Optimization and Engineering
https://doi.org/10.1007/s11081-023-09801-3

RESEARCH ARTICLE

A SDP relaxation of an optimal power flow problem for distribution networks

Vivien Desveaux[1] · **Marouan Handa**[1,2]

Received: 22 August 2022 / Revised: 22 March 2023 / Accepted: 8 April 2023

Abstract

In this work, we are interested in an optimal power flow problem with fixed voltage magnitudes in distribution networks. This optimization problem is known to be non-convex and thus difficult to solve. A well-known solution methodology consists in reformulating the objective function and the constraints of the original problem in terms of positive semi-definite matrix traces, to which we add a rank constraint. To convexify the problem, we remove this rank constraint. Our main focus is to provide a strong mathematical proof of the exactness of this convex relaxation technique. To this end, we explore the geometry of the feasible set of the problem via its Pareto-front. We prove that the feasible set of the original problem and the feasible set of its convexification share the same Pareto-front. From a numerical point of view, this exactness result allows to reduce the initial problem to a semi-definite program, which can be solved by more efficient algorithms.

Keywords Electric power distribution network · Optimal power flow · Convex relaxation · Pareto-front

Mathematics Subject Classification 90-22 · 90-26 · 90-35 · 90-90

✉ Vivien Desveaux
vivien.desveaux@u-picardie.fr

Marouan Handa
handa@utia.cas.cz

1 LAMFA, UMR CNRS 7352, Université de Picardie Jules Verne, 33 rue Saint Leu, 80039 Amiens, France

2 Institute of Information Theory and Automation, 18208 Prague, Czech Republic

Published online: 21 April 2023

1 Introduction

The optimal power flow (OPF) study in an electrical network is an essential tool in power engineering. It is defined as an optimization problem where the cost function can represent an economic objective such as power generation cost, or/and a technical objective such as the minimization of power losses in the network. The constraints often represent some physical restrictions at each bus and at each transmission line of the network.

OPF problems are widely used in the literature for different goals, see for instance Elattar and ElSayed (2019), especially:

– The minimization of power production (or importation) costs. In this case, the objective function is quadratic.
– The minimization of losses of active and/or reactive power in the transmission lines. The objective function is then non-linear, or even non-convex.
– Maintaining a constant voltage profile. The objective function is then in the form of a l^1 or l^2 distance between the voltage v and a target value v_t, i.e. $J = \| v - v_t \|$.

Some papers combine several cost functions, leading to consider multi-criteria optimization problems Shaheen et al. (2016).

To compute the fluxes through the lines of the network, we use the power flow equations [see for instance Momoh (2017)]. Those equations appear in the OPF problem as equality constraints and are non-convex. Thus, it makes the optimization problem itself non-convex and therefore very difficult and costly to solve numerically. On the other hand, there is no guarantee that an approximate solution provided by an optimization method will be a global minimum. For these reasons, it is preferable to find a convex problem that is equivalent in a certain sense.

There are several methods that can be used to convexify the problem. A first approach is to linearize the non-convex constraints under certain physical assumptions, in particular the reactive power constraints must be neglected and the voltages must be fixed. It is then possible to obtain a linear or a quadratic problem [see Castillo and Gayme (2017)].

Another approach relies on the convexification of the original problem via a convex relaxation. This consists in replacing the non-convex set of constraints by a convex set containing it. The convex relaxation is then said to be exact if the optimal value of the original problem is equal to the optimal value of the convexified problem. A first and natural possibility is to take the convex hull of the constraints set [see for instance Lavaei et al. (2012)]. However, the disadvantage of this method is the difficulty to find an algebraic representation of the convex hull that can be used in practice. An alternative is to use a larger convex set than the convex hull, and which can be represented algebraically in a simple manner.

In Bai et al. (2008), the authors reformulate the objective function and the constraints of the original problem in terms of traces of positive semi-definite matrices, to which is added a rank constraint. By removing this rank constraint, they obtain a convex problem. This method is called a SDP relaxation.

The authors of Lavaei et al. (2012, 2013) states that the SDP relaxation is exact under certain conditions of monotony of the objective function and on the voltage phase angles. In Lavaei et al. (2012), the voltage magnitudes are assumed to be fixed and to prove the exactness of the SDP relaxation, they studied the geometry of the feasible set of the problem via its Pareto-front. However, in their proof the authors makes a simplifying hypothesis that is not always satisfied, which makes the proof incomplete (Lemma 4, p. 581). In Lavaei et al. (2013), a generalization to variable voltage magnitudes is proposed, but the proof of exactness relies on the proof of the fixed voltage magnitude case. Therefore a complete proof is still an open problem in both the fixed and variable voltage magnitude cases.

The main goal of this paper is to provide a rigorous proof of the exactness of the SDP relaxation in the case of fixed voltage magnitudes. Whereas the variable voltage magnitude case is of practical greater interest, an exactness proof appears to be much more difficult to reach in this case without additional very restrictive conditions (for instance no lower bound limitation on the injected active power). In order to do so, we will use two main ingredients: the tree structure of the distribution network and a geometrical study of the feasible sets and their Pareto-front. Furthermore, the new techniques proposed in this paper can be useful tools to help reaching a proof in the variable voltage magnitude case in a future work.

The rest of this paper is organized as follows. In Sect. 2, we present the model and the physical equations governing the power distribution network. We then define the optimization problem of interest and provide different equivalent formulations of this problem. Section 3 is dedicated to the notion of Pareto-front. We give some definitions and provide some elementary properties that will be frequently used to prove the theoretical results of this paper. In Sect. 4, we introduce and study the convexified problem in the simplified case of a network of only one line, and we show some results that will be used in the following. In Sect. 5, we prove the main result of this paper which states that under some conditions, the initial problem and its convexification share the same Pareto-front, i.e. the relaxation is exact. We conclude by proving a theorem that allows to use this convexification in practice. In Sect. 6, we present some numerical tests to highlight the proven theoretical results. Finally, in Sect. 7, we give some concluding remarks and perspectives.

2 Modeling of the power distribution network

2.1 Geometry of the network

We consider a power distribution network modeled by a tree $\Sigma = (S, E)$, where $S = \{1, \cdots m\}$, $m \geq 2$, is the set of vertices that represent the nodes of the network and $E \subset \{\{i,k\} \in \mathbb{R}^2,\ i \neq k\}$ is the set of non-oriented edges that represent the transmission lines. Let us notice that since Σ is a tree, the cardinal of E is necessarily $m - 1$. Figure 1 shows an example of such a network presented in Kersting (1991).

We will use the notation $i \sim k$ to point out that the vertices i and k are adjacent, i.e. $\{i, k\} \in E$.

In order to enumerate the elements of E, we consider a bijection

$$\psi : \{1, \cdots, m-1\} \to E.$$

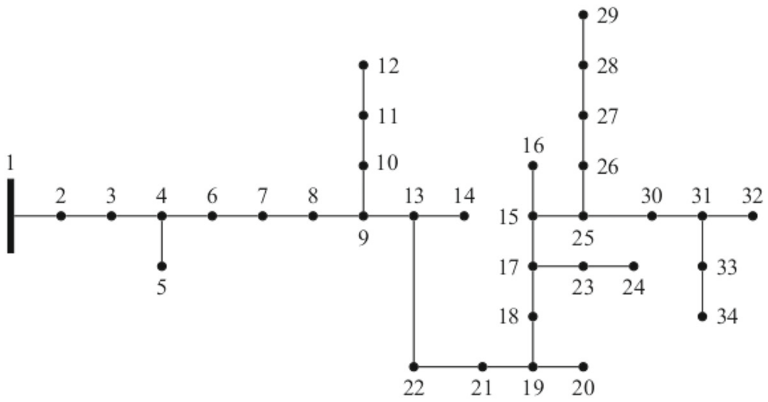


Fig. 1 Example of the graph of a distribution network of 34 nodes

In the following, we will need to consider flows in both directions in the transmission lines. To this end, for each non-oriented edge of E, we associate two oriented edges. Therefore, we define the set

$$\mathcal{E} = \left\{(i, k) \in S^2 \mid \{i, k\} \in E\right\},$$

whose cardinal is $2(m-1)$. In other words, for each edge $\{i, k\}$ in E, the set $\mathcal{E}$ contains the two couples (i, k) and (k, i).

To enumerate the elements of $\mathcal{E}$, we introduce the bijection

$$\varphi : \{1, \cdots, 2(m-1)\} \to \mathcal{E},$$

defined such that if $\psi(\ell) = \{i, k\}$, with $i < k$, then $\varphi(2\ell - 1) = (i, k)$ and $\varphi(2\ell) = (k, i)$. It means that the two orientations of the same edge are consecutive in φ, and that the order of the edges is the same in φ and in ψ.

2.2 Physical formulation

We present here the main governing equations and constraints in a AC-power distribution system. We refer to Momoh (2017) for a more detailed presentation of power engineering equations.

Let v_i be the complex voltage at node i, written as

$$v_i = V_i e^{j\theta_i}, \tag{1}$$

where V_i is the modulus of v_i, called the voltage magnitude and θ_i is the argument of v_i in $]-\pi, \pi]$, called the voltage angle. We assume in this work that the voltage magnitudes are fixed and known for each node i of the network.

To each non-oriented edge $\{i, k\} \in E$, we associate a complex admittance y_{ik}, defined by

$$y_{ik} = g_{ik} - jb_{ik},$$

where g_{ik} and b_{ik} are the conductance and the susceptance of the line $\{i, k\}$, respectively. We assume that $b_{ik} > g_{ik} > 0$. This hypothesis is classical in the literature when addressing this type of problems [see for instance Zhang et al. (2014)].

For each oriented edge $(i, k) \in \mathcal{E}$, the active power flow from node i to node k is defined by

$$F_{ik}(\theta_{ik}) = V_i^2 g_{ik} + V_i V_k \big(b_{ik} \sin(\theta_{ik}) - g_{ik} \cos(\theta_{ik})\big), \tag{2}$$

where $\theta_{ik} = \theta_k - \theta_i$.

The total amount of power that can flow through a transmission line $\{i, k\} = \psi(\ell)$ is limited by a physical capacity. This limit is called a thermal constraint and ensures that the equipment of each transmission line does not become overloaded or overheat (Conti et al. 2003; Liu et al. 2020). There are several ways to formulate this constraint [see Madani et al. (2014)], for instance

$$F_{ik} + F_{ki} \leq \overline{F}_\ell, \tag{3}$$

where $\overline{F}_\ell > 0$. The advantage of this formulation is that it can easily rewrite in terms of voltage angles. Indeed, according to (2), we have

$$F_{ik} + F_{ki} = (V_i^2 + V_k^2)g_{ik} - 2V_i V_k g_{ik} \cos(\theta_{ik}).$$

We can then distinguish three cases according to the value of $\overline{F}_\ell$:

- if $\overline{F}_\ell < (V_i - V_k)^2 g_{ik}$, the constraint (3) is not feasible;
- if $(V_i - V_k)^2 g_{ik} \leq \overline{F}_\ell < (V_i + V_k)^2 g_{ik}$, the constraint (3) is equivalent to $-\overline{\theta}_{ik} \leq \theta_{ik} \leq \overline{\theta}_{ik}$, where

$$\overline{\theta}_{ik} = \arccos\left(\frac{(V_i^2 + V_k^2)g_{ik} - \overline{F}_\ell}{2V_i V_k g_{ik}}\right);$$

- if $(V_i + V_k)^2 g_{ik} \leq \overline{F}_\ell$, the constraint (3) is automatically satisfied.

The first case is not interesting. Therefore in the following, the quantities $\overline{F}_\ell$ will be assumed to satisfy

$$\overline{F}_\ell \geq (V_i - V_k)^2 g_{ik},$$

for every edge. As a consequence, the thermal constraint rewrites as follows:

$$-\overline{\theta}_{ik} \leq \theta_{ik} \leq \overline{\theta}_{ik}, \tag{4}$$

with $\overline{\theta}_{ik} \in [0, \pi]$. Let us notice that $\overline{\theta}_{ik}$ does not depend on the direction of the line, thus $\overline{\theta}_{ki} = \overline{\theta}_{ik}$. In the following, the thermal constraint will be expressed indifferently in the form (3) or in the form (4).

