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# Convex weak concordance measures and their constructions

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ABSTRACT

Considering the framework of weak concordance measures introduced by Liebscher in 2014, we propose and study convex weak concordance measures. This class of dependence measures contains as a proper subclass the class of all convex concordance measures, introduced and studied by Mesiar et al. in 2022, and thus it also covers the well-known concordance measures as Spearman's $\rho$, Gini's $\gamma$ and Blomqvist's $\beta$. The class of all convex weak concordance measures also contains, for example, Spearman's footrule $\phi$, which is not a concordance measure. In this paper, we first introduce basic convex weak concordance measures built in general by means of a single point $(u,v) \in \bigtriangledown = \{(u,v) \in ]0,1[^2 \mid u \geq v\}$ and its transpose $(v,u)$ only. Then, based on basic convex weak concordance measures and probability measures on the Borel subsets of $\bigtriangledown$, two rather general constructions of convex weak concordance measures are proposed, discussed and exemplified. Inspired by Edwards et al., probability measures-based constructions are generalized to Borel measures on $\mathcal{B}(]0,1[^2)$-based constructions also allowing some infinite measures to be considered. Finally, it is shown that the presented constructions also cover the mentioned standard (convex weak) concordance measures $\rho$, $\gamma$, $\beta$, $\phi$ and provide alternative formulas for them.

## 1. Introduction

In statistical analysis, the stochastic dependence of random variables based on the degree of association between two random variables plays an important role. Probably, the most known and widely applied degree of association between two random variables is the Pearson correlation coefficient [21] introduced as a measure of the linear dependence between random variables. Later, in 1904, Spearman [30] proposed to consider the Pearson coefficient for the rank values of the considered random variables. This value is now called Spearman's rank correlation coefficient or simply Spearman's rho, and it assesses the monotone relation between two random variables. If we denote Spearman's rho for continuous random variables $X$ and $Y$ by $\rho(X,Y)$, then for any real functions $f$ and $g$ which are strictly increasing on the ranges $\mathrm{Ran}X$ and $\mathrm{Ran}Y$, respectively, we have $\rho(f(X),g(Y)) = \rho(X,Y)$, i.e., Spearman's rho is a scale-invariant characteristic of a random vector $(X,Y)$, and consequently, it can be determined by means of the copula $C = C_{X,Y}$ uniquely corresponding to $X$ and $Y$ (therefore we will also write $\rho(C)$ instead of $\rho(X,Y)$). Another well-known rank-dependent correlation coefficient for continuous random variables $X$ and $Y$ is Kendall's tau [14]. Among other rank-dependent

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measures of association between two random variables we recall, for example, Spearman's footrule $\phi$, Gini's gamma and Blomqvist's beta [1]. For more details we recommend, e.g., the monographs [5,20] or the survey [25]. Numerous other papers are devoted to the study of concordance measures, for example, [6,7,9,11,13,23,32] among others.

For some classes of rank-dependent measures of association, i.e., such measures which can be expressed by means of the copula of the random variables, also axiomatic approaches were proposed. Here, we will consider the system of properties characterizing "measures of concordance", introduced by Scarsini in [22], and the system of properties characterizing the so-called "weak concordance measures" proposed by Liebscher in [17]. Note that the set of all concordance measures contains measures of association as, for example, Spearman's $\rho$, Kendall's $\tau$, Gini's $\gamma$, Blomqvist's $\beta$, but it does not contain Spearman's footrule $\phi$, and on the other hand, the set of all weak concordance measures covers all concordance measures but also, for example, Spearman's footrule as a proper member.

We also recall that the set $\mathcal{C}_2$ of all 2-dimensional copulas is convex. There are even several parametrized families of copulas which are convex as, for example, the Fréchet family or the Eyraud-Farlie-Gumbel-Morgenstern family of copulas [5,20]. It is not difficult to check that both the set of all concordance measures and the set of all weak concordance measures are also convex. All above mentioned (weak) concordance measures, except of Kendall's tau, commute with the convex combinations of copulas. For example, for Spearman's rho we have

$$\rho\left(\sum_{i=1}^{k} \lambda_i C_i\right) = \sum_{i=1}^{k} \lambda_i \rho(C_i)$$

for any copulas $C_1, \dots, C_k \in \mathcal{C}_2$ and any numbers $\lambda_1, \dots, \lambda_k \in [0,1]$ with $\sum_{i=1}^{k} \lambda_i = 1$. These facts inspired us to introduce and study the so-called convex concordance measures, see [19]. A similar study of convex weak concordance measures is the topic of this paper, which is organized as follows. The next section is devoted to the necessary preliminaries concerning 2-dimensional copulas, concordance measures and weak concordance measures. At the end of this section we define the notion of convex weak concordance measures. In Section 3, we construct the simplest (basic) convex weak concordance measures $\nu_{(u,v)}$ generated, in general, by a single point $(u, v)$ and its transpose $(v, u)$. In Section 4, probability-based convex weak concordance measures are studied — two approaches based on basic convex weak concordance measures, appropriate probability measures and the Lebesgue-Stieltjes integrals resulting in convex weak concordance measures are proposed. Also, a characterization of convex weak concordance measures based on transposition-invariant Borel measures on $\mathcal{B}(]0,1[^2)$ is given. In addition, the relationship between the two probability-based proposed types of convex weak concordance measures is clarified. Finally, alternative formulas for the convex weak concordance measures $\rho$, $\gamma$ and $\phi$, following from the mentioned new approaches, are given.

## 2. Copulas and weak concordance measures

Copulas were introduced by Sklar in 1959, see [29], and since then they have been discussed in numerous publications. To avoid superfluous repetitions of well-known facts, for interested readers we recommend, e.g., the monographs [5,20], and we only recall the definition of 2-dimensional copulas and a few of their properties necessary for considerations in this paper. From the axiomatic point of view, a 2-dimensional copula is a function $C : [0,1]^2 \to [0,1]$ satisfying the boundary conditions:

- $C(0, v) = C(u, 0)$ for all $u, v \in [0,1]$;
- $C(u, 1) = u,\ C(1, v) = v$ for all $u, v \in [0,1]$;

and the 2-increasing property:

- $C(u_2, v_2) + C(u_1, v_1) - C(u_1, v_2) - C(u_2, v_1) \geq 0$ for all $u_1, u_2, v_1, v_2 \in [0,1]$ with $u_1 \leq u_2, v_1 \leq v_2$.

