Fuzzy Sets and Systems 499 (2025) 109178

Contents lists available at ScienceDirect

Fuzzy Sets and Systems

journal homepage: www.elsevier.com/locate/fss

Short communication

Some notes on the coincidence of the Choquet integral and the pan-integral

Tong Kang [a], Radko Mesiar [b,c], Yao Ouyang [d,*], Jun Li [a,e]

[a] State Key Laboratory of Media Convergence & Communication, Communication University of China, Beijing 100024, China
[b] Slovak University of Technology, Faculty of Civil Engineering, Radlinského 11, 810 05 Bratislava, Slovakia
[c] UTIA CAS, Pod Vodárenskou věží 4, 182 08 Prague, Czech Republic
[d] Faculty of Science, Huzhou University, Huzhou, Zhejiang 313000, China
[e] School of Sciences, Communication University of China, Beijing 100024, China

ARTICLE INFO

Keywords:
Monotone measures
(M)-property
Weak (M)-property
Pan-integral
Choquet integral

ABSTRACT

In this note, we provide an example to show the weak (M)-property is really weaker than the (M)-property for some finite monotone measure defined on infinite space, and hence answer an open problem which was proposed in the paper (Li et al. (2023) [9]). We prove that if a monotone measure μ is autocontinuous, then the weak (M)-property and the (M)-property of μ are equivalent. We propose the concept of *(C-P)-property* of monotone measures and show a set of sufficient and necessary conditions that the Choquet integral coincides with the pan-integral. We further study the relationships between the Choquet integral and the pan-integral in the setting of the ordered pairs of monotone measures, and obtain some interesting properties.

1. Introduction

The Choquet integral [1] and the pan-integral [18] are two types of important nonlinear integrals. As is known, the Choquet integral is based on finite chains of measurable sets and the pan-integral is related to finite measurable partitions. When the considered measures are σ-additive, these two types of integrals and the Lebesgue integral are coincident. But, for a general monotone measure, they are not coincident for all measurable functions on the considered measurable space. The relationships between these two types of integrals were studied and some meaningful results were obtained, see [3,4,8,9,11,14,16].

In [11] Mesiar et al. proposed the (M)-property of monotone measure, and used it to discuss the relationship between the Choquet integral and the pan-integral. In [16] Ouyang et al. showed that the (M)-property implies the coincidence of the Choquet integral and the pan-integral. When X is a finite space and the underlying monotone measure μ is finite, then the (M)-property is also necessary for the coincidence, see [14]. Furthermore, Li et al. [9] introduced the *weak (M)-property* of monotone measure, and proved that the weak (M)-property is a necessary and sufficient condition for the coincidence of these two kinds of integrals. According to these results we know that the (M)-property and the weak (M)-property play important roles in the studying the coincidence of the Choquet integral and the pan-integral.

There are some close relationships between the (M)-property and the weak (M)-property. The (M)-property implies the weak (M)-property. Furthermore, if X is a finite set and μ is finite, then the weak (M)-property of μ implies the (M)-property, and therefore,

* Corresponding author.
E-mail addresses: kangtong@cuc.edu.cn (T. Kang), mesiar@math.sk (R. Mesiar), oyy@zjhu.edu.cn (Y. Ouyang), lijun@cuc.edu.cn (J. Li).

https://doi.org/10.1016/j.fss.2024.109178
Received 4 August 2024; Received in revised form 25 October 2024; Accepted 2 November 2024

Available online 5 November 2024
0165-0114/© 2024 Elsevier B.V. All rights reserved, including those for text and data mining, AI training, and similar technologies.

the weak (M)-property and the (M)-property are equivalent. But, when μ is infinite, the weak (M)-property does not imply the (M) property in general, see [9]. Thus, in [9] we raised the following open problem:

For a general space (not necessarily finite set) and a finite monotone measure, does the weak (M)-property imply the (M)-property?

This note answers the above question. We present an example in Section 3 to show that for some finite monotone measures defined on infinite space the weak (M)-property does not imply the (M)-property, and hence this question has a negative answer. We also prove that if a monotone measure μ is autocontinuous from above, then the weak (M)-property and the (M)-property of μ are equivalent. In Section 4 we introduce the concept of *(C-P)-property* of monotone measures and discuss some of its properties. We prove that the superaddivity with (C-P)-property is a sufficient and necessary condition that the Choquet integral coincides with the pan-integral. In Section 5 we further study the relationships between the Choquet integral and the pan-integral in the framework of the ordered pair of monotone measures, and obtain some general results. These generalize the previous results related to the coincidence of the Choquet integral and the pan-integral involving a unique monotone measure.

