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Multi-criteria decision making beyond consistency: An alternative to AHP for real-world industrial problems

Silvia Carpitella [a,*], Václav Kratochvíl [b], Miroslav Pištěk [b]

[a] Department of Manufacturing Systems Engineering and Management, California State University, Northridge, 18111 Nordhoff St, Northridge, CA 91330, United States

[b] The Czech Academy of Sciences, Institute of Information Theory and Automation, Pod Vodárenskou věží 4, 182 08, Prague 8, Czech Republic

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ABSTRACT

The Analytic Hierarchy Process (AHP) is a widely used method for multi-criteria decision-making that relies on consistency in pairwise comparisons. However, decision-makers often struggle to provide fully consistent judgments in real-world scenarios. This article introduces a decision-making framework that operates independently of consistency. Utilizing the Skew-Symmetric Bilinear representation of preferences allows decision-makers to more accurately evaluate alternatives and criteria, making this framework more applicable in practical settings. The proposed method is validated through practical examples and an in-depth case study in the textile industry, effectively resolving a complex decision-making problem related to acquiring a data analytics tool for supplier selection. The results underscore the robustness and flexibility of this consistency-independent technique as an alternative to traditional AHP methods.

1. Introduction

One of the basic tools for supporting multi-criteria decision-making is the so-called Analytical Hierarchy Process (AHP), introduced by Saaty (1980). This tool provides a structured approach to analyzing complex decision problems where the goal is to identify the optimal alternative that aligns best with a given set of criteria. AHP provides a comprehensive, step-by-step framework guiding users through the decision-making process.

Acknowledging the cognitive challenge of comprehensively assessing relationships among more than three alternatives across various criteria, AHP adopts a pairwise comparison strategy, wherein alternatives are evaluated against each other under a single criterion. All alternatives are systematically compared pairwise for each criterion, followed by a similar evaluation for the criteria themselves. Leveraging these comparison matrices, AHP determines the most advantageous alternative concerning the specified mix of criteria.

The AHP methodology is based on two premises: that alternatives and criteria can be arranged linearly according to their importance, and that the matrices derived from pairwise comparisons consistently align with this arrangement. Thus, consistency verification plays a crucial role and so various consistency measures were introduced. They all share the same principle: If the consistency index exceeds a defined threshold, it indicates that the method's reliability is compromised, and its outcomes might be inaccurate. Typically, users are advised to revise the input data when faced with such a scenario to uphold the credibility and accuracy of the decision-making process. Indeed, an inconsistent input is inherently considered flawed, and the decision-maker internally ambivalent and irrational.

However, what if the pairwise comparison matrix is inconsistent but still valid? This situation can occur, especially when the entered alternatives cannot be linearly arranged based on their importance. This is well illustrated by the following example, inspired by the game of “intransitive dice”, see, e.g., Butler and Blavatskyy (2020).

Example 1.1. Suppose that there are three statistically independent assets. Asset D yields 9% with a probability of $1/3$ and does not yield any return otherwise. Asset E yields a negative return of -1% with a probability of $1/3$ and 5% otherwise. A risk-free asset F yields 3%; note that the expected returns of both assets D and E are also 3%. Now, given that a fund manager's preference is to choose an asset that is more likely to yield a higher return, she would prefer asset D over E, asset E over F, and finally asset F over D.[1]

Unlike a linear ordering where one is definitively superior and another inferior, in this circular arrangement, no single alternative dominates over all the others. AHP can be applied to inconsistent input,

* Corresponding author.
E-mail addresses: silvia.carpitella@csun.edu (S. Carpitella), velorex@utia.cas.cz (V. Kratochvíl), pistek@utia.cas.cz (M. Pištěk).

[1] For a more detailed discussion, see the continuation of Example 1.1, where an optimal portfolio is also constructed.

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but without a guarantee of identifying the best trade-off alternative as Saaty's method inherently assumes consistency for its theoretical foundations. In practical decision-making, human judgments seldom exhibit full consistency, especially when dealing with complex, multi-criteria problems (Munier, Hontoria, et al., 2021). Moreover, the consistency assumption imposes limitations, as it may force decision-makers to artificially adjust their preferences to fit a mathematical framework rather than reflect their true, sometimes inconsistent, judgments. As a result, AHP's reliance on consistency may hinder its ability to capture the full complexity of subjective preferences, making it less applicable in practical scenarios where inconsistencies are common. Allowing for inconsistency can deliver rational and effective outcomes by reflecting more realistic and adaptable human judgment patterns (Szczypińska & Piotrowski, 2009; Tavana, Soltanifar, & Santos-Arteaga, 2023).

There were many attempts to mitigate this limitation as researchers developed enhanced algorithms and methods to handle inconsistency more effectively (Benítez, Delgado-Galván, Izquierdo, & Pérez-García, 2012). Some of these methods focus on adjusting the pairwise comparison matrix to achieve consistency (Ishizaka & Lusti, 2004), employing mathematical and statistical techniques to modify the judgments until a satisfactory level of consistency is attained. These adjustments might involve iterative recalibration of the matrix elements, including the estimation of uncertain judgments (Benítez, Carpitella, Certa, & Izquierdo, 2019), to enhance the reliability of the results. Furthermore, other techniques such as the Best-Worst Method (BWM) (Lei, Wu, & Wu, 2022), may reduce the cognitive complexity by simplifying comparisons while offering specific preference-modification suggestions. However, rather than focusing on refining the comparison process as the BWM does, the approach presented in this article emphasizes offering more flexibility, especially when achieving full consistency is not always feasible.

Alternative approaches propose frameworks that inherently accommodate inconsistencies. Such methods often incorporate models that recognize and quantify the degree of inconsistency, then use this information to minimally adjust matrices and generate priority vectors that reflect the relative importance of the options. Techniques such as fuzzy logic (Carpitella, Ocaña-Levario, Benítez, Certa, & Izquierdo, 2018) and optimization-based methodologies (Zhang et al., 2023), along with other advanced approaches (Pascoe, 2022), are often employed to manage and interpret inconsistent data. Furthermore, researchers have explored hybrid methodologies that combine matrix adjustment techniques with novel prioritization approaches, aiming to balance the benefits of both strategies. These hybrid methods also align with the preference of experts for approaches that require fewer pairwise comparisons and less interaction, resulting in a more efficient process and higher acceptability (Tavana et al., 2023).

We argue that situations illustrated by Example 1.1 fall beyond the scope of decision-making theories that assume transitive preferences, such as the well-known Expected Utility Theory (EUT). Therefore we employed the Skew-Symmetric Bilinear (SSB) representation of preferences, see Fishburn (1982, 1988) and Kreweras (1961), a concise mathematical model allowing for more accurate and flexible assessment of non-transitive preferences. Unlike traditional methods, this approach models the inherent asymmetries in human preferences, which often do not follow a strict reciprocal pattern. The SSB representation enables us to capture these asymmetries more precisely by avoiding the assumption of reciprocal consistency. The previous publication (Carpitella, Inuiguchi, Kratochvíl, & Pištěk, 2022) addressed a partially inconsistent problem. While inconsistency was allowed for the pairwise comparison matrices of alternatives under individual criteria, the traditional consistency requirement for the pairwise comparison matrix of the individual criteria was maintained. This limitation, though manageable, constrained the flexibility of the method from Carpitella et al. (2022) in certain practical applications.

Building on this insight, we recognized that the assumption of consistency at any level might not be necessary for effective decision-making. As a result, we propose a generalized method that relaxes the consistency requirement across all pairwise comparison matrices, whether they involve criteria or alternatives. Such an approach is more flexible and thus also more applicable in real-world scenarios where full consistency is often difficult or impossible to achieve. The method works by first aggregating sub-hierarchies into individual matrices, and then combining these matrices into an upper-level aggregated preference matrix. This process mirrors the formation of hierarchies in the traditional AHP, but without the need to maintain consistency in any of the matrices. By aggregating preferences in this way, we retain the hierarchical structure fundamental to AHP, while also addressing the complexities of real-world decision-making, where preferences can be inconsistent or incomplete.

Assuming sufficient consistency, AHP first transforms pairwise comparison matrices into corresponding weight vectors and then aggregates preferences by the weighted summation of these vectors. However, for potentially inconsistent data, information loss can be reduced by reversing these two operations, see Remark 4.1. Motivated by broader applicability, we adopted the latter approach, providing a more flexible framework that may not converge to the AHP solution, see Theorem 4.2. However, when it comes to the core computational step of weight vector evaluation, consistency ensures that our solution concept, i.e., the canonical weight vector, coincides with the principal eigenvector used in AHP, see Theorem 3.2. To further illustrate how these solution concepts differ for inconsistent matrices, we conducted a numerical experiment with randomly generated pairwise comparison matrices at various consistency levels, see Fig. 1.

The article is organized as follows: Section 2 outlines the motivation behind this work and describes the existing methodologies that will be used to develop a new approach for handling inconsistency. Section 3 presents this new methodological approach by mapping a reciprocal pairwise comparison matrix (PCM) to the so-called canonical weight vector. Using this concept, Section 4 introduces an entirely new multi-criteria decision-making procedure, illustrated with several examples[2] Section 5 applies the new method to a real-world industrial case study. Finally, Section 6 provides the conclusion.

