

Bipartite secret sharing and staircases*

Laszlo Csirmaz[†]

František Matúš[‡]

Carles Padró[§]

Abstract

Bipartite secret sharing schemes have a bipartite access structure in which the set of participants is divided into two parts and all participants in the same part play an equivalent role. Such a bipartite scheme can be described by a *staircase*: the collection of its minimal points. The complexity of a scheme is the maximal share size relative to the secret size; and the κ -complexity of an access structure is the best lower bound provided by the entropy method. An access structure is κ -ideal if it has κ -complexity 1. Motivated by the abundance of open problems in this area, the main results can be summarized as follows. First, a new characterization of κ -ideal multipartite access structures is given which offers a straightforward and simple approach to describe ideal bipartite and tripartite access structures. Second, the κ -complexity is determined for a range of bipartite access structures, including those determined by two points, staircases with equal widths and heights, and staircases with all heights 1. Third, matching linear schemes are presented for some non-ideal cases, including staircases where all heights are 1 and all widths are equal. Finally, finding the Shannon complexity of a bipartite access structure can be considered as a discrete submodular optimization problem. An interesting and intriguing continuous version is defined which might give further insight to the large-scale behavior of these optimization problems.

Keywords: cryptography; multipartite secret sharing, entropy method, linear secret sharing, submodular optimization.

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1 Introduction

Secret sharing schemes serve as a natural cryptographic primitives used in group signatures, secure file storage and secure multiparty computation just to mention a few applications. The initial idea goes back to Blakley [3] and Shamir [25]. For an introduction and early bibliography see [26], for a more recent review see [1] or [4].

A *secret sharing scheme*, abbreviated as sss, involves a *secret* 0 , the set $[n] = \{1, 2, \dots, n\}$ of n *participants* and an *access structure* Γ , which is a nonempty family of subsets of $[n]$ that is closed to supersets and does not contain the empty set. A participant $i \in [n]$ is *essential* if some set $I \in \Gamma$ contains i while $I \setminus i \notin \Gamma$.

In the traditional *probabilistic framework* a secret sharing scheme consists of jointly distributed random variables $\xi_0, \xi_1, \dots, \xi_n$ taking finitely many values such that ξ_0 – the secret – is a function of the vector $\xi_I = \langle \xi_i, i \in I \rangle$ almost surely if and only if $I \in \Gamma$. In an application the dealer samples the distribution, sets the secret value to be that of ξ_0 , and communicates the share ξ_i to participant i privately. The condition ensures that based on their shares only, participants in I can recover the secret almost surely if and only if $I \in \Gamma$.

In a *linear framework* the secret sharing scheme consists of subspaces E_0, E_1, \dots, E_n of some finite linear space V such that E_0 is contained in E_I if and only if $I \in \Gamma$, where E_I is the linear

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[†]Rényi Institute, Budapest and UTIA, Prague

[‡]UTIA, Prague

[§]Polytechnic University of Catalonia, Barcelona

span of $\bigcup\{E_i : i \in I\}$. Any linear sss can be turned into the probabilistic one by setting ξ_i to be the orthogonal projection of a randomly chosen vector from V to E_i .

Widening the usual definition we consider also a *polymatroidal framework* in which case the scheme consists of a polymatroid $(0 \cup [n], h)$ such that $h(I) = h(0 \cup I)$ for each $I \in \Gamma$, see, for example, [12]. The scheme $(0 \cup [n], h)$ *realizes* Γ if $h(0 \cup I) = h(I)$ holds if and only if $I \in \Gamma$.

Polymatroidal schemes cover both probabilistic and linear ones. In the first case the rank $h(I)$ equals the Shannon entropy $\mathbf{H}(\xi_I)$, and in the second case $h(I) = \dim(E_I)$. Polymatroids obtained this way are called entropic and linear, respectively.

General polymatroidal schemes are not associated with any sort of practical realization as opposed to probabilistic and linear schemes. Their principal role is to provide a common platform to investigate the possibilities and limits of the entropy method using only Shannon information inequalities [19]. As polymatroidal schemes cover both probabilistic and linear ones, results on polymatroidal schemes (such as lower bounds) automatically carry over to the realizable cases. From now on, if not mentioned otherwise, all schemes are polymatroidal ones.

The scheme $(0 \cup [n], h)$ is *perfect* if $h(0)$ is positive and for every $I \subseteq [n]$ either $h(0 \cup I) = h(0) + h(I)$ or $h(0 \cup I) = h(I)$. Intuitively this means that any collection I of participants either determines the secret, or has no information on the secret. All schemes in this paper are assumed to be perfect. Every access structure can be realized by some perfect linear scheme, see [17].

Referring to the probabilistic framework, $h(0)$ is the *secret size* while $h(i)$ is the *share size* of the participant $i \in [n]$. In a perfect scheme $h(i) \geq h(0)$ for essential participants $i \in [n]$.

The *information ratio*, or *complexity*, of a scheme $(0 \cup [n], h)$ is the largest share size relative to the secret size. In other words, it is the maximal value of $h(i)/h(0)$ as i runs over the elements of $[n]$. For an access structure Γ its *Shannon complexity*, denoted by $\kappa(\Gamma)$, is the infimum of the information ratios of all of its perfect polymatroidal realizations [19]. When the minimization is restricted to entropic, or linear polymatroids, the corresponding infimum is denoted by $\sigma(\Gamma)$, or $\lambda(\Gamma)$, respectively. It follows that $\lambda(\Gamma) \geq \sigma(\Gamma) \geq \kappa(\Gamma) \geq 1$. By [8] we have $\kappa(\Gamma) \leq n$ for all Γ and it can happen that $\kappa(\Gamma) \geq O(n/\log n)$. The infimum is always achieved for $\kappa(\Gamma)$ (thus it is actually a minimum); while there is an access structure Γ for which $\lambda(\Gamma) = \sigma(\Gamma) = 1$ and neither of these values are taken, see [20]. A sss is *κ -ideal* if its Shannon complexity is the smallest possible, namely 1. The access structure Γ is *ideal* if it is realized by some ideal entropic polymatroid, that is, $\sigma(\Gamma) = 1$. A fundamental result by Brickell and Davenport [6] essentially states that a κ -ideal sss is actually a matroid (up to a scaling factor), which is determined uniquely by the access structure when all participants are essential. Therefore, a κ -ideal access structure is ideal if and only if that matroid has an entropic multiple. Matroid ports, a combinatorial object introduced by Lehman [18], are basically the same as κ -ideal access structures. The characterization of matroid ports by Seymour [24] implies that $\kappa(\Gamma) \geq 3/2$ if Γ is not κ -ideal [19].

An access structure Γ is *threshold* if $I \in \Gamma$ depends only on the cardinality of I . Even though only threshold secret sharing schemes are needed in most applications, some situations, as for example hierarchical organizations, require access structures in which participants are divided into several groups according to their different roles. Specifically, Γ is *multipartite*, if $[n]$ can be expressed as a disjoint union of $m \geq 1$ sets N_1, \dots, N_m such that $I \in \Gamma$ is determined solely through the cardinalities of $I \cap N_1$ through $I \cap N_m$. For $m = 2$ such an access structure is called *bipartite*. If n_1, n_2 denotes the number of elements in N_1 and N_2 , respectively, a bipartite Γ gives rise to an integer $\ell = \ell_\Gamma \geq 1$ and two sequences of integers

$$0 \leq i_1 < \dots < i_\ell \leq n_1 \quad \text{and} \quad n_2 \geq j_1 > \dots > j_\ell \geq 0, \quad (1)$$

such that $I \in \Gamma$ is equivalent to $|I \cap N_1| \geq i_k$ and $|I \cap N_2| \geq j_k$ for some $1 \leq k \leq \ell$. The sequence $(i_1, j_1), \dots, (i_\ell, j_\ell)$ is a *staircase* determining Γ , having steps of width $w_k = i_{k+1} - i_k$, and heights $h_k = j_k - j_{k+1}$. The staircase is *regular* if all widths are the same and all heights are the same. Figure 1 illustrates a staircase of length $\ell_\Gamma = 4$ for Γ bipartite with $n_1 = 6$ and $n_2 = 4$. The widths of the steps are 2, 1, 2, and the heights are 1, 1, 1. We refer the reader to the works [11, 12, 23] for further motivation, basic definitions, and a more gentle introduction to this topic.