2. Preliminaries

Let $(X, \mathcal{A})$ be a measurable space. The set of all $\mathcal{A}$-measurable functions $h : X \to [0, +\infty)$ is denoted by $\mathcal{F}^+$, and let χ_T denote the characteristic function of $T \in \mathcal{A}$.

A *monotone measure* (or non-additive measure) on $(X, \mathcal{A})$ is a set function $\nu : \mathcal{A} \to [0, +\infty]$ satisfying the conditions: (1) $\nu(\emptyset) = 0$, and (2) $\nu(S) \le \nu(T)$ whenever $S \subseteq T$ and $S, T \in \mathcal{A}$. We denote by $\mathcal{M}$ the set of all monotone measures on $(X, \mathcal{A})$.

The monotoe measure μ is called superadditive, if for any $Q, S \in \mathcal{A}$ with $Q \cap S = \emptyset$, we have

$$\mu(Q \cup S) \ge \mu(Q) + \mu(S).$$

We recall the Choquet integral [1] (see also [17]) and pan-integral [18].

Let $\mu \in \mathcal{M}$ be given and let $h \in \mathcal{F}_+$.

The *Choquet integral* of h on X w.r.t. μ, is defined by

$$\int^{Ch} h d\mu = \sup \left\{ \sum_{i=1}^{m} c_i \mu(C_i) : \sum_{i=1}^{m} c_i \chi_{C_i} \le h, (C_i)_{i=1}^m \in \mathcal{H}_{Ch}, c_i \ge 0 \right\},$$

where $\mathcal{H}_{Ch}$ is the set of all finite chains in $\mathcal{A} \setminus \{\emptyset\}$.

The *pan-integral* of h on X w.r.t. μ, is defined by

$$\int^{pan} h d\mu = \sup \left\{ \sum_{j=1}^{n} p_j \mu(P_j) : \sum_{j=1}^{n} p_j \chi_{P_j} \le h, (P_j)_{j=1}^n \in \mathcal{H}_{pan}, p_j \ge 0 \right\},$$

where $\mathcal{H}_{pan}$ is the set of all finite measurable partitions of X.

Let $\mu \in \mathcal{M}$. For any $T \in \mathcal{A}$, from the fact that $\{T\}$ is a chain and $\{T, X \setminus T\}$ is a finite partition, the following properties are obvious:

(i) $\int^{Ch} \chi_T d\mu = \mu(T)$; (ii) $\int^{pan} \chi_T d\mu \ge \mu(T)$.

The following result played an important role in the discussion of coincidence of the Choquet integral and pan-integral, see [4,9,11,14].

Proposition 2.1. *Let $\mu \in \mathcal{M}$ be fixed. For all $h \in \mathcal{F}_+$, it holds*

$$\int^{Ch} h d\mu \ge \int^{pan} h d\mu$$

if and only if μ is superadditive.

For more information relating to the Choquet and pan-integrals, see [9,11,14,17,18].

3. (M)-property and weak (M)-property of monotone measures

We recall the concepts of (M)-property and weak (M)-property of monotone measures.

Definition 3.1. (Mesiar et al. [11]) Let $\mu \in \mathcal{M}$. μ is said to have *(M)-property*, if for any $U, V \in \mathcal{A}, U \subset V$, there is $T \in \mathcal{A}, T \subset U$ such that

$$\mu(T) = \mu(U) \quad \text{and} \quad \mu(V) = \mu(T) + \mu(V \setminus T).$$

Definition 3.2. (Li et al. [9]) Let $\mu \in \mathcal{M}$. μ is said to have *weak (M)-property*, if for any $U, V \in \mathcal{A}, U \subset V, \mu(V) < \infty$ and any $\epsilon > 0$, there is $T_\epsilon \in \mathcal{A}, T_\epsilon \subset U$ such that

$$\mu(T_\epsilon) > \mu(U) - \epsilon$$

and

$$\mu(T_\epsilon) + \mu(V \setminus T_\epsilon) \le \mu(V) < \mu(T_\epsilon) + \mu(V \setminus T_\epsilon) + \epsilon.$$

In [9], by using the weak (M)-property, the coincidence of the Choquet integral and the pan-integral was further studied, and the following result was shown:

Proposition 3.3. *Let $\mu \in \mathcal{M}$. For each $h \in \mathcal{F}_+$, it holds*

$$\int^{Ch} h d\mu = \int^{pan} h d\mu$$

if and only if μ has weak (M)-property.

From Definitions 3.1 and 3.2, obviously, if μ has (M)-property, then it has weak (M)-property. Furthermore, if X is a finite set and μ is finite, then the weak (M)-property of μ implies the (M)-property, and therefore, the weak (M)-property and the (M)-property are equivalent (see [9]).

If μ is infinite, or X is infinite, then the weak (M)-property does not necessarily imply (M)-property.