For some technical reasons, we assume that

$$0 < \overline{\theta}_{ik} \leq \frac{\pi}{2}.$$

This condition is equivalent to

$$(V_i - V_k)^2 g_{ik} < \overline{F}_\ell \leq (V_i^2 + V_k^2) g_{ik}.$$

Actually, an even more restrictive assumption will be made later. The physical relevance of this restriction will then be discussed.

At each node $i \in S$, the injected active power P_i must be equal to the generated active power minus the consumed active power. These powers can also be written in terms of power flow equations as follows [see Monticelli (2012)]

$$P_i = \sum_{k \sim i} F_{ik}, \tag{5}$$

which stands as an equality constraint. Furthermore, the injected active power must satisfy the following inequality constraints

$$\underline{P}_i \leq P_i \leq \overline{P}_i, \tag{6}$$

which limit the quantity of generated or consumed power at each node Monticelli (2012).

There are many objective functions that can be considered to define the optimal power flow problem, such as the total loss in the network or the power generation cost [see Elattar and ElSayed (2019)]. Let $\boldsymbol{P} = (P_1, \cdots, P_m)^T \in \mathbb{R}^m$ be the vector of all the power injections defined by equation (5). In this work, we consider an objective function

$$J : \boldsymbol{P} \longmapsto J(\boldsymbol{P}), \tag{7}$$

strictly increasing with respect to $\mathbf{P}$, in a sense that we will be precised in Sect. 3.

Let $\boldsymbol{\theta} \in \mathbb{R}^{2(m-1)}$ be the vector containing all the phase angles of the network, such that $\boldsymbol{\theta}_\ell = \theta_{ik}$ for $\varphi(\ell) = (i, k)$. Let us notice that according to (5) and (2), and since the voltage magnitudes V_i are fixed, the active injected power P_i can be seen as a function of the phase angles θ_{ik}. In other words, $\boldsymbol{P} := \boldsymbol{P}(\boldsymbol{\theta})$ is a state variable which depends on the control variable $\boldsymbol{\theta}$. Therefore, we introduce the objective function

$$\widetilde{J}(\boldsymbol{\theta}) = J(\boldsymbol{P}(\boldsymbol{\theta})),$$

where J is the function defined by (7).

The optimal power flow problem of interest then states as follows

$$\begin{aligned} &\min_{\boldsymbol{\theta}\in\mathbb{R}^m} \widetilde{J}(\boldsymbol{\theta}), \\ s.t. \quad & -\overline{\theta}_{ik} \le \theta_{ik} \le \overline{\theta}_{ik}, \quad \forall (i,k)\in\mathcal{E}, \\ & P_i = \textstyle\sum_{k\sim i} F_{ik}, \quad \forall i \in S, \\ & \underline{P}_i \le P_i \le \overline{P}_i, \quad \forall i \in S. \end{aligned} \tag{OPF$_1$}$$

2.3 Semi-definite reformulation

The power flow equations make the problem (OPF$_1$) non-convex and therefore difficult to solve. In this section, we are going to reformulate this problem in a matrix form [see Lam et al. (2012)]. The non-convexity will then appear more clearly. On the other hand, this reformulation will also be helpful to build a convexification of the problem.

We start from expressing all the constraints of Problem (OPF$_1$) in terms of the vector $\boldsymbol{v} = (v_1, \cdots, v_m)^T$ of the complex voltages. In order to do so, we introduce the following hermitian matrices:

– for $1 \le i \le m$, the matrix $\boldsymbol{E^i}$ such that $\boldsymbol{E^i_{i,i}} = 1$ and all the other entries are equal to 0;
– the matrix $\boldsymbol{Y}$ defined by

$$\begin{cases} \forall i \in S, \quad \boldsymbol{Y}_{i,i} = \sum_{k\sim i} y_{ik}, \\ \forall (i,k)\in\mathcal{E}, \quad \boldsymbol{Y}_{i,k} = -y_{ik}, \end{cases}$$

and all other entries are equal to 0;
– for $1 \le i \le m$, the matrix $\boldsymbol{B^i}$ defined by

$$\boldsymbol{B^i} = \frac{1}{2}(\boldsymbol{Y}^*\boldsymbol{E^i} + \boldsymbol{E^i}\boldsymbol{Y});$$

– for $(i,k)\in\mathcal{E}$, the matrix $\boldsymbol{B^{ik}}$ such that

$$\begin{cases} \boldsymbol{B^{ik}_{i,i}} = g_{ik}, \\ \boldsymbol{B^{ik}_{i,k}} = -\frac{y_{ik}}{2}, \\ \boldsymbol{B^{ik}_{k,i}} = -\frac{y_{ik}}{2}, \end{cases}$$

and all other entries are equal to 0.
An immediate computation shows that $F_{ik} = \mathrm{Tr}(\boldsymbol{B^{ik}}\boldsymbol{v}\boldsymbol{v}^*)$ for all $(i,k)\in\mathcal{E}$ and $P_i = \mathrm{Tr}(\boldsymbol{B^i}\boldsymbol{v}\boldsymbol{v}^*)$, for all $i \in S$.

Now, let us consider the following optimization problem

$$\begin{aligned} \min_{\boldsymbol{W}\in\mathbb{H}_m} \quad & \widehat{J}(\boldsymbol{W}), \\ \text{s.t.} \quad & \underline{P}_i \le \operatorname{Tr}(\boldsymbol{B^i W}) \le \overline{P}_i, \quad \forall i \in S \\ & \operatorname{Tr}(\boldsymbol{B^{ik} W}) + \operatorname{Tr}(\boldsymbol{B^{ki} W}) \le \overline{F}_\ell, \quad \forall \{i,k\} = \psi(\ell) \in E, \\ & \boldsymbol{W} \succeq 0, \\ & \operatorname{rank}(\boldsymbol{W}) = 1, \end{aligned} \tag{OPF$_2$}$$

where $\mathbb{H}_m$ is the set of hermitian matrices of order m, the notation $\boldsymbol{W} \succeq 0$ means that the matrix $\boldsymbol{W}$ is positive semi-definite and

$$\widehat{J}(\boldsymbol{W}) = J(\operatorname{Tr}(\boldsymbol{B^1 W}), \cdots, \operatorname{Tr}(\boldsymbol{B^m W})).$$

Problems (OPF$_2$) and (OPF$_1$) can be linked by the following result whose proof is immediate.

Proposition 1 *Problems* (OPF$_1$) *and* (OPF$_2$) *are equivalent in the following sense:*

- *if* $\boldsymbol{\theta}$ *is a solution of* (OPF$_1$), *then noting* $\boldsymbol{v} = \boldsymbol{V} e^{j\boldsymbol{\theta}}$, *the matrix* $\boldsymbol{W} = \boldsymbol{v}\boldsymbol{v}^*$ *is a hermitian matrix of rank* 1 *and a solution of* (OPF$_2$)*;*
- *if* $\boldsymbol{W}$ *is a solution of* (OPF$_2$)*, then there exists a vector* $\boldsymbol{v} \in \mathbb{C}^m$ *such that* $\boldsymbol{W} = \boldsymbol{v}\boldsymbol{v}^*$ *and* $\boldsymbol{\theta} = \arg(\boldsymbol{v})$ *is a solution of* (OPF$_1$)*.*

Problem (OPF$_2$) is obviousely non-convex because of the rank constraint. A natural way to make it convex is therefore to remove this rank constraint, which leads to consider the following problem:

$$\begin{aligned} \min_{\boldsymbol{W}\in\mathbb{H}_m} \quad & \widehat{J}(\boldsymbol{W}), \\ \text{s.t.} \quad & \underline{P}_i \le \operatorname{Tr}(\boldsymbol{B^i W}) \le \overline{P}_i, \quad \forall i \in S, \\ & \operatorname{Tr}(\boldsymbol{B^{ik} W}) + \operatorname{Tr}(\boldsymbol{B^{ki} W}) \le \overline{F}_\ell \quad \forall \{i,k\} = \psi(\ell) \in E, \\ & \boldsymbol{W} \succeq 0. \end{aligned} \tag{$\overline{\text{OPF}_2}$}$$

This is a semi-definite program (SDP) that is much easier to solve numerically than Problem (OPF$_2$) [see for instance Bai et al. (2008)]. However, there is no guarantee that a solution of ($\overline{\text{OPF}_2}$) is of rank 1 and therefore leads to a solution of (OPF$_2$). The following of this paper is dedicated to find some conditions that ensure that this is the case and to prove it.

2.4 Set reformulation

In this section, we propose another formulation of Problem (OPF$_1$) by considering the vector of injected active power as a decision variable and by defining the constraints in terms of sets. This formulation will be more adapted to the mathematical study that will follow.

We define the following order relation: for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, we set

$$\boldsymbol{x} \leq \boldsymbol{y} \quad \Leftrightarrow \quad \forall i \in \{1, \cdots, n\},\ \boldsymbol{x}_i \leq \boldsymbol{y}_i.$$

The fluxes defined by (2) are gathered in one vector $\boldsymbol{F}(\boldsymbol{\theta}) \in \mathbb{R}^{2(m-1)}$ where the ℓ-th component is

$$\boldsymbol{F}_\ell(\boldsymbol{\theta}) = F_{ik}(\theta_{ik}), \quad \text{with } \varphi(\ell) = (i, k). \tag{8}$$

The injected active power flow vector $\boldsymbol{P}(\boldsymbol{\theta}) \in \mathbb{R}^m$ then writes

$$\boldsymbol{P}_i(\boldsymbol{\theta}) = \sum_{k \sim i} F_{ik}(\theta_{ik}). \tag{9}$$

We are now going to define a feasible set for each constraint. Concerning the injected active power constraint, we only need to introduce

$$\mathcal{P}_P = \left\{ \boldsymbol{P} \in \mathbb{R}^m,\ \underline{\boldsymbol{P}} \leq \boldsymbol{P} \leq \overline{\boldsymbol{P}} \right\}. \tag{10}$$

Next we consider the thermal constraint. For each non-oriented edge $\psi(\ell) = \{i, k\} \in E$, the feasible set for the active power flows is defined by

$$\mathcal{F}_\ell = \left\{ (F_{ik}(\theta_{ik}), F_{ki}(-\theta_{ik}))^T,\ -\overline{\theta}_{ik} \leq \theta_{ik} \leq \overline{\theta}_{ik} \right\}. \tag{11}$$

Considering the whole network, the feasible set for the active power flow reads

$$\mathcal{F} = \left\{ \boldsymbol{F}(\boldsymbol{\theta}),\ -\overline{\boldsymbol{\theta}} \leq \boldsymbol{\theta} \leq \overline{\boldsymbol{\theta}} \right\}. \tag{12}$$

We immediately see that the sets $\mathcal{F}$ and $\mathcal{F}_\ell$ are connected as follows

$$\mathcal{F} = \prod_{\ell=1}^{m-1} \mathcal{F}_\ell. \tag{13}$$

Finally, the feasible set of the injected active power for the thermal constraint is given by

$$\mathcal{P}_\theta = \left\{ \boldsymbol{P}(\boldsymbol{\theta}),\ -\overline{\boldsymbol{\theta}} \leq \boldsymbol{\theta} \leq \overline{\boldsymbol{\theta}} \right\}. \tag{14}$$

In order to link the sets $\mathcal{F}$ and $\mathcal{P}_\theta$, we introduce the connectivity matrix $\boldsymbol{A} \in M_{m,2(m-1)}(\mathbb{R})$, defined by

$$\boldsymbol{A}_{i,\ell} = \begin{cases} 1 & \text{if } \varphi(\ell) = (i, \cdot), \\ 0 & \text{otherwise,} \end{cases}$$

where the notation $\varphi(\ell) = (i, \cdot)$ means that there exists $k \in S$ such that $\varphi(\ell) = (i, k)$ (i.e. there exist $k \in S$ such that $i \sim k$). We then have

$$\boldsymbol{P}(\boldsymbol{\theta}) = \boldsymbol{A}\boldsymbol{F}(\boldsymbol{\theta}),$$

and therefore

$$\mathcal{P}_\theta = \boldsymbol{A}\mathcal{F}. \tag{15}$$

Such a construction already exists in the literature, see for instance Lavaei et al. (2012, 2013).