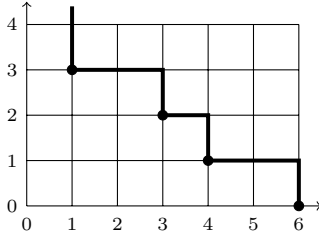


Figure 1: An example staircase

Main results

Motivated by the open problems raised in [12], the main results of this paper can be summarized as follows.

a) In Section 2 a complete characterization of κ -ideal multipartite access structures is presented. It simplifies both the description and the proof of the characterization in [11, Theorem 5.3]. Using the fact that in the bipartite and tripartite cases ideal and κ -ideal access structures coincide, we recover the results in [11] about ideal bipartite and tripartite access structures. While they used detailed case by case analysis, our result leads to the same collection of access structures in a simpler way, explaining the occurrence of exceptional cases. The method is illustrated for the bipartite case; the similar procedure for the tripartite case is left to the interested reader. In addition, by using Ingleton inequality, we present a much simpler proof for the fact that ideal and κ -ideal tripartite access structures coincide. We hope that the general characterization gives further insight into these interesting families.

b) In Section 4 the Shannon complexity $\kappa(\Gamma)$ is determined for a range of bipartite access structures including those determined by two points (that is, when $\ell_\Gamma = 2$), regular staircases with equal width and height, and staircases where all heights are $h_k = 1$ and the width sequence satisfies an additional technical assumption. Shannon complexity is the best lower bound on $\sigma(\Gamma)$ implied by the Shannon information inequalities. Computing $\kappa(\Gamma)$ is a linear optimization problem with linear constraints. The number of constraints, however, is exponential in the number of participants. The exact value of the Shannon complexity was known only for a few infinite families of graph-based access structures [5, 9, 10]. Exploiting the internal symmetry of bipartite access structures their Shannon complexity is expressed as the solution of another linear optimization problem where the number of constraints is at most quadratic in the number of participants. The solution of the reduced optimization problem is determined for the above bipartite access structures using the duality theorem of linear programming. While it is known that the Shannon complexity cannot exceed the number n of participants [8], it is an open problem whether there is a positive constant $c > 0$ such that $\kappa(\Gamma) > c \cdot n$ for some bipartite access structure Γ on $[n]$ for infinitely many n .

c) In a few cases linear schemes were found matching the corresponding Shannon bound, such as all regular staircases of height 1. The constructions are presented in Section 5.

d) Determining the Shannon complexity of bipartite access structures can be considered as discrete submodular optimization, see [13, 14]. In Section 6 a corresponding continuous optimization problem is defined. The intuition is scaling down the increasing optimization problems so that constraints separating qualified and unqualified subsets converge to constraints along a continuous curve. Results on the this continuous optimization problem could give a hint on the large scale behavior of bipartite access structures. For the continuous case, a local lower bound on the optimal value is proved which is tight in certain cases. This section concludes with some open problems in this framework.

2 Ideal multipartite secret sharing revisited

Ideal multipartite secret sharing schemes received a considerable attention, see [11] and the references therein. This section provides a characterization of κ -ideal multipartite access structures. It is indicated how this characterization can be used to generate all ideal bipartite and tripartite access structures. The main tool is the fundamental result by Brickell and Davenport [6], namely, a κ -ideal sss is a matroid (up to a scaling factor), and this matroid is determined uniquely by the access structure. Moreover a κ -ideal access structure is ideal if and only if this matroid has an entropic multiple.

Let (M, f) be an integer polymatroid which is linearly representable by subspaces of a finite dimensional vector space over the finite field \mathbb{F} . As the same representation works over any extension of \mathbb{F} , we may assume \mathbb{F} to be arbitrarily large. Fix $k \in M$, and let E_k be the subspace of dimension $f(k)$ corresponding to k witnessing the linear representability. Choose a “generic” vector $u \in E_k$ which is not in any proper subspace of E_k cut off by the subspaces E_J for $J \subseteq M$. Each such requirement discards at most $|\mathbb{F}|^{f(k)-1}$ elements of E_k , thus such a generic vector exists whenever \mathbb{F} is large enough. It is clear that for any $J \subseteq M$ the linear span of $u \cup E_J$ has dimension one more than the dimension of E_J except when E_k is a subspace of E_J . This motivates the following definition. Fix $k \in M$, $u \notin M$, and extend the rank function f to subsets of $u \cup M$ as follows: for every $J \subseteq M$ let

$$f(uJ) = \begin{cases} f(J) & \text{if } f(kJ) = f(J), \\ f(J) + 1 & \text{otherwise.} \end{cases}$$

The clearly integer polymatroid $(u \cup M, f)$ is called the *generic extension of f along $k \in M$* . The above discussion shows that whenever (M, f) is linearly representable then so is this generic extension.

Let $(0 \cup [n], h)$ be a κ -ideal sss realizing the multipartite access structure Γ with partition $[n] = N_1 \cup \dots \cup N_m$. By the result of Brickell and Davenport [6] h can be assumed to be a matroid which is invariant under every permutation π of $[n]$ that keeps N_k fixed for each $k \in [m]$. Let $([n], h')$ be the restriction of h discarding the secret 0. As h is a one-point extension of h' , it is determined uniquely by the *modular cut* (see, e.g., [22])

$$\mathcal{F} = \{F \subseteq [n] : F \text{ is a flat in } ([n], h') \text{ and } h(F) = h(0F)\}. \quad (2)$$

Lemma 2.1. *If F is a minimal flat in \mathcal{F} and $k \in [m]$, then either $N_k \subseteq F$ or $F \cap N_k = \emptyset$.*

Proof. Suppose that $F \cap N_k$ is neither empty nor equals N_k . Let π be the permutation of $[n]$ which swaps only two elements of N_k , one in $F \cap N_k$ and the other in $N_k \setminus F$. As the matroid h is invariant under π , both F and $\pi(F)$ are flats with the same rank and $\pi(F) \in \mathcal{F}$. Observe that F and $\pi(F)$ form a modular pair. This is so as $F \cup \pi(F)$ has one element more than F , and its rank is strictly bigger than that of F (as F is a flat), thus equals $h(F) + 1$. Similarly, $F' = F \cap \pi(F)$ has one element less than F , and its rank must be strictly smaller than the rank of F (as F' is an intersection of two different flats), thus $h(F') = h(F) - 1$. As both F and $\pi(F)$ are in \mathcal{F} , their intersection, F' is in \mathcal{F} as well. That contradicts the assumption that F is minimal in \mathcal{F} . \square

Lemma 2.2. *Let $i \in N_k$ and $i \notin J \subseteq 0 \cup [n]$. If $h(J) \neq h(J \cup N_k)$, then $h(iJ) = h(J) + 1$.*

Proof. Suppose $h(iJ) = h(J)$. By the multipartite symmetry the same equality holds for every $i \in N_k \setminus J$, and then $h(J) = h(J \cup N_k)$. \square

For $I \subseteq 0 \cup [m]$ the set $\bigcup \{N_i : i \in I\}$ is denoted by N_I where we take $N_0 = \{0\}$.

Lemma 2.3. *$J \subseteq 0 \cup [n]$ is independent in $(0 \cup [n], h)$ if and only if $|J \cap N_I| \leq h(N_I)$ for all $I \subseteq 0 \cup [m]$.*

Proof. The condition is clearly necessary. Sufficiency is immediate for $m = 1$ as in this case $(0 \cup [n], h)$ is the uniform matroid. Otherwise let $B = J \cap N_m$. If $B = \emptyset$, then use induction on the matroid restricted to $0 \cup [n] \setminus N_m$. If $B \neq \emptyset$, then $|B| \leq h(N_m)$ by assumption, thus $h(B) = |B|$ by the multipartite symmetry. From here induction on the contraction $(0 \cup [n], h) \setminus N_m$ gives the claim of the lemma. \square

Since the collection of independent sets determines the matroid [22], a consequence of this lemma is that the matroid $(0 \cup [n], h)$ is uniquely determined by the ranks $\{h(N_I) : I \subseteq 0 \cup [m]\}$.

Lemma 2.4. *For a partition $[n] = N_1 \cup \dots \cup N_m$, there is a one-to-one correspondence between the κ -ideal m -partite sss $(0 \cup [n], h)$ and the pairs $\langle ([m], f'), \mathcal{M} \rangle$, where $([m], f')$ is an integer polymatroid with $f'(k) \leq |N_k|$ for each $k \in [m]$ and \mathcal{M} is a modular cut in $([m], f')$.*

Proof. Consider the map φ from $0 \cup [n]$ to $0 \cup [m]$ defined by $\varphi(0) = 0$ and $\varphi(i) = k$ whenever $i \in N_k$. Let $(0 \cup [n], h)$ be a κ -ideal m -partite sss for the given partition. The factor of $(0 \cup [n], h)$ by φ is the (integer) polymatroid on the ground set $0 \cup [m]$ with the rank function $f(I) = h(\varphi^{-1}(I))$. In particular, $f(0) = 1$ and $f(k) = h(N_k) \leq |N_k|$ for $k \in [m]$. Let $([m], f')$ be the restriction of this polymatroid to $[m]$, and $\mathcal{M} = \varphi(\mathcal{F})$ where \mathcal{F} is the modular cut in (2). By Lemma 2.1 \mathcal{M} is a modular cut in $([m], f')$; this defines the corresponding integer polymatroid and modular cut. In the other direction, take the integer polymatroid $([m], f')$ and the modular cut \mathcal{M} . Let the corresponding one-element extension be $(0 \cup [m], f)$, namely

$$f(0J) = \begin{cases} f'(J) & \text{if } \text{cl}(J) \in \mathcal{M}, \\ f'(J) + 1 & \text{otherwise,} \end{cases} \quad (3)$$

where $\text{cl}(J)$ is the closure of J in $([m], f')$. For the chosen partition of $[n]$, take any m -partite secret sharing matroid $(0 \cup [n], h)$ such that its φ -factor is $(0 \cup [m], f)$. According to Lemmas 2.2 and 2.3 the ranks of $(0 \cup [n], h)$ are determined uniquely, thus there is at most one such matroid. To show the existence, starting from $(0 \cup [m], f)$ take $|N_i|$ generic extensions repeatedly along i for each $i \in [m]$, and then restrict the final extension to $0 \cup [n]$. It is easy to check that it has the desired properties. \square

The fact that every integer polymatroid is a factor of a matroid goes back to T. Helgason [15]. A similar construction using a completely different setting appeared in [7].