2. Motivation and existent methodologies

We start with a brief description of AHP method in Section 2.1, an effective tool to solve a decision problem described by PCMs. A concept of consistent user inputs, being an essential assumption of AHP, is discussed in Section 2.2. However, real-world problems frequently exhibit inconsistencies that lead to loss of information when representing a PCM by a weight vector in AHP. Such a loss may be reduced by first combining user input into an aggregated preference PCM, see Section 2.3, and then applying the theory of SSB representation of (potentially non-transitive) preferences, cf. Section 2.4. Finally, the principle of maximal entropy, see Section 2.5, will be used to enforce uniqueness of the resulting weight vector.

Let us introduce notations and definitions that will be used throughout the whole article. We say that a (binary) relation $\succ$ defined on a set S is *transitive* if $p \succ q$ and $q \succ r$ implies $p \succ r$ for all $p, q, r \in S$. An element $m \in S$ is a *maximal element* of S with respect to $\succ$ if set $\{q \in S : q \succ m\}$ is empty. Given a positive integer k, let us denote by $\mathscr{P}(k)$ the set of all probability distributions having finite support of cardinality k, i.e. $\mathscr{P}(k) = \left\{ p \in \mathbb{R}^k : p \geq 0, \sum_{i=1}^{k} p_i = 1 \right\}$. For a square matrix $\boldsymbol{X} \in \mathbb{R}^{k\times k}$, we denote the *transpose* by $\boldsymbol{X}^\intercal$, and say that $\boldsymbol{X}$ is *skew-symmetric* if $\boldsymbol{X}^\intercal = -\boldsymbol{X}$. A matrix $\boldsymbol{X}$ having only positive entries (obtained from comparisons between certain attributes following a predefined scale) is a *pairwise comparison matrix* (PCM). Such a matrix is *reciprocal*, $\boldsymbol{X} \in \mathscr{R}(k)$, if $x_{ji} = 1/x_{ij}$ for all $i, j = 1, \dots, k$. A reciprocal matrix $\boldsymbol{X}$ is *consistent* with a weight vector $\boldsymbol{w} \in \mathscr{P}(k)$ if $\boldsymbol{w}$ reflects the priorities expressed by elements of $\boldsymbol{X}$ in such a way that $x_{ij} = w_i/w_j$ for all $i, j = 1, \dots, k$.

[2] We made their source code available in Python at Carpitella, Kratochvíl, and Pištěk (2024).

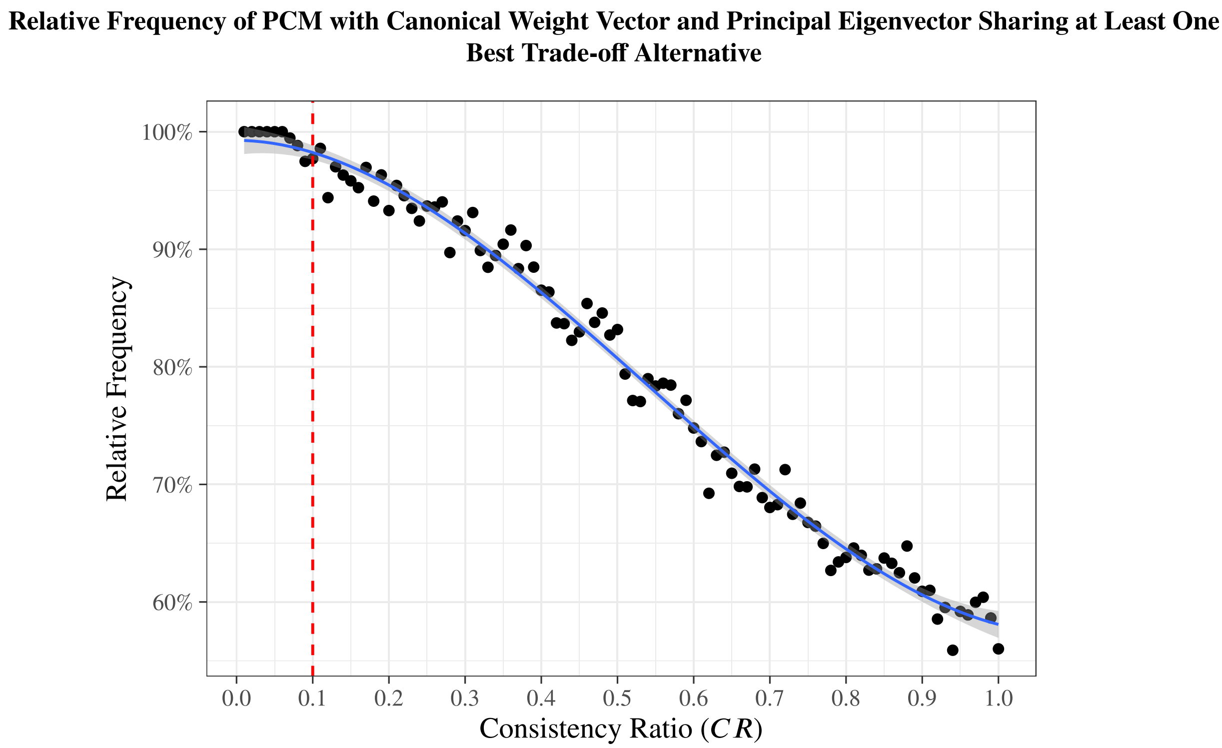

Fig. 1. Black dots represent the relative frequencies of cases where the canonical weight vector and the principal eigenvector share a best trade-off alternative (each dot represents PCMs with a CR in the corresponding part of the $[0,1]$ interval). A red dashed line marks the consistency ratio threshold ($CR = 0.1$), the general applicability limit for AHP. The blue trend curve was fitted using Generalized Linear Model smoothing, assuming a polynomial shape.

2.1. Analytic hierarchy process

AHP operates within the domain of relative measurement theory and methodology. Specifically, it focuses on aspects beyond exact scores or precise measurements of alternatives. For instance, when faced with a choice between the last two apples in a bowl, the concern is not primarily centered on their precise weight or sugar content individually. The pivotal point is the comparison between the two apples to determine which one is sweeter.

In a traditional comparison scenario, one should measure that the sugar content is precisely 6% for the bigger apple and 12% for the other. However, the crucial comparison lies in the fact that the second apple is twice as sweet, regardless of whether the sugar content results are (5%, 10%) or (7%, 14%). This highlights that often, what matters most is not the exact quantitative evaluation of the alternatives but rather their relative comparison, which is adequate for selecting the superior alternative.

Note that also the human brain faces challenges when directly evaluating multiple alternatives simultaneously. To tackle this issue, it is advantageous to break down the original problem of scoring alternatives into smaller sub-problems, specifically comparing two alternatives at a time. Dealing with evaluations that rely solely on relative comparisons among pairs (without necessarily involving physical or directly measurable aspects) is precisely the domain where AHP proves valuable. Moreover, it is an effective tool to handle multi-criteria decision-making (MCDM) scenarios where evaluations against multiple, often conflicting criteria, are essential.

In this article, we address a decision-making problem involving criteria $C_1, \dots, C_m$ and alternatives $AT_1, \dots, AT_n$. Preferences are captured by an input decision matrix $\boldsymbol{A} \in \mathscr{R}(m)$, a PCM representing the relative importance of the criteria, and several matrices pairwise comparing alternatives $\boldsymbol{B}^{(l)} \in \mathscr{R}(n)$, with $l = 1, \dots, m$. In these matrices, element $b_{ij}^{(l)}$ indicates the degree of preference for alternative AT_i over alternative AT_j with respect to the lth criterion. The objective is to derive the *final weight vector* $\boldsymbol{z} \in \mathscr{P}(n)$, with elements representing the relative score of a specific alternative $AT_1, \dots, AT_n$ in comparison to others; for all k such that $z_k = \max_i z_i$ we call AT_k the *best trade-off alternative*.

Remark 2.1. Depending on the characteristics and the complexity of the practical problem under analysis, aggregation at the criteria level may also be necessary. In group decision-making, it is indeed a frequent situation that multiple stakeholders with complementary professional backgrounds are consulted for weighting criteria, and each one is asked to fill in a dedicated PCM. Thus obtained judgment matrices $\boldsymbol{A}^{(i)} \in \mathscr{R}(m)$, $i = 1, \dots, k$, are then aggregated into a single matrix $\boldsymbol{A} \in \mathscr{R}(m)$, using, potentially, an individual weight for each decision-maker.