As we will only deal with 2-dimensional copulas, we will call them simply copulas. There are three basic copulas, namely the product copula $\Pi : [0,1]^2 \to [0,1]$ (copula of independence), the Fréchet-Hoeffding upper bound $M : [0,1]^2 \to [0,1]$ (copula of comonotone dependence) and the Fréchet-Hoeffding lower bound $W : [0,1]^2 \to [0,1]$ (copula of countermonotone dependence) which are given, respectively, by

$$\Pi(u, v) = uv, \quad M(u, v) = \min\{u, v\}, \quad W(u, v) = \max\{u + v - 1, 0\}.$$

Recall that the copulas $M$ and $W$ are the greatest and smallest elements of $\mathcal{C}_2$, respectively, i.e., for each copula $C \in \mathcal{C}_2$ we have $W \leq C \leq M$. Also recall, that all convex combinations of the three basic copulas form the Fréchet family of copulas, $\mathfrak{F} = \{\lambda_1 M + \lambda_2 W + (1 - \lambda_1 - \lambda_2)\Pi \mid \lambda_1, \lambda_2, 1 - \lambda_1 - \lambda_2 \in [0,1]\}$.

The three basic copulas also play an important role in the axiomatic definitions of concordance measures and weak concordance measures as well. In 1984, Scarsini [22] introduced concordance measures $\kappa$ as the mappings which assign to each copula $C \in \mathcal{C}_2$ a real number $\kappa(C)$ satisfying the following axioms:

(c1) for each $C \in \mathcal{C}_2$, $\kappa(C^t) = \kappa(C)$, where $C^t$ is a copula given by $C^t(u, v) = C(v, u)$ for all $(u, v) \in [0,1]^2$;
(c2) for all $C_1, C_2 \in \mathcal{C}_2$, $\kappa(C_1) \leq \kappa(C_2)$ whenever $C_1 \leq C_2$;

(c3) for each $C \in \mathcal{C}_2$, $\kappa(C^-) = -\kappa(C)$, where $C^-$ is a copula given by $C^-(u,v) = v - C(1-u,v)$ for all $(u,v) \in [0,1]^2$;
(c4) $\kappa(M) = 1$;
(c5) for each $\{C_n\}_{n=1}^{\infty} \subset \mathcal{C}_2$, if $C_n \underset{n\to\infty}{\longrightarrow} C$ then $\lim_{n\to\infty} \kappa(C_n) = \kappa(C)$.

Note that in (c1), $C^t \in \mathcal{C}_2$ is the transpose of $C$, satisfying the property that if $C = C_{X,Y}$, then $C^t = C_{Y,X}$. In (c3), $C^- \in \mathcal{C}_2$ is a reflected copula with the property that if $C = C_{X,Y}$ then $C^- = C_{-X,Y}$. Also observe that in (c5) the pointwise convergence of $\{C_n\}_{n=1}^{\infty}$ is considered. Moreover, combining (c3) and (c4) gives immediately $\kappa(W) = -1$, and similarly, due to the property $\Pi^- = \Pi$, (c3) ensures $\kappa(\Pi) = 0$. Because of (c2) and the property $W \le C \le M$ for each $C \in \mathcal{C}_2$, it holds $-1 \le \kappa(C) \le 1$.

As mentioned in the previous section, a possible property of concordance measures is convexity [19]. If a concordance measure $\kappa$ commutes with convex combinations of copulas then, for example, for each Fréchet copula $C = \lambda_1 M + \lambda_2 W + (1-\lambda_1-\lambda_2)\Pi$ (with any $\lambda_1, \lambda_2, 1-\lambda_1-\lambda_2 \in [0,1]$) we have $\kappa(C) = \lambda_1 - \lambda_2$.

In particular, the above mentioned concordance measures, i.e., Spearman's $\rho$, Kendall's $\tau$, Gini's $\gamma$ and Blomqvist's $\beta$ (when applied to population) are given, respectively, by:

$$\rho(C) = 12 \int_{[0,1]^2} C(u,v)\,du\,dv - 3;$$

$$\tau(C) = 4 \int_{[0,1]^2} C(u,v)\,dC(u,v) - 1;$$

$$\gamma(C) = 4 \int_0^1 (C(u,u) + C(u,1-u))\,du - 2;$$

$$\beta(C) = 4C\left(\frac{1}{2}, \frac{1}{2}\right) - 1.$$

For more details we recommend, e.g., [5,20,25].

The Spearman footrule $\phi$ was introduced in 1906, see [31]. It measures a rank-dependent distance between random variables, and hence it can also be determined by means of the copula $C$ of the considered two continuous random variables $X$ and $Y$:

$$\phi(C) = 1 - 3 \int_{[0,1]^2} |u-v|\,dC(u,v) = 6 \int_0^1 C(u,u)\,du - 2,$$

and can be seen as a kind of concordance measures. For an overview of several fields of its application we refer, e.g., to [26]. Using the last formula for the copula $W$ yields

$$\phi(W) = 6 \int_0^1 \max(2u-1,0)\,du - 2 = -0.5,$$

which shows that $\phi$ does not satisfy the property $\phi(W) = -1$ following from the axioms of the concordance measures, i.e., Spearman's footrule is not a measure of concordance in the sense of Scarsini's axioms. One can see that $\phi$ satisfies the properties in (c1), (c2), (c4) and (c5), but it fails to satisfy the property in (c3). In [17], Liebscher highlighted similar properties also for some other measures of association, which are now called weak concordance measures.

**Definition 2.1.** A mapping $\nu : \mathcal{C}_2 \to \mathbb{R}$ is called a weak concordance measure if it satisfies the axioms (w1) - (w5), where

$$(\text{w1}) = (\text{c1}),\ (\text{w2}) = (\text{c2}),\ (\text{w4}) = (\text{c4}),\ (\text{w5}) = (\text{c5}),\ \text{and}\ (\text{w3}):\ \nu(\Pi) = 0.$$

Stress that axiom (w3) is weaker than (c3) since (c3) implies (w3), as already mentioned above.

Evidently, each concordance measure is also a weak concordance measure, and Spearman's footrule $\phi$ is a proper weak concordance measure. In [19, Definition 3.1], we introduced the notion of convex concordance measures. Let us stress that in this context, the convexity of a concordance measure means commuting with the operation of the convex sum. In a similar way, we now define convex weak concordance measures:

**Definition 2.2.** A weak concordance measure $\nu : \mathcal{C}_2 \to \mathbb{R}$ is convex if for all $C_1, C_2 \in \mathcal{C}_2$ and each $\lambda \in [0,1]$ we have

$$\nu(\lambda C_1 + (1-\lambda)C_2) = \lambda\nu(C_1) + (1-\lambda)\nu(C_2). \tag{1}$$

Condition (1) can be seen as the linearity of $\nu$ applied to convex combinations of copulas. Equivalently, it means that $\nu$ is of degree 1, as introduced for concordance measures by Edwards and Taylor in [8].