This leads to consider the following optimization problem

$$\begin{aligned} &\min_{P \in \mathbb{R}^m} J(\boldsymbol{P}) \\ &\text{s.t. } P \in \mathcal{P}_\theta \cap \mathcal{P}_P \end{aligned} \tag{OPF$_3$}$$

Proposition 2 *Problems* (OPF$_1$) *and* (OPF$_3$) *are equivalent in the following sense:*

- *If $\boldsymbol{\theta}$ is a solution of* (OPF$_1$)*, then $\boldsymbol{P}(\boldsymbol{\theta})$ defined by (8) and (9) is a solution of* (OPF$_3$).
- *If $\boldsymbol{P}$ is a solution of* (OPF$_3$)*, then there exists $\boldsymbol{\theta} \in \mathbb{R}^{2(m-1)}$ such that $\boldsymbol{P} = \boldsymbol{P}(\boldsymbol{\theta})$ and $\boldsymbol{\theta}$ is a solution of* (OPF$_1$).

Once again, we omit the proof that is immediate.

3 Pareto-front

In order to convexify Problem (OPF$_1$), we are going to study the geometry of the feasible set for Problem (OPF$_3$). The main tool we will use to locate potential solutions is the Pareto-front. This section is dedicated to recall this notion and its main properties.

First, we introduce the following order relation

$$\boldsymbol{x} \prec \boldsymbol{y} \quad \Leftrightarrow \quad \forall i \in \{1, \cdots, n\},\ \boldsymbol{x}_i \leq \boldsymbol{y}_i \text{ and } \exists i \in \{1, \cdots, n\},\ \boldsymbol{x}_i < \boldsymbol{y}_i.$$

We can then define the notion of Pareto-optimality.

Definition 1 Let $\mathcal{A} \subset \mathbb{R}^n$. A vector $\boldsymbol{x} \in \mathcal{A}$ is said to be Pareto-optimal in $\mathcal{A}$ if

$$\left\{\boldsymbol{y} \in \mathcal{A}, \boldsymbol{y} \prec \boldsymbol{x}\right\} = \varnothing.$$

The set of all vectors $\boldsymbol{x} \in \mathcal{A}$ which are Pareto-optimal in $\mathcal{A}$ is called the Pareto-front of $\mathcal{A}$ and is denoted by $\mathcal{O}(\mathcal{A})$.

Let us notice that the Pareto-front of a set $\mathcal{A}$ is always a subset of the topological frontier of $\mathcal{A}$.

The following definition allows to link the Pareto-front with optimization problems.

Definition 2 A function $f : \mathcal{A} \subset \mathbb{R}^n \to \mathbb{R}$ is said to be strictly increasing on $\mathcal{A}$ if for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{A}$ such that $\boldsymbol{x} \prec \boldsymbol{y}$, we have $f(\boldsymbol{x}) < f(\boldsymbol{y})$.

We then have the following classical result.

Proposition 3 *Let $f : \mathcal{A} \subset \mathbb{R}^n \to \mathbb{R}$ be a strictly increasing function. If $\boldsymbol{x}^\star = \arg\min_{\boldsymbol{x}\in\mathcal{A}} f(\boldsymbol{x})$, then $\boldsymbol{x}^\star \in \mathcal{O}(\mathcal{A})$.*

Proof By absurd, suppose that $x^\star \notin \mathcal{O}(\mathcal{A})$. Then there exists $x' \in \mathcal{A}$ such that $x' \prec x^\star$. Since f is strictly increasing, we have $f(x') < f(x^\star)$. That contradicts the optimality of $x^\star$. □

Next, we establish a result that ensures that under certain conditions, nested sets have the same Pareto-front. Up to our knowledge, this result does not exist in the literature, although it is quite elementary.

Proposition 4 *Let $\mathcal{A} \subset \mathcal{B} \subset \mathbb{R}^n$.*

1. *We have $\mathcal{A} \cap \mathcal{O}(\mathcal{B}) \subset \mathcal{O}(\mathcal{A})$.*
2. *If in addition $\mathcal{B}$ is a compact set and $\mathcal{O}(\mathcal{B}) \subset \mathcal{A}$, then $\mathcal{O}(\mathcal{A}) = \mathcal{O}(\mathcal{B})$.*

Proof

1. Let $\boldsymbol{x} \in \mathcal{A} \cap \mathcal{O}(\mathcal{B})$. If $\boldsymbol{x} \notin \mathcal{O}(\mathcal{A})$, then there exists $\boldsymbol{x}^\star \in \mathcal{A}$ such that $\boldsymbol{x}^\star \prec \boldsymbol{x}$. Since $\mathcal{A} \subset \mathcal{B}$, we have $\boldsymbol{x}^\star \in \mathcal{B}$. This contradicts $\boldsymbol{x} \in \mathcal{O}(\mathcal{B})$ and therefore we have $\mathcal{A} \cap \mathcal{O}(\mathcal{B}) \subset \mathcal{O}(\mathcal{A})$.
2. Since $\mathcal{O}(\mathcal{B}) \subset \mathcal{A}$, the statement 1. reformulates as $\mathcal{O}(\mathcal{B}) \subset \mathcal{O}(\mathcal{A})$. Conversely, let $\boldsymbol{x} \in \mathcal{O}(\mathcal{A})$ and let us suppose by absurd that $\boldsymbol{x} \notin \mathcal{O}(\mathcal{B})$. We introduce the set

$$\mathcal{C} = \mathcal{B} \cap \left\{ \boldsymbol{y} \in \mathbb{R}^n,\ \boldsymbol{y} \leq \boldsymbol{x} \right\},$$

which is compact as an intersection of a compact and a closed set. Let $f : \mathcal{C} \to \mathbb{R}$ be the function defined by $f(\boldsymbol{y}) = \sum_{i=1}^n y_i$. The function f is continuous on $\mathcal{C}$, therefore there exists $\boldsymbol{x}^\star \in \mathcal{C}$ such that

$$\boldsymbol{x}^\star = \arg\min_{\boldsymbol{y}\in\mathcal{C}} f(\boldsymbol{y}).$$

If $\boldsymbol{x}^\star$ does not belong to $\mathcal{O}(\mathcal{B})$, then there exists $\boldsymbol{x}' \in \mathcal{B}$ such that $\boldsymbol{x}' \prec \boldsymbol{x}^\star$. Hence, we have $\boldsymbol{x}' \leq \boldsymbol{x}^\star \leq \boldsymbol{x}$ and consequently $\boldsymbol{x}' \in \mathcal{C}$. In addition, f is strictly increasing, therefore we have $f(\boldsymbol{x}') < f(\boldsymbol{x}^\star)$, which contradicts the optimality of $\boldsymbol{x}^\star$. As a consequence, $\boldsymbol{x}^\star \in \mathcal{O}(\mathcal{B})$. Since $\boldsymbol{x}$ does not belong to $\mathcal{O}(\mathcal{B})$, we cannot have $\boldsymbol{x}^\star = \boldsymbol{x}$. Hence we have $\boldsymbol{x}^\star \prec \boldsymbol{x}$. Now, since $\boldsymbol{x}^\star \in \mathcal{O}(\mathcal{B}) \subset \mathcal{O}(\mathcal{A}) \subset \mathcal{A}$, this contradicts $\boldsymbol{x} \in \mathcal{O}(\mathcal{A})$. Therefore $\boldsymbol{x} \in \mathcal{O}(\mathcal{B})$ and $\mathcal{O}(\mathcal{A}) \subset \mathcal{O}(\mathcal{B})$.

□

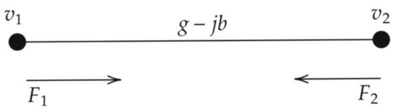


Fig. 2 Network of one transmission line

4 Distribution network of one transmission line

The objective of the following sections is to build a relevant convexification of Problem (OPF$_3$), then to prove that this convexification is exact under some conditions. This type of proof will be based on two main elements:

- the geometrical properties of the feasible sets at the local level of a transmission line;
- the tree structure of the network.

In this section, we are interested in a network constituted of only one transmission line. In a first instance, we don't take into account the injected power constraints. The purpose is to introduce some notations, to prove some geometric results on the feasible set and to come up with a convexification of the problem in this very simplified case. We will then extend these results to the whole network in the following sections.

We consider a graph constituted of a single edge $\{1, 2\}$ (see Fig. 2). For simplicity of notations, the coefficients b_{12} and g_{12} will simply be noted b and g. The active power flow F_{12} and F_{21} will respectively be noted by F_1 and F_2. Finally, the angle θ_{12} will be noted θ.

We introduce the angle

$$\alpha = \arctan\left(\frac{b}{g}\right) \in \left]0, \frac{\pi}{2}\right[. \tag{16}$$

4.1 The feasible set $\mathcal{F}$ of the active power flow

We consider the function $\boldsymbol{F}(\theta) = (F_{12}(\theta), F_{21}(\theta))^T$, where

$$\begin{aligned} F_{12}(\theta) &= V_1^2 g + V_1 V_2 b \sin(\theta) - V_1 V_2 g \cos(\theta), \\ F_{21}(\theta) &= V_2^2 g - V_1 V_2 b \sin(\theta) - V_1 V_2 g \cos(\theta). \end{aligned}$$

We define

$$\mathcal{A} = \left\{\boldsymbol{F}(\theta) \in \mathbb{R}^2, \theta \in [-\pi, \pi]\right\}.$$

Let us notice that $\mathcal{A}$ is an ellipse whose axis of symmetry is the line $y = x$, see Fig. 3 (a).

The feasible set of the active power flow is defined by

$$\mathcal{F} = \left\{\boldsymbol{F}(\theta) \in \mathbb{R}^2, \theta \in [-\bar{\theta}, \bar{\theta}]\right\}. \tag{17}$$

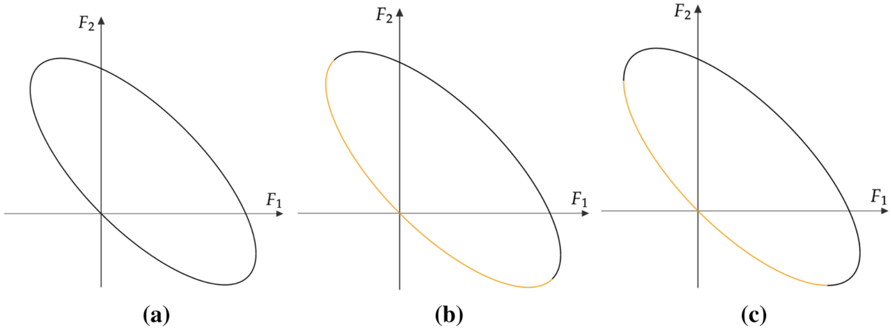


Fig. 3 **a** The set $\mathcal{A}$, **b** the set $\mathcal{F}$, **c** the set $\mathcal{O}(\mathcal{F})$

This set is an arc of the ellipse $\mathcal{A}$ (see the example in Fig. 3 (b), with $\overline{\theta} = \pi/2$).

By studying the variations of functions F_1 and F_2, we can easily see that the Pareto-front of the set $\mathcal{F}$ is

$$\mathcal{O}(\mathcal{F}) = \left\{ \boldsymbol{F}(\theta) \in \mathbb{R}^2, \theta \in \left[-\max(\overline{\theta}, \alpha), \min(\overline{\theta}, \alpha) \right] \right\}, \tag{18}$$

see Fig. 3 (c).

4.2 Convexification of the set $\mathcal{F}$

The purpose here is to build an exact convexification of the set $\mathcal{F}$ which can be used in practice. A natural idea is to consider the convex hull of $\mathcal{F}$. However, the convex hull does not have a simple algebraic characterization and is therefore difficult to use in practice for our optimization problem of interest.

We will build another convexification based on a positive semi-definite hermitian matrix formulation which has a very simple algebraic characterization.

