its neighbors.

Eq. (8) can be rewritten as

$$u_i = \sum_{j=1}^{N} e_{ij} \Big(K(x_{i,\tau_i} - x_{j,\tau_i}) + K(x_{j,\tau_i} - x_{j,\tau_j}) \Big). \tag{24}$$

Define also $\hat{u}_i = \sum_{j=1}^{N} e_{ij} K(x_{j,\tau_i} - x_{j,\tau_j})$ and vector $\hat{u} = (\hat{u}_1^T, \ldots, \hat{u}_N^T)^T$. Let also $\tilde{e} = (e_{1,\tau_1}^T, \ldots, e_{N,\tau_N}^T)$. Then (13) attains the form

$$\dot{e} = (I_N \otimes A)e + (\bar{L} \otimes BK)\tilde{e} + (\bar{L} \otimes B)\hat{u} + G(w - \mathbf{1} \otimes \bar{w}). \tag{25}$$

We can consider the sum of the last two terms as a disturbance.

Assume now for the sake of simplicity that the delays are heterogeneous but $w = 0$. For the subsequent analysis, matrix

$$\widetilde{\Lambda}(d'') = \left(\begin{array}{c|cc} \Lambda(0, d'') & \begin{matrix} d''B & Q^T \\ \varepsilon d''B & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \hline * & \begin{matrix} -\gamma^2 I_p & 0 \\ * & -I_n \end{matrix} \end{array} \right) \tag{26}$$

will be useful.

The H_∞-stability conditions for multi-agent systems are similar to the conditions for H_∞-stability of large-scale interconnected systems. Inequality (22) is, however, replaced by

$$\left(\begin{array}{c|cc} \Phi & \begin{array}{c}\bar{L}\otimes B\\ \varepsilon\bar{L}\otimes B\\ 0\\ 0\end{array} & \begin{array}{c}I_N\otimes Q^T\\ 0\\ 0\\ 0\end{array} \\ \hline * & \begin{array}{c}-\gamma^2 I_{pN}\\ *\end{array} & \begin{array}{c}0\\ -I_{nN}\end{array} \end{array}\right) < 0 \tag{27}$$

in the case of heterogeneous delays. Now, the diagonalization procedure from the previous section is applied. Its output is the equivalence of the condition (27) and inequalities $\widetilde{\Lambda}(d_i) < 0$ for all $i = 2, \ldots, N$. Thanks to the linearity of matrix-valued function $\widetilde{\Lambda}$, one arrives at

Proposition 4.4. Consider system (25), and assume moreover that $w = 0$, $e(t) = 0$ for $t \in [-\bar{\tau}, 0]$. Assume also the delays in all agents satisfy Assumption 2.6. Let also $\widetilde{\Lambda}(d_2) < 0$, $\widetilde{\Lambda}(d_N) < 0$ and (20) hold and let $K = YQ^{-1}$. Then there exists a constant $\varkappa > 0$ so that $\int_0^T \|e(s)\|^2 ds \le \varkappa \int_0^T \|\hat{u}(s)\|^2 \, \mathrm{d}s$.

The heterogeneity of delays prevents the synchronization error from converging to zero. Fortunately, the H_∞-control can be used to estimate the limit of the norm of the synchronization error (for time increasing to infinity). The estimate was also derived in the above papers.

To sum up, we obtain

Theorem 4.5. (Rehák and Lynnyk [27]) Assume there exist a scalar $\varepsilon > 0$, $n \times n$-dimensional matrices $P > 0$, $R > 0$, $S > 0$, Q non-singular and M and a $m \times n$-dimensional matrix Y. Let also Assumption 2.6 holds and suppose also $0 > \left(\begin{smallmatrix} -R & M \\ * & -R \end{smallmatrix}\right)$.

1. If $0 > \Lambda'(d_1, 1)$, $0 > \Lambda'(d_N, 1)$ then the large-scale interconnected system (9) is asymptotically stabilized by the feedback (7) with
$$K = YQ^{-1}. \tag{28}$$

2. If $0 > \widetilde{\Lambda}(d_2)$, $0 > \widetilde{\Lambda}(d_N)$ and $w = 0$ then the disagreement dynamics (25) is H_∞-stable with $K = YQ^{-1}$. In other words, there exists a constant $\varkappa > 0$ (that depends on γ, K, B and $\bar{L}$) such that, if $e(t) = 0$ for all $t \in [-\bar{\tau}, 0]$, one has:
$$\int_0^T \|e(s)\|^2 \, \mathrm{d}s \le \varkappa \int_0^T \|\hat{u}(s)\|^2 \, \mathrm{d}s. \tag{29}$$
Furthermore, if $\hat{u} = 0$, then system (25) is asymptotically stable.