Observe that the convexity of $\rho, \gamma, \beta$ and $\phi$ follows immediately from the additivity and homogeneity of the Riemann integral. For Kendall's $\tau$ we have, e.g., $\tau\left(\frac{1}{2}M + \frac{1}{2}\Pi\right) = \frac{5}{12}$, but $\frac{1}{2}\tau(M) + \frac{1}{2}\tau(\Pi) = \frac{1}{2}$, hence $\tau$ is not a convex (weak) concordance measure. Note that $\tau$ is of degree 2 in the sense of [8].

## 3. Single point-generated convex weak concordance measures

In this section, inspired by the idea of Blomqvist's beta and our previous work [19], we will look for the simplest convex weak concordance measures $\nu$ determined by the value of a considered copula $C$ at a fixed single point $(u, v)$ and its transpose $(v, u)$. As all copulas coincide on the whole boundary of the unit square, such a point has to be located in $]0,1[^2$. By (w1), the value $C(v, u)$ has the same influence on $\nu(C)$, thus with no loss of generality we can start from a point $(u, v) \in \bigtriangledown = \{(u, v) \in ]0,1[^2 \mid u \geq v\}$. A weak concordance measure generated by a fixed point $(u, v) \in \bigtriangledown$ will be denoted by $\nu_{(u,v)}$.

Given any fixed point $(u, v) \in \bigtriangledown$, we assign to each copula $C \in \mathcal{C}_2$ a quantity $V_{(u,v)}(C) = C(u, v) + C(v, u)$. Evidently, the mapping $V_{(u,v)} : \mathcal{C} \to \mathbb{R}$ satisfies the axioms (w1), (w2) and (w5). Also, due to the fact that $V_{(u,v)}(\Pi) = 2uv < 2v = V_{(u,v)}(M)$, we can normalize $V_{(u,v)}$ into $\nu_{(u,v)}$ as follows:

$$\nu_{(u,v)}(C) = \frac{V_{(u,v)}(C) - V_{(u,v)}(\Pi)}{V_{(u,v)}(M) - V_{(u,v)}(\Pi)}. \tag{2}$$

In this way, we have guaranteed that $\nu_{(u,v)}$ satisfies (w4) because $\nu_{(u,v)}(M) = 1$, and also (w3): $\nu_{(u,v)}(\Pi) = 0$. The properties given in (w1), (w2) and (w5) remain valid, too.

**Theorem 3.1.** *Let $(u, v) \in \bigtriangledown$. Then the mapping $\nu_{(u,v)} : \mathcal{C} \to \mathbb{R}$ given by*

$$\nu_{(u,v)}(C) = \frac{C(u, v) + C(v, u) - 2uv}{2v - 2uv}, \tag{3}$$

*is a convex weak concordance measure.*

**Proof.** From the above discussion it follows that $\nu_{(u,v)}$ given by (2) is a weak concordance measure. As $V_{(u,v)}(C) = C(u, v) + C(v, u)$, $V_{(u,v)}(\Pi) = 2uv$ and $V_{(u,v)}(M) = 2v$, $\nu_{(u,v)}(C)$ can be written immediately in the form (3). Moreover, for any $C_1, C_2 \in \mathcal{C}_2$ and any $\lambda \in [0, 1]$, we have

$$\begin{aligned}
&\nu_{(u,v)}(\lambda C_1 + (1-\lambda)C_2) \\
&= \frac{\lambda(C_1(u, v) + C_1(v, u)) + (1-\lambda)(C_2(u, v) + C_2(v, u)) - 2uv}{2v - 2uv} \\
&= \lambda \frac{C_1(u, v) + C_1(v, u) - 2uv}{2v - 2uv} + (1-\lambda)\frac{C_2(u, v) + C_2(v, u) - 2uv}{2v - 2uv} \\
&= \lambda \nu_{(u,v)}(C_1) + (1-\lambda)\nu_{(u,v)}(C_2),
\end{aligned}$$

which completes the proof. □

Note that for $(u, v) = \left(\frac{1}{2}, \frac{1}{2}\right)$, Eq. (3) gives the Blomqvist beta, i.e., $\nu_{(1/2,1/2)} = \beta$.

**Example 3.1.** Consider any point $(u, u) \in \bigtriangledown$. Then for each $C \in \mathcal{C}_2$, we get

$$\nu_{(u,u)}(C) = \frac{C(u, u) - u^2}{u - u^2}.$$

Considering $C = W$, we have

$$\nu_{(u,u)}(W) = \begin{cases} -\frac{u}{1-u} & \text{if } u \in \left]0, \frac{1}{2}\right], \\ -\frac{1-u}{u} & \text{if } u \in \left[\frac{1}{2}, 1\right[, \end{cases}$$

which yields $\nu_{(u,u)}(W) \in [-1, 0[$, and thus, for any $C \in \mathcal{C}_2$ and $u \in ]0, 1[$, $\nu_{(u,u)}(C) \in [-1, 1]$. Moreover, $\{\nu_{(u,u)}(C) \mid C \in \mathcal{C}_2\} = [-1, 1]$ if and only if $u = \frac{1}{2}$, and in that case, $\nu_{(1/2,1/2)} = \beta$ is recovered.

However, in general, there is no lower bound for (convex) weak concordance measures as the following example shows.

**Example 3.2.** Consider any point $(u, v) \in \bigtriangledown$ with $u + v \leq 1$, and the copula $W$. Then

$$\nu_{(u,v)}(W) = \frac{-2uv}{2v - 2uv} = -\frac{u}{1-u} \in ]-\infty, 0[.$$

Further, for any $\alpha \in ]1, \infty[$, let $u = \frac{\alpha}{1+\alpha}$, $v = \frac{1}{1+\alpha}$. Then $(u, v) \in \bigtriangledown$, $u + v = 1$, and thus $\nu_{(u,v)}(W) = -\alpha$, which confirms that there is no lower bound for (convex) weak concordance measures.

## 4. Probability- and measure-based convex weak concordance measures

It is easy to check that convex combinations of convex weak concordance measures $\nu_{(u_1,v_1)}, \dots, \nu_{(u_k,v_k)}$, i.e., functions $\nu : \mathcal{C}_2 \to \mathbb{R}$ given as

$$\nu(C) = \sum_{i=1}^{k} \lambda_i \nu_{(u_i,v_i)}(C), \tag{4}$$

where $\lambda_i \in [0, 1]$, $\sum_{i=1}^{k} \lambda_i = 1$, are again convex weak concordance measures.