We note $\mathbb{H}_m^+$ the set of $m \times m$ positive semi-definite hermitian matrices. We introduce the set

$$\mathcal{H} = \Big\{ \boldsymbol{W} \in \mathbb{H}_2^+, \boldsymbol{W}_{1,1} = V_1^2, \boldsymbol{W}_{2,2} = V_2^2,$$
$$-\tan(\overline{\theta}) \operatorname{Re}\left(\boldsymbol{W}_{1,2}\right) \leq \operatorname{Im}\left(\boldsymbol{W}_{1,2}\right) \leq \tan(\overline{\theta}) \operatorname{Re}\left(\boldsymbol{W}_{1,2}\right) \Big\}.$$

One can easily check that a matrix $\boldsymbol{W}$ of order 2 belongs to $\mathcal{H}$ if and only if there exists $\alpha \in [0, V_1 V_2]$ and $\theta \in [-\overline{\theta}, \overline{\theta}]$ such that

$$\boldsymbol{W} = \begin{pmatrix} V_1^2 & \alpha e^{j\theta} \\ \alpha e^{-j\theta} & V_2^2 \end{pmatrix}. \tag{19}$$

Moreover, such a matrix is of rank 1 if and only if $\alpha = V_1 V_2$.

The set $\mathcal{F}$ can then be written in terms of hermitian matrices of $\mathcal{H}$ as follows

$$\mathcal{F} = \{\operatorname{Re}(\operatorname{diag}(\boldsymbol{W}\boldsymbol{K})), \boldsymbol{W} \in \mathcal{H}, \operatorname{rank}(\boldsymbol{W}) = 1\}, \tag{20}$$

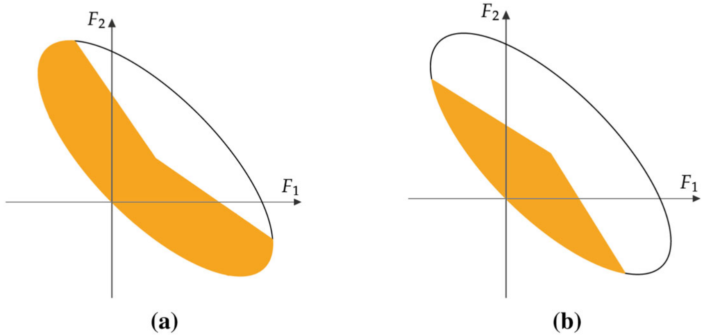


Fig. 4 **a** The set $\overline{\mathrm{conv}}(\mathcal{F})$ with $\overline{\theta} > \frac{\pi}{2}$, **b** The set $\overline{\mathrm{conv}}(\mathcal{F})$ with $\overline{\theta} \leq \frac{\pi}{2}$

where the matrix $\boldsymbol{K}$ is defined by

$$\boldsymbol{K} = \begin{pmatrix} g + jb & -g - jb \\ -g - jb & g + jb \end{pmatrix}.$$

By removing the rank constraint in (20), we obtain a convexification of $\mathcal{F}$, i.e. a convex set containing $\mathcal{F}$, which will be denoted by

$$\overline{\mathrm{conv}}(\mathcal{F}) = \{\mathrm{Re}(\mathrm{diag}(\boldsymbol{W}\boldsymbol{Y})), \boldsymbol{W} \in \mathcal{H}\}. \tag{21}$$

Let us notice that thanks to the hypothesis $\overline{\theta} \leq \frac{\pi}{2}$, the set $\overline{\mathrm{conv}}(\mathcal{F})$ is actually convex. Indeed, this set is a sector of an ellipse contained in a half-ellipse, see Fig. 4. Without this hypothesis, the set $\overline{\mathrm{conv}}(\mathcal{F})$ would not be convex (except in the case $\overline{\theta} = \pi$ where $\overline{\mathrm{conv}}(\mathcal{F})$ is the entire ellipse).

Once again, the study of variations of F_1 and F_2 shows immediately that $\mathcal{O}(\overline{\mathrm{conv}}(\mathcal{F})) = \mathcal{O}(\mathcal{F})$. This can also be seen in Fig. 4. This result will be later extended to the whole network.

4.3 Technical results

We present here some technical results which will be used in the following and which are obtained by making a more restrictive assumption on $\overline{\theta}$.

First, let us notice that according to Eqs. (17) and (18), if $\overline{\theta} \leq \alpha$, we have

$$\mathcal{O}(\overline{\mathrm{conv}}(\mathcal{F})) = \mathcal{O}(\mathcal{F}) = \mathcal{F}. \tag{22}$$

Now we present a characterization of the elements of $\overline{\mathrm{conv}}(\mathcal{F})\backslash\mathcal{F}$.

Lemma 1 *Let us assume that* $\overline{\theta} \leq \alpha$. *Let* $(x, y) \in \overline{\mathrm{conv}}(\mathcal{F})\backslash\mathcal{F}$. *Then there exists* $\delta > 0$ *such that* $(x - \delta, y) \in \mathcal{F}$.

In addition, for all $\varepsilon \in [0, \delta]$, *we have* $(x - \varepsilon, y) \in \overline{\mathrm{conv}}(\mathcal{F})$.

Proof Since $\overline{\theta} \leq \alpha$, we have $\mathcal{O}(\overline{\text{conv}}(\mathcal{F})) = \mathcal{F}$. The function F_2 is strictly decreasing on $[-\overline{\theta}, \overline{\theta}]$, therefore there exists a unique $\theta \in [-\overline{\theta}, \overline{\theta}]$ such that $F_2(\theta) = y$. Since $(x, y) \notin \mathcal{F}$, we cannot have $F_1(\theta) = x$. If $F_1(\theta) > x$, then we have $(x, y) \prec \boldsymbol{F}(\theta)$, which contradicts $\mathcal{O}(\overline{\text{conv}}(\mathcal{F})) = \mathcal{F}$. Therefore, we have necessarily $F_1(\theta) < x$. By setting $\delta = x - F_1(\theta)$, we have $(x - \delta, y) = \boldsymbol{F}(\theta) \in \mathcal{F}$.

The second statement is immediate by convexity of $\overline{\text{conv}}(\mathcal{F})$. □

Finally, we prove a property that allows to reverse the sense of inequalities between the two components of vectors of $\mathcal{F}$ and $\overline{\text{conv}}(\mathcal{F})$.

Lemma 2 *We assume that $\overline{\theta} \leq \alpha$. Let $(x^\star, y^\star) \in \mathcal{F}$ and $(x, y) \in \overline{\text{conv}}(\mathcal{F})$. Then, we have*

1. *if $x \leq x^\star$, then $y \geq y^\star$;*
2. *if $x < x^\star$, then $y > y^\star$;*
3. *if $y \leq y^\star$, then $x \geq x^\star$;*
4. *if $y < y^\star$, then $x > x^\star$.*

Proof Since $\overline{\theta} \leq \alpha$, thanks to (22), we have $(x^\star, y^\star) \in \mathcal{O}(\overline{\text{conv}}(\mathcal{F}))$. If $x \leq x^\star$ and $y < y^\star$, then we have $(x, y) \prec (x^\star, y^\star)$, which contradicts $(x^\star, y^\star) \in \mathcal{O}(\overline{\text{conv}}(\mathcal{F}))$. The other statements can be proved in a similar way. □

4.4 Consideration of injected active power constraints

Here, we consider in addition the constraints

$$\underline{P}_i \leq F_i(\theta) \leq \overline{P}_i, \;\; i \in \{1, 2\}.$$

The feasible set is then written in the form $\mathcal{F} \cap \mathcal{F}_P$ where $\mathcal{F}$ is defined by (17) and

$$\mathcal{F}_P = \left\{ \boldsymbol{F} \in \mathbb{R}^2, \;\; \underline{\boldsymbol{P}} \leq \boldsymbol{F} \leq \overline{\boldsymbol{P}} \right\}.$$

Note that since $\mathcal{F}_P$ is convex, the set $\overline{\text{conv}}(\mathcal{F}) \cap \mathcal{F}_P$ is a simple convexification of $\mathcal{F} \cap \mathcal{F}_P$. In order to prove that this convexification is exact, we would like to have

$$\mathcal{O}(\mathcal{F} \cap \mathcal{F}_P) = \mathcal{O}(\overline{\text{conv}}(\mathcal{F}) \cap \mathcal{F}_P). \tag{23}$$

However, we can easily see that this equality is not always satisfied. To illustrate this failure, let us consider the following setting $\overline{\theta} = \pi/2$, $b = 5$, $g = 2$ and $V_1 = V_2 = 1$. Figure 5 shows different scenarios for the set $\mathcal{F} \cap \mathcal{F}_P$, depending on the values of $\underline{P}_i$ and $\overline{P}_i$. In the cases (a) and (b), the sets $\mathcal{O}(\overline{\text{conv}}(\mathcal{F}) \cap \mathcal{F}_P)$ and $\mathcal{O}(\mathcal{F} \cap \mathcal{F}_P)$ are both equal to $\mathcal{F} \cap \mathcal{F}_P$ and so (23) holds. However, it is not true in case (c), as we can see more clearly in Fig. 6.

We observe that those problematic cases only appear when some elements of $\mathcal{F}$ are not Pareto-optimal and the values of $\underline{P}_i$ are large enough. To avoid this, a classical assumption is to suppose that all the points of $\mathcal{F}$ are Pareto-optimal in $\mathcal{F}$ [see for

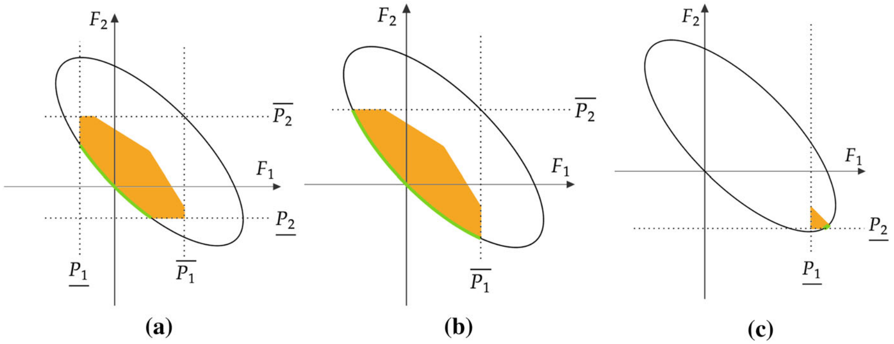


Fig. 5 Some possible cases for the set $\mathcal{F} \cap \mathcal{F}_P$; the orange surface corresponds to $\overline{\text{conv}}(\mathcal{F}) \cap \mathcal{F}_P$; the green curve corresponds to $\mathcal{F} \cap \mathcal{F}_P$

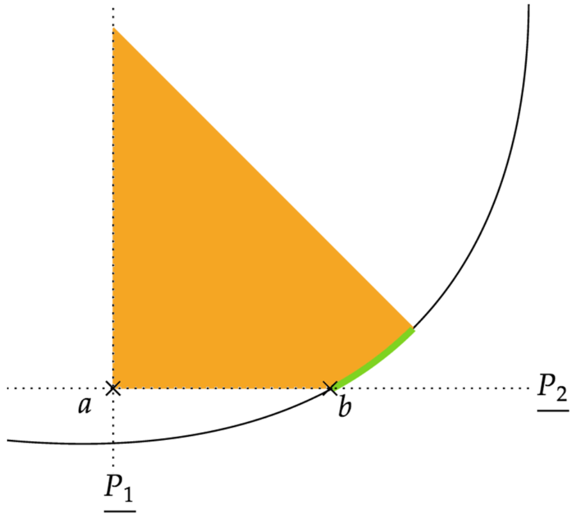


Fig. 6 Zoom of the figure 5 (c); we have $\mathcal{O}(\overline{\text{conv}}(\mathcal{F}) \cap \mathcal{F}_P) = \{a\}$ and $\mathcal{O}(\mathcal{F} \cap \mathcal{F}_P) = \{b\}$

instance Lavaei et al. (2013)]. This assumption is equivalent to $\overline{\theta} \leq \alpha$. In Lavaei et al. (2013); Huang et al. (2019), authors explain that this assumption does always hold in practice for distribution networks.

