On cardinalities of different degrees of Belief functions conjunctive conflictness

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Abstract

This paper examines mutual conflict behavior between belief function structures across different discernment frame sizes (Ω). Through experiments on Ω_2 to Ω_6, we observe that as frame size increases, non-conflicting pairs and higher-order hidden conflicts become exceedingly relatively rare despite of exponential grows of cardinalities of their classes. The super-exponential growth of possible belief structures complicates exhaustive analysis, leading us to employ random sampling. Our findings reveal that cardinality of class of first-degree hidden conflicts (HC_1) grows faster than cardinality of non-conflicts as frame size increases, highlighting the challenges and implications for applying belief function theory in complex decision-making scenarios.

1 Introduction

The theory of belief functions was developed to better express uncertainty in information, extending beyond traditional methods such as second-order probability, which represents the probability of a probability—or, more precisely, our confidence in the likelihood of a single phenomenon.

In real life, we constantly deal with uncertainty. Consider, for instance, the weather forecast many of us check on our smartphones. You may have noticed that different websites or applications often provide varying predictions. These discrepancies arise from the different models used and the data available to those models, resulting in different sources of information that are typically inconsistent with one another. When faced with this situation, how do we decide which prediction to trust? Instead of choosing just one, belief functions allow us to combine all available sources of information using rules such as Dempster's rule or its non-normalized version, often referred to as the Conjunctive rule.

While combining conflicting information using these methods is possible, there are well-known examples where such combinations lead to paradoxical or meaningless results. A famous example is the combination of medical diagnoses, such as cancer and the flu, where conflicting evidence can produce highly counter-intuitive outcomes, such as assigning an unreasonably high degree of belief to an almost impossible event. This highlights the importance of quantifying the degree of inconsistency—often referred to as the magnitude of conflict—between belief functions.

The simplest definition of conflict comes directly from the Conjunctive rule, where the conflict is quantified by the probability mass assigned to the empty set by the combination rule. The critical difference between the Conjunctive rule and Dempster's rule lies in how they handle this mass: the Conjunctive rule retains it in the empty set, whereas Dempster's rule proportionally redistributes it among all non-empty sets in the resulting combination.

This definition of conflict is highly dependent on the structure of the belief functions involved. The existence of conflict depends on the structure, and its magnitude is influenced by the size of the assigned probability masses. In this article, we focus on the structures of belief functions and their impact on conflict. Specifically, we explore the likelihood that two random structures will generate a conflict, the nature of higher-level hidden conflicts, the numbers of conflicting and non-conflicting pairs, and the technical feasibility of examining these scenarios. These topics form the core of the following discussion.

2 Basic Notions

This section will recall some basic notations needed in this paper.

Assume a finite frame of discernment Ω with elements denoted usually by ω_i, i.e., $\{\omega_1, \omega_2, \ldots, \omega_n\}$ and their sets by capital letters. In the case of $|\Omega| = n$, we will highlight this fact using a subscript as Ω_n. $\mathcal{P}(\Omega) = \{X | X \subseteq \Omega\}$ is a *power-set* of Ω. $\mathcal{P}(\Omega)$ is often denoted also by 2^Ω, e.g., in Pichon et al. (2019).

A *basic belief assignment (bba)* is a mapping $m : \mathcal{P}(\Omega) \longrightarrow [0,1]$ such that $\sum_{A \subseteq \Omega} m(A) = 1$. The values of the bba are called *basic belief masses (bbm)*. $m(\emptyset) = 0$ is usually assumed. We sometimes speak about m as of a mass function.

There are other equivalent representations of m: A *belief function (BF)* is a mapping $Bel : \mathcal{P}(\Omega) \longrightarrow [0,1]$, $Bel(A) = \sum_{\emptyset \neq X \subseteq A} m(X)$. Because there is a unique correspondence between m and corresponding Bel we often speak about m as of a belief function.