Formally, a convex weak concordance measure $\nu$ given by (4) can be seen as the Lebesgue integral with respect to the probability measure $\Lambda$ defined on the set of all Borel subsets of $\bigtriangledown$ (denoted by $\mathcal{B}(\bigtriangledown)$) by $\Lambda(E) = \sum_{(u_i,v_i)\in E} \lambda_i$, i.e.,

$$\nu(C) = \int_{\bigtriangledown} \nu_{(u,v)}(C)\, d\Lambda(u, v). \tag{5}$$

Let

$$p_i = \frac{\frac{\lambda_i}{2v_i - 2u_i v_i}}{\sum_{j=1}^{k} \frac{\lambda_j}{2v_j - 2u_j v_j}}.$$

Then $\sum_{i=1}^{k} p_i = 1$, i.e., we have introduced another probability measure $P^*$ on $(\bigtriangledown, \mathcal{B}(\bigtriangledown))$ given by $P^*((u_i, v_i)) = p_i$. Using the Lebesgue integral, we first introduce for each $C \in \mathcal{C}_2$ the quantity

$$V(C) = \int_{\bigtriangledown} V_{(u,v)}\, dP^*(u, v),$$

and then its normalization $\eta(C) = \frac{V(C) - V(\Pi)}{V(M) - V(\Pi)}$, which can be rewritten as

$$\begin{aligned}
\eta(C) &= \frac{\int_{\bigtriangledown} V_{(u,v)}\, dP^*(u,v) - \int_{\bigtriangledown} 2uv\, dP^*(u,v)}{\int_{\bigtriangledown} 2v\, dP^*(u,v) - \int_{\bigtriangledown} 2uv\, dP^*(u,v)} \\
&= \frac{\sum_{i=1}^{k} V_{(u_i,v_i)}(C) p_i - \sum_{i=1}^{k} 2u_i v_i p_i}{\sum_{i=1}^{k} 2v_i p_i - \sum_{i=1}^{k} 2u_i v_i p_i} \\
&= \frac{\sum_{i=1}^{k} V_{(u_i,v_i)}(C) \cdot \frac{\lambda_i}{2v_i - 2u_i v_i} - \sum_{i=1}^{k} 2u_i v_i \cdot \frac{\lambda_i}{2v_i - 2u_i v_i}}{\sum_{i=1}^{k} (2v_i - 2u_i v_i) \cdot \frac{\lambda_i}{2v_i - 2u_i v_i}} \\
&= \sum_{i=1}^{k} \lambda_i \cdot \frac{V_{(u_i,v_i)}(C) - 2u_i v_i}{2v_i - 2u_i v_i} = \sum_{i=1}^{k} \lambda_i \nu_{(u_i,v_i)}(C) = \nu(C),
\end{aligned} \tag{6}$$

i.e., $\eta = \nu$, which implies that $\eta$ is a convex weak concordance measure.

Considering an arbitrary probability measure $P$ defined on all Borel subsets of $\bigtriangledown$, we can generalize formulas (5) and (6), respectively, as follows:

$$\nu_P(C) = \int_{\bigtriangledown} \nu_{(u,v)}(C)\, dP(u, v), \tag{7}$$

$$\nu^P(C) = \frac{\int\limits_{\bigtriangledown} V_{(u,v)}(C)\, dP(u,v) - \int\limits_{\bigtriangledown} 2uv\, dP(u,v)}{\int\limits_{\bigtriangledown} 2v\, dP(u,v) - \int\limits_{\bigtriangledown} 2uv\, dP(u,v)}. \tag{8}$$

**Theorem 4.1.** *Let $P$ be a probability measure defined on the set of all Borel subsets of $\bigtriangledown$. Then $\nu^P$ given in (8) is a convex weak concordance measure.*

**Proof.** Because of the uniform continuity of copulas and the fact that the denominator in (8) is positive for any $C$, $\nu^P$ is well defined for any copula $C$.

To verify that $\nu^P$ satisfies axioms (w1)-(w5), we first observe that due to the fact that $V_{(u,v)}(C) = V_{(u,v)}(C^t)$ for any $C \in \mathcal{C}_2$ and $(u,v) \in \bigtriangledown$, $\nu^P$ satisfies axiom (w1). Similarly, the relation $C_1 \le C_2$ implies $V_{(u,v)}(C_1) \le V_{(u,v)}(C_2)$ for each $(u,v) \in \bigtriangledown$, which ensures satisfying (w2). It is trivial to show that $\nu^P$ also satisfies axioms (w3) and (w4). Concerning axiom (w5), observe that the pointwise convergence of copulas ensures the related uniform convergence, see [4,15]. Therefore, for each $\{C_n\}_{n=1}^{\infty} \subset \mathcal{C}_2$, if $C_n \underset{n\to\infty}{\longrightarrow} C$, also $V_{(u,v)}(C_n) \underset{n\to\infty}{\longrightarrow} V_{(u,v)}(C)$ uniformly for each $(u,v) \in \bigtriangledown$, and thus $\int\limits_{\bigtriangledown} V_{(u,v)}(C_n)\, dP(u,v) \underset{n\to\infty}{\longrightarrow} \int\limits_{\bigtriangledown} V_{(u,v)}(C)\, dP(u,v)$.

Consequently, $\nu^P(C_n) \underset{n\to\infty}{\longrightarrow} \nu^P(C)$, i.e., $\nu^P$ also satisfies (w5), thus $\nu^P$ is a convex weak concordance measure. □

We have shown that for each probability measure $P$ on $(\bigtriangledown, \mathcal{B}(\bigtriangledown))$, $\nu^P$ given in (8) is well defined and is a convex weak concordance measure. This is no longer true for $\nu_P$ given in (7), because the Lebesgue-Stieltjes integral on the right-hand side of (7) need not exist, see the following example.

**Example 4.1.** Let $P$ be a probability measure on $(\bigtriangledown, \mathcal{B}(\bigtriangledown))$ whose support is the set $S = \{(u,v) \mid v = 1-u, u \in \left[\frac{3}{4}, 1\right[\}$ and density $p(u,v) = 4$, $(u,v) \in S$. Then

$$\nu_P(W) = \int_{\frac{3}{4}}^{1} \frac{-2u(1-u)}{2(1-u) - 2u(1-u)} \cdot 4\, du = \int_{\frac{3}{4}}^{1} \frac{-4u}{1-u}\, du = -\infty,$$

which shows that $\nu_P$ is not a weak concordance measure. For illustration, applying (8) for computing $\nu^P(W)$, we get

$$\nu^P(W) = \frac{-\int_{\frac{3}{4}}^{1} 2u(1-u)\cdot 4\, du}{\int_{\frac{3}{4}}^{1} 2(1-u)\cdot 4\, du - \int_{\frac{3}{4}}^{1} 2u(1-u)\cdot 4\, du} = -5.$$

Here we stress the difference between the order of operations applied in formulas (7) and (8). While in (7) the function $V_{(u,v)} = C(u,v) + C(v,u)$ is first normalized and then the obtained result is integrated, in (8), $V_{(u,v)}$ is first integrated and then normalized.