Let us notice that under this condition, equality (23) does not necessary hold. But it does not hold only when the feasible set $\mathcal{F} \cap \mathcal{F}_P$ is empty, as we can see in Fig. 7. This alternative is sufficient to apply in the framework of our optimization problem.

In the following section, we generalize and prove relation (23) in the case of a general network of $m \geq 2$ nodes.

5 General distribution network

We start by generalizing the definitions and notations of Sect. 4.2 to the case of a whole network.

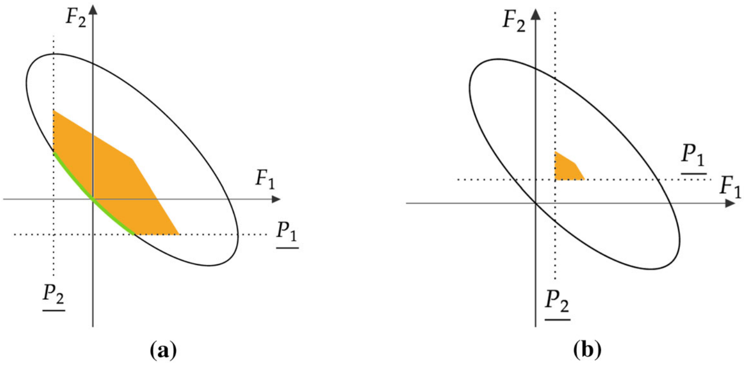


Fig. 7 **a** The relation $\mathcal{O}(\mathcal{F} \cap \mathcal{F}_P) = \mathcal{O}(\overline{\text{conv}}(\mathcal{F}) \cap \mathcal{F}_P)$ holds, **b** the set $\mathcal{F} \cap \mathcal{F}_P$ is empty

For each non-oriented edge $\psi(\ell) = \{i, k\}$, we reproduce the convexification of the power flow feasible set $\mathcal{F}_\ell$, defined by (11). We thus introduce

$$\overline{\text{conv}}(\mathcal{F}_\ell) = \{\text{Re}(\text{diag}(\boldsymbol{W}\boldsymbol{K}^{\boldsymbol{ik}})),\ \boldsymbol{W} \in \mathcal{H}_{ik}\}, \tag{24}$$

where the set $\mathcal{H}_{ik}$ is defined by

$$\mathcal{H}_{ik} = \Big\{\boldsymbol{W} \in \mathbb{H}_2^+,\ \boldsymbol{W}_{1,1} = V_i^2,\ \boldsymbol{W}_{2,2} = V_k^2,$$
$$-\tan(\overline{\theta}_{ik})\,\text{Re}(\boldsymbol{W}_{1,2}) \leq \text{Im}(\boldsymbol{W}_{1,2}) \leq \tan(\overline{\theta}_{ik})\,\text{Re}(\boldsymbol{W}_{1,2})\Big\},$$

and the matrix $\boldsymbol{K}^{\boldsymbol{ik}}$ is given by

$$\boldsymbol{K}^{\boldsymbol{ik}} = \begin{pmatrix} g_{ik} + jb_{ik} & -g_{ik} - jb_{ik} \\ -g_{ik} - jb_{ik} & g_{ik} + jb_{ik} \end{pmatrix}.$$

According to relations (13) and (15), it is natural to define the convexification of the sets $\mathcal{F}$ and $\mathcal{P}_\theta$ by

$$\overline{\text{conv}}(\mathcal{F}) = \prod_{\ell=1}^{m-1} \overline{\text{conv}}(\mathcal{F}_\ell) \tag{25}$$

and

$$\overline{\text{conv}}(\mathcal{P}_\theta) = \boldsymbol{A}\,\overline{\text{conv}}(\mathcal{F}). \tag{26}$$

The sets $\overline{\text{conv}}(\mathcal{F})$ and $\overline{\text{conv}}\,(\mathcal{P}_\theta)$ are convex as a product of convex sets and the image by a linear map of a convex set.

Those definitions lead us to consider the following optimization problem

$$\begin{aligned} \min_{\boldsymbol{P}\in\mathbb{R}^m} \; & J(\boldsymbol{P}) \\ \text{s.t. } & \boldsymbol{P} \in \overline{\text{conv}}(\mathcal{P}_\theta) \cap \mathcal{P}_P, \end{aligned} \tag{$\overline{\text{OPF}_3}$}$$

where $\mathcal{P}_P$ is defined by (10).

As stated previously, we obtain ($\overline{\text{OPF}_2}$) as a convexification of (OPF_2). On the other hand, Problems (OPF_2) and (OPF_3) are both equivalent to (OPF_1). Moreover, ($\overline{\text{OPF}_2}$) and ($\overline{\text{OPF}_3}$) are equivalent as stated in the following result, whose proof is straightforward and not given here.

Proposition 5 *Problems ($\overline{\text{OPF}_2}$) and ($\overline{\text{OPF}_3}$) are equivalent in the following sense:*

- *If $\boldsymbol{P}$ is a solution of ($\overline{\text{OPF}_3}$), then for every edge $\{i, k\} \in E$, there exists $\alpha_{ik} \in [0, V_i V_k]$ and $\theta_{ik} \in [-\overline{\theta}_{ik}, \overline{\theta}_{ik}]$ such that we have*

$$P_i = \sum_{k\sim i}(V_i^2 g_{ik} + \alpha_{ik} b_{ik} \sin(\theta_{ik}) - \alpha_{ik} g_{ik} \cos(\theta_{ik})).$$

  The matrix $\boldsymbol{W}$ defined by

$$\begin{cases} \boldsymbol{W}_{i,i} = V_i^2, & \text{if } i \in S, \\ \boldsymbol{W}_{i,k} = \alpha_{ik} e^{j\theta_{ik}}, & \text{if } (i,k) \in \mathcal{E}, \\ \boldsymbol{W}_{i,k} = 0, & \text{else}, \end{cases} \tag{27}$$

  is then a solution of ($\overline{\text{OPF}_2}$).
  In addition, if $\boldsymbol{P} \in \mathcal{P}_\theta$, then the matrix $\boldsymbol{W}$ is of rank 1.
- *If $\boldsymbol{W}$ is a solution of ($\overline{\text{OPF}_2}$), then by defining*

$$P_i = \text{Tr}(\boldsymbol{B^i} \boldsymbol{W}), \tag{28}$$

  where $\boldsymbol{B^i}$ is defined by (2.3), the vector $\boldsymbol{P}$ is a solution of ($\overline{\text{OPF}_3}$).
  In addition, if the matrix $\boldsymbol{W}$ is of rank 1, then $\boldsymbol{P} \in \mathcal{P}_\theta$. In this case, there exists $\boldsymbol{v} \in \mathbb{C}^m$ such that $\boldsymbol{W} = \boldsymbol{v}\boldsymbol{v}^*$ and we have $\boldsymbol{P} = \boldsymbol{P}(\arg(\boldsymbol{v}))$.

In order to prove that the convexification introduced earlier is exact, we need the following assumption.

Assumption $\mathbf{A_1}$ *For every edge $\{i, k\} \in E$, we have*

$$\overline{\theta}_{ik} \leq \arctan\left(\frac{b_{ik}}{g_{ik}}\right).$$

As introduced in the previous section, the convexification of the feasible set $\mathcal{P}$ is $\overline{\text{conv}}\,(\mathcal{P}_\theta) \cap \mathcal{P}_P$, where $\overline{\text{conv}}(\mathcal{P}_\theta)$ is defined by (26). In this section, in a first instance, we establish that $\mathcal{O}\,(\overline{\text{conv}}\,(\mathcal{P}_\theta) \cap \mathcal{P}_P) = \mathcal{O}(\mathcal{P})$, under certain conditions that will be precised. This result is difficult to prove and will require some preliminary results. We will then deduce the main theorem of this paper, which ensures that the convexification ($\overline{\text{OPF}_2}$) is exact in this framework.

As a first preliminary result, we give a characterization of the vectors of $\mathcal{O}(\overline{\text{conv}}(\mathcal{P}_\theta) \cap \mathcal{P}_P)$.

Lemma 3 *Let us suppose that Assumption A_1 holds. Let $\boldsymbol{P} \in \mathcal{O}(\overline{\text{conv}}(\mathcal{P}_\theta) \cap \mathcal{P}_P)$ and $\boldsymbol{F} \in \overline{\text{conv}}(\mathcal{F})$ be such that $\boldsymbol{P} = \boldsymbol{AF}$.*

If a vertex $i \in S$ satisfies $P_i > \underline{P}_i$, then for every vertex $k \in S$ such that $k \sim i$, we have $(F_{ik}, F_{ki}) \in \mathcal{F}_\ell$, with $\psi(\ell) = \{i, k\}$ and $i < k$.

Proof By absurd, we suppose that $P_i > \underline{P}_i$ and that there exists $k \in S$ with $k \sim i$, such that $(F_{ik}, F_{ki}) \notin \mathcal{F}_\ell$. We will construct a vector $\boldsymbol{P}^\star \in \overline{\text{conv}}\,(\mathcal{P}_\theta) \cap \mathcal{P}_P$ such that $\boldsymbol{P}^\star \prec \boldsymbol{P}$. This will contradict the assumption that $\boldsymbol{P}$ belongs to the Pareto-front of this set.

We have $(F_{ik}, F_{ki}) \in \overline{\text{conv}}\,(\mathcal{F}_\ell) \setminus \mathcal{F}_\ell$, then according to Lemma 1, there exists $\delta > 0$ such that $(F_{ik} - \delta, F_{ki}) \in \mathcal{F}_\ell$ and for all $\varepsilon \in [0, \delta]$, we have $(F_{ik} - \varepsilon, F_{ki}) \in \overline{\text{conv}}\,(\mathcal{F}_\ell)$.

We set $\varepsilon = \min\left(\delta, P_i - \underline{P}_i\right) > 0$. Let $\boldsymbol{F}^\star$ be defined by

$$F^\star_{qr} = \begin{cases} F_{qr} & \text{if } (q, r) \neq (i, k), \\ F_{ik} - \varepsilon & \text{if } (q, r) = (i, k), \end{cases}$$

and $\boldsymbol{P}^\star = \boldsymbol{AF}^\star$. Then we have

$$P^\star_i = \sum_{n \sim i} F^\star_{in} = F_{ik} - \varepsilon + \sum_{\substack{n \sim i \\ n \neq k}} F_{in} = P_i - \varepsilon$$

and for all $n \neq i$, we have $P^\star_n = P_n$. We deduce that $\boldsymbol{P}^\star \prec \boldsymbol{P}$.

Now, let us prove that $\boldsymbol{P}^\star \in \overline{\text{conv}}\,(\mathcal{P}_\theta) \cap \mathcal{P}_P$. We have $\boldsymbol{F}^\star \in \overline{\text{conv}}(\mathcal{F})$, therefore $\boldsymbol{P}^\star \in \overline{\text{conv}}\,(\mathcal{P}_\theta)$. On the other hand, since $\underline{P}_i \leq P_i - \varepsilon = P^\star_i < P_i \leq \overline{P}_i$, we deduce $\boldsymbol{P}^\star \in \mathcal{P}_P$.

Hence, we have proved that $\boldsymbol{P}^\star \in \overline{\text{conv}}\,(\mathcal{P}_\theta) \cap \mathcal{P}_P$. □

The following Lemma allows to find a condition for a vector of $\overline{\text{conv}}(\mathcal{F})$ to be in $\mathcal{F}$. This result relies on the tree structure of the network.

Lemma 4 *Let us suppose that assumption A_1 holds. Let $\boldsymbol{F}^\star \in \mathcal{F}$ and $\boldsymbol{F} \in \overline{\text{conv}}(\mathcal{F})$ such that for every $i \in S$, we have*

$$\sum_{k \sim i} F_{ik} \leq \sum_{k \sim i} F^\star_{ik}. \tag{29}$$

Then, we have $F_{ik} = F^\star_{ik}$, for every $(i, k) \in \mathcal{E}$.

Proof We fix arbitrarily a root r of the tree Σ. Let us notice that for a vertex $i \in S\backslash\{r\}$, of parent k, relation (29) writes

$$F_{ik} - F_{ik}^{\star} \leq \sum_{n \text{ child of } i} \left(F_{in}^{\star} - F_{in}\right). \tag{30}$$

First, we show by a bottom-up induction, that for every vertex $i \in S\backslash\{r\}$, of parent k, we have $F_{ik} \leq F_{ik}^{\star}$. Let $i \in S\backslash\{r\}$, of parent k.