The correspondence expressed in Lemma 2.4 can be turned into a procedure which enumerates all κ -ideal access structures. The correctness of the procedure is immediate from the lemma.

Theorem 2.5. *The procedure outlined below generates all κ -ideal m -partite access structures on $[n] = N_1 \cup \dots \cup N_m$.*

1. Take any integer polymatroid $([m], f')$ with $f'(k) \leq |N_k|$, and take a modular cut \mathcal{M} in $([m], f')$.
2. Let $(0 \cup [m], f)$ be the corresponding one-point extension as defined in (3).
3. Starting from $(0 \cup [m], f)$ add N_i generic elements along i for each $i \in [m]$. Restrict the final polymatroid to $0 \cup [n]$. The result is a matroid $(0 \cup [n], h)$; it is κ -ideal, m -partite, and the corresponding access structure is $\{I \subseteq [n] : h(I) = h(0I)\}$.

Note that if the polymatroid $(0 \cup [m], f)$ is linearly representable, then the same applies to the matroid $(0 \cup [n], h)$. Consequently the corresponding access structure can be realized by an ideal linear sss.

2.1 Ideal bipartite access structures

For bipartite access structures the procedure of Theorem 2.5 can be detailed as follows. Take an integer polymatroid (M, f') on the two-element set $M = \{1, 2\}$. The polymatroid (M, f') is determined by the integer ranks $a = f'(1)$, $b = f'(2)$ and $c = f'(12)$, where $c \leq a + b$ (here 12 is the two-element set $\{1, 2\}$). Assume neither a nor b is zero and $a, b < c$ (thus 1, 2 and 12 are all flats). If $c < a + b$, then (M, f') has four non-trivial modular cuts:

$$\begin{aligned} \mathcal{M}_1 &= \{1, 12\}, & \mathcal{M}_2 &= \{2, 12\}, \\ \mathcal{M}_3 &= \{12\}, & \mathcal{M}_4 &= \{1, 2, 12\}, \end{aligned}$$

If $c = a + b$, then $\{1, 2\}$ is a modular pair, thus \mathcal{M}_4 is not a modular cut. The one-point extensions $(\{0, 1, 2\}, f)$ are integer polymatroids on three elements, consequently they are linearly representable; see [21]. The generic extensions created in step 3 are also linearly representable, thus every κ -ideal bipartite access structure admits an ideal linear sss.

Let us compute the ranks in the generic extension $(0 \cup N_1 \cup N_2, h)$. For $I_1 \subseteq N_1$ and $I_2 \subseteq N_2$ we have

$$\begin{aligned} h(I_1) &= \min\{|I_1|, a\}, \\ h(I_2) &= \min\{|I_2|, b\}, \\ h(I_1 \cup I_2) &= \min\{h(I_1) + h(I_2), c\}. \end{aligned}$$

If f was generated by \mathcal{M}_1 , then

$$\begin{aligned} h(0 \cup I_1) &= h(I_1) && \iff h(I_1) = a, \\ h(0 \cup I_2) &= h(I_2) + 1 \\ h(0 \cup I_1 \cup I_2) &= h(I_1 \cup I_2) && \iff h(I_1) = a \text{ or } h(I_1 \cup I_2) = c, \end{aligned}$$

and similarly for the other cases. In summary, the access structures corresponding to the modular cuts $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4$ are:

$$\begin{aligned} \Gamma_1 &= \{I_1 \cup I_2 : |I_1| \geq a \text{ or } (|I_1| \geq c - b \text{ and } |I_1| + |I_2| \geq c)\}, \\ \Gamma_2 &= \{I_1 \cup I_2 : |I_2| \geq b \text{ or } (|I_2| \geq c - a \text{ and } |I_1| + |I_2| \geq c)\}, \\ \Gamma_3 &= \{I_1 \cup I_2 : |I_1| \geq c - b \text{ and } |I_2| \geq c - a \text{ and } |I_1| + |I_2| \geq c\}, \\ \Gamma_4 &= \{I_1 \cup I_2 : |I_1| \geq a \text{ or } |I_2| \geq b \text{ or } |I_1| + |I_2| \geq c\}. \end{aligned}$$

If $c = a + b$ then Γ_4 is missing as it would be the same access structure which is generated by a , b , and $c - 1$. Figure 2 illustrates the four types of ideal bipartite access structures. Qualified subsets correspond to the lattice points in the shaded area.

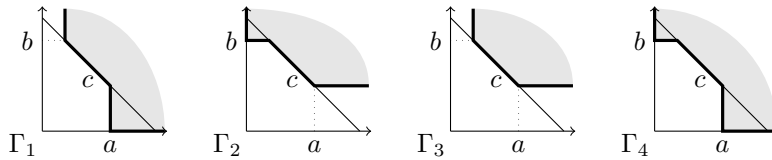


Figure 2: Ideal bipartite access structures corresponding to the modular cuts

2.2 Ideal tripartite access structures

Tripartite κ -ideal access structures can be generated similarly to the bipartite case. One starts from an integer polymatroid (M, f') on three elements, extends it to $(0 \cup M, f)$ using a modular cut, and then adds generic elements. As (M, f') is on three elements, it is linearly representable.

We claim that $(0 \cup M, f)$ is also linearly representable, thus all tripartite κ -ideal access structures are, in fact, ideal. This claim has been proved first in [11, Theorem 19].

An integer polymatroid on four elements $\{a, b, c, d\}$ has a linearly representable multiple if and only if it satisfies all instances of the Ingleton inequality $\text{lng}(a, b, c, d) \geq 0$, see [21]. The Ingleton expression is a linear combination of ten ranks as follows [16]:

$$\begin{aligned} \text{lng}(a, b, c, d) = & -f(a) - f(b) - f(cd) - f(abc) - f(abd) + \\ & + f(ab) + f(ac) + f(ad) + f(bc) + f(bd), \end{aligned} \quad (4)$$

where, as usual, brackets around singletons and the union signs are omitted. The Ingleton expression is invariant for swapping the first pair and the second pair of arguments, respectively, which means that it has six different instances. The following inequalities hold in every polymatroid:

$$\begin{aligned} \text{lng}(a, b, c, d) + f(a) + f(c) - f(ac) &\geq 0, \\ \text{lng}(a, b, c, d) &\geq f(a) - f(ac), \\ \text{lng}(a, b, c, d) &\geq f(c) - f(ac). \end{aligned}$$

For example, the first inequality can be written equivalently as

$$\delta(ab, bc) + \delta(ad, bd) + \delta(c, d) \geq 0,$$

where $\delta(I, J) = f(I) + f(J) - f(I \cup J) - f(I \cap J)$ is the non-negative modular defect of I and J . Similar rearrangements work for the other two inequalities.

Let us return to the claim that $(0 \cup M, f)$ is linearly representable. Due to the symmetry of the Ingleton expression we can assume that the secret 0 is either a or c . As f is integer, $\text{lng}(a, b, c, d) < 0$ means $\text{lng}(a, b, c, d) \leq -1$, and then $f(a) + f(c) - f(ac) \geq 1$ by the first inequality. Also, $f(0) = 1$ implies that either $f(a) = 1$ or $f(c) = 1$, and then either $f(a) - f(ac) \geq 0$ or $f(c) - f(ac) \geq 0$. In both cases $\text{lng}(a, b, c, d) \geq 0$ according to the second and third inequality. Consequently all Ingleton expressions are non-negative proving that $(0 \cup M, f)$ is linearly representable, as claimed.

3 Definitions and basic tools

This section introduces the basic tools which will be used in Section 4 to provide lower bounds on the Shannon complexity of some bipartite access structures.