We now proceed to prove the fact that $\nu^P$ given in (8), which is a convex weak concordance measure for each probability measure $P$ on $(\bigtriangledown, \mathcal{B}(\bigtriangledown))$, can be expressed as $\nu_Q$ for some other probability measure $Q$ on $(\bigtriangledown, \mathcal{B}(\bigtriangledown))$ (compare also (4) and (6)).

**Theorem 4.2.** *Let $P$ be a probability measure on $(\bigtriangledown, \mathcal{B}(\bigtriangledown))$. Then there is a probability measure $Q$ on $(\bigtriangledown, \mathcal{B}(\bigtriangledown))$ such that for each $C \in \mathcal{C}_2$ we have $\nu^P(C) = \nu_Q(C)$, where $\nu^P$ and $\nu_Q$ are convex weak concordance measures given, respectively, by (8) and (7).*

**Proof.** Consider a probability measure $Q$, absolutely continuous with respect to $P$, whose Radon-Nikodym derivative $g = \frac{dQ}{dP}$ is given by

$$g(u,v) = \frac{v - uv}{\int\limits_{\bigtriangledown} (s - rs)\, dP(r,s)}, \quad (u,v) \in \bigtriangledown, \tag{9}$$

i.e., for any Borel subset $B \in \bigtriangledown$,

$$Q(B) = \frac{\int\limits_{B} (v - uv)\, dP(u,v)}{\int\limits_{\bigtriangledown} (s - rs)\, dP(r,s)}.$$

Let $C \in \mathcal{C}_2$. Then applying (7) for $Q$, we obtain

$$\begin{aligned} \nu_Q(C) &= \int_{\nabla} \nu_{(u,v)}(C)\, dQ(u,v) = \int_{\nabla} \frac{V_{(u,v)}(C) - 2uv}{2v - 2uv} \cdot g(u,v)\, dP(u,v) \\ &= \int_{\nabla} \frac{V_{(u,v)}(C) - 2uv}{2v - 2uv} \cdot \frac{v - uv}{\int_{\nabla} (s - rs)\, dP(r,s)}\, dP(u,v) \\ &= \frac{1}{2\int_{\nabla} (s - rs)\, dP(r,s)} \int_{\nabla} (V_{(u,v)}(C) - 2uv)\, dP(u,v) \\ &= \frac{\int_{\nabla} V_{(u,v)}(C)\, dP(u,v) - \int_{\nabla} 2uv\, dP(u,v)}{\int_{\nabla} 2v\, dP(u,v) - \int_{\nabla} 2uv\, dP(u,v)} = \nu^P(C), \end{aligned}$$

i.e., $\nu_Q(C) = \nu^P(C)$ as claimed. □

**Example 4.2.** Consider the probability measure $P$ on $(\nabla, \mathcal{B}(\nabla))$ with density $p(u,v) = 6v$, $(u,v) \in \nabla$. Then the corresponding convex weak concordance measure $\nu^P$ introduced in (8) can be determined as follows:

As $dP(u,v) = 6v\, du dv$, for the integrals involved in (8) we have $\int_{\nabla} 2uv\, dP(u,v) = \int_0^1 \int_0^u 12uv^2\, du dv = \frac{4}{5}$, $\int_{\nabla} 2v\, dP(u,v) = \int_0^1 \int_0^u 12v^2\, du dv = 1$, and after substituting these values into (8), for any $C \in \mathcal{C}_2$, we get

$$\nu^P(C) = 30 \int_{\nabla} (C(u,v) + C(v,u)) \cdot v\, du dv - 4. \tag{10}$$

In particular,

$$\nu^P(W) = 30 \int_{1/2}^{1} \int_{1-u}^{u} 2(u + v - 1) \cdot v\, du dv - 4 = -0.875,$$

which shows that $\nu^P$ is a proper convex weak concordance measure.

The same result as given in Eq. (10), can be obtained by using the representation of $\nu^P$ as $\nu_Q$ as described in Theorem 4.2, namely

$$\nu^P(C) = \nu_Q(C) = \int_{\nabla} \frac{C(u,v) + C(v,u) - 2uv}{2v - 2uv}\, dQ(u,v),$$

where $Q$ is a probability on $(\nabla, \mathcal{B}(\nabla))$ such that

$$dQ(u,v) = 10(1-u)v\, dP(u,v) = 60(1-u)v^2\, du dv,\ (u,v) \in \nabla,$$

as can be verified by a direct computation.

The construction given in (8) can be written in a form similar to that given by Edwards et al. in [7, Theorem 0.6]. Indeed, for any probability measure $P$ on $\mathcal{B}(\nabla)$ there is a finite regular Borel measure $\mu$ on $\mathcal{B}(]0,1[^2)$ that is $t$-invariant, i.e., $\mu(A^t) = \mu(A)$ for each $A \in \mathcal{B}(]0,1[^2)$, where $A^t = \{(u,v) \in ]0,1[^2 \mid (v,u) \in A\}$, and such that $\mu(B) = P(B) + P(B \cap \nabla^t)$ for each $B \in \mathcal{B}(\nabla)$. Then, for any $C \in \mathcal{C}_2$,

$$\nu^P(C) = \frac{\int_{]0,1[^2} (C - \Pi)\, d\mu}{\int_{]0,1[^2} (M - \Pi)\, d\mu}.$$

The following characterization result for convex weak concordance measures can be proved using the same arguments as in the proof of Theorem 0.6 in [7], the only difference being in replacing $D_4$-invariance of the measures considered in [7] by $t$-invariance of the measures in our case.

**Theorem 4.3.** *Let $\mu$ be a Borel measure on $\mathcal{B}(]0,1[^2)$. Then $\nu^{(\mu)} : \mathcal{C}_2 \to \mathbb{R}$ given by*

$$\nu^{(\mu)}(C) = k \int_{]0,1[^2} (C - \Pi)\, d\mu, \tag{11}$$

*is a convex weak concordance measure for some $k > 0$ if and only if $\mu$ is positive, regular, t-invariant, satisfying the property $\int_{]0,1[^2} (M - \Pi)\, d\mu \in ]0, \infty[$, and $k = \frac{1}{\int_{]0,1[^2} (M-\Pi)\, d\mu}$.*

Obviously, $\nu^{(\mu)} = \nu^P$ for a probability measure $P$ on $\mathcal{B}(\bigtriangledown)$ if and only if $\mu(]0,1[^2) \in ]0, \infty[$. On the other hand, if $\mu$ satisfies the constraints of Theorem 4.3 and $\mu(]0,1[^2) = \infty$, then there is no probability measure $P$ on $\mathcal{B}(\bigtriangledown)$ such that $\nu^{(\mu)} = \nu^P$ (compare with [7, Proposition 0.7]). Note that for any $\mu$ satisfying the constraints of Theorem 4.3 and any positive constant $r$, $r \cdot \mu$ also satisfies Theorem 4.3 and $\nu^{(r\cdot\mu)} = \nu^{(\mu)}$.