Let m be a belief function defined on Ω and $A \subseteq \Omega$. If $m(A) > 0$ we say A is a *focal element* of m. The set of focal elements is denoted by $\mathcal{F}_m$ (or simply $\mathcal{F}$ for short), and we call it a *structure* of m. We say that a focal element $X \in \mathcal{F}$ is proper if $X \neq \Omega$. In the case of $m_{vac}(\Omega) = 1$ we speak about the *vacuous BF* (VBF) and about *a non-vacuous BF* otherwise. If all focal elements have a non-empty intersection, we speak about *consistent*

BF. If focal elements are nested, we speak about consonant BF.

The (non-normalized) conjunctive rule of combination $\odot$, see e.g. Smets (2005), is defined by:

$$(m_1 \odot m_2)(A) = \sum_{X \cap Y = A; X, Y \subseteq \Omega} m_1(X) m_2(Y)$$

for any $A \subseteq \Omega$. $\kappa = \sum_{X \cap Y = \emptyset; X, Y \subseteq \Omega} m_1(X) m_2(Y)$ is usually considered to represent a conflict of respective belief functions when $\kappa > 0$. By normalization of $m_{12} = m_1 \odot m_2$ we obtain Dempster's rule $\oplus$, see Shafer (1976). To simplify formulas, we often use $\bigodot_1^3 m = m \odot m \odot m$, and also $\bigodot_1^k (m_1 \odot m_2) = (m_1 \odot m_2) \odot \ldots \odot (m_1 \odot m_2)$, where $(m_1 \odot m_2)$ is repeated k-times.

3 Hidden conflict

Let us assume conjunctively non-conflicting belief functions m_1 and m_2, i.e., $(m_1 \odot m_2)(\emptyset)$ $= m_{12}(\emptyset) = 0$. In the case that there exists $k \geq 1$ such that $(\bigodot_1^{k+1} m_{12})(\emptyset) > 0$, then we say that there is a hidden conflict of degree k between m_1 and m_2. Note that k is the smallest with this property. We can formalize this in the following definition.

Definition 1 *Assume two BFs m_1 and m_2 such that for some $k > 0$ $(\bigodot_1^k (m_1 \odot m_2))(\emptyset) = 0$. If there further holds $(\bigodot_1^{k+1} (m_1 \odot m_2))(\emptyset) > 0$ we say that there is a* conflict of BFs m_1 *and* m_2 hidden in the k-th degree *(hidden conflict of k-th degree, abbreviated as HC_k). If there is already $(\bigodot_1^{k+1} (m_1 \odot m_2))(\emptyset) = (m_1 \odot m_2))(\emptyset) > 0$ for $k = 0$ then there is a conflict of respective BFs which is not hidden or we can say that it is* conflict hidden in degree zero *(HC_0).*

Arnaud Martin called $(\bigodot_1^k m)(\emptyset)$ *auto-conflict* of of k-th order of m in Osswald and Martin (2006). Thus conflict of m_1 and m_2 hidden in the k-th degree is auto-conflict of m_{12} hidden in k-th degree, specially positive a_{k+1} of combined $m_{12} = m_1 \odot m_2$ hidden by zero $a_k(m_{12})$ (i.e., $a_{k+1}(m_{12}) > 0$ where $a_k(m_{12}) = 0$), see also our contribution in CJS'17 Daniel and Kratochvíl (2017).

Hidden conflict and its degrees are extensions of the classic Shafer's definition of conflict. It is not an alternative definition or approach but a more detailed classification of situations where $m(\emptyset) = 0$.

This technical definition defines all degrees of hidden conflict using repeated combinations[1]. However, the original observation of hidden conflict came from the analysis of conflict *Conf*, defined in Daniel (2014), and its comparison with conjunctive conflict in situations where $Conf(m_1, m_2) > 0$ while $(m_1 \odot m_2)(\emptyset) = 0$. For better insight into

[1] Repeated combination is only a technical tool here: we are interested only in m_1, m_2, and m_{12}, not in bbms of any repeated combination either of them, with the exceptipon of bbm assigned to empty set by the non-normalized conjunctive rule $\odot$, i.e., bbm which is normalized when Dempster's rule is applied. Regardles of this, considering two or more numerically same BFs does not mean anything about their depenency, we can assume that both / all of them come from indepenent sources.

this, we refer to the Introductory and Little Angel examples published in Daniel and Kratochvíl (2020), for brief presentation of these examples see Appendix 1.