Remark 4.6. If the delays are heterogeneous and an external disturbance acts on the system, it is not difficult to see that there exists a constant $\varkappa > 0$ so that for the disagreement dynamics (25) holds $\int_0^T \|e(s)\|^2 ds \le \varkappa \int_0^T (\|\hat{u}(s)\|^2 + \|w(s) - \mathbf{1} \otimes \bar{w}(s)\|) \, \mathrm{d}s$.

Large-scale systems with non-identical delays in the controls in each subsystem can achieve asymptotic stabilization; however, multi-agent systems with non-identical delays in the control signal cannot be fully synchronized. The H_∞ control design methods can be used to find a bound on the limit of the norm of error for $t \to \infty$.

5. NONLINEAR LARGE-SCALE AND MULTI-AGENT SYSTEMS

In this section, nonlinear large-scale systems and nonlinear multi-agent systems are studied. In both cases, it is assumed that the subsystems or agents admit the full exact feedback linearization; see the Appendix for details.

5.1. Stabilization of nonlinear large-scale systems

Assume there exist functions f, g, λ satisfying Assumption A.1. Furthermore, suppose that matrix L has the same properties as presented in Subsection 2.1. Let there also exist vector $\tilde{a} \in \mathbb{R}^n$ so that the interconnection term $\mathcal{I}_i$ is defined as

$$\mathcal{I}_i = \sum_{j=1}^{N} l_{ij} \tilde{a} \lambda(x_j). \tag{30}$$

The ith subsystem of a nonlinear large-scale system is defined as

$$\dot{x}_i = f(x_i) + g(x_i) u_i + \mathcal{I}_i, \; x_i(0) = x_{i,0}, \; i = 1, \dots, N. \tag{31}$$

Let $\xi_i = \mathcal{T}(x_i)$. It is worth noting that, thanks to the definition of transformation $\mathcal{T}$, one has $\xi_{i,1} = \lambda(x_i)$. As a result, for the interconnection term $\mathcal{I}_i$ holds

$$\mathcal{I}_i = \sum_{j=1}^{N} l_{ij} \tilde{a} \xi_{j,1}. \tag{32}$$

The exact feedback linearization converts the system $\dot{x} = f(x) + g(x)u$ into a linear form. Hence the transformed subsystem (without interconnections) reads (with v_i being the transformed input of the ith subsystem)

$$\dot{\xi}_i = A \xi_i + B v_i, \; i = 1, \dots, N. \tag{33}$$

On the other hand, taking the interconnections into account makes the situation somewhat complicated. From the definition of the subsystem and from (52) follows that

$$\dot{\xi}_i = A \xi_i + B v_i + \frac{\partial}{\partial x} \mathcal{T}\Big(\mathcal{T}^{-1}(\xi_i)\Big) \sum_{j=1}^{N} l_{ij} \tilde{a} \xi_{j,1}. \tag{34}$$

In line with the procedure presented in [21], assume there exist matrices $D' \in \mathbb{R}^{n \times n}$, $E' \in \mathbb{R}^{n \times n}$ and measurable matrix-valued functions $F_i : [0, \infty) \to \mathbb{R}^{n \times n}$, $i = 1, \dots, N$, such that $\|F_i(t)\| \leq 1$ and

$$\frac{\partial}{\partial x} \mathcal{T}\Big(\mathcal{T}^{-1}(\xi_i)\Big) = D' F_i(t) E' \tag{35}$$

for all $t \geq 0$.