**Example 4.3.** Let $m$ be the standard Lebesgue measure on $\mathcal{B}(]0,1[)$ and let $\mu : \mathcal{B}(]0,1[^2) \to \mathbb{R}$ be given by

$$\mu(A) = \int_{\{x \in ]0,1[\, |(x,x)\in A\}} \frac{1}{x}\, dm(x).$$

Then $\mu$ satisfies all constraints of Theorem 4.3, $\mu(]0,1[^2) = \infty$, and

$$\int_{]0,1[^2} (M - \Pi)\, d\mu = \int_0^1 \frac{x - x^2}{x}\, dx = \frac{1}{2}.$$

Hence $k = 2$ and $\nu^{(\mu)}$ given by

$$\nu^{(\mu)}(C) = 2 \int_0^1 \frac{C(x,x)}{x}\, dx - 1$$

is a convex weak concordance measure. Observe, e.g., that $\nu^{(\mu)}(W) = 1 - \ln 4$.

Now, consider an $a \in ]0, 1/2]$ and an ordinal sum copula $C_a$, $C_a = (\langle a, 1, \Pi\rangle)$, see [5,20]. Then

$$\|C_a - \Pi\|_\infty = a(1-a) \text{ and } \nu^{(\mu)}(C_a) = -\left(\frac{a}{2} + \frac{a \ln a}{1-a}\right),$$

i.e.,

$$\frac{\nu^{(\mu)}(C_a)}{\|C_a - \Pi\|_\infty} = -\frac{\frac{1}{2} + \frac{\ln a}{1-a}}{1-a} \underset{a\to 0^+}{\longrightarrow} \infty.$$

On the other hand, for each probability measure $P$ on $\mathcal{B}(\bigtriangledown)$, we have

$$|\nu^P(C_a)| \le \frac{2\|C_a - \Pi\|_\infty}{\int_{\bigtriangledown} 2v\, dP(u,v) - \int_{\bigtriangledown} 2uv\, dP(u,v)},$$

and thus $\{\frac{\nu^P(C_a)}{\|C_a - \Pi\|_\infty}\}_{a\in ]0,1/2]}$ is bounded. Therefore, $\nu^{(\mu)}$ cannot be obtained in the form (8), $\nu^{(\mu)} \neq \nu^P$ for each probability measure $P$ on $\mathcal{B}(\bigtriangledown)$.

**Example 4.4.** Let $\mu$ be a discrete measure on $\mathcal{B}(]0,1[^2)$ given by

$$\mu(A) = \sum_{(1/n,1/n)\in A, n\ge 2} \frac{1}{n}.$$

Then $\mu(]0,1[^2) = \infty$ and

$$\nu^{(\mu)}(C) = \frac{\sum_{n=2}^{\infty} \frac{C\left(\frac{1}{n},\frac{1}{n}\right)}{n} - \sum_{n=2}^{\infty} \frac{1}{n^3}}{\sum_{n=2}^{\infty}\left(\frac{1}{n^2} - \frac{1}{n^3}\right)}$$

is a convex weak concordance measure.

## 5. Representation of basic convex weak concordance measures

In this section, we will look at the representations of the basic convex weak concordance measures — Spearman's $\rho$, Gini's $\gamma$, Blomqvist's $\beta$, and Spearman's footrule $\phi$ — via $\nu^P$ and $\nu_Q$, introduced in (8), (7) and Theorem 4.2. We also add some other illustrative examples.

**Proposition 5.1.** *Let $P$, $Q$ be probability measures on $(\bigtriangledown, \mathcal{B}(\bigtriangledown))$, with densities (w.r.t. the Lebesgue measure), respectively, $p(u,v)=2$, and $q(u,v)=24(v-uv)$, $(u,v)\in\bigtriangledown$. Then for Spearman's $\rho$ we have $\rho=\nu^P=\nu_Q$.*

**Proof.** The probability $P$ is uniformly distributed over $\bigtriangledown$. As $dP(u,v)=2\,du\,dv$, applying (8), for any $C\in\mathcal{C}_2$, we obtain:

$$\nu^P(C)=\frac{\int_{\bigtriangledown} 2V_{(u,v)}(C)\,du\,dv-\int_{\bigtriangledown} 4uv\,du\,dv}{\int_{\bigtriangledown} 4v\,du\,dv-\int_{\bigtriangledown} 4uv\,du\,dv}$$

$$=\frac{\int_{\bigtriangledown} V_{(u,v)}(C)\,du\,dv-\int_0^1\int_0^u 2uv\,dv\,du}{\int_0^1\int_0^u 2v\,dv\,du-\int_0^1\int_0^u 2uv\,dv\,du}=\frac{\int_{\bigtriangledown} V_{(u,v)}(C)\,du\,dv-\int_0^1 u^3\,du}{\int_0^1 u^2\,du-\int_0^1 u^3\,du}$$

$$=\frac{\int_{\bigtriangledown}(C(u,v)+C(v,u))\,du\,dv-\frac{1}{4}}{\frac{1}{3}-\frac{1}{4}}=12\int_{[0,1]^2} C(u,v)-3=\rho(C).$$

For determining $\nu_Q(C)$ it is enough to observe the relation between $Q$ and $P$, namely that $dQ(u,v)=24(v-uv)\,du\,dv=12(v-uv)\cdot 2\,du\,dv=g(u,v)\,dP(u,v)$, see Eq. (9) in the proof of Theorem 4.2, which ensures the equality $\nu_Q=\nu^P$, and together with the previous part of the proof, we get the claim. □

Let us complement that $\rho=\nu^{(m)}$, $m$ being the standard Lebesgue measure on $\mathcal{B}(]0,1[^2)$.

**Proposition 5.2.** *Let $P$, $Q$ be probability measures on $(\bigtriangledown, \mathcal{B}(\bigtriangledown))$, with densities (w.r.t. the Lebesgue measure), respectively,*

$$p(u,v)=\begin{cases}\frac{1}{2} & \text{if } v=u,\ u\in]0,1[,\\ 1 & \text{if } v=1-u,\ u\in\left]\frac{1}{2},1\right[,\end{cases}$$

$$q(u,v)=\begin{cases}4(v-uv) & \text{if } v=u,\ u\in]0,1[,\\ 8(v-uv) & \text{if } v=1-u,\ u\in\left]\frac{1}{2},1\right[.\end{cases}$$

*Then for Gini's $\gamma$ we have $\gamma=\nu^P=\nu_Q$.*

**Proof.** The support of both probability measures $P$, $Q$ is the set $S=\{(u,v)\mid v=u, u\in]0,1[\}\cup\{(u,v)\mid v=1-u, u\in\left]\frac{1}{2},1\right[\}$.