From a large amount of results about hidden conflicts and their degrees, we recall the following principal theorem:

Theorem 1 *Hidden conflict of non-vacuous BFs on $\Omega_n, n > 1$ is always of a degree less or equal to $n-2$; i.e., the condition*

$$(\circledcap_1^{n-1}(m_1 \circledcap m_2))(\emptyset) = 0 \tag{1}$$

always means full non-conflictness of respective BFs, and no hidden conflict exists.

For more detail on the limitation of the degree of hidden conflicts in characteristic situations, see Daniel and Kratochvíl (2020) and for comparison with related Pichon's approach, see Daniel and Kratochvíl (2022).

Analogously to degrees of hidden conflicts, degrees of non-conflictness were defined in Daniel and Kratochvíl (2019). Analogously, to distinguishing internal conflict(s) of individual BFs from mutual conflict between them, also internal hidden conflicts are defined, and mutual hidden conflicts distinguished Daniel and Kratochvíl (2020), internal hidden conflict was presented for the first time in CJS 2017 in Pardubice by Daniel and Kratochvíl (2017) in fact. Considering this, a hidden conflict of two BFs is hidden internal conflict of their combination.

Preparing the actual presentation of hidden conflict we have the following observed:
1. Consistent BFs have no internal conflict nor hidden internal conflict, i.e., their auto-conflict any order is always equal to zero.
2. Non-consistent BFs always have some internal conflict, either hidden or non-hidden, i.e., there is always positive auto-conflict of some order.
3. BFs m_1 and m_2 with consistent $m_1 \circledcap m_2$ are in no conflict nor hidden conflict of any degree. (Such m_is are always consistent itselves.)
4. BFs m_1 and m_2 with non-consistent $m_1 \circledcap m_2$ are always in hidden conflict of some degree greater or equal to zero. (regardless whether m_1 and/or m_2 are/is consistent).

Lemma 1 *(i) Two BFs m_1 and m_2 are in a hidden conflict of a positive degree whenever $(m_1 \circledcap m_2)(\emptyset) = 0$ and $m_1 \circledcap m_2$ is not consistent.*
(ii) Specially, hidden conflict of the first degree appears whenever $(m_1 \circledcap m_2)(\emptyset) = 0$ and $m_1 \circledcap m_2$ has positive auto-conflict (of the second order: $a(m_1 \circledcap m_2) = a_2(m_1 \circledcap m_2) > 0$).

Proof. Proofs follow Theorem 5 from Daniel and Kratochvíl (2022) and the previous observations.

4 Conflict analysis

In our previous work Daniel and Kratochvíl (2022), we explored the concept of conjunctive conflict, specifically focusing on the amount of probabilistic mass that the non-normalized conjunctive rule assigns to the empty set. We found that the existence or absence of