There exist numerous methodologies for estimating a weight vector from the pairwise comparison matrix, comparative studies exploring these methodologies can be found in Refs. Choo and Wedley (2004), Cook and Kress (1988) and Ishizaka and Lusti (2006). Among these approaches, one of the most renowned methods was proposed by Saaty himself within the framework of AHP, suggesting that the weight vector should be derived from the principal (or the Perron–Frobenius) eigenvector of the given PCM (by employing, typically, the power method, see, e.g., Horn and Johnson (2012) and Peretti (2014)). Thus, this approach assumes that all input PCMs can be accurately represented by their corresponding weight vectors (i.e., the input PCMs are sufficiently consistent, as discussed in Section 2.2). Given this assumption, we denote by $\boldsymbol{w} \in \mathscr{P}(m)$ the vector of *criteria weights* derived from matrix $\boldsymbol{A}$, and by $\boldsymbol{v}^{(l)} \in \mathscr{P}(n)$ the *local priorities* that are the vectors of weights derived from matrices $\boldsymbol{B}^{(l)}$ for all $l = 1, \dots, m$. Then, vector $\boldsymbol{z}_{AHP} \in \mathscr{P}(n)$ of final weights is obtained as multiplication of matrix $\boldsymbol{V}$ whose columns rows are vectors of local priorities $\boldsymbol{v}^{(l)}$, and vector $\boldsymbol{w}$ of criteria weights:

$$\boldsymbol{z}_{AHP} = \boldsymbol{V}\boldsymbol{w}. \tag{1}$$

Despite the availability of various simpler methods (e.g., the Geometric Mean Method, see Crawford and Williams (1985)), the above-sketched method remains one of the most popular and frequently employed techniques in this domain due to its efficiency in achieving accurate results.

2.2. Consistency

We will discuss the consistency of matrix $\boldsymbol{A} \in \mathscr{R}(m)$; the same considerations however apply also to all matrices $\boldsymbol{B}^{(l)}$. For a fully rational[3] decision-maker matrix $\boldsymbol{A}$ is consistent, i.e. there exists a weight vector $\boldsymbol{w} \in \mathscr{P}(m)$ such that $a_{ij} = w_i/w_j$ for all i, j. Consequently, the following relationship can be derived:

$$a_{ij} \cdot a_{jk} = \frac{w_i}{w_j} \cdot \frac{w_j}{w_k} = \frac{w_i}{w_k} = a_{ik}, \quad forall \quad i, j = 1, \dots, m \tag{2}$$

In other words, every direct comparison a_{ik} precisely aligns with all indirect comparisons $a_{ij}a_{jk}$ for all j. Essentially, a decision-maker capable of providing fully consistent pairwise comparisons does not encounter contradictions within their comparisons. A matrix that conforms to this transitivity condition is considered consistent within AHP framework. Even though the natural inconsistency of human thinking tends to escalate when confronted with an increasing number of alternatives, consistency remains a crucial aspect for checking the quality of the elicited judgments, and solution methods are often based on the assumption of at least partial consistency in assignments. The measure of inconsistency is typically quantified using a specific *inconsistency index*, with various indices available due to each method necessitating its particular definition. For further insight into these indices and their comparative analysis, we suggest a survey article (Brunelli, Canal, & Fedrizzi, 2013).

Adhering to Saaty's original method, we employ his *consistency ratio* CR to incorporate a degree of allowable inconsistency. This is done by evaluating the difference between the largest eigenvalue of matrix A and the dimension of the matrix n. To facilitate comparison across different sizes of matrices, the resulting value is normalized by the so-called *random index* RI_n as follows:

$$CR(A) = \frac{1}{RI_n} \frac{\lambda_{max}(A) - n}{n-1}.$$

Saaty provided the values of RI_n by evaluating a large set of randomly generated matrices in his foundational work (Saaty, 2000). In practical applications, a matrix with a value of $CR \le 0.1$ is typically considered acceptable for inclusion in calculations. This threshold implies that the pairwise comparison estimates exhibit around 10% inconsistency as if they were randomly generated. It is worth noting that reciprocal matrices of order 2 are inherently consistent.

The consistency ratio CR is also commonly used to evaluate the degree of transitivity in PCMs within the AHP framework. However, it is important to highlight that a PCM representing intransitive preferences can still exhibit CR values below Saaty's inconsistency threshold of 0.1, indicating an acceptable level of inconsistency, see Siraj, Mikhailov, and Keane (2015). Indeed, CR is more designed to measure the above-defined cardinal consistency, whereas transitivity is more related to the concept of ordinal consistency, see e.g. Zhou, Hu, Fan, Cheng, and Liu (2024). The *Ordinal Consistency Index* (OCI) was introduced already by Saaty (1980), providing a different perspective on how consistent rankings correspond to (expected) transitive relations. A recently introduced concept of a transitivity threshold offers further insight into preference misclassification and enhances reliability without requiring judgment revisions (Amenta, Lucadamo, & Marcarelli, 2020). That work also examined other consistency indices generalizing CR and OCI, such as the Salo–Hamalainen index and the Geometric Consistency Index. However, we did not incorporate these transitivity thresholds in our experiments.

We often cannot discard a given matrix A in real-world scenarios even if it did not pass the consistency test. This might be due to limitations such as the unavailability of means to modify matrix A (for instance, when there is no expert available to revise the matrix) or the fundamental nature of the problem not satisfying the transitivity assumption. When a standard method such as AHP is employed to solve such problems, using a matrix that fails the consistency test can potentially lead to misleading outcomes. This discrepancy arises because consistency serves as a fundamental assumption of the method,[4] and its violation compromises the theoretical foundation of the entire methodology. Consequently, applying such a method on inconsistent data can significantly impact the reliability and accuracy of the obtained solutions.

2.3. Aggregated preference matrix

Assuming consistency of all PCM matrices (to a high enough degree), AHP first transforms each matrix to the respective weight vector and then aggregates these vectors, cf. Section 2.1 above. However, this may lead to high information loss for inconsistent PCM matrices. Therefore, we choose to aggregate the involved PCM matrices first. Numerous variants of aggregation are discussed in Blagojevic, Srdjevic, Srdjevic, and Zoranovic (2016); the most common is the aggregation of individual judgments and the aggregation of individual priorities (Abel, Mikhailov, & Keane, 2015; Ramanathan & Ganesh, 1994), but also models based on consensus convergence (Lehrer & Wagner, 2012) and 'soft' consensus computations (Wu & Xu, 2012) have been applied. This article will employ element-wise (weighted) geometric mean.

Given any n and m, let us consider reciprocal PCM matrices $\boldsymbol{R}^{(i)} \in \mathscr{R}(n)$, $i = 1, \dots, m$ and a vector of weights $\boldsymbol{w} \in \mathscr{P}(m)$ assigning a relative relevance to each $\boldsymbol{R}^{(i)}$. The *aggregated preference matrix* $\boldsymbol{P}$ then reads

$$\boldsymbol{P} = \boldsymbol{R}^{\boldsymbol{w}} := \prod_i \left(\boldsymbol{R}^{(i)}\right)^{w_i}, \tag{3}$$

where both product $\prod$ and power $\left(\boldsymbol{R}^{(i)}\right)^{w_i}$ are performed element-wise.[5] Note that matrix $\boldsymbol{P}$ is reciprocal, $\boldsymbol{P} \in \mathscr{R}(n)$. This is important in Section 4 where Eq. (3) is employed to aggregate the evaluation of alternatives by individual criteria $\boldsymbol{B}^{(l)}$. Similarly, if multiple experts are involved in evaluating the criteria, cf. Remark 2.1, thus obtained PCMs $\boldsymbol{A}^{(k)}$ with the respective weights u_k may be aggregated into a reciprocal matrix $\boldsymbol{A}^{\boldsymbol{u}}$.

2.4. Skew-symmetric bilinear representation of non-transitive preferences

The discussion in Section 2.2 highlights the need for a tool capable of handling potentially intransitive expert evaluations. Next, we introduce such a tool, avoiding the normative debate on the non-transitivity of preferences. Readers interested in this debate can find relevant references in Carpitella et al. (2022).

Next, let us briefly revise the concept of rationality in the sense of EUT. Given a positive integer k, a (preference) relation $\succ$ on $\mathscr{P}(k)$ is said to be rational if there is vector $\boldsymbol{x} \in \mathscr{P}(k)$ representing $\succ$ as follows:

$$\boldsymbol{p} \succ \boldsymbol{q} \quad \iff \quad \boldsymbol{x}^\intercal \boldsymbol{p} > \boldsymbol{x}^\intercal \boldsymbol{q} \quad \text{for all} \quad \boldsymbol{p}, \boldsymbol{q} \in \mathscr{P}(k).$$

However, such a model of preferences may never account for possible non-transitivity of individual preferences. To this end, the theory of SSB representation of preferences has been proposed (Fishburn, 1982). We

[3] It is important to stress that rationality in the sense of EUT is meant here, which is arguably too restrictive a notion to be used in the domain of AHP, see Section 2.4 below.

[4] One may observe that if a matrix $\boldsymbol{M}$ is a PCM that is consistent with a vector $\boldsymbol{w}$, the principal eigenvector of $\boldsymbol{M}$ is proportional to $\boldsymbol{w}$.

[5] Since element-wise logarithm of an aggregated preference matrix will be used in the definition of the optimal distribution of preference in (5), one may equivalently use (weighted) arithmetic mean of the corresponding logarithms there.

Table 1
The SSB preferences $\boldsymbol{X}$ of the fund manager from Example 1.1.