As for $P$, the mass $\frac{1}{2}$ is uniformly distributed on each of the two parts of $S$. Let $C\in\mathcal{C}_2$. For determining $\nu^P(C)$, see (8), we need to evaluate the following integrals:

(a) $\int_{\bigtriangledown} V_{(u,v)}(C)\,dP(u,v)$

$$=\int_0^1 2C(u,u)\cdot\frac{1}{2}\,du+\int_{\frac{1}{2}}^1 (C(u,1-u)+C(1-u,u))\,du$$

$$=\int_0^1 C(u,u)\,du+\int_{\frac{1}{2}}^1 C(u,1-u)\,du+\int_0^{\frac{1}{2}} C(t,1-t)\,dt$$

$$=\int_0^1 (C(u,u)+C(u,1-u))\,du,$$

(b) $\int_{\bigtriangledown} 2uv\,dP(u,v)=\int_0^1 2u^2\cdot\frac{1}{2}\,du+\int_{\frac{1}{2}}^1 2u(1-u)\,du=\frac{1}{2},$

and finally,

(c) $\int_{\bigtriangledown} 2v\, dP(u,v) = \int_0^1 2u \cdot \frac{1}{2}\, du + \int_{\frac{1}{2}}^1 2(1-u)\, du = \frac{3}{4}$.

Substituting these partial results (a) - (c) into Eq. (8) gives

$$\nu^P(C) = \frac{\int_0^1 (C(u,u) + C(u,1-u))\, du - \frac{1}{2}}{\frac{3}{4} - \frac{1}{2}}$$

$$= 4\int_0^1 (C(u,u) + C(u,1-u))\, du - 2 = \gamma(C),$$

as required.

The representation of $\gamma$ via $\nu_Q$ can be shown either by using Theorem 4.2 or by a direct computation. We omit the details. □

It can also be shown that $\gamma = \nu^{(\mu)}$, where $\mu$ is a probability measure on $\mathcal{B}(]0,1[^2)$ related to the copula $C = \frac{M+W}{2}$.

**Proposition 5.3.** *Let $P$, $Q$ be probability measures on $(\bigtriangledown, \mathcal{B}(\bigtriangledown))$. Let for each $(u,v) \in \bigtriangledown$, their densities (w.r.t. the Lebesgue measure) be given, respectively, by*

$$p(u,v) = \begin{cases} 1 & \text{if } v = u,\ u \in ]0,1[, \\ 0 & \text{otherwise,} \end{cases}$$

$$q(u,v) = \begin{cases} 6(u-u^2) & \text{if } v = u,\ u \in ]0,1[, \\ 0 & \text{otherwise.} \end{cases}$$

*Then Spearman's footrule $\phi$ can be represented as $\phi = \nu^P = \nu_Q$.*

**Proof.** The proof is a matter of simple computations, we omit the details. □

Note that the probability measure $P$ considered for Spearman's footrule $\phi$ is related to the strongest copula $M$. Similarly, one can derive a convex weak concordance measure $\nu^P$ with $P$ related to an arbitrarily chosen fixed exchangeable copula $E$. Put, for example, $E = W$. Then the support of $P$ is the set $\{(u,v) \mid u \in [1/2, 1[, v = 1-u\}$, and the related density is $p(u,v) = p(u,1-u) = 2$. Then

$$\nu^P(C) = 12\int_0^1 C(u,1-u)\, du - 2. \tag{12}$$

Let us add that $\phi = \nu^{(P)}$ for the same $P$, i.e., $\phi = \nu^P = \nu^{(P)}$.

As already mentioned above, for Blomqvist's $\beta$ we have $\beta = \nu_{(1/2,1/2)}$. Observe that this can be written as $\beta = \nu^P = \nu_P$, where $P$ is the Dirac measure concentrated at the point $(1/2, 1/2)$.

We add a more general example covering Spearman's footrule.

**Example 5.1.** For any $a \in ]0,1]$, consider a probability measure $P_a$ on $(\bigtriangledown, \mathcal{B}(\bigtriangledown))$, uniformly distributed over its support $S_a = \{(u,v) \mid v = au, u \in ]0,1]\}$, i.e., $p_a(u,v) = 1$ for $(u,v) \in S_a$. Then $\int_{\bigtriangledown} 2uv\, dP_a(u,v) = \int_0^1 2au^2\, du = \frac{2a}{3}$ and $\int_{\bigtriangledown} 2v\, dP_a(u,v) = \int_0^1 2au\, du = a$.

Therefore, for any $C \in \mathcal{C}_2$, $\nu^{P_a}(C)$ defined by (8) can be written as

$$\nu^{P_a}(C) = \frac{\int_0^1 (C(u,au) + C(au,u))\, du - \frac{2a}{3}}{a - \frac{2a}{3}} = \frac{3}{a}\int_0^1 (C(u,au) + C(au,u))\, du - 2.$$

In particular, for $C = W$ we have

$$\nu^{P_a}(W) = \frac{3}{a}\int_{\frac{1}{a+1}}^1 2(u + au - 1)\, du - 2 = 1 - \frac{3}{a+1} \in \left]-2, -\frac{1}{2}\right],$$

i.e., $\nu^{P_a}(W)$ is a proper convex weak concordance measure. Note, that for $a = 1$, $\nu^{P_1} = \phi$, and $\nu^{P_1}(W) = -\frac{1}{2}$ as mentioned before. Observe that if $a = \frac{1}{2}$ then $\nu^{P_{1/2}}(W) = -1$, however, $\nu^{P_{1/2}}$ does not fulfil (c3).

## 6. Relations between single point-determined convex concordance measures and convex weak concordance measures

In [19], we have studied concordance measures $\kappa$ whose values $\kappa(C)$, $C \in \mathcal{C}_2$, depend on the value of $C$ at some fixed point $(u, v)$ only. From similar reasons as explained here in Section 3, it is enough to consider a point $(u, v) \in \triangle = \{(u, v) \in ]0, 1[^2 \mid 0 < v \le u \le 1/2\}$, but because of the required properties of $\kappa$, in general, it is necessary to add seven other points derived from $(u, v)$, and define the function $K : \mathcal{C}_2 \to \mathbb{R}$ given by

$$K_{(u,v)}(C) = C(u, v) + C(v, u) + C(1-u, v) + C(u, 1-v) + C(v, 1-u) + C(1-v, u) + C(1-u, 1-v) + C(1-v, 1-u).$$

We have proved, see [19, Theorem 3.1], that the function $\kappa_{(u,v)} : \mathcal{C}_2 \to \mathbb{R}$,

$$\kappa_{(u,v)}(C) = \frac{K_{(u,v)}(C) - 2}{4v} \tag{13}$$

is a convex concordance measure, see also [7, Example 0.10]. As each convex concordance measure is also a convex weak concordance measure, it can be interesting to establish the relationship between the measures $\kappa_{(u,v)}$ and $\nu_{(u,v)}$.