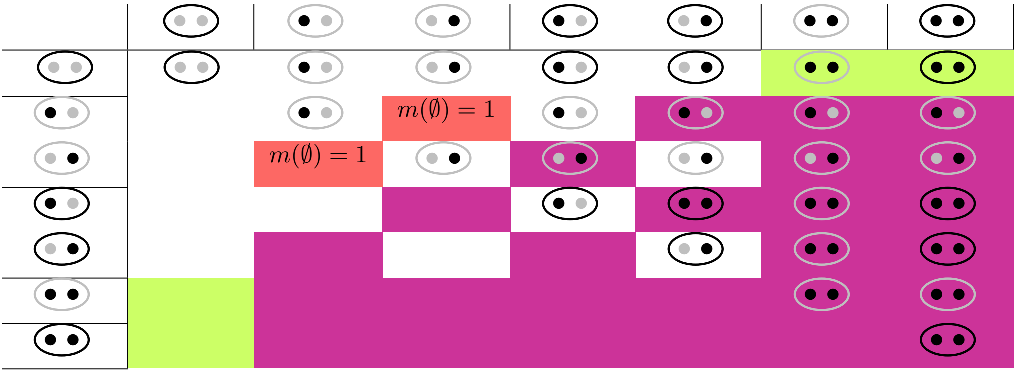

Table 1: All possible conjunctive combinations of belief structures and classes of non-conflictness on Ω_2: White – non-conflict, Green – hidden conflict, Red+Magenta – conflict. The table presents all possible combinations of belief function structures for Ω_2. The different structures are represented along the x and y axes. For Ω_2, there are two singletons and one set of cardinality 2, which represents the entire frame of discernment Ω_2. In the table, sets that are included in a particular structure are shaded black, while those not included are shaded grey.

conjunctive conflict is determined solely by the belief functions' structure. In contrast, the magnitude of the conflict depends on the individual probability masses. However, our primary interest lies in conflict's mere existence or non-existence rather than its magnitude in this study. Therefore, we concentrated on analysing the structures of belief functions.

Given the size of the frame of discernment Ω, we can enumerate the number of unique belief function structures. This enables us to calculate all possible combinations of these structures and determine how many are conflicting, hidden conflicting, or non-conflicting. In our previous study Daniel and Kratochvíl (2022), we performed this analysis for the three smallest frames, where $|\Omega| = 2, 3, 4$. The complete set of structure combinations for $|\Omega| = 2$ is illustrated in Table 1 and for $|\Omega_3| = 3$ by bitmap in Appendix 2, while the counts of conflicting and non-conflicting structures for different cardinalities are summarized in Table 2.

	NC	C i.e., HC_0	HC_1	HC_2	HC_3
Ω_2	17	28	4	–	–
Ω_3	649	14720	756	4	–
Ω_4	258.785	1.071.676.416	1.738.492	2.592	4

Table 2: Number of conflicting belief structure couples in different degrees hiddeness of conflictness/non-conflictness

We can immediately see that the most frequent cases are not hidden conflicts (C, i.e., hidden in degree zero HC_0). There are always four singular cases where hidden conflict of degree $n-1$ (HC_{n-1}) appears for Ω_n. Cardinalities of all other classes rapidly increase with the cardinality of the frame of discernment n.

For these small frames, it is interesting to note that the cardinality of HC_1, and thus the class of hidden conflicts in general, grows significantly faster than the cardinality of the non-conflict class (NC). There are significantly fewer hidden conflicts on Ω_2, but already more on Ω_3, and significantly more hidden conflicts than non-conflicts on Ω_4.

The cardinality of HC_2 is less than that of NC on Ω_3 and Ω_4. Nevertheless, it also grows quicker than the cardinality of NC: it is about 160 times less on Ω_3, while only about 100 times less on Ω_4.

5 Random sampling approach

As we attempt to extend our analysis to higher dimensions, we encounter a significant computational challenge due to the super-exponential growth in the number of structures. For example, with Ω_5, there are 31 possible focal elements, resulting in 2^{31} possible structures and 2^{62} combinations of these structures. The sheer magnitude of these numbers makes it infeasible to compute all possible combinations using current technology, and the problem only worsens with Ω_6 and beyond.

One potential solution is to employ random sampling. By selecting a sufficiently large random sample of structure combinations, we can estimate the distribution of different classes of conflicts, including different degrees of hidden conflict.

First, we validated our approach by performing random generation for Ω_2, Ω_3, and Ω_4 to ensure its accuracy. For these cases, we converted the results from Table 2 into percentages. Table 3 compares our randomly sampled results and the original results obtained from an exhaustive search of all possible combinations of the listed structures. The first three rows of Table 3 represent the exact results from the complete search, while the next three rows show the outcomes based on random sampling.