The following example is taken from [9], which indicates that for some infinite monotone measures, (M)-property is really stronger than weak (M)-property.

Example 3.4. Let X be a finite set and $\mathcal{A} = 2^X$. Define $\mu : 2^X \to [0, \infty]$ by

$$\mu(A) = \begin{cases} +\infty & \text{if } A = X, \\ |A| & \text{if } A \ne X. \end{cases}$$

Then $\mu \in \mathcal{M}$ and it has weak (M)-property. However, μ has no (M)-property. Let $A \subset X$, $A \ne \emptyset$ and $A \ne X$, then for any $T \subset A$, if $\mu(T) = \mu(A)$, then $T = A$. Thus $\mu(X) = \infty > |X| = \mu(T) + \mu(X \setminus T)$.

In [9], an open problem has been raised: for a general space (not necessarily finite set) and a finite monotone measure, does the weak (M)-property imply the (M)-property?

Now we present a new example to show that this question has a negative answer, i.e., there is some finite monotone measure μ defined on infinite space with the weak (M)-property, but μ has not the (M)-property.

Example 3.5. Let $X = [0, 1]$, $\mathcal{B}([0, 1])$ be the σ-algebra of all Borel subsets of $[0, 1]$. Define $\mu : \mathcal{B}([0, 1]) \to [0, 1]$ by

$$\mu(A) = \begin{cases} 0 & \text{if } m(A \cap [0, \frac{1}{2}]) = 0 \\ m(A) & \text{otherwise,} \end{cases}$$

where m is the Borel measure.

We claim that μ has weak (M)-property. In fact, let $U \subset V$ and $\epsilon > 0$ be arbitrarily given. If $\mu(U) = 0$ we take $T = \emptyset$ then

$$\mu(U) = \mu(T) \ \text{ and } \ \mu(V) = \mu(T) + \mu(V \setminus T).$$

If $\mu(U) > 0$, then it holds necessarily that $m(U \cap [0, \frac{1}{2}]) > 0$. Take $T_\epsilon \subset U$ satisfying

(i) $m(T_\epsilon \cap [0, \frac{1}{2}]) = \max\left\{m(U \cap [0, \frac{1}{2}]) - \frac{\epsilon}{2}, \frac{1}{2} m(U \cap [0, \frac{1}{2}])\right\}$;

(ii) $T_\epsilon \cap [\frac{1}{2}, 1] = U \cap [\frac{1}{2}, 1]$.

Then

$$\mu(T_\epsilon) = m(T_\epsilon) \ge m(U) - \frac{\epsilon}{2} > \mu(U) - \epsilon$$

and

$$m((V \setminus T_\epsilon) \cap [0, \frac{1}{2}]) = \min\{\frac{\epsilon}{2}, \frac{1}{2} m(U \cap [0, \frac{1}{2}]\} > 0.$$

Thus we have $\mu(V \setminus T_\epsilon) = m(V \setminus T_\epsilon)$ which implies that

$$\mu(V) = m(V) = m(T_\epsilon) + m(V \setminus T_\epsilon) = \mu(T_\epsilon) + \mu(V \setminus T_\epsilon),$$

i.e., μ has weak (M)-property.

To see that μ has no (M)-property, it is enough to take $U = [0, \frac{1}{2}]$ and $V = [0, 1]$. For any $T \subset U$ such that $\mu(T) = \mu(U) = \frac{1}{2}$, we have $m((V \setminus T) \cap [0, \frac{1}{2}]) = 0$ and which implies that $\mu(V \setminus T) = 0$. Thus $\mu(V) = 1 > \frac{1}{2} = \mu(T) + \mu(V \setminus T)$.

In the following, we present some characteristics of (M)-property and weak (M)-property.

The monotone measure ν on $(X, \mathcal{A})$ is called *null-additive* [18], if for any $A, N \in \mathcal{A}$, $\nu(N) = 0$ implies $\nu(A \cup N) = \nu(A)$; *autocontinuous from above* [18], if for any $A \in \mathcal{A}, (N_k)_{k=1}^{\infty} \subset \mathcal{A}$, $\lim_{k\to+\infty} \nu(N_k) = 0$ implies

$$\lim_{k\to+\infty} \nu(A \cup N_k) = \nu(A). \tag{3.1}$$

Proposition 3.6. *Let $\nu \in \mathcal{M}$ be finite and autocontinuous from above. If ν has weak (M)-property then it has (M)-property, thus the (M)-property and the weak (M)-property are equivalent.*

Proof. Let $U \subset V$ be given. Since ν has weak (M)-property, for any n there is $T_n \subset U$ such that $\nu(T_n) > \nu(U) - \frac{1}{n}$ and

$$\nu(T_n) + \nu(V \setminus T_n) \le \nu(V) < \nu(T_n) + \nu(V \setminus T_n) + \frac{1}{n}.$$