	D	E	F
D	0	1/9	−1/3
E	−1/9	0	1/3
F	1/3	−1/3	0

say that a relation $\succ$ on $\mathscr{P}(k)$ admits a *skew-symmetric bilinear* (SSB) *representation* $\boldsymbol{X} \in \mathbb{R}^{k\times k}$ if matrix $\boldsymbol{X}$ is skew-symmetric and

$$p \succ q \iff p^{\intercal}\boldsymbol{X}q > 0 \quad \text{for all} \quad p, q \in \mathscr{P}(k).$$

A possible non-transitivity of thus represented $\succ$ may seem to be an insurmountable obstacle in the domain of decision-making. However, the well-known Minimax Theorem, see, e.g., Von Neumann and Morgenstern (1953), implies that there exists a maximal element.

Theorem 2.2. *Let matrix $\boldsymbol{X}$ be an SSB representation of a preference relation $\succ$ defined on $\mathscr{P}(k)$, then there exists a maximal element $\boldsymbol{m}$ of $\mathscr{P}(k)$ w.r.t. $\succ$, that may be characterized by the following equivalent optimality conditions:*

(i) $\boldsymbol{m}^{\intercal}\boldsymbol{X}\boldsymbol{q} \geq 0$ *for all* $\boldsymbol{q} \in \mathscr{P}(k)$,
(ii) $\boldsymbol{X}\boldsymbol{m} \leq 0$.

Since $\boldsymbol{m}^{\intercal}\boldsymbol{X}\boldsymbol{q}$ in condition (i) above amounts to the probability of $\boldsymbol{m}$ yielding a more preferred alternative than $\boldsymbol{q}$ (as elements x_{ij} are proportional to the scale of preference of alternative i over j), we see that maximal element $\boldsymbol{m}$ yields a more preferred alternative more (or equally) likely than any other probability vector in $\mathscr{P}(k)$. Note that generalizations of Theorem 2.2 may be found in, e.g., Fishburn (1988) and Pištěk (2018, 2019).

Finally, let us observe that Theorem 2.2 provides us with a solution to Example 1.1.

Continuation of Example 1.1. For any pair of assets $X, Y \in \{D, E, F\}$ we denote by $\phi(X, Y)$ the probability that asset X yields a higher return than asset Y. One may verify that $\phi(D, E) = 5/9$ and $\phi(E, F) = \phi(F, D) = 2/3$. Now, the likelihood of asset X yielding a higher return than asset Y may be expressed by $\phi(X, Y) - \phi(Y, X)$. Thus, the preferences of the fund manager may be represented by the SSB matrix $\boldsymbol{X}$ in Table 1.

By solving inequality (ii) of Theorem 2.2, we see that for the given SSB preferences $\boldsymbol{X}$ there is a maximal element $\boldsymbol{m} = (3/7, 3/7, 1/7)$. Thus, when managing for example €7M, the fund manager should invest €3M in both assets D and E, and €1M in asset F. This allocation is the most likely to yield a higher return than any other possible portfolio.

2.5. Principle of maximum entropy

In the previous section, we have seen that given an SSB representation of (potentially non-transitive) preferences, a maximal element exists, cf. Theorem 2.2. A less crucial but still important issue is the possible non-uniqueness of such an element. To provide a decision-maker with one optimal solution, it is fruitful to recall that all maximal elements are (discrete) probability distributions and no additional relevant information is available. However, this is precisely the setting when the distribution with maximum entropy should be chosen as the most unbiased one. The well-known *principle of maximum entropy* (PME), a method used to select a probability distribution that best represents a given set of information, is discussed in a vast literature, see, e.g. Csiszar (1991) and Jaynes (2003). Given any k, the *Shannon's entropy* is a functional $H(\cdot)$ defined on $\mathscr{P}(k)$ by

$$H(\boldsymbol{v}) = -\sum_{i=1}^{k} v_i \log(v_i).$$

For the applicability of PME, strict concavity of H is essential, implying the following theorem.

Theorem 2.3. *Let Q be a convex and closed subset of $\mathscr{P}(k)$, then there is a unique maximizer of H on Q.*

Shannon entropy, widely used across diverse fields, originally stemmed from signal processing theory. In Information theory, Shannon entropy denotes the minimum limit at which a signal cannot be compressed any further. Different notions of entropy tailored for specific domains exist. For an in-depth comparison of over twenty entropy definitions applied to time-series data in various fields, we recommend referring to the survey article by Ribeiro et al. (2021).

3. Canonical weight vector for reciprocal matrix

This section explains the main novelty of our approach, partly following the same path as in Carpitella et al. (2022). We start with a reciprocal PCM matrix that may be highly inconsistent. Since element-wise logarithm of such a matrix is skew-symmetric, one may write down a system of linear inequalities determining all maximal preferred elements (in the sense of SSB representation). To deal with possible non-uniqueness, the *optimal distribution of preference* is newly defined as the unique maximal preferred element that maximizes entropy. On such a basis, we then define the *canonical weight vector* and show that for a reciprocal consistent PCM, this vector corresponds to the weight vector of AHP, see Theorem 3.2. Note that the technique presented below may be of general interest unrelated to AHP.

Let k be any integer, we will consider a reciprocal PCM matrix $\boldsymbol{R} \in \mathscr{R}(k)$ and search for a vector of weights in $\mathscr{P}(k)$ that well-represents the comparisons in $\boldsymbol{R}$ in the sense of SSB representation. First, one needs a skew-symmetric matrix $\boldsymbol{X}$ such that x_{ij}, if positive, represents the scale of preference of i over j; we propose to use element-wise logarithm

$$\boldsymbol{X} = \log \boldsymbol{R}. \tag{4}$$

Indeed, matrix X is skew-symmetric using reciprocity[6] of $\boldsymbol{R}$, the sign of x_{ij} indicates if i is preferred to j or vice versa, and the absolute value of x_{ij} corresponds to the scale of such a preference. A maximal preferred element with respect to matrix $\boldsymbol{X}$, denoted by $\boldsymbol{m} \in \mathscr{P}(k)$, has to satisfy $\boldsymbol{X}\boldsymbol{m} \leq 0$, see Theorem 2.2. Let us recall that such $\boldsymbol{m} \in \mathscr{P}(k)$ leads to a more preferred outcome more (or equally) likely than any other probability distribution in $\mathscr{P}(k)$. As there may be several maximal elements, we use Theorem 2.3 and define *the optimal distribution of preference* $\widetilde{\boldsymbol{m}}$ as the maximal preferred element that, moreover, maximizes the entropy.

$$\widetilde{\boldsymbol{m}} = \operatorname*{arg\,max}_{\boldsymbol{m}\in\mathscr{P}(k),\, \boldsymbol{X}\boldsymbol{m}\leq 0} H(\boldsymbol{m}). \tag{5}$$

The Python implementation of (5) can be found in Carpitella et al. (2024) as the `find_max_entropy_SSB_max()` function. Note that we are employing the Python package *scipy.optimize* and the fact that Theorem 2.2 closely relates SSB representation to convex polytopes in computational geometry and optimization.

The following result, which was partially observed in Carpitella et al. (2022), shows that measuring the (expected) degree of preference on the Saaty's scale (i.e. using the original PCM $\boldsymbol{R}$ instead of $\boldsymbol{X}$), distribution $\widetilde{\boldsymbol{m}}$ is preferred (or indifferent) to any other distribution in $\mathscr{P}(k)$.

Lemma 3.1. *It holds $\boldsymbol{R}^{\intercal}\widetilde{\boldsymbol{m}} \geq 1$, and so $\widetilde{\boldsymbol{m}}^{\intercal}\boldsymbol{R}p \geq 1$ for all $p \in \mathscr{P}(k)$.*

Proof. The optimality condition $(\log \boldsymbol{R})\widetilde{\boldsymbol{m}} \leq 0$ and reciprocity of $\boldsymbol{R}$ imply $(\log \boldsymbol{R}^{\intercal})\widetilde{\boldsymbol{m}} \geq 0$. The statement then stems from the Jensen inequality. □

[6] One might argue that our method fits a reciprocal PCM better than AHP, where only the positivity of matrix elements is required to derive the principal (or the Perron–Frobenius) eigenvector that is used in Section 2.1.

Denoting $\pi = \boldsymbol{R}^{\intercal}\widetilde{\boldsymbol{m}}$, value $1/\pi_i \le 1$ is the intensity of preference of alternative i in comparison to the optimal distribution $\widetilde{\boldsymbol{m}}$. This motivates the following definition of the *canonical weight vector* $\boldsymbol{\kappa}(\boldsymbol{R})$ for a reciprocal matrix $\boldsymbol{R}$.[7]

$$\boldsymbol{\kappa}(\boldsymbol{R}) = \frac{1/\pi}{|1/\pi|} \quad \text{where} \quad \pi = \boldsymbol{R}^{\intercal}\widetilde{\boldsymbol{m}}. \tag{6}$$

One immediately has $|\boldsymbol{\kappa}(\boldsymbol{R})| = 1$ and $\boldsymbol{\kappa}(\boldsymbol{R}) > 0$. We defined canonical vector $\boldsymbol{\kappa}(\boldsymbol{R})$ to indicate the relative score of individual alternatives.[8] The validity of such interpretation is further supported by showing that for a consistent matrix $\boldsymbol{R}$, vector $\boldsymbol{\kappa}(\boldsymbol{R})$ is the weight vector reflecting priorities of $\boldsymbol{R}$.