Let $Z \in \mathbb{R}^{n\times(n-1)}$ be a zero matrix and let matrix $\tilde{A} \in \mathbb{R}^{n\times n}$ and function $\mathcal{F}$ be defined as

$$\tilde{A} = \left(\tilde{a}|Z\right), \quad \mathcal{F}(t) = \operatorname{diag}(F_1(t), \ldots, F_N(t)).$$

Let $\zeta = (\xi_1^T, \ldots, \xi_N^T)^T)$, $\omega = (v_1, \ldots, v_N)^T$. Then, the overall system in the transformed coordinates obeys

$$\dot{\zeta} = (I_N \otimes A)\zeta + (I_N \otimes B)\omega + (I_N \otimes D')\mathcal{F}(t)(L \otimes E'\tilde{A})\zeta. \tag{36}$$

As one can see, the problem of stabilization of a nonlinear large-scale interconnected system has been converted into the problem of robust stabilization of a linear uncertain system.

The only remaining issue is how to adopt the robust control algorithm in such a way that the resulting control design procedure is independent of the number of subsystems. This is conducted in a similar way as in the previous section. Thanks to the well-known properties of the Kronecker product, one has $(L \otimes E'\tilde{A}) = (L \otimes I_n)(I_N \otimes E'\tilde{A})$. Let us define the matrix-valued function $\tilde{\mathcal{F}}$ and scalar μ by

$$\mu = \sup_{t\geq 0} \|\mathcal{F}(t)(L \otimes I_n)\|, \quad \tilde{\mathcal{F}}(t) = \frac{1}{\mu}\mathcal{F}(t)(L \otimes I_n), \tag{37}$$

The scalar μ exists since function $\mathcal{F}$ is bounded. Then, with help of (37), one can rewrite (36) as

$$\dot{\zeta} = (I_N \otimes A)\zeta + (I_N \otimes B)\omega + \mu(I_N \otimes D')\tilde{\mathcal{F}}(t)(I_N \otimes E'\tilde{A})\zeta. \tag{38}$$

This set of equations is coupled through the matrix-valued function $\tilde{\mathcal{F}}$. However, throughout the process of the robust control design (which can be applied as $\|\tilde{\mathcal{F}}(t)\| = 1$ for all $t \geq 0$, hence the problem is in the standard setting of robust control problems), this function is removed thanks to the application of the Young inequality, provided that the Lyapunov function used to design the control law is sought in the form $V = \zeta^T(I_N \otimes P)\zeta$ where $P \in \mathbb{R}^{n\times n}$, $P > 0$, one can obtain a decentralized control law $\omega_i = K\zeta_i$, $i = 1, \ldots, N$ (where matrix K is equal for all subsystems). As the particular control design procedure is quite standard, the detailed description of this design procedure is omitted.

After the control for the large-scale system in the transformed coordinated has been designed, it is necessary to express this control in the original coordinates by (51). This yields

$$u_i = \frac{1}{\Psi(\mathcal{T}(x_i))}\Big(K\mathcal{T}(x_i) - \Phi(\mathcal{T}(x_i))\Big), \tag{39}$$

This control asymptotically stabilizes the interconnected large-scale system composed of N subsystems (31).

5.2. Synchronization of nonlinear multi-agent systems

The multi-agent system to be considered in this section is composed of a set of N agents in form

$$\dot{x}_i = f(x_i) + g(x_i)u_i, \ i = 1, \ldots, N. \tag{40}$$

where functions f, g satisfy Assumption A.1 for a (chosen by the designer of the control law) output λ. As a result, the exact feedback linearization can be conducted and the transformation $\mathcal{T}$ is equal for all agents.

Remark 5.1. The output λ is chosen in the process of the controller design. Hence the transformation $\mathcal{T}$ and, subsequently, the result is also dependent on this choice. On the other hand, Assumption A.1 guarantees the existence of at least one function λ that can be chosen as the output.

For the feedback design, we assume that only the delayed values of the state variables are available.

The goal is to find a continuous function $k : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ so that, with input signal u_i of the ith agent defined as

$$u_i = k(x_i, \sum_{j=1}^{N} \epsilon_{i,j}(x_i - x_j)) \tag{41}$$

the condition (3) is satisfied.

Applying the exact feedback linearization of every agent we get (with $\xi_i = \mathcal{T}(x_i)$ and $v_i = \Psi(\xi_i)u_i + \Phi(\xi_i)$)

$$\dot{\xi}_i = A\xi_i + Bv_i. \tag{42}$$

Hence, a linear multi-agent system appears.