Denote $T = \bigcup_{n=1}^{\infty} T_n$. Then we have that $T \subset U$, $\nu(T) = \nu(U)$ and

$$\nu(V) < \nu(T_n) + \nu(V \setminus T_n) + \frac{1}{n} \le \nu(T) + \nu((V \setminus T) \cup (T \setminus T_n)) + \frac{1}{n}. \tag{3.2}$$

On the other hand, since μ has weak (M)-property, it is superadditive. Thus we have

$$\nu(T \setminus T_n) \le \nu(U \setminus T_n) \le \nu(U) - \nu(T_n) < \frac{1}{n}.$$

Let $n \to \infty$ in Eq. (3.2), by the autocontinuity of ν we have $\nu(V) \le \nu(T) + \nu(V \setminus T)$. Since ν is superadditive, it then follows that $\nu(V) = \nu(T) + \nu(V \setminus T)$, i.e., ν has (M)-property. □

Note that the autocontinuity implies the null-additivity ([18]), and the null-additivity together with (M)-property imply the additivity, see [16]. Thus we have the following result.

Corollary 3.7. *Let $\nu \in \mathcal{M}$ be finite and autocontinuous from above. If ν has weak (M)-property then it is additive.*

When X is a countable set and $\nu \in \mathcal{M}$ is finite and continuous, then the null-additivity is equivalent to the autocontinuity from above, see [18]. Thus we obtain the following corollary:

Corollary 3.8. *Let X be a countable set and $\nu \in \mathcal{M}$ be finite and continuous. If ν is null-additive, then the (M)-property and the weak (M)-property of ν are equivalent.*

It should be pointed out that a monotone measure satisfying weak (M)-property and null-additivity may not be additive. It is enough to see Example 3.5, where μ is null-additive and has weak (M)-property, but it is not additive.

4. A new sufficient and necessary condition for coincidence of the Choquet and pan-integrals

In this section, we present a new sufficient and necessary condition that the Choquet integral coincides with the pan-integral. Let us begin with the following concept.

Definition 4.1. Let $\nu \in \mathcal{M}$ be fixed. If for every $\epsilon > 0$ and every $(C_i)_{i=1}^{r} \in \mathcal{H}_{Ch}$ with $\nu(C_i) < \infty$ and $c_i \ge 0, i = 1, 2, \ldots, r$, there is a $(P_j)_{j=1}^{s} \in \mathcal{H}_{pan}$ with $p_j \ge 0, j = 1, 2, \ldots, s$, such that

$$\sum_{i=1}^{r} c_i \chi_{C_i} \ \ge \ \sum_{j=1}^{s} p_j \chi_{P_j} \tag{4.1}$$

and

$$\sum_{i=1}^{r} c_i \nu(C_i) - \epsilon \ < \ \sum_{j=1}^{s} p_j \nu(P_j), \tag{4.2}$$

then we say that ν has (C-P)-property.

Note 4.2. (i) In the above definition, if $\nu(C_i) = \infty$ and $c_i > 0$ for some i, then there is a $(P_j)_{j=1}^{s} \in \mathcal{H}_{pan}$ with $p_j \ge 0, j = 1, 2, \ldots, s$, such that Eq. (4.1) holds and $\sum_{i=1}^{r} c_i \nu(C_i) = \sum_{j=1}^{s} p_j \nu(P_j) = \infty$. In fact, without loss of generality, we can suppose that $\nu(C_1) = \infty$ and $c_1 > 0$. In this case we need only take $P_1 = C_1, P_2 = X \setminus C_1$ and $p_1 = c_1, p_2 = 0$.

(ii) It is not difficult to see that ν has (C-P)-property if it is subadditive.

Proposition 4.3. *Let $\nu \in \mathcal{M}$ be fixed. For all $h \in \mathcal{F}_+$, it holds*

$$\int^{Ch} h d\nu \leq \int^{pan} h d\nu \tag{4.3}$$

if and only if ν has (C-P)-property.

Proof. Suppose that ν has (C-P)-property. We prove that the Eq. (4.3) holds.