Consider the rank function f of a sss polymatroid representing a bipartite access structure on $N_1 \cup N_2$. All polymatroidal constraints on the rank function are linear, thus one can incorporate all symmetries of the access structure into the constraints (by taking the average over all automorphisms of the access structure). In this way the rank function $f(J)$ depends only on the numbers $|J \cap N_1|$ and $|J \cap N_2|$. This idea is detailed in [12] where it is shown that all machinery can be explained in terms of so-called *multipartite polymatroids*. In the bipartite case the rank function $f(i, j)$ is defined on $\mathbb{N} \times \mathbb{N}$, the set of non-negative lattice points. Constraints resulting from the polymatroidal axioms are listed in (5) where i and j run over the non-negative integers. These constraints can also be considered as definition: if a real function f defined on the non-negative lattice points satisfies all these constraints, then it is a *discrete submodular function*.

$$\begin{aligned} f(i, j) \geq 0, \quad f(0, 0) = 0 & \quad \text{non-negativity} \\ \left. \begin{aligned} f(i+1, j) &\geq f(i, j) \\ f(i, j+1) &\geq f(i, j) \end{aligned} \right\} & \quad \text{monotonicity} \\ \left. \begin{aligned} f(i, j) - f(i-1, j) &\geq f(i+1, j) - f(i, j) \\ f(i, j) - f(i, j-1) &\geq f(i, j+1) - f(i, j) \end{aligned} \right\} & \quad \text{submodularity - 1} \\ f(i+1, j) - f(i, j) &\geq f(i+1, j+1) - f(i, j+1) & \quad \text{submodularity - 2} \end{aligned} \quad (5)$$

Next to these constraints additional *strong inequalities* express the additional requirement that the polymatroid should be a sss for the access structure Γ . It turns out that this requirement is

equivalent to require that the difference between the left and right hand side in some inequalities in (5) is at least one, depending on whether (any, or all) of the subsets J_{ij} identified by the arguments i and j , that is, $|J_{ij} \cap N_1| = i$ and $|J_{ij} \cap N_2| = j$, is qualified or not. In (6) below rather than using such a verbal description, we use the notation $f^\bullet(i, j)$ to indicate that the subset J_{ij} corresponding to the argument (i, j) is qualified, and $f^\circ(i, j)$ to indicate that J_{ij} is unqualified.

$$\begin{aligned}
 & \left. \begin{aligned} f^\bullet(i+1, j) &\geq f^\circ(i, j) + 1 \\ f^\bullet(i, j+1) &\geq f^\circ(i, j) + 1 \end{aligned} \right\} \text{strong monotonicity} \\
 & \left. \begin{aligned} f^\bullet(i, j) - f^\circ(i-1, j) &\geq f^\bullet(i+1, j) - f^\bullet(i, j) + 1 \\ f^\bullet(i, j) - f^\circ(i, j-1) &\geq f^\bullet(i, j+1) - f^\bullet(i, j) + 1 \end{aligned} \right\} \text{strong submodularity - 1} \\
 & f^\bullet(i+1, j) - f^\circ(i, j) \geq f^\bullet(i+1, j+1) - f^\bullet(i, j+1) + 1 \quad \text{strong submodularity - 2}
 \end{aligned} \tag{6}$$

Let H and V denote the first horizontal and vertical values at the origin, respectively:

$$H = f(1, 0), \quad V = f(0, 1). \tag{7}$$

With these notation the Shannon complexity of the bipartite access structure Γ is

$$\kappa(\Gamma) = \inf_f \{ \max(H, V) : f \text{ satisfies (5) and (6) } \}. \tag{8}$$

The aim of this Section and Section 4 is to find, or give a good estimate for, this value.

Let us fix the bipartite access structure Γ and a function f which satisfies the constraints in (5) and (6). Arguments of f are the lattice points in the non-negative quadrant. These points are denoted by A_1, B_2 , etc., and with an abuse of notation, they also denote the value of f at that point. Qualified and unqualified arguments are denoted by solid and hollow dots, respectively. Figure 3 illustrates three horizontally consecutive lattice points A_1, A_2 , and A_3 such that A_1 is unqualified, and A_2 and A_3 are qualified. Monotonicity constraints from (5) give

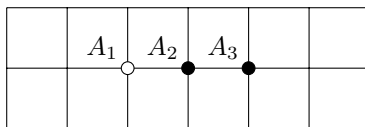


Figure 3: Lattice points representing f

$$A_1 \leq A_2 \leq A_3.$$

while the first line of submodularity-1 in (5) translates to

$$2A_2 \geq A_1 + A_3.$$

As A_1 is unqualified, and both A_2 and A_3 are qualified, the stronger inequalities from (6) also hold:

$$A_2 \geq A_1 + 1, \quad \text{and} \quad 2A_2 \geq A_1 + A_3 + 1$$

Submodularity-1 actually says that the function f , going from left to right (first line), or going from bottom up (second line), is concave. Lemma 3.1 is an easy consequence of this concavity and

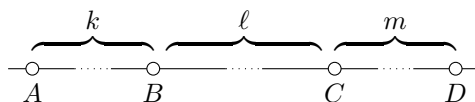


Figure 4: Consequences of concavity

it refers to Figure 4. The lattice points A, B, C and D are on a horizontal (or vertical) line going from left to right (or from bottom up). The distance between A and B , B and C , C and D are k, ℓ, m , respectively. In particular, $k = \ell = m = 1$ if A, B, C, D are consecutive nodes.

Lemma 3.1. *With the notation of Figure 4,*

- a) $\frac{B - A}{k} \geq \frac{C - B}{\ell},$
- b) $\frac{B - A}{k} \geq \frac{D - C}{m},$
- c) *if A is unqualified, B is qualified, and there are s qualified nodes between A and B (not including B), then $\frac{B - A}{k} \geq \frac{C - B}{\ell} + \frac{k - s}{k};$*
- d) *if A and B are unqualified, C and D are qualified, then $\frac{B - A}{k} \geq \frac{D - C}{m} + 1.$ \square*

Claim b) is immediate from a); c) is a strong version of a); and d) is a strong version of b). There are other strong versions of a) and b) depending on how many qualified nodes are between certain pairs. As these versions can also be proved similarly, they will be used without any reference.

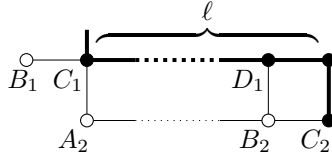


Figure 5: Single increment

Lemma 3.2 refers to Figure 5. Nodes C_1 , D_1 , and C_2 are qualified, and nodes B_1 , A_2 , and B_2 are not. The distance between C_1 and D_1 (between A_2 and B_2) is $\ell - 1$; B_1C_1 , C_1A_2 , etc., have length 1.

Lemma 3.2. *With the notation of Figure 5 and assuming $\ell \geq 2$,*

- a) $C_1 - B_1 \geq (C_2 - B_2) + 1 - \frac{V - 1}{\ell - 1},$
- b) $H \geq C_1 - B_1$, and $C_2 - B_2 \geq 1.$

Proof. By claim c) of Lemma 3.1 we have

$$C_1 - B_1 \geq \frac{D_1 - C_1}{\ell - 1} + 1,$$

and by a) of the same Lemma,

$$\frac{B_2 - A_2}{\ell - 1} \geq C_2 - B_2.$$

To finish the proof one has to observe that $D_1 \geq B_2 + 1$ by strong monotonicity, and $C_1 \leq A_2 + V$ by submodularity. \square

Corollary 3.3. *Suppose Γ has a step of width $w = w_k = i_{k+1} - i_k \geq 2$ such that $i_k \neq 0$. Then $\kappa(\Gamma) \geq 2 - 1/w$.*

Proof. Denote the point (i_k, j_k) by C_1 , and the point $(i_{k+1}, j_k + 1)$ by C_2 . With this choice Lemma 3.2 gives

$$H \stackrel{(b)}{\geq} C_1 - B_1 \stackrel{(a)}{\geq} (C_2 - B_2) + 1 - \frac{V - 1}{w - 1} \stackrel{(b)}{\geq} 2 - \frac{V - 1}{w - 1}.$$

By rearranging $(w - 1)H + V \geq 2w - 1$, thus either H or V must be at least $(2w - 1)/w$, as was claimed. \square

4 Shannon complexity of some bipartite access structures

The bound given by Corollary 3.3 is tight for some bipartite access structures, namely their Shannon complexity is $\kappa = 2 - 1/w$. To show that this is the case, it is enough to present a particular submodular function f on the non-negative grid with $\max\{H, V\} \leq \kappa$ which satisfies all constraints in (5) and (6). Rather than giving the values of f at the grid points, it is more convenient to give values at horizontal and vertical edges, which are the difference of the function values at the edge endpoints. As $f(0,0)$ is zero, these differences determine f uniquely. Properties (5) and (6) can be expressed in terms of these differences in an equivalent form:

- | | | |
|---|--------------------------|-----|
| (a) edge values are non-negative, | monotonicity | |
| (b) on each 1×1 square, the sum of left and top edges equals the sum of bottom and right edges, | consistency | |
| (c) values are decreasing from left to right, and from bottom up (both for vertical and horizontal edges), | submodularity | |
| (d) an edge between a qualified and an unqualified vertex has value at least 1, | strong monotonicity | (9) |
| (e) the increment between two adjacent horizontal (vertical) edges is at least one if the second edge has two qualified endpoints, and the first edge has only one, | strong submodularity - 1 | |
| (f) in an 1×1 square with three qualified nodes the left edge is at least 1 more than the right edge. | strong submodularity - 2 | |