Matrix $\bar{L}$ defined as in the previous text allows us to find a compact form of the multi-agent system: let

$$\zeta = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_N \end{pmatrix}, \ \ \omega = \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix}. \tag{43}$$

Then

$$\dot{\zeta} = (I_N \otimes A)\zeta + (I_N \otimes B)\omega. \tag{44}$$

As this system is linear, it is straightforward to design a linear synchronizing controller for it. In this case, we are looking for a matrix $K \in \mathbb{R}^{1\times n}$ such that multi-agent system (40) is synchronized if the control signal of the ith agent is equal to $v_i = K\xi_i$. Then, since the exact feedback linearization is a diffeomorphism, this implies synchronization of the original system. Thus, we get

$$\dot{\zeta} = (I_N \otimes A + \bar{L} \otimes BK)\zeta. \tag{45}$$

This is a standard compact description of a linear multi-agent system; to be specific, it is free of any uncertainties. Thus, finding such a matrix K guaranteeing synchronization of system (45) is a standard matter; therefore, this topic is not elaborated in this paper in more detail. The algorithms to find such a matrix are described, e. g., in [6].

In the original coordinates, the control signal is given by

$$u_i = \frac{1}{\Psi(\mathcal{T}(x_i))}\Big[K\Big(\mathcal{T}(x_j) - \mathcal{T}(x_i)\Big) - \Phi(\mathcal{T}(x_i))\Big], \tag{46}$$

hence by a similar relation as in the case of interconnected large-scale systems.

A nonlinear multi-agent system can be converted to a linear one through the exact feedback linearization. The construction of a synchronizing controller for the linearized system does not require any further robustness. Moreover, one has some degree of freedom by choosing the auxiliary output λ for the exact feedback linearization - the only requirement is the validity of Assumption A.1.

On the other hand, the application of the exact feedback linearization-based method to the stabilization of the large-scale system, the output λ appears is the definitions of the interconnections between the subsystems, it cannot be arbitrarily chosen. Moreover, to stabilize the large-scale system, some extra robustness of the controller is required.

6. EXAMPLE

The results of Sec. 4 suggest that the synchronization of a multi-agent system with not equal delays is not achievable. This example demonstrates this effect.

As an example, we consider four linear oscillators with inputs, the input is denoted as $\tilde{u}_i$

$$\dot{x}_{1,i} = x_{2,i},\ \dot{x}_{2,i} = -x_{1,i} + \tilde{u}_i,\ i = 1, \ldots, 4. \tag{47}$$

To define the large-scale system, it is necessary to define the interconnection terms $\mathcal{I}_i = 0.1(x_{1,i-1} + x_{1,i+1})$ for $i = 2, 3$, $\mathcal{I}_1 = 0.1(x_{1,4} + x_{1,2})$ and $\mathcal{I}_4 = 0.1(x_{1,1} + x_{1,3})$. Then, for the large-scale system, let $\tilde{u}_i = \mathcal{I}_i + u_i$. The control u_i is given as $u_i = Kx_i$ where the control gain K is obtained by (28).

In our example, we suppose the delays in the control are uniformly bounded by the constant $\bar{\tau} = 0.25$s. Hence the initial conditions are defined on the interval $[-0.25, 0]$s. Solution of the LMI problem from Section 4 yields the control gain $K = (0.1194,\ -1.0959)$.

The multi-agent system is composed of four systems described by Eq. (47) (agents). We suppose the agents are interconnected with the ring topology: $u_i = \tilde{u}_i = K(2x_i - x_{i+1} - x_{i-1})$ for $i = 2, 3$, $u_1 = K(2x_1 - x_2 - x_4)$ and $u_4 = K(2x_4 - x_1 - x_2)$. The bound on the maximal delay in the control was in this case also set to 0.25s. The control gain computed by the LMIs presented in Section 4 is $K = (0.0233,\ -0.2019)$.

The first state of all four oscillators is shown in the following figures. Note that, in case all delays are equal to $0.25s$ in all oscillators, the full synchronization can be achieved as can be seen from Figure 1. However, if the delays satisfy the given bounds but are not equal (we set in the example $\tau_1 = 0.1s$, $\tau_2 = 0.15s$, $\tau_3 = 0s$ and $\tau_4 = 0.25s$), full synchronization is not achieved. This is illustrated by Figure 2. On the other hand, the large-scale interconnected system can be stabilized even if the delays are $\tau_1 = 0.1s$,

$\tau_2 = 0.15s$, $\tau_3 = 0s$ and $\tau_4 = 0.25s$, the large-scale system can be asymptotically stabilized. This means $x(t) \to 0$ for $t \to \infty$. as seen from Figure 3.