	NC	C	HC_1	HC_2	HC_3
Ω_2	34.7	57.1	8.2	–	–
Ω_3	4.02	91.26	4.69	0.025	–
Ω_4	0.024	99.814	0.162	0.00024	0.000000373
Ω_2 sampling	35.41	56.22	8.37	–	–
Ω_3 sampling	4.09	91.17	4.72	0.025	–
Ω_4 sampling	0.023	99.817	0.159	0.001	0.000

Table 3: Percentage representation of conflict levels across all possible combinations of Belief function structures for a given frame of discernment

The generation of random structures was carried out in two steps. First, we determined the number of focal elements for the generated structure. Then, we randomly selected the corresponding number of distinct subsets from Ω, forming the desired structure. The number of focal elements was generated such that the probability of selecting a given number corresponded to the frequency distribution of focal elements among all possible structures on Ω. As is well-known from combinatorics, the most probable number of focal elements is approximately $2^{|\Omega|}/2$, which aligns with the highest number of possible combinations of subsets. Due to applying this two-step generation, we validated it as presented in Table 3.

Encouraged by our initial success, we proceeded to experiment with the Ω_5 frame. Given the vast number of possible combinations of the structures involved, we conducted eight separate experiments, each consisting of 100 million random combinations. We divided these into eight batches to assess the consistency of the intermediate results. The results from these individual experiments are remarkably consistent and are summarized in Table 4.

no.	NC	C	HC_1	HC_2	HC_3	HC_4
1	1	99999968	31	0	0	0
2	0	99999969	31	0	0	0
3	1	99999960	39	0	0	0
4	0	99999961	39	0	0	0
5	0	99999963	37	0	0	0
6	1	99999950	49	0	0	0
7	0	99999953	47	0	0	0
8	0	99999965	35	0	0	0

Table 4: Results of random sampling for Ω_5

For Ω_6, the results were even more compelling. Given the enormous number of possible structures and their combinations, we conducted 4 billion trials, organized into 160 sets of 25 million pairs each. Despite the extensive sampling, we did not find a single non-conflicting pair or a pair with a hidden conflict - summarized in Table 5. In other words, all pairs exhibited conflict. This suggests that the probability of encountering a pair/couple of non-conflicting belief functions for a larger frame of discernment is exceedingly close to zero, and the same holds for couples in hidden conflict of various degrees.

	NC	C	HC_1	HC_2	HC_3	HC_4	HC_5
Ω_6	0	4.000.000.000	0	0	0	0	0

Table 5: Results of random sampling for Ω_6

6 Summary and Results Analysis

Presenting the results of our experiments on Ω_5, we can confirm our observation that the cardinality of HC_1 grows much faster than NC, already reaching about a hundred times greater in this case. Unfortunately, the 100 million generated samples were insufficient to obtain a HC_i case for $i > 1$.

	NC	C	HC_1	HC_2	HC_3	HC_4	HC_5
Ω_2	17	28	4	–	–	–	–
Ω_3	649	14 720	756	4	–	–	–
Ω_4	258 785	1 071 676 416	1 738 492	2 592	4	–	–
Ω_5	$1.725e10$	$4.611e18$	$1.771e12$	$(>> 3e3)$	$(>> 4)$	4	–
Ω_6	?	$8.507e37$	?	?	?	?	4

Table 6: Numbers of conflicting belief structure couples in different degrees of conflictness/non-conflictness. There are precise number for $\Omega_2 - \Omega_4$, and estimation for entire space for Ω_5 and Ω_6

Since we know that the cardinality of HC_{n-1} is equal to 4 and that all HC_i values are greater for $i > n-1$, we have marked 0^+ in the cases where no samples were generated, although they could theoretically exist. Similarly, for Ω_6, even 25 million generated pairs were not enough to encounter anything other than a conflict that is not hidden.