Theorem 3.2. *Let a pairwise comparison matrix $\boldsymbol{R} \in \mathscr{R}(k)$ be consistent, then, for all $i, j = 1, \dots, k$,*

$$r_{ij} = \frac{\boldsymbol{\kappa}(\boldsymbol{R})_i}{\boldsymbol{\kappa}(\boldsymbol{R})_j}.$$

Proof. Let us denote by $\boldsymbol{w} \in \mathscr{P}(k)$ the weight vector satisfying $r_{ij} = w_i/w_j$ for all $i, j = 1, \dots, k$ due to consistency of matrix $\boldsymbol{R}$. Next, let $\widetilde{w} := \max_i w_i$ be the maximal weight, and $\mathcal{I} := \{i \in 1, \dots, k \,:\, w_i = \widetilde{w}\}$ the set of indices attaining $\widetilde{w}$. We will evaluate $\boldsymbol{\kappa}(\boldsymbol{R})$ according to (5) and (6), and show that it equals to $\boldsymbol{w}$. Since we have $\log r_{ij} = \log w_i - \log w_j$ for all i, j, inequality $\log \boldsymbol{Rm} \le 0$ is equivalent to $\log \widetilde{w} \le \sum_{j=1}^{k} m_j \log w_j$. This implies that $\boldsymbol{m}_i > 0$ if and only if $i \in \mathcal{I}$, and so the entropy maximizer $\widetilde{\boldsymbol{m}}$ in (5) has to be the uniform distribution on $\mathcal{I}$. Thus, for all i, one has

$$\pi_i = \left(\boldsymbol{R}^{\intercal}\widetilde{\boldsymbol{m}}\right)_i = \sum_{j=1}^{k} r_{ji}\widetilde{m}_j = \frac{1}{|\mathcal{I}|}\sum_{j\in\mathcal{I}} r_{ji} = \frac{1}{|\mathcal{I}|}\sum_{j\in\mathcal{I}} \frac{\widetilde{w}}{w_i} = \frac{\widetilde{w}}{w_i}.$$

Thus, $1/\pi_i = w_i/\widetilde{w}$, and so using $\sum_i w_i = 1$, definition (6) implies that $\boldsymbol{\kappa}(\boldsymbol{R})_i = w_i$ for all $i = 1, \dots, k$. □

Having a clear interpretation and satisfying Theorem 3.2, the concept of the canonical weight vector allows for a significant generalization of Carpitella et al. (2022) in the following section.

3.1. Numerical comparison of canonical weight vector and principal eigenvector

Before proceeding, let us address how often the canonical weight vector leads to the same decision as the principal (Perron–Frobenius) eigenvector, or more specifically, how frequently there is at least one common element in the corresponding sets of the best trade-off alternatives. According to Theorem 3.2, both vectors are equal for fully consistent PCMs, and so the sets of the best trade-off alternatives coincide. However, as the consistency ratio (CR) increases (i.e., as PCMs become less consistent), these vectors start to differ and the likelihood of obtaining intersecting sets of the best trade-off alternatives is expected to decrease.

The following numerical experiment confirmed this expectation, as illustrated by the results in Fig. 1. We randomly generated 50,000 PCMs of dimension 5×5 with varying CRs to evenly cover the interval $[0, 1]$ divided into 100 equal parts of 0.01 width. For each PCM, we recorded whether there is at least one common best trade-off alternative identified by both the canonical weight vector and the principal eigenvector. As expected, this was the case for sufficiently consistent PCMs (with $CR \le 0.06$). As inconsistency increased, differences gradually emerged; however, for $CR \le 0.1$, the relative frequency remained above 95%. Finally, for highly inconsistent PCMs (CR near 1), the relative frequency of a shared best trade-off alternative decreases to approximately 58%.

[7] The Python implementation of (6) can be found in Carpitella et al. (2024) as `kappa()` function.

[8] Note that in a typical case, there is only one maximal preferred element, cf. Carpitella et al. (2022)[Remark 2.2], and so the optimal distribution of preference $\widetilde{\boldsymbol{m}}$ has form $(0, \dots, 1, \dots, 0)$. Thus, the best trade-off alternative is indicated, however, with no comparison to other alternatives.

Table 2
$\boldsymbol{A}$: criteria pairwise comparison.

	Age	Charisma	Education	Experience	CR
Age	1	1/5	1/3	1/7	
Charisma	5	1	3	1/3	0.04435
Education	3	1/3	1	1/4	
Experience	7	3	4	1	

4. Consistency-independent decision technique

Now we are ready to introduce a new method to cope with the (potential) inconsistency of experts' judgments,

4.1. Consistency-independent alternative to AHP

We will follow the steps of Carpitella et al. (2022, Section 3) with one significant improvement: the consistency of the pairwise comparison of criteria (represented by matrix $\boldsymbol{A}$) is no longer assumed, and the respective vector of evaluation criteria weights is calculated using the approach of Section 3 (which also provides a clear interpretation to $\boldsymbol{\kappa}(\boldsymbol{A})$). Since such a method is entirely independent of any consistency assumptions. Consequently, we will denote the resulting vector of final weights by $\boldsymbol{z}_{CI} \in \mathscr{P}(n)$, with the index CI standing for *consistency independent.*

First, for a given reciprocal matrix $\boldsymbol{A} \in \mathscr{R}(m)$ we calculate the canonical weight vector $\boldsymbol{\kappa}(\boldsymbol{A}) \in \mathscr{P}(m)$. In the second step, the obtained vector is used as a weight to aggregate matrices $\boldsymbol{B}^{(l)} \in \mathscr{R}(n)$ using the (weighted) geometric mean, cf. (3), to obtain the aggregated preference matrix $\boldsymbol{B}^{\boldsymbol{\kappa}(\boldsymbol{A})} \in \mathscr{R}(n)$. Since such a matrix is again reciprocal, we may evaluate the vector of final weights $\boldsymbol{z}_{CI} \in \mathscr{P}(n)$ as follows

$$\boldsymbol{z}_{CI} = \boldsymbol{\kappa}\left(\boldsymbol{B}^{\boldsymbol{\kappa}(\boldsymbol{A})}\right). \tag{7}$$

Remark 4.1. It is important to note that, alternatively, one could introduce vectors $\boldsymbol{v}^{(l)} = \boldsymbol{\kappa}\left(\boldsymbol{B}^{(l)}\right)$, where $l = 1, \dots, m$, combine them into a matrix $\boldsymbol{V}$, and then compute the vector of final weights as $\boldsymbol{V}\boldsymbol{\kappa}(\boldsymbol{A})$, similarly to Eq. (1) used in AHP. However, representing potentially highly inconsistent matrices $\boldsymbol{B}^{(l)}$ with vectors $\boldsymbol{v}^{(l)}$ may result in a high information loss. Eq. (7) addresses this issue by aggregating all PCMs $\boldsymbol{B}^{(l)}$ into matrix $\boldsymbol{B}^{\boldsymbol{\kappa}(\boldsymbol{A})}$ first, and then performing only one linearization into the vector of final weights $\boldsymbol{z}_{CI}$. This approach reduces information loss and thus provides broader applicability. Note that this is not an issue in AHP where, assuming full consistency of all PCMs, vectors $\boldsymbol{v}^{(l)}$ hold the same information as matrices $\boldsymbol{B}^{(l)}$.

Finally, let us observe that if k experts are involved in calculating criteria weights, as described in Remark 2.1, we obtain PCMs $\boldsymbol{A}^{(i)} \in \mathscr{R}(m)$, $i = 1, \dots, k$, as well as a weight vector $\boldsymbol{u} \in \mathscr{P}(k)$ associating a weight to each expert. The above method may still be employed; it suffices to use $\boldsymbol{A}^{\boldsymbol{u}} \in \mathscr{R}(m)$ instead of $\boldsymbol{A}$ in formula (7).

4.2. Consistent case: Relation to analytic hierarchy process

Next, we will examine how the above-introduced technique relates to AHP for a fully consistent problem. In other words, let us assume that $\boldsymbol{A} \in \mathscr{R}(m)$ and $\boldsymbol{B}^{(l)} \in \mathscr{R}(n)$ for all $l = 1, \dots, m$ be such that, given positive vectors $\boldsymbol{w} \in \mathscr{P}(m)$ and $\boldsymbol{v}^{(l)} \in \mathscr{P}(n)$ for all $l = 1, \dots, m$, it holds

$$a_{kl} = \frac{w_k}{w_l}, \qquad \text{and} \qquad b_{ij}^{(l)} = \frac{v_i^{(l)}}{v_j^{(l)}}. \tag{8}$$

In the case of AHP, formula (1) implies that the holistic evaluation is made by the weighted arithmetic mean of local priorities, i.e. $z_{AHP} = \sum_l w_l v_j^{(l)}$. Next, let us evaluate z_{CI} for the given setting.

Table 3
$\boldsymbol{B}^{(k)}$: alternative pairwise comparison in each criterion.