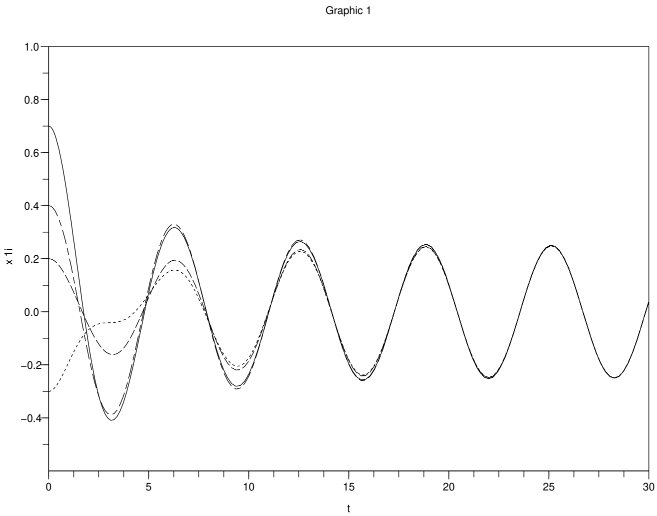

Fig. 1. Multi-agent system, state $x_{1,i}$, delays equal.

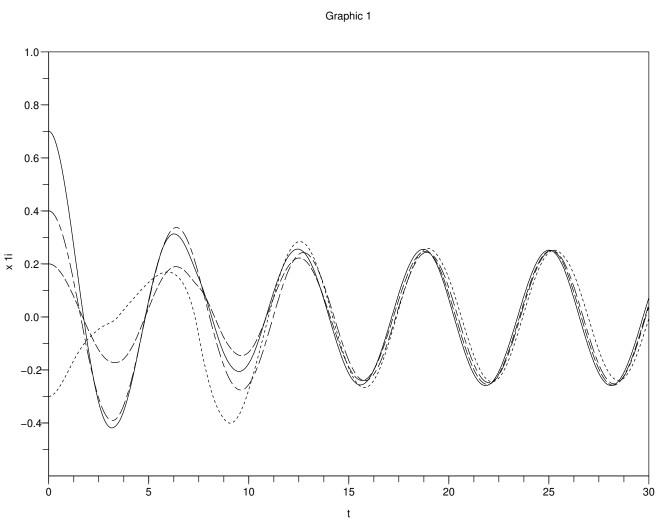

Fig. 2. Multi-agent system, state $x_{1,i}$, delays different.

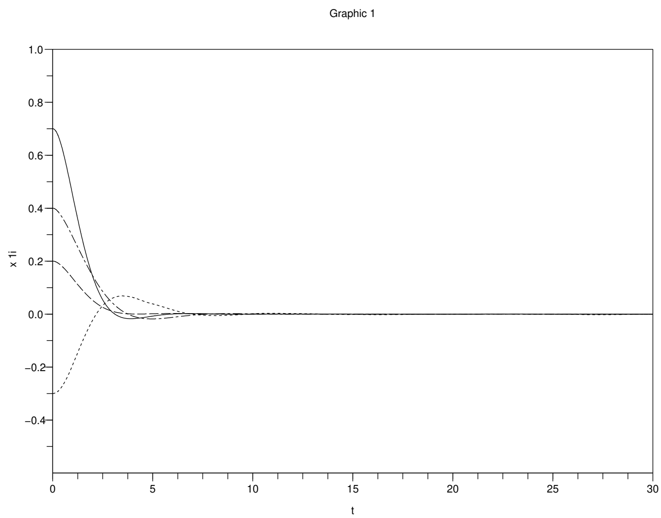

Fig. 3. Large-scale system, state $x_{1,i}$, delays different.

7. CONCLUSIONS

The paper delivers a comparison of various aspects of the multi-agent and large-scale interconnected systems. First, the problems of synchronization of a linear multi-agent system and the stabilization of a linear large-scale system, in both cases with heterogeneous delays in the control loop were investigated. This pair of problems share several similarities: both problems lead to a solution of a set of LMIs whose size is reduced up to the size of one subsystem/agent; this reduction is more or less analogous. The second problem studied in this paper was the problem of synchronization of a nonlinear multi-agent system followed by the problem of stabilization of a nonlinear large-scale system.