Analyzing our results, it is clear that all classes of conflict/non-conflict increase with n. The largest class is always $C \sim HC_0$: the class of pairs with a classic unhidden conjunctive conflict. The cardinality of HC_i decreases from a maximum at $i = 0$ down to 4 for HC_{n-1}. The second-largest class, HC_1, is greater than NC (the class of non-conflicting pairs) for $n \geq 3$ and grows faster than NC as n increases.

As we are interesting only in the belief structures, we have sizes of no conflicts nor hidden conflicts here. Nevertheless, we should notice one exception which is structural: i.e., *full conflict* where all focal elements of m_1 have empty intersection with all focal elements of m_2, there appears $m_{12}(\emptyset) = 0$, the case where the conjunctive rule gives no other information and Dempster's rule is not applicable. There are two such cases on Ω_2, see red fields in Table 1, 36 on Ω_3, and 1 154 on Ω_4. This class also grows with the size of frame, but we can see that its cardinality incomparably less with the class of all conflicts HC_0 and also less than cardinalities of NC, HC_1 and on Ω_4 less than HC_2. I.e., cardinality of FC is less than cardinalities all classes investigated here, with the exception of HC_{n-1} which is always equal to 4 in any frame.

	NC	C	HC_1	HC_2	HC_3	HC_4	HC_5
Ω_2	34.7	57.1	8.2	–	–	–	–
Ω_3	4.02	91.26	4.69	0.025	–	–	–
Ω_4	0.024	99.814	0.162	0.00024	3.73e-10	–	–
Ω_5	$3.75e{-9}$	100^-	$3.85e{-7}$	0^+	0^+	$8.67e{-15}$	–
Ω_6	0^+	100^-	0^+	0^+	0^+	0^+	$4.70e{-36}$

Table 7: Actual (for $\Omega_2 - \Omega_4$) and estimated ($\Omega_5 - \Omega_6$) percentages of conflict types among all possible combinations of structures for a given frame of discernment size

As the actual cardinalities of these rapidly growing classes are difficult to conceptualize, it is more intuitive to compare the percentages of belief structure pairs in particular conflict classes (see Table 7) or in direct comparison of the classes: for comparison of HC_i

classes with class of non-conflicting pairs and of class fully conflicting pairs with the other classes see Table 8. We can see rapid increase relative comparison of HC_1/N with the size of the frame, while decrease of any comparison of FC, especially with the class of all conflicting pairs C.

Since we did not observe any HC_i situations for $i > 1$ within our random generation for Ω_5 and Ω_6, but we know such situations exist, we have marked the corresponding fields in Table 7 for Ω_5 and Ω_6 with 0^+.

	HC_1/NC	HC_2/NC	HC_3/NC	FC / NC	FC / C	FC/HC_1	FC/HC_2
Ω_2	0.235	–	–	1.176e-1	7.143e-2	5.000e-1	–
Ω_3	1.166	0.006	–	5.547e-2	2.446e-3	4.762e-2	9.000e0
Ω_4	6.718	0.010	1.545e-5	4.459e-3	1.077e-6	6.638e-4	4.452e-1
Ω_5	102.666	(?)	(?)	(?)	(?)	(?)	(?)

Table 8: Increasing of relative frequencies of HC_i/NC with size of frame (columns 1–3). Relative frequencies of full conflict in comparison with conflicting/non-conflicting classes decreasing with size of frame (columns 4–7)

7 Conclusion

In this study, we investigated the behavior of conflict of couples of belief function structures, particularly focusing on the probability and distribution of conflicts across discernment frames Ω of different sizes. Through extensive experiments on Ω_2 to Ω_6, we observed that the cardinality of **all** conflict classes, particularly non-hidden conflicts (C), increases significantly with the size of Ω. Notably, our experiments on Ω_5 and Ω_6 revealed that despite the growing of cardinalities of all degrees of hidden conflicts, that even non-conflicting pairs and pairs of small positive higher-order degree of hidden conflicts ($i = 1, 2$) are exceedingly rare; confirming the hypothesis that the probability of encountering such pairs is almost negligible as the frame size grows and completely general[2] belief functions are considered.