Given any $h_0 \in \mathcal{F}_+$. We consider the Choquet integral of h_0 w.r.t. ν, and divide the proof into two situations:

Case I: $\int^{Ch} h_0 d\nu = \infty$. From the definition of the Choquet integral, for any $K > 0$, there is $(C_i)_{i=1}^r \in \mathcal{H}_{Ch}$ with $c_i \geq 0, i = 1, 2, \ldots, r$, such that $\sum_{i=1}^r c_i \chi_{C_i} \leq h_0$ and $\sum_{i=1}^r c_i \nu(C_i) > 2K$. Now there are two subcases:

(a) If $\sum_{i=1}^r c_i \nu(C_i) < \infty$, from that ν has (C-P)-property, there exist $(P_j)_{j=1}^s \in \mathcal{H}_{pan}$ with $p_j \geq 0, j = 1, 2, \ldots, s$, such that $\sum_{j=1}^s p_j \chi_{P_j} \leq \sum_{i=1}^r c_i \chi_{C_i} \leq h_0$

$$\sum_{i=1}^r c_i \nu(C_i) - K \ < \ \sum_{j=1}^s p_j \nu(P_j), \tag{4.4}$$

and hence $\sum_{j=1}^s p_j \nu(P_j) > K$. This implies $\int^{Ch} h_0 d\nu = \infty = \int^{pan} h_0 d\nu$.

(b) If $\nu(C_i) = \infty$, it follows from Note 4.2(i) that there exist $(P_j)_{j=1}^s \in \mathcal{H}_{pan}$ with $p_j \geq 0, j = 1, 2, \ldots, s$, such that $\sum_{j=1}^s p_j \chi_{P_j} \leq \sum_{i=1}^r c_i \chi_{C_i} \leq h_0$ and

$$\sum_{i=1}^r c_i \nu(C_i) \ = \ \sum_{j=1}^s p_j \nu(P_j) = \infty.$$

Therefore, $\int^{Ch} h_0 d\nu = \infty = \int^{pan} h_0 d\nu$.

Case II: $\int^{Ch} h_0 d\nu < \infty$. Then for any $\epsilon > 0$, there is $(C_i)_{i=1}^r \in \mathcal{H}_{Ch}$ with $c_i \geq 0, i = 1, 2, \ldots, r$, such that $\sum_{i=1}^r c_i \chi_{C_i} \leq h_0$ and

$$\int^{Ch} h_0 d\nu < \sum_{i=1}^r c_i \nu(C_i) + \epsilon. \tag{4.5}$$

By the (C-P)-property of ν, there exist $(P_j)_{j=1}^s \in \mathcal{H}_{pan}$ with $p_j \geq 0, j = 1, 2, \ldots, s$, such that $\sum_{j=1}^s p_j \chi_{P_j} \leq \sum_{i=1}^r c_i \chi_{C_i} \leq h_0$

$$\sum_{i=1}^r c_i \nu(C_i) - \epsilon \ < \ \sum_{j=1}^s p_j \nu(P_j). \tag{4.6}$$

Therefore, from Eqs. (4.5) and (4.6) we know

$$\int^{Ch} h_0 d\nu < \sum_{i=1}^r c_i \nu(C_i) - \epsilon + 2\epsilon < \sum_{j=1}^s p_j \nu(P_j) + 2\epsilon < \int^{pan} h_0 d\nu + 2\epsilon. \tag{4.7}$$

Letting $\epsilon \to 0$ we get $\int^{Ch} h_0 d\nu \leq \int^{pan} h_0 d\nu$ as desired.

Conversely, assume that Eq. (4.3) holds for all $h \in \mathcal{F}_+$. Given $\epsilon > 0$, $(C_i)_{i=1}^r \in \mathcal{H}_{Ch}$ with $\nu(C_i) < \infty$ and $c_i \geq 0, i = 1, 2, \ldots, r$.

Denote $\tilde{h} = \sum_{i=1}^r c_i \chi_{C_i}$. Then, for the pan-integral of $\tilde{h}$, there are two cases:

(a) If $\int^{pan} \tilde{h} d\nu < \infty$, then there exist $(P_j)_{j=1}^s \in \mathcal{H}_{pan}$ with $p_j \geq 0, j = 1, 2, \ldots, s$, such that

$$\sum_{i=1}^r c_i \chi_{C_i} = \tilde{h} \ \geq \ \sum_{j=1}^s p_j \chi_{P_j} \tag{4.8}$$

and

$$\int^{pan} \tilde{h} d\mu < \sum_{j=1}^s p_j \mu(P_j) + \epsilon.$$

Therefore,

$$\sum_{i=1}^r c_i \mu(C_i) - \epsilon \leq \int^{Ch} \tilde{h} d\mu - \epsilon \leq \int^{pan} \tilde{h} d\mu - \epsilon < \sum_{j=1}^s p_j \mu(P_j). \tag{4.9}$$