(a) $\boldsymbol{B}^{(1)}$: Age

	Tom	Dick	Harry	CR
Tom	1	1/3	5	
Dick	3	1	9	0.02795
Harry	1/5	1/9	1	

(b) $\boldsymbol{B}^{(2)}$: Charisma

	Tom	Dick	Harry	CR
Tom	1	5	9	
Dick	1/5	1	4	0.06852
Harry	1/9	1/4	1	

(c) $\boldsymbol{B}^{(3)}$: Education

	Tom	Dick	Harry	CR
Tom	1	3	1/5	
Dick	1/3	1	1/7	0.06239
Harry	5	7	1	

(d) $\boldsymbol{B}^{(4)}$: Experience

	Tom	Dick	Harry	CR
Tom	1	1/4	4	
Dick	4	1	9	0.03548
Harry	1/4	1/9	1	

Theorem 4.2. *Given matrices $\boldsymbol{A}, \boldsymbol{B}^{(l)}$ defined by (8) with $\boldsymbol{w} \in \mathscr{P}(m)$ and $\boldsymbol{v}^{(l)} \in \mathscr{P}(n)$ for all $l = 1, \ldots, m$, vector z_{CI} satisfies*

$$z_{CI} = \frac{\boldsymbol{v}^{\boldsymbol{w}}}{|\boldsymbol{v}^{\boldsymbol{w}}|}. \tag{9}$$

Proof. Following (7), we have $z_{CI} = \kappa\left(\boldsymbol{B}^{\kappa(\boldsymbol{A})}\right)$. Consistency of $\boldsymbol{A}$ and Theorem 3.2 imply that $\kappa(\boldsymbol{A}) = \boldsymbol{w}$. Based on (8) we calculate elements of matrix $\boldsymbol{B}^{\boldsymbol{w}}$; for all $i, j = 1, \ldots, n$ we obtain

$$b_{ij}^{\boldsymbol{w}} = \prod_l \left(b_{ij}^{(l)}\right)^{w_l} = \frac{\prod_l (v_i^{(l)})^{w_l}}{\prod_l (v_j^{(l)})^{w_l}} = \frac{(\boldsymbol{v}^{\boldsymbol{w}})_i}{(\boldsymbol{v}^{\boldsymbol{w}})_j}. \tag{10}$$

Since vector $\boldsymbol{v}^{\boldsymbol{w}}$ reflects the priorities expressed by elements $b_{ij}^{\boldsymbol{w}}$, matrix $\boldsymbol{B}^{\boldsymbol{w}}$ is consistent. Then, Theorem 3.2 implies that $\kappa\left(\boldsymbol{B}^{\boldsymbol{w}}\right)$ has to be proportional to $\boldsymbol{v}^{\boldsymbol{w}}$. After a normalization one arrives at (9). □

The above theorem may be interpreted in the same way as Carpitella et al. (2022, Theorem 3.1); from the formula (9) we can observe that in the newly proposed method, the holistic evaluation is made by combining local priorities using the weighted geometric mean (or, equivalently, the weighted arithmetic mean of logarithms). This approach reminds us of the Weighted Product Model, see, e.g. Triantaphyllou (2000), where evaluations by criteria and the weights of criteria are not exactly given.

4.3. Examples

For practical implementation and further exploration, both examples discussed are available in Python. The complete source code can be accessed and experimented with at Carpitella et al. (2024). This provides an excellent opportunity to interact with and validate the methodologies presented.

4.3.1. Leader example

To demonstrate the typical problem, let us consider a decision-making scenario by Wikipedia contributors (2023) involving three alternatives and four criteria. This can be exemplified using a scenario called "Tom, Dick, and Harry", which is based on a real-world situation of selecting a new leader for a company facing the retirement of its founder. In this example, there are three potential candidates for the leadership role — Tom, Dick, and Harry — and they are assessed based on four different criteria: Age, Charisma, Education, and Experience. The problem involves comparing these criteria in pairs and recording the preferences in a matrix form. These preferences are denoted as a_{ij} and are arranged in an input matrix referred to as $\boldsymbol{A}$ (see Table 2).

In the same way, the preference of each candidate regarding each criterion can be assessed. These preferences are recorded in separate matrices, denoted as $\boldsymbol{B}^{(l)}$, where l represents each of the four criteria (Age, Charisma, Education, Experience). To illustrate the consistency, CR is added as well (see Table 3).

Solution:

Step 1: Compute canonical weight vector $\kappa(\boldsymbol{A})$ (see Table 4).

Table 4
$\kappa(\boldsymbol{A})$.

	Experience	Education	Charisma	Age
$\kappa(\boldsymbol{A})$	0.579	0.145	0.193	0.083

Table 5
Aggregated matrix $\boldsymbol{B}^{\kappa(\boldsymbol{A})}$.

	Tom	Dick	Harry	CR
Tom	1.000	0.654	3.088	
Dick	1.528	1.000	4.223	0.00132
Harry	0.324	0.237	1.000	

Table 6
Vector of final weights $z_{CI} = \kappa\left(\boldsymbol{B}^{\kappa(\boldsymbol{A})}\right)$ and results of other methods.

	Tom	Dick	Harry
z_{CI}	0.346	0.529	0.125
z_{AHP}	0.319	0.520	0.161
BWM	0.392	0.543	0.065
TOPSIS	0.342	0.733	0.188
PROMETHEE	0.386	1.035	−1.421

Step 2: Aggregate preference matrices $\boldsymbol{B}^{(k)}$ using the canonical weight vector $\kappa(\boldsymbol{A})$ to get $\boldsymbol{B}^{\kappa(\boldsymbol{A})}$ (see Table 5)

Step 3: Compute the vector of final weights as the canonical weight vector $z_{CI} = \kappa\left(\boldsymbol{B}^{\kappa(\boldsymbol{A})}\right)$ - see Table 6.

Note that the full implementation of our method, along with additional examples, is available in our GitHub repository (Carpitella et al., 2024). In Table 6, we compare the results of various decision-making methods applied to the Leader example, including AHP, BWM (Rezaei, 2015), TOPSIS (Tzeng & Huang, 2011), and PROMETHEE (Brans, Vincke, & Mareschal, 1986). Although all methods agree on the overall ranking — placing Dick as the top candidate, followed by Tom and Harry — each method employs a different approach, leading to variations in the scoring:

- **Our method** and **AHP** produce similar, balanced results in this consistent case. However, AHP is more rigid and sensitive to inconsistencies in the input data, requiring a consistency ratio check, whereas our method handles inconsistency directly.
- **BWM** yields slightly more polarized results due to its focus on comparing only the best and worst criteria, making it more efficient but also leading to larger gaps between candidates.
- **TOPSIS** emphasizes closeness to an ideal solution, amplifying differences between candidates. As a result, Dick receives a higher score compared to other methods, though the overall ranking remains consistent.
- **PROMETHEE** generates the most extreme results, strongly favoring Dick and penalizing Harry, reflecting its sensitivity to the degree of dominance one candidate has over others.

Each method exhibits different sensitivity to inconsistency, but the final ranking remains unchanged. For further details, including source

Table 7
Key criteria identified through a synthesis of existing literature.

ID	Criterion	Description	Reference
C_1	Quality assurance	Tool's capability in assessing supplier quality. This criterion evaluates the chosen data analytics tool's ability to assess and enhance supplier quality by analyzing historical data on materials or services provided, adherence to specifications, and relevant certifications.	Ali, Nipu, and Khan (2023), Haq, Moazzam, Khan, and Ahmed (2023) and Janeiro, Pereira, Ferreira, Sá, and Silva (2020)
C_2	Cost efficiency	Tool's cost optimization capability. This criterion examines the chosen data analytics tool's capability to optimize costs by analyzing various cost factors, including initial purchase price, shipping costs, potential volume discounts, and long-term cost projections for supplier selections.	Ali et al. (2023) and Hamdan, Cheaitou, Shikhli, and Alsyouf (2023)
C_3	Delivery reliability	Tool's delivery performance analysis. This criterion focuses on the chosen data analytics tool's effectiveness in analyzing and improving delivery reliability. It assesses the tool's capacity to evaluate supplier performance in terms of on-time deliveries and the ability to handle rush orders efficiently.	Ali et al. (2023) and Talay, Oxborrow, and Brindley (2020)
C_4	Supplier reputation	Tool's trustworthiness and stability assessment. This criterion measures the data analytics tool's capacity to assess and enhance supplier reputation. It considers the tool's ability to evaluate factors such as market reputation, financial stability, and past collaborations to ensure trustworthy and stable supplier selections.	Ali et al. (2023) and Talay et al. (2020)
C_5	Environmental impact	Tool's sustainability assessment. This criterion evaluates the chosen data analytics tool's capability to assess and enhance environmental sustainability practices within the supply chain. It assesses the tool's effectiveness in analyzing factors such as waste reduction, ethical sourcing, and carbon footprint reduction for environmentally responsible supplier selections.	Bhandari et al. (2022), Colasante and D'Adamo (2021), Karnad and Udiaver (2022) and Rahman, Bari, Ali, and Taghipour (2022)

code and calculations, please refer to the GitHub repository (Carpitella et al., 2024).

4.3.2. Car example

A more comprehensive and well-known example, often called the 'Car example' (contributors, 2021), represents a family deciding to purchase a new car. This example expands the decision-making process by breaking down broader criteria into more specific elements. For instance, the *cost criterion* is subdivided into *purchase price, fuel costs, maintenance costs*, and *resale value*. Corresponding matrices represent each sub-criteria, denoted as $\boldsymbol{B}_C^{(l)}$ where l is an index representing each sub-criterion. Matrix $\boldsymbol{A}_C$ contains pairwise comparisons of given sub-criteria.