- If i is an external vertex of Σ (i.e. i has no child), then (30) implies directly that $F_{ik} \leq F_{ik}^{\star}$.
- If i is an internal vertex of Σ (i.e. i has at least one child), such that for every child n of i, we have $F_{ni} \leq F_{ni}^{\star}$, then Lemma 2 ensures that $F_{in} \geq F_{in}^{\star}$. Relation (30) then implies that $F_{ik} \leq F_{ik}^{\star}$.

The property is then true for every vertex $i \in S\backslash\{r\}$. Next, we show by a top-down induction that for every internal vertex $i \in S$ and for every children n of i, we have $F_{in} = F_{in}^{\star}$ and $F_{ni} = F_{ni}^{\star}$.

- If i is the root r of Σ, relation (30) writes

$$\sum_{n \text{ child of } i} \left(F_{in}^{\star} - F_{in}\right) \geq 0.$$

  From the above, we know that for every child n of i, we have $F_{ni} \leq F_{ni}^{\star}$. Therefore, according to Lemma 2, we have $F_{in} \geq F_{in}^{\star}$. We deduce that $F_{in} = F_{in}^{\star}$. Lemma 2 then implies that $F_{ni} \geq F_{ni}^{\star}$, therefore $F_{ni} = F_{ni}^{\star}$.
- If i is not the root and its parent k satisfies $F_{ki} = F_{ki}^{\star}$, then according to Lemma 2, we have $F_{ik} \geq F_{ik}^{\star}$. From the above, for every child n of i, we have $F_{ni} \leq F_{ni}^{\star}$, therefore according to Lemma 2, $F_{in} \geq F_{in}^{\star}$. Relation (30) then writes

$$0 \leq F_{ik} - F_{ik}^{\star} \leq \sum_{n \text{ child of } i} \left(F_{in}^{\star} - F_{in}\right) \leq 0$$

  We deduce that $F_{in} = F_{in}^{\star}$. Lemma 2 ensures that $F_{ni} \geq F_{ni}^{\star}$, hence $F_{ni} = F_{ni}^{\star}$.

We have thus proven that for every $(i, k) \in \mathcal{E}$, we have $F_{ik} = F_{ik}^{\star}$. □

Thanks to these lemmas, we can now prove that the feasible set $\mathcal{P}_\theta \cap \mathcal{P}_P$ and its convexification $\overline{\text{conv}}\,(\mathcal{P}_\theta) \cap \mathcal{P}_P$ share the same Pareto-front, provided the set $\mathcal{P}_\theta \cap \mathcal{P}_P$ is non-empty.

Theorem 5 *We suppose that Assumption A_1 holds. If $\mathcal{P}_\theta \cap \mathcal{P}_P \neq \emptyset$, then we have*

$$\mathcal{O}(\mathcal{P}_\theta \cap \mathcal{P}_P) = \mathcal{O}\left(\overline{\text{conv}}\,(\mathcal{P}_\theta) \cap \mathcal{P}_P\right).$$

Proof If $\mathcal{O}\left(\overline{\text{conv}}\left(\mathcal{P}_\theta\right)\cap\mathcal{P}_P\right)\subset\mathcal{P}_\theta\cap\mathcal{P}_P$, then Proposition 4 ensures that $\mathcal{O}(\mathcal{P}_\theta\cap\mathcal{P}_P)=\mathcal{O}\left(\overline{\text{conv}}\left(\mathcal{P}_\theta\right)\cap\mathcal{P}_P\right)$. As a consequence, it is sufficient to prove the first inclusion.

Let $\boldsymbol{P}\in\mathcal{O}\left(\overline{\text{conv}}\left(\mathcal{P}_\theta\right)\cap\mathcal{P}_P\right)$ and $\boldsymbol{F}\in\overline{\text{conv}}(\mathcal{F})$ such that $\boldsymbol{P}=\boldsymbol{A}\boldsymbol{F}$. By absurd, we suppose that $\boldsymbol{P}\notin\mathcal{P}_\theta\cap\mathcal{P}_P$. The proof will consist in constructing a vector $\boldsymbol{P}'\in\overline{\text{conv}}\left(\mathcal{P}_\theta\right)\cap\mathcal{P}_P$ such that $\boldsymbol{P}'\prec\boldsymbol{P}$, which will contradict the assumption.

Since $\boldsymbol{P}\notin\mathcal{P}_\theta$, there exists an edge that we note $\{1,2\}=\psi(1)\in E$, such that $(F_{12},F_{21})\notin\mathcal{F}_1$. Let us notice that Lemma 3 ensures that $P_1=\underline{P}_1$ and $P_2=\underline{P}_2$.

By removing the edge $\{1,2\}$ of the graph Σ, we obtain two connected components. We note S_1 and S_2 the set of vertices of the connected component containing the vertex 1 and 2, respectively.

Finally, since $\mathcal{P}_\theta\cap\mathcal{P}_P\neq\varnothing$, there exists $\boldsymbol{P}^\star\in\mathcal{P}_\theta\cap\mathcal{P}_P$ and $\boldsymbol{F}^\star\in\mathcal{F}$ such that $\boldsymbol{P}^\star=\boldsymbol{A}\boldsymbol{F}^\star$.

1st step: there exists a vertex $i\in S$ such that $P_i>\underline{P}_i$.
Let us suppose this is not the case, in other words that every vertex $i\in S$ satisfies $P_i=\underline{P}_i$. Since $\underline{P}_i\leq P_i^\star$, we deduce that for every $i\in S$, we have

$$\sum_{k\sim i}F_{ik}\leq\sum_{k\sim i}F_{ik}^\star.$$

Lemma 4 then ensures that $F_{ik}=F_{ik}^\star$ for every oriented edge $(i,k)\in\mathcal{E}$. Therefore, we have $(F_{12},F_{21})=\left(F_{12}^\star,F_{21}^\star\right)\in\mathcal{F}_1$, which is impossible.

2nd step: determination of the part of the graph where we will modify P.
We call a feasible path every path of vertices $q_1\sim\cdots\sim q_r$ such that

- $q_1\in\{1,2\}$;
- $q_2\notin\{1,2\}$;
- $P_{q_r}>\underline{P}_{q_r}$;
- for all $1\leq i\leq r-1$, we have $P_i=\underline{P}_i$.

We consider the sub-graph $\widetilde{\Sigma}=(\widetilde{S},\widetilde{E})$ of Σ, connected and maximal, such that every vertex $i\in\widetilde{S}$ satisfying $P_i>\underline{P}_i$ belongs to a feasible path in $\widetilde{\Sigma}$. In other words, by browsing the graph from 1 or 2, as soon as we meet a vertex such that $P_i>\underline{P}_i$, we remove all its descendants. The feasible paths and the construction of the graph $\widetilde{\Sigma}$ is illustrated for a simple example in Fig. 8.

Let us notice that $\widetilde{\Sigma}$ admits two types of leaves:

- the leaves i satisfying $P_i=\underline{P}_i$, in which case, this is a leaf from the original tree Σ;
- the leaves i satisfying $P_i>\underline{P}_i$, in which case, this is not necessarily a leaf of the original tree Σ; according to the first step and by maximality of $\widetilde{\Sigma}$, there exists at least one leaf of this type in $\widetilde{\Sigma}$.

The vertices $i\in\widetilde{\Sigma}$ which are not leaves necessarily satisfy $P_i=\underline{P}_i$.

We now prove that there exists a feasible path $q_1\sim\cdots\sim q_r$ in $\widetilde{\Sigma}$ such that for all $i=1,\cdots,r-1$, we have $F_{q_iq_{i+1}}<F_{q_iq_{i+1}}^\star$. By absurd, let us suppose that for all feasible path in $\widetilde{\Sigma}$, there exists $s\in\{1,\cdots,r-1\}$ such that $F_{q_sq_{s+1}}\geq F_{q_sq_{s+1}}^\star$.

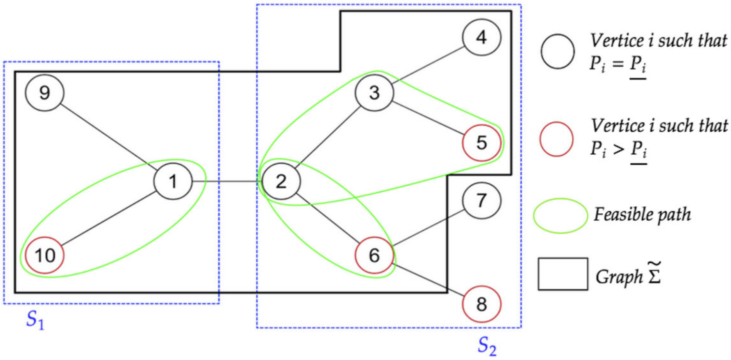


Fig. 8 Example of graph $\widetilde{\Sigma}$ and the feasible paths

We consider then the subgraph $\widehat{\Sigma} = (\widehat{S}, \widehat{E})$ of $\widetilde{\Sigma}$ in which we have removed all the vertices beyond q_{s+1} in the feasible paths.

Let us consider a leaf i in $\widehat{\Sigma}$, whose only neighbor is noted by k. We have two possibilities:

- either i corresponds to a q_{s+1} described previously and then satisfies $F_{ki} \geq F^\star_{ki}$;
- or i is also a leaf of $\widetilde{\Sigma}$ satisfying $P_i = \underline{P}_i$. It is therefore a leaf from the original tree Σ. As a consequence, we have $P_i \leq P^\star_i$, thus $F_{ik} \leq F^\star_{ik}$. Lemma 2 then ensures that $F_{ki} \geq F^\star_{ki}$.

We consider now the vertex 1 as the root of $\widehat{\Sigma}$. For an internal vertex $i \in \widehat{S} \cap S_2$, of parent k, we prove by a bottom-up induction that $F_{ik} \leq F^\star_{ik}$.

- If all the children of i are external vertices of $\widehat{\Sigma}$: since i is an internal vertex of $\widehat{\Sigma}$ different from the root, it is not a leaf of $\widehat{\Sigma}$. Therefore, we have $P_i = \underline{P}_i$, thus

$$F_{ik} + \sum_{n \text{ child of } i} F_{in} \leq F^\star_{ik} + \sum_{n \text{ child of } i} F^\star_{in},$$

and then

$$F_{ik} - F^\star_{ik} \leq \sum_{n \text{ child of } i} \left(F^\star_{in} - F_{in}\right).$$

A child n of i, which is an external vertex by assumption, and therefore a leaf of $\widehat{\Sigma}$, satisfies according to the above $F_{in} \geq F^\star_{in}$, thus we have $F_{ik} \leq F^\star_{ik}$.
- If all the children n of i satisfy $F_{ni} \leq F^\star_{ni}$, then Lemma 2 ensures that $F_{in} \geq F^\star_{in}$ and we deduce in the same fashion that $F_{ik} \leq F^\star_{ik}$.

The result is then true for all the vertices of $\widehat{S} \cap S_2$ and in particular for the vertex 2, therefore $F_{21} \leq F^\star_{21}$.

The same result can be obtained by considering the vertex 2 as a root and the set $\widehat{S} \cap S_1$. We deduce that $F_{12} \leq F^\star_{12}$. Lemma 2 then ensures that $F_{21} \geq F^\star_{21}$

and $F_{12} \geq F^\star_{12}$, therefore $(F_{12}, F_{21}) = \left(F^\star_{12}, F^\star_{21}\right)$, which contradicts the fact that $(1, 2) \notin \mathcal{F}_1$.

It follows that there exists a feasible path $q_1 \sim \cdots \sim q_r$ in $\widetilde{\Sigma}$ such that for all $i = 1, \cdots, r-1$, we have $F_{q_i q_{i+1}} < F^\star_{q_i q_{i+1}}$ and thus $F_{q_{i+1} q_i} > F^\star_{q_{i+1} q_i}$ according to Lemma 2. We will modify $\boldsymbol{P}$ on the vertices of this feasible path.