Figure 6 shows the non-zero edge values for a submodular function f (the values are multiples of $1/3$). It realizes the bipartite access structure Γ defined by the points $(2,4)$ and $(5,2)$. Qualified

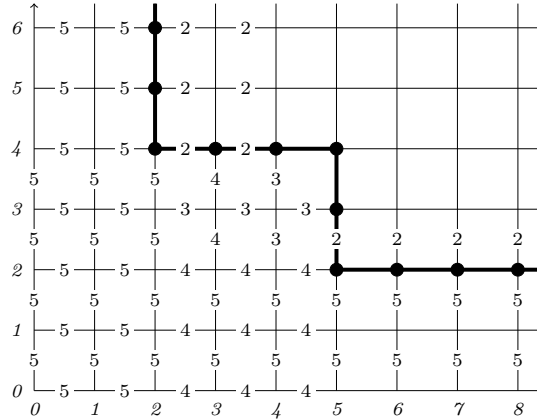


Figure 6: Submodular function by differences; values are multiples of $1/3$

and unqualified nodes are separated by the solid line. The value of f at any grid point is the sum of the differences along any shortest “Manhattan” path from the point to the origin. Conditions in (9) clearly hold. For example, (9d) requires that values between adjacent unqualified and qualified vertices should be at least one; such edges are $(4,2)-(5,2)$, $(4,3)-(5,3)$, $(4,3)-(4,4)$, or $(8,1)-(8,2)$. (9e) requires a difference of 1 or more for certain edge pairs such as $(4,3)-(5,3)-(6,3)$, or $(1,y)-(2,y)-(3,y)$ for all $y \geq 4$. There is only one square where (9f) applies, the one with diagonal points $(4,4)$ and $(5,3)$. Γ has a single step of width $w = 3$, thus Corollary 3.3 gives $\kappa(\Gamma) \geq 2 - 1/3$. As $H = V = 5/3$ and f realizes Γ , we also have $\kappa(\Gamma) \leq 5/3$, thus $\kappa(\Gamma) = 5/3$. This construction generalizes for every single-step bipartite access structure.

Theorem 4.1. *Suppose Γ is defined by two points (i_1, j_1) and (i_2, j_2) where $0 < i_1$; $w = i_2 - i_1 \geq h = j_1 - j_2$. If $w \geq 2$ then $\kappa(\Gamma) = 2 - 1/w$.*

Proof. By Corollary 3.3, the Shannon complexity of Γ is at least $2 - 1/w$. The submodular function defined by the non-zero edge values on Figure 7 has complexity $(2w - 1)/w$, thus it gives

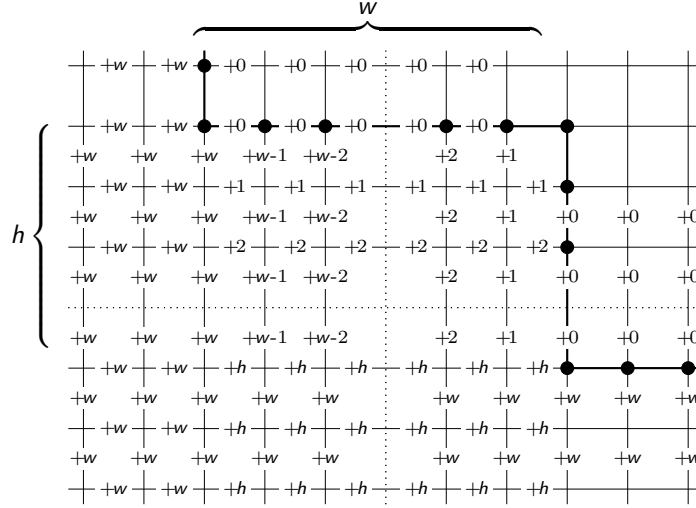


Figure 7: Single step access structure, values are multiplies of $1/w$

the required upper bound. Edge numbers are multiplies of $1/w$ and numbers preceded by a + sign should be increased by $w - 1$, e.g., $+w$ means $w + w - 1 = 2w - 1$ (edge values at the bottom left corner). Similarly, $+0 = w - 1$, $+h = h + w - 1$, etc. The bottom row and the leftmost column can be repeated until the bottom left vertex becomes the origin. It is a routine to check that all conditions in (9) actually hold. \square

Theorem 4.2. *Let Γ be a regular staircase with the same width and height $w = h \geq 2$ such that for some $1 \leq k \leq \ell_\Gamma$ the point (i_k, j_k) has positive coordinates. Then $\kappa(\Gamma) = 2 - 1/w$.*

Proof. The additional condition that (i_k, j_k) is not on any of the coordinate axes guarantees that Corollary 3.3 can be applied, and gives $\kappa(\Gamma) \geq 2 - 1/w$. For the other direction Figure 8 shows part of the non-zero edge values of a submodular function for the regular staircase with $h = w = 5$.

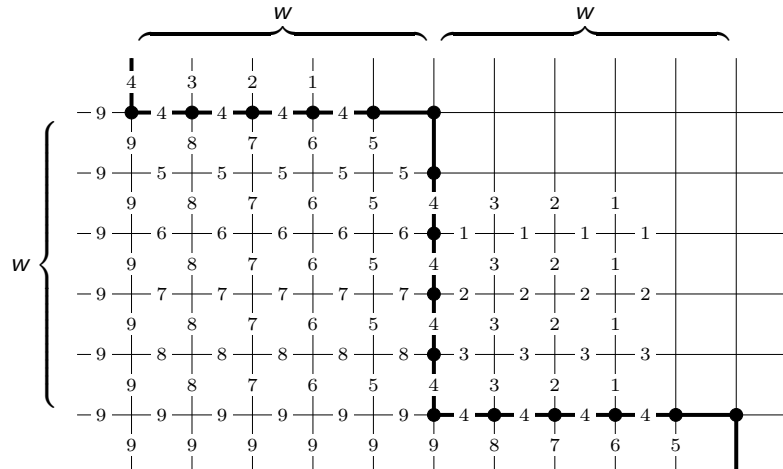


Figure 8: Regular staircase with same width and height w

Values are multiplies of $1/w$. The given pattern should be repeated by shifting it down and right (up and left) by w until it fills the non-negative quadrant. Conditions in (9) clearly hold. The pattern easily generalizes for every regular staircase with equal width and height. \square

Theorem 4.3. *Suppose all heights of the staircase Γ are 1, the first point (i_1, j_1) is not on the y -axis, and all widths are $w_k \geq 2$. Then*

$$\kappa(\Gamma) \geq \kappa_0 = 1 + \frac{\ell_\Gamma - 1}{1 + \sum_k \frac{1}{w_k - 1}} \quad (10)$$

Proof. Let $\ell = \ell_\Gamma$ and denote the points $(i_1, j_1), \dots, (i_\ell, j_\ell)$ by C_1, \dots, C_ℓ , see Figure 9. By

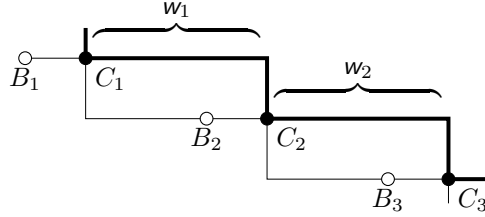


Figure 9: Staircase with heights 1

Lemma 3.2 we have

$$H \geq C_1 - B_1 \geq (C_2 - B_2) + 1 - \frac{V - 1}{w_1 - 1},$$

for each $1 \leq k \leq \ell - 1$

$$C_k - B_k \geq (C_{k+1} - B_{k+1}) + 1 - \frac{V - 1}{w_k - 1},$$

and finally

$$C_\ell - B_\ell \geq 1.$$

Adding them up we get

$$H \geq \ell - \sum_{k=1}^{\ell-1} \frac{V - 1}{w_k - 1}, \quad (11)$$

or

$$H + V \sum \frac{1}{w_k - 1} \geq \ell + \sum \frac{1}{w_k - 1}.$$

Consequently either H or V must be at least κ_0 . \square

Under an additional technical assumption the lower bound κ_0 in Theorem 4.3 is tight. The proof is by exhibiting an appropriate discrete submodular function.

Theorem 4.4. *With the assumptions of Theorem 4.3, if, additionally, $w_k \geq \kappa_0$ for all widths, then $\kappa(\Gamma) = \kappa_0$.*

Proof. The structure of non-zero edge values are sketched on Figure 10. The $+$ symbol before x , y , etc., indicates $+1$, for example, $+y$ means $y + 1$. The value V is the “vertical” value between the origin and $(0, 1)$. There are sequences of vertical edges marked by \otimes between a V and a 1 edge; their values should be computed so that they form an arithmetical progression starting with V and ending with 1.