(b) If $\int^{pan} \tilde{h} d\nu = \infty$, we take K such that $K > \max\{\epsilon, \sum_{i=1}^r c_i \nu(C_i)\}$, then there exist $(P_j)_{j=1}^s \in \mathcal{H}_{pan}$ with $p_j \geq 0, j = 1, 2, \ldots, s$, such that

$$\sum_{i=1}^{r} c_i \chi_{C_i} = \tilde{h} \geq \sum_{j=1}^{s} p_j \chi_{P_j} \tag{4.10}$$

and

$$\sum_{j=1}^{s} p_j \mu(P_j) > K - \varepsilon.$$

Therefore,

$$\sum_{i=1}^{r} c_i \mu(C_i) - \epsilon \leq K - \epsilon < \sum_{j=1}^{s} p_j \mu(P_j). \tag{4.11}$$

The proof is complete. □

Combining Proposition 2.1 and Proposition 4.3, we obtain a new sufficient and necessary condition that the Choquet integral coincides with the pan-integral.

Proposition 4.4. *Let $\mu \in \mathcal{M}$ be fixed. Then the following are equivalent:*
(i) μ is superadditive and has (C-P)-property;
(ii) for all $h \in \mathcal{F}_+$, it holds

$$\int^{Ch} h d\mu = \int^{pan} h d\mu.$$

Proposition 3.3 and Proposition 4.4 imply the following result.

Proposition 4.5. *Let $\mu \in \mathcal{M}$ be fixed. Then the following are equivalent:*
(i) μ has weak (M)-property;
(ii) μ is superadditive and has (C-P)-property.

From Propositions 3.3, 4.4 and 4.5, we see that each of the subadditivity, (M)-property and weak (M)-property of a monotone measure implies the (C-P)-property. However, the following example shows that the converse is not true.

Example 4.6. Let $X = \{a, b, c, d\}$ and $\mathcal{A} = 2^X$. Define the monotone measure μ as $\mu(\emptyset) = 0, \mu(X) = 3, \mu(A) = 1$ for $A \neq \emptyset, X$. Then μ is neither subadditive nor superadditive, thus μ has no (M)-property and weak (M)-property. Note that μ is subadditive w.r.t. singletons, *i.e.*, $\mu(A) \leq \sum_{i \in A} \mu(\{i\})$, thus μ has (C-P)-property.

5. Coincidence of the Choquet and pan-integrals related to the ordered pair of monotone measures

In this section, we consider the relations between the Choquet integral and the pan-integral in the setting of the ordered pairs (λ, ν) of monotone measures λ and ν.

We recall some properties of the pan-integrals (see [7,10]).

Given $\nu \in \mathcal{M}$, corresponding to the pan-integral w.r.t. ν, the monotone measure ν_{pan} is defined as

$$\nu_{pan}(T) = \int^{pan} \chi_T d\nu, \quad T \in \mathcal{A}. \tag{5.1}$$

Obviously, $\nu_{pan} \geq \nu$. We have the following result: $\nu_{pan} = \nu$, i.e.,

$$\nu(T) = \int^{pan} \chi_T d\nu, \quad T \in \mathcal{A}, \tag{5.2}$$

if and only if ν is superadditive (see Proposition 2 in [7]).

Proposition 5.1. *(i) For all $h \in \mathcal{F}^+$, it holds*

$$\int^{pan} h d\nu = \int^{pan} h d\nu_{pan}, \tag{5.3}$$

in particular, for any $T \in \mathcal{A}$, $\nu_{pan}(T) = \int^{pan} \chi_T d\nu_{pan}$ holds.

(ii) For all $h \in \mathcal{F}^+$,

$$\int^{Ch} h dv = \int^{Ch} h dv_{pan} \tag{5.4}$$

if and only if v is superadditive.

Proof. For the proof of (i), we refer to [10].

(ii) The sufficiency follows from the fact that $v_{pan} = v$ if v is a superadditive measure. Now, suppose that the Eq. (5.4) holds for all $h \in \mathcal{F}^+$. If v is not superadditive, then there exist two subsets S, T of X such that $S \cap T = \emptyset$ and $v(S \cup T) < v(S) + v(T)$.

Define

$$h(x) = \begin{cases} 1 & \text{if } x \in S \\ 2 & \text{if } x \in T, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \int^{Ch} f dv_{pan} &= v_{pan}(S \cup T) + v_{pan}(T) \\ &\geq v(S) + 2v(T) \\ &> v(S \cup T) + v(T) \\ &= \int^{Ch} f dv. \end{aligned}$$

This is a contradiction. Therefore, v is superadditive. □

The following result is a generalization of Proposition 2.1 (when $m = v$, it goes back to Proposition 2.1).