3rd step: construction of $\boldsymbol{P}'$.

Without loss of generality, we assume that $q_1 = 1$. We set

$$
\begin{aligned}
F'_{12} &= F_{12} - \delta, \\
F'_{21} &= F_{21}, \\
F'_{q_i q_{i+1}} &= \lambda_i F_{q_i q_{i+1}} + (1 - \lambda_i)\, F^\star_{q_i q_{i+1}}, \quad \forall 1 \leq i \leq r-1, \\
F'_{q_{i+1} q_i} &= \lambda_i F_{q_{i+1} q_i} + (1 - \lambda_i)\, F^\star_{q_{i+1} q_i}, \quad \forall 1 \leq i \leq r-1,
\end{aligned}
$$

where $\delta > 0$ and the numbers $\lambda_i \in]0, 1[$ will be fixed in the following. For every other line $\{i, k\}$ of E, we set $F'_{ik} = F_{ik}$ and $F'_{ki} = F_{ki}$.

We then define $\boldsymbol{P}' = \boldsymbol{A}\boldsymbol{F}'$ and we have

$$
\begin{aligned}
P'_1 - P_1 &= -\delta + (1 - \lambda_1)\left(F^\star_{q_1 q_2} - F_{q_1 q_2}\right), \\
P'_{q_r} - P_{q_r} &= (1 - \lambda_{r-1})\left(F^\star_{q_r q_{r-1}} - F_{q_r q_{r-1}}\right),
\end{aligned}
$$

and for $2 \leq i \leq r-1$,

$$
P'_{q_i} - P_{q_i} = (1 - \lambda_{i-1})\left(F^\star_{q_i q_{i-1}} - F_{q_i q_{i-1}}\right) + (1 - \lambda_i)\left(F^\star_{q_i q_{i+1}} - F_{q_i q_{i+1}}\right).
$$

The objective is to have $\boldsymbol{P}' \prec \boldsymbol{P}$ and $\boldsymbol{P}' \in \mathcal{P}_P$, thus $\underline{P}_i \leq P'_i \leq P_i$ for every $i \in S$. The vertices q_i, for $1 \leq i \leq r-1$, satisfy $P_i = \underline{P}_i$, therefore we must enforce $P'_i = P_i$. This implies that

$$
\begin{aligned}
\lambda_1 &= 1 - \frac{\delta}{F^\star_{q_1 q_2} - F_{q_1 q_2}}, \\
\lambda_i &= 1 - (1 - \lambda_{i-1})\,\frac{F_{q_i q_{i-1}} - F^\star_{q_i q_{i-1}}}{F^\star_{q_i q_{i+1}} - F_{q_i q_{i+1}}}, \quad \text{for } 2 \leq i \leq r-1.
\end{aligned}
$$

Let us notice that for δ small enough, λ_i belongs to $]0, 1[$ and can be as close to 1 as we want.

Since $F^\star_{q_r q_{r-1}} - F_{q_r q_{r-1}} < 0$, we deduce that $P'_{q_r} < P_{q_r}$. By construction, all the other vertices $i \in \Sigma \setminus \{q_r\}$ satisfy $P'_i = P_i$, thus we have $\boldsymbol{P}' \prec \boldsymbol{P}$.

4th step: verification that $\boldsymbol{P}' \in \overline{\text{conv}}\,(\mathcal{P}_\theta) \cap \mathcal{P}_P$.

Let ℓ_i be such that $\psi(\ell_i) = \{q_i, q_{i+1}\}$. We first notice that by convexity of $\overline{\text{conv}}\left(\mathcal{F}_{\ell_i}\right)$, we have $\left(F'_{q_i q_{i+1}}, F'_{q_{i+1} q_i}\right) \in \overline{\text{conv}}\left(\mathcal{F}_{\ell_i}\right)$ for all $1 \leq i \leq r-1$. On the other hand, since $(F_{12}, F_{21}) \in \overline{\text{conv}}\,(\mathcal{F}_1) \setminus \mathcal{F}_1$, Lemma 3 ensures that $\left(F'_{12}, F'_{21}\right)$ belongs to $\overline{\text{conv}}\,(\mathcal{F}_1)$

at least for δ small enough. Therefore, for δ small enough, we have $\boldsymbol{P}' \in \overline{\text{conv}}\,(\mathcal{P}_\theta)$. To prove that $\boldsymbol{P}' \in \mathcal{P}_P$, it is sufficient to prove that $P'_{q_r} > \underline{P}_{q_r}$. We have

$$P'_{q_r} - \underline{P}_{q_r} = P_{q_r} - \underline{P}_{q_r} + (1 - \lambda_{r-1}) \left(F^\star_{q_r q_{r-1}} - F_{q_r q_{r-1}} \right).$$

We have seen that by choosing δ small enough, we can make λ_{r-1} as close to 1 as we want. Since $P_{q_r} > \underline{P}_{q_r}$, we have $P'_{q_r} > \underline{P}_{q_r}$ for δ small enough, thus $\boldsymbol{P}' \in \mathcal{P}_P$.

We have thus proven that for δ small enough, $\boldsymbol{P}'$ belongs to $\overline{\text{conv}}\,(\mathcal{P}_\theta) \cap \mathcal{P}_P$. Since $\boldsymbol{P}' \prec \boldsymbol{P}$, the initial assumption is contradicted. □

Now, we conclude this section with a result allowing to connect the solutions of the convexified problem ($\overline{\text{OPF}_2}$) to the solutions of the physiscal problem (OPF_1). This result has direct numerical applications, since it allows in practice to solve the convex problem ($\overline{\text{OPF}_2}$) and obtain either a solution of Problem (OPF_1), or the non-existence of solutions for this problem.

Theorem 6 *We suppose that Assumption A_1 holds. Then we distinguish three cases:*

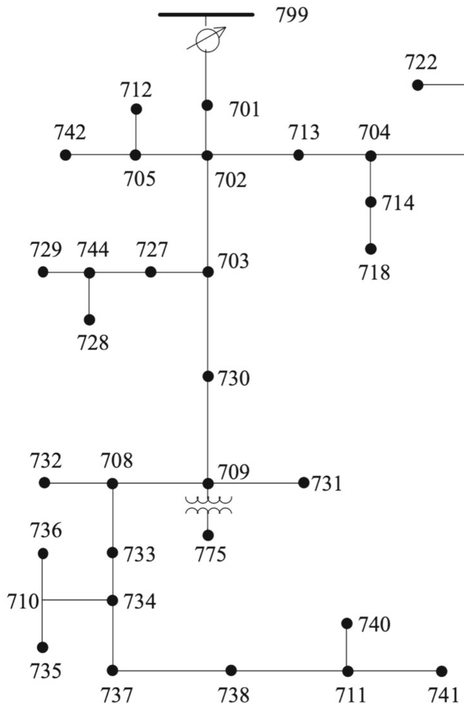
1. *if ($\overline{\text{OPF}_2}$) admits a solution $\boldsymbol{W}$ of rank 1, then there exists $\boldsymbol{v} \in \mathbb{C}^m$ such that $\boldsymbol{W} = \boldsymbol{v}\boldsymbol{v}^*$ and $\arg(\boldsymbol{v})$ is a solution of (OPF_1);*
2. *if ($\overline{\text{OPF}_2}$) admits a solution $\boldsymbol{W}$ of a rank at least 2, then (OPF_1) is infeasible;*
3. *if ($\overline{\text{OPF}_2}$) is infeasible, then (OPF_1) is infeasible.*

Proof

1. If ($\overline{\text{OPF}_2}$) admits a solution $\boldsymbol{W}$ of rank 1, then there exists $\boldsymbol{v} \in \mathbb{C}^m$ such that $\boldsymbol{W} = \boldsymbol{v}\boldsymbol{v}^*$ and Proposition 5 ensures that $\boldsymbol{P} = \boldsymbol{P}(\arg(\boldsymbol{v}))$ is a solution of ($\overline{\text{OPF}_3}$) and that $\boldsymbol{P} \in \mathcal{P}_\theta$. Therefore, we have $\mathcal{P}_\theta \cap \mathcal{P}_P \neq \varnothing$ and according to Theorem 5, we have
$$\mathcal{O}(\overline{\text{conv}}(\mathcal{P}_\theta) \cap \mathcal{P}_P) = \mathcal{O}(\mathcal{P}_\theta \cap \mathcal{P}_P).$$
Since the cost function J is strictly increasing, we have $\boldsymbol{P} \in \mathcal{O}(\overline{\text{conv}}(\mathcal{P}_\theta) \cap \mathcal{P}_P)$ and therefore $P \in \mathcal{O}(\mathcal{P})$. We deduce that $\boldsymbol{P}$ is a solution of (OPF_3) and thus $\arg(\boldsymbol{v})$ is a solution of (OPF_1).
2. If ($\overline{\text{OPF}_2}$) admits a solution $\boldsymbol{W}$ of rank greater than 2, by absurd, let us suppose that (OPF_1) is feasible. Then (OPF_3) is also feasible, therefore $\mathcal{P}_\theta \cap \mathcal{P}_P \neq \varnothing$. Theorem 5 ensures that
$$\mathcal{O}(\overline{\text{conv}}(\mathcal{P}_\theta) \cap \mathcal{P}_P) = \mathcal{O}(\mathcal{P}_\theta \cap \mathcal{P}_P).$$
In other words, every solution of ($\overline{\text{OPF}_3}$) belongs to $\mathcal{P}_\theta$. It follows from Proposition 5 that every solution of ($\overline{\text{OPF}_2}$) is of rank 1, which contradicts the assumption.
3. If ($\overline{\text{OPF}_2}$) is infeasible, by absurd, let us suppose that (OPF_1) is feasible. Then (OPF_3) is also feasible. As a consequence, we have $\mathcal{P}_\theta \cap \mathcal{P}_P \neq \varnothing$. Since $\mathcal{P}_\theta \cap \mathcal{P}_P \subset \overline{\text{conv}}(\mathcal{P}_\theta) \cap \mathcal{P}_P$, we have $\overline{\text{conv}}(\mathcal{P}_\theta) \cap \mathcal{P}_P \neq \varnothing$, thus ($\overline{\text{OPF}_3}$) is also feasible. It follows from Proposition 5 that ($\overline{\text{OPF}_2}$) is feasible, which is a contradiction.

□

Fig. 9 IEEE 37 node test feeder



6 Numerical tests

We present here some numerical examples to highlight Theorem 6 proved previously and to test if this theoretical result can be extended to take into account reactive powers.

6.1 IEEE 37 node test feeder

We consider the distribution network of 37 nodes whose structure is given in Fig. 9. The network and the line data are provided in Schneider et al. (2017). The admittance matrix is computed using the code provided in Group (2010). In order to make our tests more relevant, we provide data values of the lower (upper) bound of the injected active powers, thermal constraint and of the voltages, displayed in Table 1.

The $\overline{F}_\ell$ are chosen so that assumption A_1 holds. For every line $\{i, k\}$, we compute the angle α_{ik} defined by (16) from admittance y_{ik}. We then set the thermal constraint as $\overline{F}_\ell = F_{ik}(\alpha_{ik}) + F_{ki}(\alpha_{ki})$. Note that we then have $\overline{\theta}_{ik} = \alpha_{ik}$.

The computations for Problem ($\overline{\text{OPF}_2}$) are carried out by the primal-dual interior points method, with the function `Mosek` using `Yalmip` toolbox on Matlab®, see Löfberg (2004); ApS (2019). To solve Problem (OPF_1), we use the function `FMINCON` which relies on a SQP-BFGS type algorithm [see Waltz et al. (2006)].