**Proposition 6.1.** *Let $(u, v) \in \triangle = \{(u, v) \in ]0, 1[^2 \mid 0 < v \le u \le 1/2\}$. Let $P$ be a probability measure on $(\bigtriangledown, \mathcal{B}(\bigtriangledown))$ with support $S = \{(u, v), (1-u, v), (1-v, u), (1-v, 1-u)\}$, distributed as follows:*

$$P((u, v)) = \frac{1-u}{2}, \; P((1-u, v)) = P((1-v, u)) = \frac{u}{2}, \; P((1-v, 1-u)) = \frac{1-u}{2}.$$

*Then $\kappa_{(u,v)}(C) = \nu_P(C) = \int_{\bigtriangledown} \nu_{(u,v)}(C)\, dP(u, v)$.*

**Proof.** Clearly, if $(u, v) \in \triangle$, then $S \subset \bigtriangledown$, and $P$ is a well-defined discrete probability on $(\bigtriangledown, \mathcal{B}(\bigtriangledown))$. Using (7) and (3), we get

$$\begin{aligned}\nu_P(C) &= \int_{\bigtriangledown} \nu_{(u,v)}(C)\, dP(u, v) \\ &= \frac{C(u, v) + C(v, u) - 2uv}{2v - 2uv} \cdot \frac{1-u}{2} + \frac{C(1-u, v) + C(v, 1-u) - 2(1-u)v}{2v - 2(1-u)v} \cdot \frac{u}{2} \\ &+ \frac{C(1-v, u) + C(u, 1-v) - 2(1-v)u}{2u - 2(1-v)u} \cdot \frac{u}{2} \\ &+ \frac{C(1-v, 1-u) + C(1-u, 1-v) - 2(1-v)(1-u)}{2(1-u) - 2(1-v)(1-u)} \cdot \frac{1-u}{2},\end{aligned}$$

which can be simplified into the form $\frac{K_{(u,v)}(C)-2}{4v} = \kappa_{(u,v)}(C)$, as required. □

Finally, let us note that in a similar way one can show that for any probability measure $P$ on $(\triangle, \mathcal{B}(\triangle))$ there is a probability measure $Q$ on $(\bigtriangledown, \mathcal{B}(\bigtriangledown))$ such that the convex concordance measure $\kappa^P : \mathcal{C}_2 \to \mathbb{R}$ introduced in [19] and given by

$$\kappa^P(C) = \frac{\int_{\Delta} K_{(u,v)}(C)\, dP(u, v) - 2}{\int_{\Delta} K_{(x,y)}(M)\, dP(u, v) - 2},$$

see [19, Theorem 4.1], can be represented via $\nu_Q$, i.e., $\kappa^P = \nu_Q$, where $\nu_Q$ is a convex weak concordance measure defined in Eq. (7).

## 7. Concluding remarks

Following the idea of weak concordance measures introduced by Liebscher in [17], in this paper, we have defined and studied convex weak concordance measures, in particular, the construction methods for this kind of stochastic dependence parameters. At first, we have constructed the simplest type of convex weak concordance measures determined by at most two symmetric points $(u, v), (v, u) \in ]0, 1[^2$. The obtained result generalizes the Blomqvist $\beta$ considering the only point $(1/2, 1/2)$ of the 2-copulas domain. Then, based on probability measures and basic pointwisely generated convex weak concordance measures, we have introduced more complicated types of convex weak concordance measures, covering and representing, among others, the standard concordance measures as Spearman's $\rho$, Gini's $\gamma$, but also Spearman's footrule $\phi$ which is a proper weak concordance measure. Recall that convex concordance measures, i.e., polynomial concordance measures of degree 1, were completely characterized in three equivalent ways in [8], one of them being an integral representation similar to our formula (8) bringing representation $\nu^P$. Inspired by [7, Theorem 0.6], we have given a characterization of convex weak concordance measures by means of transposition-invariant Borel measures $\mu$ on $\mathcal{B}(]0,1[^2)$. Also, an example of a convex weak concordance measure $\kappa$ not admitting the representation $\kappa = \nu^P$, see (8), has been added.

Our work brings several new open problems. As a challenging problem for the future research we also see the statistical interpretation of some (convex) weak concordance measures, as well as the proposal and a deep study of non-convex weak concordance measures. As an example, observe that the power $\beta^3$ is a (weak) concordance measure which is not convex, see also Manstavičius [18]. Similarly, if Spearman's footrule $\phi$ is considered, then the power $\phi^i$ is a proper weak concordance measure for each odd integer $i > 1$ which is not convex. Another interesting problem concerns the geometric influence of a probability $P$ on the convex weak concordance measure $\nu^P$. For example, in Example 4.2 based on the density $p(u,v) = 6v$, $(u,v) \in \bigtriangledown$, one can see a big impact of high values of $v$ on the values of $\nu^P$, see formula (10). Among other open problems, we can briefly mention the newly proposed weak concordance measures in the view of their statistical interpretation and proposals of the related estimators. Note that for the basic convex concordance measures $\nu_{(u,v)}$ and a copula $C \in \mathcal{C}_2$, the corresponding estimator could be given by

$$\widehat{\nu_{(u,v)}}(C) = \frac{\hat{C}(u,v) + \hat{C}(v,u) - 2uv}{2v - 2uv},$$

where $\hat{C}$ stands for the empirical estimate of the copula $C$. Inspired by the recent results concerning some well-known concordance measures (such as $\rho$ and $\tau$) in connection with the Markov product of copulas [4], a similar study could be of interest for weak concordance measures $\nu^P$. Also, the investigation of constraints between the pairs of (convex weak) concordance measures ($\rho$ and $\tau$ [24], $\rho$ and $\phi$ [16]) indicate a possibility of a similar investigation for $\nu^{P_1}$ and $\nu^{P_2}$ ($\nu^{(\mu_1)}$ and $\nu^{(\mu_2)}$). The application of Spearman's footrule in dimension reduction principle discussed by Fuchs in [10] opens a problem of applying some other convex weak concordance measures in dimension reduction, as, e.g., that one given in (12) based on $W$. Several other results known for concordance measures, as, for example, those given in [2,3,12,27,28], also suggest possible directions in investigating similar applications of (convex) weak concordance measures.

## CRediT authorship contribution statement

**Radko Mesiar:** Conceptualization, Funding acquisition, Methodology, Project administration, Writing – review & editing. **Anna Kolesárová:** Formal analysis, Methodology, Writing – original draft, Writing – review & editing. **Ayyub Sheikhi:** Formal analysis, Investigation, Validation. **Svitlana Shvydka:** Formal analysis, Investigation, Validation.

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