Proposition 5.2. *Given the ordered pair $(m, v) \in \mathcal{M} \times \mathcal{M}$. Then, for any $h \in \mathcal{F}^+$,*

$$\int^{Ch} h dm \geq \int^{pan} h dv \tag{5.5}$$

if and only if the following condition holds: for any $S_j \in \mathcal{A}, j = 1, 2, \dots, n,$ $S_k \cap S_l = \emptyset$ $(1 \leq k, l \leq n, k \neq l), n \in \mathbb{N}$,

$$m\Big(\bigcup_{j=1}^{n} S_j\Big) \geq \sum_{j=1}^{n} v(S_j). \tag{5.6}$$

For generalized coincidence of the Choquet integrals and the pan-integrals related to the ordered pair of monotone measures, we have the following results, which covers Proposition 4.4 when $m = v$.

Proposition 5.3. *Let $(m, v) \in \mathcal{M} \times \mathcal{M}$. Then the following are equivalent:*

(i) for all $h \in \mathcal{F}^+$,

$$\int^{Ch} h dm = \int^{pan} h dv; \tag{5.7}$$

(ii) $m = v_{pan}$ and m has (C-P)-property.

Proof. $(i) \Rightarrow (ii)$. For any $A \in \mathcal{A}$, we have

$$m(A) = \int^{Ch} \chi_A dm = \int^{pan} \chi_A dv = v_{pan}(A),$$

i.e., $m = v_{pan}$. Thus, from Proposition 5.1(i), then

$$\int^{Ch} h dm = \int^{pan} h dv = \int^{pan} h dv_{pan} = \int^{pan} h dm \tag{5.8}$$

holds for all $h \in \mathcal{F}^+$. From Proposition 4.4, it implies m has (C-P)-property.

$(ii) \Rightarrow (i)$. Note that v_{pan} is superadditive, in fact, from Eq. (5.1) and Proposition 5.1(i), we have

$$v_{pan}(A) = \int^{pan} \chi_A dv = \int^{pan} \chi_A dv_{pan} = (v_{pan})_{pan}(A), \tag{5.9}$$

i.e., $v_{pan} = (v_{pan})_{pan}$. This implies that v_{pan} is superadditive, and hence m is superadditive. Therefore, from that m has (C-P)-property, the Eq. (5.7) holds for all $h \in \mathcal{F}^+$. □

Proposition 5.3 implies the following result.

Proposition 5.4. *Let $(m, v) \in \mathcal{M} \times \mathcal{M}$. If for all $h \in \mathcal{F}^+$*

$$\int^{Ch} h dm = \int^{pan} h dv, \tag{5.10}$$

then

$$\int^{Ch} h dm = \int^{pan} h dm \tag{5.11}$$

holds for all $h \in \mathcal{F}^+$, and hence m has weak (M)-property. Moreover, we have

$$\int^{Ch} h dv \leq \int^{pan} h dv \tag{5.12}$$

holds for all $h \in \mathcal{F}^+$, and hence v has (C-P)-property.

Proof. Eq. (5.11) is a direct consequence of Proposition 5.3 and (i) of Proposition 5.1. Then the weak (M)-property of m follows from Proposition 3.3.

Also, by Proposition 5.3 we have

$$\int^{Ch} h dv \leq \int^{Ch} h dv_{pan} = \int^{Ch} h dm = \int^{pan} h dv,$$

i.e., (5.12) holds. The (C-P) property of v follows from Proposition 4.3, 5.1 and 4.3. □

From Proposition 3.3, we can deduce the following result:

Corollary 5.5. *Let $v \in \mathcal{M}$. Then*

(i) If v has weak (M)-property, then v_{pan} has weak (M)-property.

(ii) If v_{pan} has weak (M)-property then v has (C-P)-property.

Similar to the above discussions, for the case of the Choquet integral and the concave integral (see [6,5]), we can obtain the following result:

Proposition 5.6. *Let $m \in \mathcal{M}$ be given. If there exists some $v \in \mathcal{M}$ such that for all $h \in \mathcal{F}^+$*

$$\int^{Ch} h dm = \int^{cav} h dv, \tag{5.13}$$

then $m = v_{cav}$, and

$$\int^{Ch} h dm = \int^{cav} h dm \tag{5.14}$$

holds for all $h \in \mathcal{F}^+$, and hence m is supermodular (convex).

Note: The *concave integral* of h on X w.r.t. v, is defined by

$$\int^{cav} h dv = \sup \left\{ \sum_{i=1}^{n} d_i v(D_i) : \sum_{i=1}^{n} d_i \chi_{D_i} \leq h, (D_i)_{i=1}^{n} \in \mathcal{H}_{cav}, d_i \geq 0 \right\},$$

where $\mathcal{H}_{cav}$ is the set of all finite families of sets in $\mathcal{A} \setminus \{\emptyset\}$, see [6,5].

$$\nu_{cav}(T) \triangleq \int^{cav} \chi_T d\nu,\ T \in \mathcal{A}.$$

6. Concluding remarks

We have answered an open problem in [9] concerning the (M)-property of monotone measures, and shown a new equivalence condition that the Choquet integral and the pan-integral coincide. The generalized coincidence versions of the Choquet and pan-integrals involving the ordered pair of monotone measures have been also presented.