Assume $V \geq 1$ and that all edge values are non-negative. The consistency condition in (9) clearly holds everywhere except around the edges marked by \otimes . For the block under w_k the consistency requires

$$V + (w_k - 1)z = (w_k - 1)(y + 1) + 1,$$

that is,

$$z = y + \frac{w_k - V}{w_k - 1}. \quad (12)$$

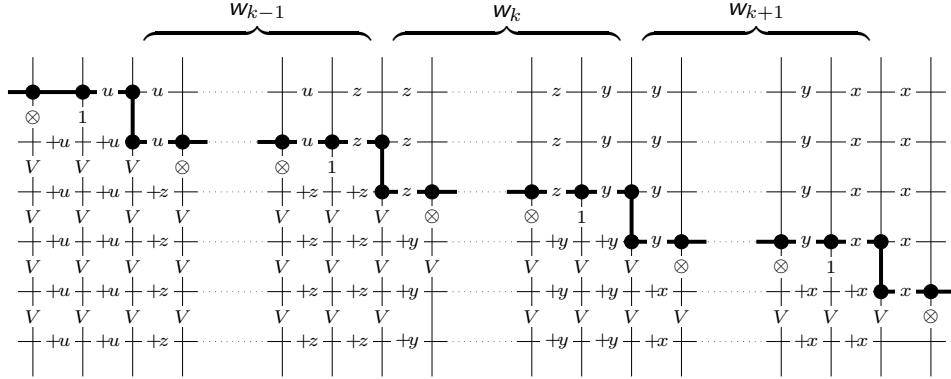


Figure 10: Submodular function for the height 1 staircase

For the submodularity property we also need $z \geq y$, that is, $V \leq w_k$. If both of them are satisfied then all requirements in (9) hold.

After the last staircase step the horizontal edge values can be chosen to be zero ($x = 0$ in the figure). Other horizontal edge values are determined by (12) and by the $+1$ increment, thus the edge between $(0, 0)$ and $(1, 0)$ has the value

$$H = 1 + \sum_{k < \ell} \frac{w_k - V}{w_k - 1} = \ell - (V - 1) \sum_{k < \ell} \frac{1}{w_k - 1}.$$

Choosing $V = \kappa_0$ we get $H = \kappa_0$, which gives the required submodular function. \square

In Theorem 4.4 the assumption that all steps have width at least κ_0 is necessary. The next theorem shows that if some intermediate stepsize is smaller than κ_0 , then the Shannon complexity is strictly larger than κ_0 . It happens, for example, when the width sequence is $(3, 3, 2, 3)$, when $\kappa_0 = 2 + 7/49$ while the Shannon complexity is $2 + 7/34 > \kappa_0$.

Theorem 4.5. *With the assumptions of Theorem 4.3 suppose that some intermediate stepsize is smaller than κ_0 . Then $\kappa(\Gamma) > \kappa_0$.*

Proof. Use the notation of Figure 11. Let $\Delta = w_{k-1} + w_k + w_{k+1} - 1$, this is the distance between A_2 and E_2 , or Z_3 and D_3 . The next two inequalities were actually proved in Lemma 3.2:

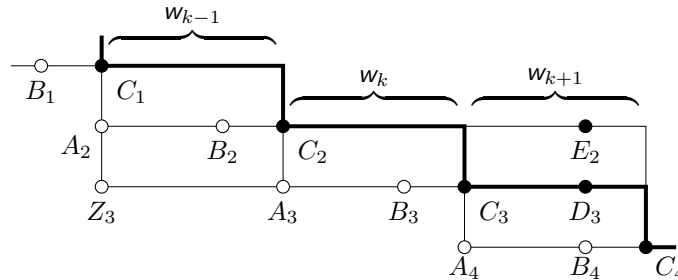


Figure 11: Case of a small stepsize

$$C_1 - B_1 \geq \frac{B_2 - A_2}{w_{k-1} - 1} + 1 - \frac{V - 1}{w_{k-1} - 1}$$

$$\frac{D_3 - C_3}{w_{k+1} - 1} \geq \frac{B_4 - A_4}{w_{k+1} - 1} - \frac{V - 1}{w_{k+1} - 1},$$

and the following three ones follow from Lemma 3.1 easily:

$$\begin{aligned}\frac{B_2 - A_2}{w_{k-1} - 1} &\geq \frac{E_2 - A_2}{\Delta} + 1 - \frac{w_{k-1}}{\Delta}, \\ \frac{D_3 - Z_3}{\Delta} &\geq \frac{D_3 - C_3}{w_{k+1} - 1} + \frac{w_{k-1} + w_k}{\Delta}, \\ \frac{B_4 - A_4}{w_{k+1} - 1} &\geq C_4 - B_4.\end{aligned}$$

Finally, we have

$$\frac{E_2 - A_2}{\Delta} \geq \frac{D_3 - Z_3}{\Delta} - \frac{V}{\Delta}$$

since $E_2 \geq D_3$ and $A_2 \leq Z_3 + V$. Adding these inequalities up we get

$$\begin{aligned}C_1 - B_1 &\geq (C_4 - B_4) + 1 - \frac{V - 1}{w_{k-1} - 1} + \\ &+ 1 - \frac{V - 1}{w_{k+1} - 1} + 1 - \frac{V - 1 + (w_{k-1} + w_{k+1})}{w_k - 1 + (w_{k-1} + w_{k+1})}.\end{aligned}\tag{13}$$

Looking back at the proof of Theorem 4.3, we see that using (13), the right hand side of the inequality (11) changes by

$$\frac{V - 1}{w_k - 1} - \frac{V - 1 + (w_{k-1} + w_{k+1})}{w_k - 1 + (w_{k-1} + w_{k+1})}.$$

When $w_k < V$, this amount is positive (the second term is closer to 1 than the first one), thus the inequality in (11) is strict. Consequently $\kappa(\Gamma) > \kappa_0$ which proves the theorem. \square

When some stepsize is below κ_0 , then instead of (11) one can use the improved estimate (13) to get a better lower bound on $\kappa(\Gamma)$. In some cases it gives the exact value, but not in every case.

5 Some linear bipartite schemes

We were able to create linear schemes with optimal complexity for a very sparse set of non-ideal bipartite access structures. For these structures the linear, the entropic, and Shannon complexities are the same.

Theorem 5.1. *Let Γ be the regular staircase with height $h = 1$, width $w \geq 2$, and length $\ell = \ell_\Gamma \geq 2$ such that the first point (i_1, j_1) is not on the y -axis. There is a linear scheme for Γ with complexity*

$$1 + \frac{\ell - 1}{1 + \frac{\ell - 1}{w - 1}},$$

which matches the lower bound κ_0 on $\kappa(\Gamma)$ from Theorem 4.3.

Proof. As explained in [12, Theorem 5], the scheme will be an integer linear combination of schemes h_1 and h_2 defined below, both realizing Γ . As both h_1 and h_2 can be represented over any finite field, this combination gives the required linear secret sharing scheme. The schemes distribute shares corresponding to some secret among the participants in $N_1 \cup N_2$ such that qualified subsets of the regular staircase can recover the secret, while unqualified subsets have no information on the secret. The share size (relative to the secret size), however, will not be uniform. In the first scheme h_1 participants in N_1 get single size shares, while participants in N_2 get shares of size w . In the second scheme h_2 it is the other way around: participants in N_2 get single size shares, while participants in N_1 get shares of size ℓ .

The idea is to combine several independent instances of these schemes – assuming that they share the same secret size, which can be done in this case. Executing α copies of h_1 and β copies of h_2 distributes $\alpha + \beta$ many independent secrets. The total share size of a participant from N_1 is $\alpha + \beta \cdot w$ times the size of a single secret, while for participants from N_2 this number is $\alpha \cdot \ell + \beta$. Choosing $\alpha = w - 1$ and $\beta = \ell - 1$ balances these numbers to be

$$(w - 1) + (\ell - 1)w = (w - 1)\ell + (w - 1) = (\ell - 1)(w - 1) + \ell + w - 2.$$

Since the this combined scheme distributes $\alpha + \beta = \ell + w - 2$ many secrets, its complexity is

$$1 + \frac{(\ell - 1)(w - 1)}{\ell + w - 2} = 1 + \frac{\ell - 1}{1 + \frac{\ell - 1}{w - 1}},$$

matching the value stated in the Theorem. It remains to describe the two sub-schemes.

Scheme 1 is an adaptation of the ideal bipartite scheme Γ_3 from Section 2.1, but see also [23]. Let $t = i_k + w \cdot j_k$ (as Γ is a regular staircase with height 1, this amount is independent of k) and consider the following ideal bipartite access structure on $N_1 + w \cdot N_2$ participants: a qualified set requires at least i_1 participants from the first set, at least $w \cdot j_\ell$ participants from the second set, and at least t participants all together, see Figure 2. To get h_1 form groups of size w from the second set, assigning all shares of a group to a single participant from N_2 .

In h_1 a the share size is one for a participant from N_1 , and w for a participant from N_2 , as was claimed. To show that h_1 realizes the access structure Γ , observe first that any qualified set in h_1 must have at least i_1 participants from the first set, and at least j_ℓ participants from the second set. When this holds, taking i participants from N_1 and j participants from N_2 , $i \geq i_1$ and $j \geq j_\ell$ will hold, and this group forms a h_1 -qualified set iff the $i + jw$ many shares they possess is above the threshold t . But this happens iff $i \geq i_k$ and $j \geq j_k$ for some k , that is, if and only the point (i, j) is in Γ .