Table 1 Bus parameters values

Nodes	$\underline{P}_i$	$\overline{P}_i$	$\underline{Q}_i$	$\overline{Q}_i$	$\widehat{V}_i$
701	−1.2023	1.2023	−0.1000	0.1000	1
702	−0.0940	−0.0740	−0.1420	0.6100	0.9992
703	−0.0100	0.0100	−0.1000	0.1000	0.9999
704	−0.0010	0.0010	−0.1000	0.1000	0.9996
705	−0.0100	0.0100	−0.1000	0.1000	0.9998
706	−0.0100	0.0100	−0.1000	0.1000	0.9999
707	−0.0100	0.0100	−0.1000	0.1000	0.9999
708	−0.0100	0.0100	−0.1000	0.1000	1
709	−0.0100	0.0100	−0.1000	0.1000	1
710	−0.0100	0.0100	−0.1000	0.1000	1
711	−0.0100	0.0100	−0.1000	0.1000	1
712	−0.0100	0.0100	−0.1000	0.1000	0.9998
713	−0.0010	0.0100	−0.1053	0.0947	0.9994
714	−0.0213	−0.0013	−0.1053	0.0947	0.9997
718	−0.0100	0.0100	−0.1000	0.1000	0.9999
720	−0.0010	0.0100	−0.1000	0.1000	0.9998
722	−0.0213	−0.0013	−0.1053	0.0947	0.9999
724	−0.0100	0.0100	−0.1000	0.1000	0.9999
725	−0.0100	0.0100	−0.1000	0.1000	0.9999
727	−0.0156	0.0044	−0.1028	0.0972	1
728	−0.0156	0.0044	−0.1028	0.0972	1
729	−0.0268	−0.0068	−0.1084	0.0916	0.9998
730	−0.0100	0.0100	−0.1000	0.1000	1
731	−0.0100	0.0100	−0.1000	0.1000	1
732	−0.0100	0.0100	−0.1000	0.1000	1
733	−0.0156	0.0044	−0.1028	0.0972	1
734	−0.0100	0.0100	−0.1000	0.1000	1
735	−0.0156	0.0044	−0.1028	0.0972	1
736	−0.0213	−0.0013	−0.1053	0.0947	0.9998
737	−0.0100	0.0100	−0.1000	0.1000	1
738	−0.0100	0.0100	−0.1000	0.1000	1
740	−0.0268	−0.0068	−0.1083	0.0917	0.9998
741	−0.0213	−0.0013	−0.1053	0.0947	0.9999
742	−0.0100	0.0100	−0.1000	0.1000	0.9998
744	−0.0100	0.0100	−0.1000	0.1000	1
775	−0.0156	0.0044	−0.1028	0.0972	1
799	−0.0100	0.0100	−0.1000	0.3798	0.9649

Table 2 Comparison of the optimal values of Problems (OPF_1) and ($\overline{\text{OPF}_2}$) for different values of $\underline{P}_{15}$

$\underline{P}_{15}$	(OPF_1) Optimal value	($\overline{\text{OPF}_2}$) Optimal value	Rank of $\boldsymbol{W}$
0.6	0.2106	0.2106	1
0.7	0.2711	0.2711	1
0.8	0.3376	0.3376	1
0.9	0.4092	0.4092	1
1.0	Infeasible	0.4861	4
1.1	Infeasible	0.5861	24
2.5	Infeasible	1.9861	35
2.6	Infeasible	2.3569	37
2.7	Infeasible	Infeasible	–
2.8	Infeasible	Infeasible	–

For all the numerical tests, we consider the objective function

$$J(\boldsymbol{P}) = \sum_{i=1}^{37} P_i,$$

which represents the total active power losses in the network. This function is strictly increasing with respect to $\boldsymbol{P}$, in the sense of Definition 2.

6.2 Illustration of Theorem 6

In order to highlight Theorem 6, the values of $\underline{P}_i$, $i \in \{1, \cdots, 37\} \backslash \{15\}$ are set as in Table 1 and the value of $\underline{P}_{15}$ varies from 0.6 to 2.8, in an increasing manner, in order to make this constraint more and more tight. Also, we set $\overline{P}_{15} = 3$. The obtained results are displayed in Table 2.

We remark that up to the value $\underline{P}_{15} = 0.9$, the matrix $\boldsymbol{W}$ solution of ($\overline{\text{OPF}_2}$) is of rank 1 and both problems share the same optimal value. For the values between $\underline{P}_{15} = 1$ and $\underline{P}_{15} = 2.6$, the matrix $\boldsymbol{W}$ is of rank greater than 1 and in this case the algorithm used for solving Problem (OPF_1) fails to find a feasible solution. Finally, for the values $\underline{P}_{15} = 2.7$ and $\underline{P}_{15} = 2.8$, the respective algorithms of both problems do not find a feasible solution. These results are consistent with the three cases of Theorem 6.

6.3 Consideration of the reactive power constraints

In many practical OPF problems, the consideration of reactive powers constraints is essential. We want to know if Theorem 6 can handle these additional constraints or if more assumptions are required. In this paper, we did not study this case theoretically. However, we will include these additional constraints in the following numerical experience.

Table 3 Comparison of the optimal values of Problems (OPF_1) and ($\overline{\mathrm{OPF}_2}$) for different values of $\underline{Q}_1$

$\underline{Q}_2$	(OPF_1)	($\overline{\mathrm{OPF}_2}$)	
	Optimal value	Optimal value	Rank of $\boldsymbol{W}$
0.03	0.0176	0.0176	1
0.04	0.0177	0.0177	1
0.05	0.080	0.080	1
0.06	0.086	0.086	1
0.09	Infeasible	0.0485	3
0.10	Infeasible	0.0585	3
0.47	Infeasible	Infeasible	–
0.48	Infeasible	Infeasible	–

The reactive power flow equation is given by

$$G_{ik} = V_i^2 b_{ik} - V_i V_k \big(g_{ik} \sin(\theta_{ik}) + b_{ik} \cos(\theta_{ik})\big), \tag{31}$$

where θ_{ik} is the phase angle. The injected reactive power can be defined from (31) by

$$Q_i = \sum_{k \sim i} G_{ik}.$$

The reactive power constraints of interests write:

$$\underline{Q}_i \leq Q_i \leq \overline{Q}_i, \text{ for all nodes } i.$$

In the following, we set the values of $\underline{Q}_i$, $i \in \{1, \cdots, 37\} \backslash \{2\}$ as in Table 1 and we vary the value of $\underline{Q}_2$ between 0.03 and 0.48 in an increasing manner. The value of $\overline{Q}_2$ is set to 0.6. All the other parameters are set as in Table 1. The obtained results are displayed in Table 3.

For the values between $\underline{Q}_2 = 0.03$ and $\underline{Q}_2 = 0.06$, the solution $\boldsymbol{W}$ of ($\overline{\mathrm{OPF}_2}$) is of rank 1 and both problems share the same optimal value. For the values $\underline{Q}_2 = 0.09$ and $\underline{Q}_2 = 0.10$, the matrix $\boldsymbol{W}$ is of rank greater than 1 and the algorithm used to solve Problem (OPF_1) fails to find a feasible solution. Finally, for the values $\underline{Q}_2 = 0.47$ and $\underline{Q}_2 = 0.48$, the algorithms of both problems do not find a feasible solution.

These results shows a similar behavior as Theorem 6. However, to prove it theoretically, some additional assumptions may be required. For instance, in Zhang et al. (2014) a study that takes into account the constraints on the reactive power was carried out, but the amplitudes of the voltages were all equal to 1 and the $\underline{Q}_i$ were assumed to be small enough.

7 Conclusion

We have presented an AC-OPF problem with fixed voltage magnitudes in a distribution network. This problem is known to be non-convex because of the power flow equations. In order to make it convex, we applied the method used in Bai et al. (2008) which consists in reformulating the cost function and the constraints in terms of traces of semi-definite matrices. This leads to a new formulation of the problem in which the non-convexity appears as a rank constraint. By removing this rank constraint, we obtain a convex relaxation of the main problem.

In order to find the conditions under which this relaxation can be exact, we used the notion of Pareto-optimality which is a very adapted tool for our study. We started by studying the problem in the case of a one transmission line network, then extended our results to the case of a whole network.

The main result we have proved is that the constraint set of the problem of interest share the same Pareto-front of its convexified relaxation. Since the cost function is strictly increasing, the relaxation is exact. We then proved a theorem which allows to exploit the convex relaxation in practice to solve the main problem on interest. We concluded by giving some numerical tests to highlight the last theorem of this work.

Some interesting extensions of this work could be:

- to extend the main result in the case of variable voltages magnitudes. Some additional assumptions might be required in order to obtain a theorem similar to Theorem 6.
- reactive powers constrains, i.e. to add the constraints $\underline{Q}_i > -\infty$ and $\overline{Q}_i < +\infty$. In the case of fixed voltage magnitudes, authors of Zhang et al. (2014) shows that the feasible set of the injected reactive power constrains is a simple linear transformation of the feasible set of the injected reactive power. Therefore, by adding an assumption equivalent to $\underline{Q}_i = -\infty$, they showed in this case that Theorem 6 holds.

Acknowledgements This work was achieved within the project VERTPOM supported by the French Environment and Energy Management Agency (ADEME), as part of the "Future Investment Program of the French government". The revision time of the second author was supported by Czech Science Foundation project No. 22-15524 S.

Declarations

Conflicts of interest All authors declare no conflicts of interest in this paper.

References

ApS M (2019) MOSEK optimization toolbox for MATLAB. Version 10.0. https://docs.mosek.com/latest/toolbox/index.html

Bai X, Wei H, Fujisawa K, Wang Y (2008) Semidefinite programming for optimal power flow problems. Int J Electr Power Energy Syst 30(6–7):383–392

Castillo A, Gayme DF (2017) Evaluating the effects of real power losses in optimal power flow-based storage integration. IEEE Trans Control Netw Syst 5(3):1132–1145

Conti S, Raiti S, Tina G, Vagliasindi U (2003) Distributed generation in LV distribution networks: voltage and thermal constraints. In: 2003 IEEE Bologna power tech conference proceedings, IEEE, vol 2
Elattar EE, ElSayed SK (2019) Modified JAYA algorithm for optimal power flow incorporating renewable energy sources considering the cost, emission, power loss and voltage profile improvement. Energy 178:598–609
Group DTFW (2010) Distribution test feeders. https://cmte.ieee.org/pes-testfeeders/resources/
Huang S, Filonenko K, Veje CT (2019) A review of the convexification methods for ac optimal power flow. In: 2019 IEEE electrical power and energy conference (EPEC), IEEE, pp 1–6
Kersting WH (1991) Radial distribution test feeders. IEEE Trans Power Syst 6(3):975–985
Lam AY, Zhang B, David NT (2012) Distributed algorithms for optimal power flow problem. In: 2012 IEEE 51st IEEE conference on decision and control (CDC), pp 430–437
Lavaei J, Tse D, Zhang B (2012) Geometry of power flows in tree networks. In: 2012 IEEE power and energy society general meeting, IEEE, pp 1–8
Lavaei J, Tse D, Zhang B (2013) Geometry of power flows and optimization in distribution networks. IEEE Trans Power Syst 29(2):572–583
Liu X, Fotouhi A, Auger DJ (2020) Optimal energy management for formula-e cars with regulatory limits and thermal constraints. Appl Energy 279:115805
Löfberg J (2004) Yalmip: a toolbox for modeling and optimization in matlab. In: Proceedings of the CACSD conference, Taipei, Taiwan
Madani R, Sojoudi S, Lavaei J (2014) Convex relaxation for optimal power flow problem: mesh networks. IEEE Trans Power Syst 30(1):199–211
Momoh JA (2017) Electric power system applications of optimization. CRC Press, Boca Raton
Monticelli A (2012) State estimation in electric power systems: a generalized approach. Springer, Berlin
Schneider KP, Mather B, Pal B, Ten CW, Shirek GJ, Zhu H, Fuller JC, Pereira JLR, Ochoa LF, de Araujo LR et al (2017) Analytic considerations and design basis for the IEEE distribution test feeders. IEEE Trans Power Syst 33(3):3181–3188
Shaheen AM, El-Sehiemy RA, Farrag SM (2016) Solving multi-objective optimal power flow problem via forced initialised differential evolution algorithm. IET Gener Transm Distrib 10(7):1634–1647
Waltz RA, Morales JL, Nocedal J, Orban D (2006) An interior algorithm for nonlinear optimization that combines line search and trust region steps. Math Program 107(3):391–408
Zhang B, Lam AY, Domínguez-García AD, Tse D (2014) An optimal and distributed method for voltage regulation in power distribution systems. IEEE Trans Power Syst 30(4):1714–1726