Note that the discussions in Sections 4 and 5 concern the decomposition systems $\mathcal{H}_{Ch}, \mathcal{H}_{pan}$ and $\mathcal{H}_{cav}$ on $(X, \mathcal{A})$. In further researches, we will generalize these results to the situation of decomposition integrals introduced by Even and Lehrer [2] (see also [3,12]). As a special case, we look for the necessary and sufficient conditions that the concave integral and the pan-integral coincide on general spaces (the research on this topic has been partially discussed in [3,4,8,13,15]). We also study the coincidences of other kinds of decomposition integrals, for example, for a fixed monotone measure ν and different decomposition systems $\mathcal{H}_1$ and $\mathcal{H}_2$, we investigate the equivalence $\int_{\mathcal{H}_1} h d\nu \vee \int_{\mathcal{H}_2} h d\nu = \int_{\mathcal{H}_1 \cup \mathcal{H}_2} h d\nu$ for all $h \in \mathcal{F}^+$.

CRediT authorship contribution statement

Tong Kang: Writing – review & editing, Writing – original draft, Methodology, Investigation, Conceptualization. **Radko Mesiar:** Writing – review & editing, Methodology, Investigation, Funding acquisition, Conceptualization. **Yao Ouyang:** Writing – review & editing, Writing – original draft, Methodology, Investigation, Conceptualization. **Jun Li:** Writing – review & editing, Writing – original draft, Methodology, Investigation, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgement

This work was supported by the Fundamental Research Funds for the Central Universities, and R. Mesiar was supported by the grant VEGA 1/0036/23.

Data availability

No data was used for the research described in the article.

References

[1] G. Choquet, Theory of capacities, Ann. Inst. Fourier 5 (1954) 131–295.
[2] Y. Even, E. Lehrer, Decomposition integral: unifying Choquet and the concave integrals, Econ. Theory 56 (2014) 33–58.
[3] T. Kang, J. Li, On equivalence of decomposition integrals based on different monotone measures, Fuzzy Sets Syst. 457 (2023) 142–155.
[4] T. Kang, L. Yan, J. Li, Coincidences of nonlinear integrals related to ordered pair of non-additive measures, Int. J. Approx. Reason. 152 (2023) 124–135.
[5] E. Lehrer, A new integral for capacities, Econ. Theory 39 (2009) 157–176.
[6] E. Lehrer, R. Teper, The concave integral over large spaces, Fuzzy Sets Syst. 159 (2008) 2130–2144.
[7] J. Li, R. Mesiar, P. Struk, Pseudo-optimal measures, Inf. Sci. 180 (2010) 4015–4021.
[8] J. Li, R. Mesiar, Y. Ouyang, On the coincidence of measure-based decomposition and superdecomposition integrals, Fuzzy Sets Syst. 457 (2023) 125–141.
[9] J. Li, R. Mesiar, Y. Ouyang, L. Wu, On the coincidence of the pan-integral and the Choquet integral, Fuzzy Sets Syst. 467 (2023) 108577.
[10] J. Li, Y. Ouyang, M. Yu, Pan-integrals based on optimal measures, in: V. Torra, et al. (Eds.), LNAI, vol. 10571, 2017, pp. 40–50.
[11] R. Mesiar, J. Li, Y. Ouyang, On the equality of integrals, Inf. Sci. 393 (2017) 82–90.
[12] R. Mesiar, A. Stupňanová, Decomposition integrals, Int. J. Approx. Reason. 54 (2013) 1252–1259.
[13] Y. Ouyang, J. Li, R. Mesiar, Relationship between the concave integrals and the pan-integrals on finite spaces, J. Math. Anal. Appl. 424 (2015) 975–987.
[14] Y. Ouyang, J. Li, R. Mesiar, On the equivalence of the Choquet, pan- and concave integrals on finite spaces, J. Math. Anal. Appl. 456 (2017) 151–162.
[15] Y. Ouyang, J. Li, R. Mesiar, Coincidences of the concave integral and the pan-integral, Symmetry 9 (6) (2017) 90.
[16] Y. Ouyang, J. Li, R. Mesiar, A sufficient condition of equivalence of the Choquet and the pan-integral, Fuzzy Sets Syst. 355 (2020) 100–105.
[17] E. Pap, Null-Additive Set Functions, Kluwer, Dordrecht, 1995.
[18] Z. Wang, G.J. Klir, Generalized Measure Theory, Springer, New York, 2009.