Table 1 summarizes the differences between the problems of stabilization of the large-scale interconnected system and synchronization of multi-agent systems.

A. EXACT FEEDBACK LINEARIZATION

For the sake of completeness, the most important facts about the exact feedback linearization are stated here, albeit without proofs and further details. A more extensive treatment can be found e. g. in [11]. These facts presented in this Appendix are useful for both the stabilization problem of a large-scale system as well as for the problem of synchronization of a nonlinear multi-agent system.

	Large-scale interconnected system	Multi-agent system
homogeneous delays	stabilization possible	full synchronization possible
heterogeneous delays	stabilization possible	full synchronization not possible
nonlinear systems	robust control design needed	no additional robustness required
nonlinear systems	output λ is determined by the problem	output λ can be defined by the control designer

Tab. 1. Comparison of both problems.

Assume we are functions $f : \mathbb{R}^n \to \mathbb{R}^n$, $\lambda : \mathbb{R}^n \to \mathbb{R}$ as well as a continuous function $g : \mathbb{R}^n \to \mathbb{R}^n$ sufficiently smooth so that the Lie derivative $\mathcal{L}_f^{n-1}\lambda(x)$ exists (the Lie derivative $\mathcal{L}$ is defined as $\mathcal{L}_f\lambda(x) = \nabla\lambda(x).f(x)$ and $\mathcal{L}_f^k\lambda(x) = \mathcal{L}_f\mathcal{L}_f^{k-1}\lambda(x)$). Then, define the auxiliary system

$$\dot{x} = f(x) + g(x)u, \ y = \lambda(x). \tag{48}$$

Assumption A.1. System (48) has relative degree n (for the definition of the relative degree, see [11]).

This assumption guarantees that all the states of the system affect the output - the controlled system has no hidden dynamics. It can be relaxed to the minimum-phase requirement (this means that the hidden dynamics may be present but is supposed to be asymptotically stable). For our presentation, such a generalization would not be too useful; the synchronization of minimum-phase multi-agent systems is described in [29].

The exact feedback input-output linearization defines the transformation $\mathcal{T}$ of the state variables x of (1) into new coordinates ξ as follows:

$$\xi = \mathcal{T}(x) = (\lambda(x), \mathcal{L}_f\lambda(x), \dots, \mathcal{L}_f^{n-1}\lambda(x))^T. \tag{49}$$

Let matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$ defined by

$$A = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Then, Assumption A.1 ensures the existence of continuous functions $\Phi, \Psi : \mathbb{R}^n \to \mathbb{R}$, $\Psi(\xi) \neq 0$ for any $\xi \in \mathbb{R}^n$ on some neighborhood U of the origin so that system (48) is transformed into the linear form

$$\dot{\xi}_i = A\xi + Bv, \quad v = \Psi(\xi)u + \Phi(\xi). \tag{50}$$

This transformation is used to handle nonlinear large-scale as well as multi-agent systems in the subsequent text.

The process to design a control law can be summarized as follows: after finding the transformation $\mathcal{T}$, the controller design is conducted for the linearized system (50). This yields a control law in form $v = \varphi(\xi)$ for some function $\varphi : \mathbb{R}^n \to \mathbb{R}$. Since $\Psi(\xi) \neq 0$, one arrives to the following expression of the control signal in the original coordinates:

$$u = \frac{1}{\Psi(\mathcal{T}(x)}\Big(\varphi(\mathcal{T}(x)) - \Phi(\mathcal{T}(x))\Big). \tag{51}$$

Remark A.2. This section was intentionally kept short. For more details see [28].

Let $\frac{\partial}{\partial x}\mathcal{T}(x)$ be the Jacobi matrix of the mapping $\mathcal{T}$. One can see that since $\xi = \mathcal{T}(x)$, we have

$$A\xi + Bv = \dot{\xi} = \frac{\partial}{\partial x}\mathcal{T}(x)\dot{x} = \frac{\partial}{\partial x}\mathcal{T}(x)\Big(f(x) + g(x)u\Big). \tag{52}$$

This relation is used to derive Eq. (34).

(Received November 4, 2022)

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