Scheme 2 is constructed as follows. The secret is a sum of two independent values. The first one is distributed among members of N_1 using an i_1 out of N_1 threshold scheme. The second value is distributed using a j_1 out of $|N_2| + \ell - 1$ threshold scheme. $|N_2|$ of the shares are given to members of N_2 ; the remaining $\ell - 1$ shares are distributed among members of N_1 as follows: one share is distributed using an i_2 out of N_1 threshold scheme, the second one by an i_3 out of N_1 threshold scheme, and the last one by an i_ℓ out of N_1 threshold scheme. Every member of N_2 gets a single share, while members of N_1 get ℓ shares.

Now we claim that h_2 also realizes Γ . A qualified set in h_2 must recover both secret values. Recovering the first one requires at least i_1 members from N_1 . Recovering the second value requires j_1 shares. Those shares might come from j_1 members from N_2 . They might also come from $j_1 - 1$ members from N_2 , and the missing share can be recovered by i_2 members of N_1 . Similarly, the second value can be recovered by $j_1 - k$ members from N_2 and at least i_k members from N_1 for any $k \leq \ell$. It shows that elements of Γ are qualified in h_2 . The reverse follows from the fact that recovering the second secret value by $j_1 - k$ members from N_2 requires at least i_k members from N_2 \square

The rest of this section describes a linear scheme for a particular bipartite access structure. For more clarity the construction uses vector spaces over reals rather than over some finite field. This can be done as, by a compactness argument, polymatroids representable over the reals are also representable over some finite field whose characteristics can be chosen to be arbitrarily large.

In the constructions vectors contain unspecified variables. Their values should be chosen to be *generic*, by which we mean that considering all vectors as rows of a huge matrix, if the determinant of any $k \times k$ submatrix is not a constant (that is, the determinant contains at least one of the unspecified variable), then the determinant should differ from zero. This can always be achieved, for example, by choosing all unspecified values to be algebraically independent.

Proposition 5.2. *The complexity of the bipartite access structure Γ defined by the points $(0, 3)$, $(1, 1)$, $(3, 0)$ is $3/2$.*

Proof. As $\kappa(\Gamma) \geq 3/2$ by Corollary 3.3, it is enough to construct a linear scheme with this complexity. We will work in the 7-dimensional vector space \mathbb{R}^7 , as explained above. Participant $a \in N_1$ will be assigned the 3-dimensional subspace E_a , and participant $b \in N_2$ will be assigned another 3-dimensional subspace E_b . The secret is a 2-dimensional subspace E_0 . This arrangement realizes the above bipartite access structure if

- (a) E_0 is in the linear hull of the subspaces assigned to any three participants from N_1 – or any three participants from N_2 .
- (b) for every $a \in N_1$ and $b \in N_2$ the linear hull of $E_a \cup E_b$ contains E_0 ;
- (c) the linear hull of any two subspaces assigned to participants from N_1 (or both from N_2) intersect E_0 trivially.

The subspace assigned to $a \in N_1$ and $b \in N_2$, respectively, will be spanned by the rows of the following matrices with seven columns:

$$E_a = \begin{pmatrix} 1 & 0 & \alpha_1 & \alpha_3 & \alpha_4 & 0 & 0 \\ 1 & 0 & \alpha_2 & 0 & 0 & \alpha_3 & \alpha_4 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad E_b = \begin{pmatrix} 0 & 1 & \beta_1 & \beta_3 & 0 & \beta_4 & 0 \\ 0 & 1 & \beta_2 & 0 & \beta_3 & 0 & \beta_4 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

while the secret space is spanned by the row vectors of

$$E_0 = \begin{pmatrix} \zeta_1 & \zeta_2 & \zeta_3 & 0 & 0 & 0 & 0 \\ \zeta_4 & \zeta_5 & \zeta_6 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $\alpha_i, \beta_j, \zeta_k$ are the generic variables as explained above.

(a) We show that the linear span of any three subspaces assigned to members of N_1 contain E_0 ; the case for N_2 is similar. The linear span of the row vectors

$$\begin{pmatrix} 1 & 0 & \alpha_1 & \alpha_3 & \alpha_4 & 0 & 0 \\ 1 & 0 & \alpha'_1 & \alpha'_3 & \alpha'_4 & 0 & 0 \\ 1 & 0 & \alpha''_1 & \alpha''_3 & \alpha''_4 & 0 & 0 \end{pmatrix}$$

contains the vector $(10\gamma_1 0000)$ for some generic γ_1 , and similarly, the linear span of the row vectors

$$\begin{pmatrix} 1 & 0 & \alpha_2 & 0 & 0 & \alpha_3 & \alpha_4 \\ 1 & 0 & \alpha'_2 & 0 & 0 & \alpha'_3 & \alpha'_4 \\ 1 & 0 & \alpha''_2 & 0 & 0 & \alpha''_3 & \alpha''_4 \end{pmatrix}$$

contains the vector $(10\gamma_2 0000)$ for another generic γ_2 . (Actually, in both cases the same linear combination can be used.) Thus the vectors $(100 0000)$, $(010 0000)$ and $(001 0000)$ are in the linear space spanned by $E_a, E_{a'}$ and $E_{a''}$, and then so is E_0 .

(b) One participant from the first group and one from the second one determine the secret. The linear span of their subspaces contains the row vectors of the matrix

$$\begin{pmatrix} 0 & 0 & \alpha_1 & \alpha_3 & \alpha_4 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 & 0 & \alpha_3 & \alpha_4 \\ 0 & 0 & \beta_1 & \beta_3 & 0 & \beta_4 & 0 \\ 0 & 0 & \beta_2 & 0 & \beta_3 & 0 & \beta_4 \end{pmatrix}$$

Taking their linear combination with coefficients $(\beta_3, \beta_4, -\alpha_3, -\alpha_4)$ one gets the vector $(00\gamma 0000)$ for some generic γ , thus E_0 is indeed inside their linear span.

(c) Let E be the 3-dimensional subspace of vectors with the last four coordinate equal to zero. As E_0 is a generic 2-dimensional subspace of E , multiples of $(010 0000)$ intersect E_0 trivially. We

claim that the span of the remaining four vectors

$$\begin{pmatrix} 1 & 0 & \alpha_1 & \alpha_3 & \alpha_4 & 0 & 0 \\ 1 & 0 & \alpha_2 & 0 & 0 & \alpha_3 & \alpha_4 \\ 1 & 0 & \alpha'_1 & \alpha'_3 & \alpha'_4 & 0 & 0 \\ 1 & 0 & \alpha'_2 & 0 & 0 & \alpha'_3 & \alpha'_4 \end{pmatrix}$$

intersect E trivially. Indeed, no nontrivial linear combination makes the last four coordinates zero as (α_3, α_4) and (α'_3, α'_4) are linearly independent. Now E_0 is a subspace of E , therefore the linear span of E_a and E_b intersects E_0 trivially, as was required. \square

6 Continuous submodular optimization

Estimating the Shannon complexity of bipartite access structures can be considered to be a discrete variant of a continuous submodular optimization as has been discussed in, e.g., [13, 14]. The intuition is scaling down the non-negative lattice so that the edge size becomes negligible and take a bird's eye view. The continuous analog of a bipartite rank function is thus a real function defined on the non-negative quadrant satisfying conditions reflecting the conditions in (5) for discrete rank functions, see Definition 6.1. These rank functions turn out to be continuous and non-decreasing, consequently have both left and right partial derivatives, see Proposition 6.3.

An access structure Γ specifies the qualified and unqualified points. For the continuous case considered here Γ is defined by a strictly decreasing continuous curve connecting points on the coordinate axes. Unqualified points are below the curve, and qualified points are above and to the right of the curve.

For intuition how to specify whether a rank function $f \in \mathcal{G}$ realizes an access structure Γ we turn to part d) of Lemma 3.1. It claims

$$(B - A)/k \geq (D - C)/m + 1$$

assuming A, B, C, D are, in this order, lattice points on a line, where A and B are unqualified and C, D are qualified. Let (u, v) be a boundary point of Γ , and choose A, B, C, D on a line parallel to the x axis so that (u, v) is between B and C . If all the points tend to (u, v) , the fraction $(B - A)/k$ tends to the left partial derivative of f at (u, v) while $(D - C)/m$ tends to the right partial derivative. Thus, in the limit, the above inequality says that $f_x^-(u, v) \geq f_x^+(u, v) + 1$. Accordingly, Definition 6.5 stipulates that the rank function $f \in \mathcal{G}$ realizes Γ if at every internal boundary point of Γ , both partial derivatives of f should drop by at least 1.

Finally, the complexity of $f \in \mathcal{G}$, corresponding to the maximal share size $\max\{H, V\}$ in the discrete case, is clearly should be $\max\{f_x^+(0, 0), f_y^+(0, 0)\}$. The continuous version of finding the Shannon complexity of a bipartite access structure thus can be spelled out as follows.