Unlike the simpler pairwise comparison matrix for each criterion, this scenario creates a sub-problem for each detailed criterion. The approach to solving such problems remains similar to the standard AHP method. The process is recursive, addressing each detailed criterion as a separate task. For example, in the case of the cost sub-problem, we get the missing global cost pairwise comparing matrix as the weighted geometric mean of $\boldsymbol{B}_C^{(l)}$ using $\boldsymbol{\kappa}\left(\boldsymbol{A}_C\right)$, i.e. $\boldsymbol{B}_C{}^{\boldsymbol{\kappa}\left(\boldsymbol{A}_C\right)}$. To see the implementation of this problem and its solution using our method, please visit our GitHub repository (Carpitella et al., 2024), where you will find more details, including source code and calculations.

5. Industrial application

To fully grasp the complexities of the following example, we recommend first reading the detailed problem description provided in this paper. Subsequently, the complete implementation in Python can be accessed and analyzed at Carpitella et al. (2024). This allows readers to see the detailed calculations and validate the presented methodologies.

We herein discuss the case of a business company operating in the textile manufacturing sector based in Italy. The company's core business refers to the design, manufacturing, and distribution of textile products such as fabrics, garments, and related items. The core processes performed by the company can be summarized into the following four macro-categories.

Table 8
Alternatives (data analytic tools) to be ranked.

ID	Description
AT_1	Software offering advanced data visualization and reporting capabilities for supplier performance analysis.
AT_2	Software specializing in predictive analytics for supplier performance forecasting, using AI and machine learning.
AT_3	Software providing cost optimization algorithms for supplier negotiation and procurement efficiency.
AT_4	Software focusing on real-time supply chain analytics and monitoring to enhance delivery reliability.
AT_5	Software offering comprehensive supplier reputation analysis through sentiment analysis and social media monitoring, contributing to environmental impact and supplier reputation assessment.
AT_6	Software providing powerful data analysis and visualization tools for assessing supplier quality and cost efficiency.
AT_7	Software offering data preparation and blending capabilities to improve supplier data quality and analysis.
AT_8	Software specializing in interactive data exploration and reporting for supplier reputation assessment and cost-effectiveness analysis.

- Product development and sourcing. This macro process focuses on creating products by researching market trends, designing products, and sourcing materials from suppliers.
- Manufacturing and quality assurance. The company efficiently manufactures textile items through various production stages in this macro process and ensures product quality and safety.
- Sales and marketing. This macro process involves making products available to customers through different sales channels and promoting the brand and products through marketing efforts.
- Operations and management. This macro process refers to overall business operations, including financial management, human resources, technology infrastructure, sustainability practices, and ongoing research and development to stay competitive and innovative in the textile industry.

Table 9
Evaluations of criteria.

(a) DM_1: supply chain analyst

DM_1	C_1	C_2	C_3	C_4	C_5	CR
C_1	1	3	2	4	4	
C_2	1/3	1	4	5	4	
C_3	1/2	1/4	1	1/6	5	0.2842
C_4	1/4	1/5	6	1	5	
C_5	1/4	1/4	1/5	1/5	1	

(b) DM_2: market trend analyst

DM_2	C_1	C_2	C_3	C_4	C_5	CR
C_1	1	3	3	4	1	
C_2	1/3	1	1/5	1/4	1/2	
C_3	1/3	5	1	2	3	0.1965
C_4	1/4	4	1/2	1	4	
C_5	1	2	1/3	1/4	1	

(c) DM_3: sustainability and ethical compliance auditor

DM_3	C_1	C_2	C_3	C_4	C_5	CR
C_1	1	6	6	7	1/8	
C_2	1/6	1	1/5	1/4	1/2	
C_3	1/6	5	1	5	1/3	0.4664
C_4	1/7	4	1/5	1	1/9	
C_5	8	2	3	9	1	

(d) Aggregated PCM $\boldsymbol{A}$

$\boldsymbol{A}$	C_1	C_2	C_3	C_4	C_5	CR
C_1	1	3.7798	3.3019	4.8203	0.7937	
C_2	0.2646	1	0.5429	0.6786	1.0000	
C_3	0.3029	1.8420	1	1.1856	1.7100	0.1114
C_4	0.2075	1.4736	0.8434	1	1.3050	
C_5	1.2599	1.0000	0.5848	0.7663	1	

We specify that overseeing all core processes, from design to distribution, is essential for Italian companies operating in the textile sector, as this ensures quality control and faster responses to market changes. Moreover, this approach safeguards the "Made in Italy" brand's reputation for craftsmanship and style. Monitoring digital transformation trends and adapting to a rapidly changing technological environment is equally critical, as it enhances efficiency and competitiveness in an evolving industry landscape.

In the context of our case study, optimizing the supplier selection process is a paramount objective for the company, aiming at optimizing product quality, reducing costs, and enhancing overall supply chain efficiency by sourcing various materials and services from various suppliers providing textiles, accessories, and logistics services. Given the industry's heavy reliance on data-driven decision-making, the company is facing the decision-making problem of selecting the most suitable data analytics tool for supplier selection. This selection has to be carried out by comprehensively assessing suppliers' performance and other factors such as reputation. AHP methodology emerges as a suitable technique, facilitating a systematic evaluation of how each tool may align with the different criteria essential to textile manufacturing while ensuring efficient resource allocation.

The proposed Consistency-Independent Multiple Criteria Decision Technique demonstrates significant practical value in addressing complex, real-world challenges within the textile manufacturing sector. Specifically, it is applied to support decision-making in supplier selection, a process where traditional AHP often encounters difficulties due to inconsistencies from evaluations' subjective and dynamic nature. Our approach effectively handles these inconsistencies, producing results comparable to AHP when cardinal consistency is present while offering greater flexibility when cardinal and ordinal inconsistencies are unavoidable.

In the supplier selection process, multiple criteria — such as cost, quality, delivery time, and sustainability — are often evaluated subjectively by decision-makers. Inconsistencies in the pairwise comparison matrices can arise from differing stakeholder priorities, but our method accommodates these variations without requiring the extensive adjustments typically needed to ensure ordinal consistency (Wu & Tu, 2021). The ability to process naturally inconsistent data without compromising decision quality is crucial for companies operating in fast-evolving industries like textile manufacturing, where balancing short-term procurement needs with long-term strategic objectives is critical.

For those interested in ordinal consistency, we measured OCI of the input matrices and found an average OCI of approximately 0.2 for $\boldsymbol{A}$ (defined in Table 9) and 0.13 for $\boldsymbol{B}^{(i)}$ (defined in Table 10). In the context of AHP, an OCI of 0.2 is generally considered acceptable for practical decision-making. Readers seeking further details on ordinal consistency can refer to Zhou et al. (2024).

It is also important to acknowledge that implementing any of the analyzed tools represents a substantial financial investment. First, the tool acquisition cost is a significant portion of this expense. Licensing or subscription fees for these tools can be substantial, varying based on features and user access. Moreover, deploying and maintaining the requisite infrastructure and hardware leads to considerable expenses. Another aspect refers to the need to invest in training and skill development, something that is imperative when implementing data analytics tools. Employees must acquire the proficiency to effectively utilize these tools, often necessitating training programs or recruiting skilled personnel, generating both temporal and financial costs. Lastly, customization and integration with existing systems have to be taken into account, as tailoring the tool to align with the company's specific requirements often entails additional development or integration efforts, further augmenting implementation expenses.

In this context, well-informed decision-making is important in light of the anticipated benefits, alongside the inherent complexity and substantial financial commitment tied to this challenge. This reiterates the necessity for structured evaluation processes such as AHP. By performing an AHP-based selection, the company can prudently allocate resources to the tool that offers the most significant value in elevating supplier selection processes. Our objective is to support the company in making informed decisions through iterative applications of the AHP methodology. Specifically, we aim to compare different tools against five predefined criteria ($m = 5$) that are logistic aspects relevant to the textile industry that have been adapted from existing literature and formalized in Table 7. The set of alternatives considered for ranking includes eight data analytics tools ($n = 8$) that have been pre-screened by the company as capable of enhancing its supplier selection process. We do not report the names of the considered software programs, but provide a comprehensive description of each of these alternatives in Table 8 in terms of their performance and achievable results.

A decision-making group comprising three stakeholders has been engaged in the process of determining the vector of criteria weights using AHP technique. Each of these individuals has been attributed equal importance, considering an equitable distribution of weights in the aggregation of individual judgments. The inclusion of three decision-makers (DMs) in this process has been based on their respective experience and complementary viewpoints. This selection aims to acquire the most precise understanding of the problem under examination. The experts within this group have the following roles: DM_1 serves as a supply chain analyst, DM_2 specializes as a market trend analyst, and DM_3 is a sustainability and ethical compliance auditor. Each of these decision-makers was tasked with completing a PCM by evaluating and comparing the criteria concerning the primary decision-making objective. The input evaluations from the three PCMs are presented in Table 9. The last columns of these matrices display the corresponding consistency ratios. In all instances, it is observed that the CR exceeds the threshold of 0.1, indicating a level of inconsistency in the judgments provided by the decision-makers which exceeds the limit considered in traditional AHP. Subsequently, the opinions and assessments of the decision-makers will be aggregated to produce the

Table 10
Evaluation of alternatives with respect to criteria and CR values.