Optimization Problem. For an access structure Γ , determined by the curve α , determine the optimal complexity of continuous rank functions realizing Γ .

The rest of this section is organized as follows. First, the family of continuous rank functions is defined, followed by propositions establishing some of their basic properties. The main result is Theorem 6.7 giving a general lower bound for this Optimization Problem in terms of the curve α . This bound is tight when α is linear. The section concludes with a few remarks and open problems.

Definition 6.1. The family \mathcal{G} of *continuous bipartite rank functions* consists of real functions f defined on the non-negative quadrant $[0, +\infty)^2$ satisfying conditions a)–d) below.

- a) f is pointed, that is, $f(0, 0) = 0$;
- b) f is non-decreasing: for $0 \leq u_1 \leq u_2$ and $0 \leq v_1 \leq v_2$ we have $f(u_1, v_1) \leq f(u_2, v_2)$;

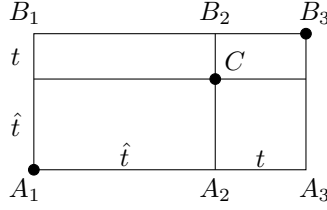


Figure 12: $f \in \mathcal{G}$ is concave in positive directions

c) f is concave separately in both coordinates: for $0 < t < 1$ and $\hat{t} = 1 - t$,

$$\begin{aligned} f(tu_1 + \hat{t}u_2, v) &\geq tf(u_1, v) + \hat{t}f(u_2, v), \\ f(u, tv_1 + \hat{t}v_2) &\geq tf(u, v_1) + \hat{t}f(u, v_2); \end{aligned}$$

d) f is submodular: for $0 \leq u_1 \leq u_2$ and $0 \leq v_1 \leq v_2$

$$f(u_1, v_2) + f(u_2, v_1) \geq f(u_1, v_1) + f(u_2, v_2). \quad (14)$$

The class \mathcal{G} is closed for non-negative linear combinations. Moreover, if $f \in \mathcal{G}$ and M is a non-negative constant, then $\min(f, M) \in \mathcal{G}$. Consequently $f(u, v) = \min\{c_u u + c_v v, M\}$ is in \mathcal{G} for every positive c_u, c_v and M .

The right and left partial derivatives of $f \in \mathcal{G}$, if exist, are denoted by f_x^+, f_y^+ and f_x^-, f_y^- , respectively. Some properties of functions in \mathcal{G} , similar to those of discrete bipartite rank functions, follow from the definition above.

Proposition 6.2. *f is concave and increasing along any positive direction: if $0 \leq u_1 \leq u_2$, $0 \leq v_1 \leq v_2$, $0 \leq t \leq 1$ and $\hat{t} = 1 - t$, then*

$$f(tu_1 + \hat{t}u_2, tv_1 + \hat{t}v_2) \geq tf(u_1, v_1) + \hat{t}f(u_2, v_2).$$

Proof. Refer to Figure 12 where A_1, B_2 are the points with coordinates (u_1, v_1) and (u_2, v_2) , respectively. Concavity along the x and y coordinates give

$$\begin{aligned} tf(A_1) + \hat{t}f(A_3) &\leq f(A_2), \\ tf(B_1) + \hat{t}f(B_3) &\leq f(B_2), \\ tf(A_2) + \hat{t}f(B_2) &\leq f(C). \end{aligned}$$

Multiplying the first inequality by t , the second one by \hat{t} , and using (14) to get

$$f(A_1) + f(B_3) \leq f(B_1) + f(A_3),$$

the required inequality follows. \square

Note that the function $f(u, v) = \min(u, 1) \cdot \min(v, 1)$ satisfies properties a)–c) of Definition 6.1, while does not satisfy d) as f is not concave in the $(1, 1)$ direction.

Proposition 6.3. *Partial derivatives of $f \in \mathcal{G}$ exist (allowing the value $+\infty$ at the boundary), they are non-negative and non-increasing in both coordinates.*

Proof. For example, if $u_1 < u_2$, then

$$\begin{aligned} f_y^+(u_1, v) &= \lim_{v' \rightarrow v+0} \frac{f(u_1, v') - f(u_1, v)}{v' - v} \\ &\geq \lim_{v' \rightarrow v+0} \frac{f(u_2, v') - f(u_2, v)}{v' - v} = f_y^+(u_2, v), \end{aligned}$$

where the inequality holds by (14). \square

Similar reasoning gives

Proposition 6.4. *If $u \rightarrow u' - 0$, then $f_x^+(u, v) \rightarrow f_x^-(u', v)$; if $v \rightarrow v' + 0$, then $f_y^+(u, v) \rightarrow f_y^-(u, v')$; and similarly for other cases. \square*

Definition 6.5. The rank function $f \in \mathcal{G}$ realizes the access structure Γ if at every internal boundary point (u, v) of Γ (that is, when both u and v are positive) we have

$$\begin{aligned} f_x^-(u, v) &\geq 1 + f_x^+(u, v), \\ f_y^-(u, v) &\geq 1 + f_y^+(u, v). \end{aligned} \tag{15}$$

We consider only access structures which are defined by the graph of a continuous, strictly decreasing curve α such that $\alpha(0) = a > 0$ and $\alpha(b) = 0$ for some $b > 0$. The point (x, y) is qualified if either $x \geq b$, or if $x \geq 0$ and $y \geq \alpha(x)$. In this case internal points of the boundary are the points $(x, \alpha(x))$ for $0 < x < b$.

Lemma 6.6. *Suppose α is as above, it is derivable everywhere and $f \in \mathcal{G}$ satisfies the constraints (15) in internal points of the graph of α . Then $f_x^+(0, 0) \geq \sup\{-\alpha'(v) : 0 < v < b\}$, where α' is the derivative of α .*

Proof. Let $0 \leq u < v$, in this case $\alpha(v) < \alpha(u)$. Then

$$[v - u] f_x^+(u, \alpha(v)) \geq f(v, \alpha(v)) - f(u, \alpha(v))$$

We also have

$$\begin{aligned} f(v, \alpha(u)) - f(u, \alpha(u)) &\geq 0, \\ f(u, \alpha(u)) - f(u, \alpha(v)) &\geq [\alpha(u) - \alpha(v)] f_y^-(u, \alpha(u)) \geq \\ &\geq [\alpha(u) - \alpha(v)] (1 + f_y^+(u, \alpha(u))), \end{aligned}$$

and

$$[\alpha(u) - \alpha(v)] f_y^+(v, \alpha(v)) \geq f(v, \alpha(u)) - f(v, \alpha(v)).$$

Adding them up we get

$$[v - u] f_x^+(u, \alpha(v)) \geq [\alpha(u) - \alpha(v)] (1 + f_y^+(u, \alpha(u)) - f_y^+(v, \alpha(v))).$$

Now f_y^+ is non-increasing in both directions, thus $f_y^+(u, \alpha(u)) \geq f_y^+(v, \alpha(u))$, which means

$$f_x^+(u, \alpha(v)) \geq \frac{\alpha(u) - \alpha(v)}{v - u} [1 + f_y^+(v, \alpha(u)) - f_y^+(v, \alpha(v))].$$

Limiting $u \rightarrow v - 0$ the left hand side becomes $f_x^-(v, \alpha(v))$, and on the right hand side we have $f_y^+(v, \alpha(u)) \rightarrow f_y^+(v, \alpha(v))$. Therefore $f_x^-(v, \alpha(v)) \geq -\alpha'(v)$, which immediately gives the claim. \square

Recall that the complexity of the rank function $f \in \mathcal{G}$ is $\max\{f_x^+(0, 0), f_y^+(0, 0)\}$.

Theorem 6.7. *Let α is strictly decreasing, derivable everywhere, $\alpha(0) = a > 0$ and $\alpha(b) = 0$ for some $b > 0$. The complexity of every $f \in \mathcal{G}$ realizing the access structure defined by α is at least*

$$\sup\{-\alpha'(v), -1/\alpha'(v) : 0 < v < b\}.$$

Proof. The inequality $f_y^+(0, 0) \geq \sup\{-1/\alpha'(v) : 0 < v < b\}$ is equivalent to

$$f_x^+(0, 0) \geq \sup\{-\alpha'(v) : 0 < v < b\},$$

proved in Lemma 6.6, by symmetry that exchanges the arguments of f and α with its inverse α^{-1} . Consequently the maximum of the two sups is a lower bound on the complexity. \square

The bound provided by Theorem 6.7 is tight when α is linear, it is attained by $f(u, v) = \min\{c_u u + c_v v, M\}$ for some positive c_u, c_v, M . It is interesting to note that the proof of Lemma 6.6 used only the local behavior of f at the curve points $(u, \alpha(u))$ without considering any global accumulation effect. It would be interesting to know whether this is typical or not. There are many other open questions, like: when the above bound is tight; whether the infimum is attained, or uniquely attained; whether the solution depends continuously on α ; what is the relation between the discrete and continuous cases; etc.

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