(a) $B^{(1)}$

	AT_1	AT_2	AT_3	AT_4	AT_4	AT_6	AT_7	AT_8	CR
AT_1	1	1/2	1/8	1/7	2	1/4	1/6	1/2	
AT_2	2	1	1/9	1/6	3	1/2	1/2	1/2	
AT_3	8	9	1	2	9	4	3	5	
AT_4	7	6	1/2	1	8	3	2	6	0.0373
AT_4	1/2	1/3	1/9	1/8	1	1/3	1/6	1/3	
AT_6	4	2	1/4	1/3	3	1	2	2	
AT_7	6	2	1/3	1/2	6	1/2	1	3	
AT_8	2	2	1/5	1/6	3	1/2	1/3	1	

(b) $B^{(2)}$

	AT_1	AT_2	AT_3	AT_4	AT_4	AT_6	AT_7	AT_8	CR
AT_1	1	1/2	1/8	1/7	2	1/4	1/2	1/2	
AT_2	2	1	1/9	1/6	3	1/2	1/2	1/2	
AT_3	8	9	1	2	9	4	3	5	
AT_4	7	6	1/2	1	8	3	2	2	0.0745
AT_4	1/2	1/3	1/9	1/8	1	1/3	1/6	1/3	
AT_6	4	2	1/4	1/3	3	1	2	2	
AT_7	2	2	1/3	1/2	6	1/2	1	1/6	
AT_8	2	2	1/5	1/2	3	1/2	6	1	

(c) $B^{(3)}$

	AT_1	AT_2	AT_3	AT_4	AT_4	AT_6	AT_7	AT_8	CR
AT_1	1	1/2	1/8	1/7	2	1/4	1/2	1/2	
AT_2	2	1	1/9	1/6	3	1/2	1/2	1/2	
AT_3	8	9	1	2	9	4	3	5	
AT_4	7	6	1/2	1	8	3	2	2	0.0745
AT_4	1/2	1/3	1/9	1/8	1	1/3	1/6	1/3	
AT_6	4	2	1/4	1/3	3	1	2	2	
AT_7	2	2	1/3	1/2	6	1/2	1	1/6	
AT_8	2	2	1/5	1/2	3	1/2	6	1	

(d) $B^{(4)}$

	AT_1	AT_2	AT_3	AT_4	AT_4	AT_6	AT_7	AT_8	CR
AT_1	1	1/2	1/8	1/7	1/8	1/4	1/2	1/2	
AT_2	2	1	1/4	1/6	1/3	2	2	2	
AT_3	8	4	1	2	1/2	4	3	5	
AT_4	7	6	1/2	1	1/3	3	2	2	0.0913
AT_4	8	3	2	3	1	6	7	8	
AT_6	4	1/2	1/4	1/3	1/6	1	2	2	
AT_7	2	1/2	1/3	1/2	1/7	1/2	1	1/6	
AT_8	2	1/2	1/5	1/2	1/8	1/2	6	1	

(e) $B^{(5)}$

	AT_1	AT_2	AT_3	AT_4	AT_4	AT_6	AT_7	AT_8	CR
AT_1	1	1/2	1/8	1/7	2	1/4	1/2	1/2	
AT_2	2	1	1/4	1/2	5	2	2	3	
AT_3	8	4	1	2	8	4	3	5	
AT_4	7	2	1/2	1	7	3	2	2	0.0933
AT_4	1/2	1/5	1/8	1/7	1	1/6	1/7	1/8	
AT_6	4	1/2	1/4	1/3	6	1	2	2	
AT_7	2	1/2	1/3	1/2	7	1/2	1	1/6	
AT_8	2	1/3	1/5	1/2	8	1/2	6	1	

Table 11
Solution vector.

	AT_1	AT_2	AT_3	AT_4	AT_5	AT_6	AT_7	AT_8
z_{CI}	0.046	0.056	**0.366**	0.183	0.063	0.091	0.122	0.073
z_{AHP}	0.034	0.069	**0.324**	0.217	0.063	0.106	0.095	0.092

PCM reported in Table 9(d), thereby treating the decision-making as a unified entity. The aggregation of the PCM has been achieved through the application of the weighted geometric mean, as outlined in Eq. (3). As previously specified, an equal weight has been uniformly assigned to each stakeholder in this process. This allocation is based on the acknowledgment that decision-makers are considered to have equal mutual significance in the analyzed problem.

Table 10 presents an evaluation of the alternatives under each of the five designated criteria. These evaluations have been collected through various brainstorming sessions with an external data analytics specialist, to compare the suitability of the tools within the problem of reference. The last columns of the table provide the corresponding consistency ratios CR. As it is possible to observe, in all these cases, AHP-based consistency of judgments is verified as all recorded CR values remain below the prescribed threshold of 0.1.

The results presented in Table 11 support the choice of AT_3 as a suitable tool for supplier selection in the context of our case study company, focusing on the need to leverage cost optimization algorithms for supplier negotiation and procurement efficiency. These are indeed essential components of successful supply chain management, enabling the company to negotiate better terms with suppliers, obtain favorable pricing, and optimize core procurement processes to minimize costs while maintaining quality and reliability. The selected alternative would offer a comprehensive suite of procurement and supply chain management tools that can be effectively tailored for the textile industry, providing robust analytics capabilities. Additionally, the cloud-based platform would enable real-time data access and collaboration among departments and partners, promoting agility in a fast-paced industry. All of these elements would positively impact the bottom line by reducing expenses while ensuring that the company remains competitive in the marketplace.

However, it is essential to consider potential constraints in implementing the proposed solution. A major constraint refers to the initial investment required for setup and training. To overcome this, the company can consider the development of a phased implementation plan, also providing step-by-step training to employees to maximize the platform's utilization. Additionally, data integration challenges may arise due to existing systems and processes. Employing data migration and integration solutions can help streamline this process and ensure seamless data flow across platforms. By proactively addressing these constraints, the company could fully exploit the capabilities of the selected data analytics tool to drive its process of supplier selection.

As a second choice, selecting AT_4 as a tool focused on real-time supply chain analytics and monitoring to enhance delivery reliability can be evaluated by the company. While AT_3 offers robust analytics capabilities for procurement and supply chain management, AT_4 would be more effective in providing comprehensive visibility into the entire supply chain, offering insights into supplier performance, quality, and reliability. The company could be supported in analyzing historical supplier data, tracking key performance indicators, and conducting predictive analyses to identify potential risks and opportunities associated with different suppliers.

We emphasize that the ultimate selection will be at the discretion of the company. However, based on the input evaluations used in this case study, it appears that the option least fitting for satisfying the needs of the company is AT_1, despite its advanced data visualization and reporting capabilities for supplier performance analysis. The main reason is that this alternative includes functionalities that may be already covered by other available alternatives, diminishing its value proposition within the context of our company's requirements and preferences.

6. Conclusion

The present research begins by providing a comprehensive overview of AHP, a fundamental tool for facilitating multi-criteria decision-making processes. By systematically comparing alternatives and criteria through pairwise evaluations, AHP aids decision-makers in identifying the most advantageous alternative that best aligns with specified criteria. However, the reliance on consistency within the pairwise comparison matrices poses a challenge, particularly when alternatives cannot be linearly arranged based on importance.

After focusing on partially inconsistent problems within matrices of alternatives in our previous research, the present article aims to address the consistency issue by introducing a comprehensive method based on a Skew-symmetric Bilinear representation that does not necessitate consistency assumptions at any level, offering a generalized approach applicable to all reciprocal pairwise comparison matrices within the AHP framework. We develop various practical examples to validate results, as well as a real case study dealing with inconsistent matrices of criteria within the process of supply chain management for a business company operating in the textile industrial sector. Findings reveal that the proposed method remains compatible with traditional AHP solutions when dealing with consistent matrices, providing a versatile approach for naturally inconsistent decision-making problems.

CRediT authorship contribution statement

Silvia Carpitella: Writing – review & editing, Writing – original draft, Visualization, Validation, Resources, Project administration, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Václav Kratochvíl:** Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Resources, Project administration, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Miroslav Pištěk:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Resources, Project administration, Methodology, Investigation, Formal analysis, Data curation, Conceptualization.

Data availability

The data are available in our GitHub repository (Carpitella et al., 2024), where the entire method is also implemented, along with some additional examples.

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Silvia Carpitella Ph.D. in Technological Innovation Engineering and Ph.D. in Mathematics. Her main research interests refer to decision support systems, treatment of uncertainty affecting human evaluations, mathematical modeling, process management and optimization.

Václav Kratochvíl Research assistant at the Department of Decision-Making Theory in the Institute of Information Theory and Automation, the Czech Academy of Sciences. Among his professional interests are Bayesian networks, polyhedral geometry, and Data-mining.

Miroslav Pištěk Research fellow in the field of Decision-Making Theory. Among his professional interests are utility theory, intransitive preferences, and analysis of Nash Equilibria.