The results underscore the computational challenges posed by the super-exponential growth in the number of possible belief structures and their combinations, making exhaustive searches infeasible for larger frames. Our use of random sampling provided valuable insights. However, the limitations of this method became apparent when no higher-order hidden conflicts were observed in larger frames despite their theoretical existence.

Furthermore, our analysis highlighted the disproportionate growth of certain conflict classes, particularly HC_1, which quickly surpasses non-conflicting pairs as Ω increases.

These findings suggest that as the frame of discernment expands, the likelihood of encountering meaningful, non-conflicting belief combinations diminishes, raising important questions about the practical implications of belief function theory in large-scale applications.

[2]It is, of course, different if some restricted class of belief function is considered: either from the point of view of their structure or the limitation of the number or size of focal elements.

In conclusion, while belief functions offer a robust framework for managing uncertainty, our findings indicate that the prevalence of conflict, particularly in larger frames, necessitates careful consideration in practical applications. Future work could explore alternative methods for managing or mitigating conflicts in belief structures, especially as the scale of analysis increases. Additionally, further research into the theoretical underpinnings of conflict distribution may yield new insights that can enhance the utility of belief function theory in complex decision-making scenarios.

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Appendix 1: Hidden Conflict Examples

In accordance to the body of this study, only belief structures are important for existence and degree of conjunctive (hidden) conflict. Hence for our examples graphical presentations in Figures 1 and 2, are more important than any specific numeric belief masses.

In our examples, we would like to illustrate how a hidden conflict is revealed. Note that because of the commutativity of $\textcircled{\cap}$, we can rewrite $\big(\textcircled{\cap}_1^3(m_1\textcircled{\cap}m_2)\big)$ into $\big(\textcircled{\cap}_1^3(m_1)\textcircled{\cap}\textcircled{\cap}_1^3(m_2)\big)$, etc. Once a positive mass is assigned to the empty set, it cannot be removed by $\textcircled{\cap}$. Let us highlight the first occurrence of a positive mass on an empty set to clarify the examples.

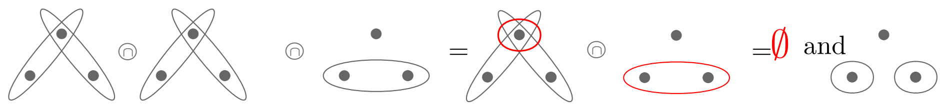


Figure 1: Arising of a hidden conflict: focal elements of m', m', m''; $m'\textcircled{\cap}m'$, m''; and of $m'\textcircled{\cap}m'\textcircled{\cap}m''$. Where ω_1 is the top one element, ω_2 and ω_3 clock-wise numbered.

Introductory Example. Let us assume Bel', Bel'' on Ω_3 where $\mathcal{F}' = \{\{\omega_1,\omega_2\}, \{\omega_1,\omega_3\}\}$ and $\mathcal{F}'' = \{\{\omega_2,\omega_3\}\}$ (see Fig. 1). Then $(m'\textcircled{\cap}m'')(\emptyset) = 0$. But $(m'\textcircled{\cap}m'\textcircled{\cap}m'')(\emptyset) > 0$ (as highlighted in Figure 1, where conflicting focal elements are drawn in red), which implies $\textcircled{\cap}_1^2(m'\textcircled{\cap}m'')(\emptyset) > 0$ as well. Thus, there is a conflict hidden in the 1st degree. For detail, see Daniel and Kratochvíl (2020); and for an example of numeric bmms also Table 9. In comparison with $(m'\textcircled{\cap}m'\textcircled{\cap}m'')(\emptyset) = (m'\textcircled{\cap}m''\textcircled{\cap}\ m'\textcircled{\cap}m'')(\emptyset) = 0.48$, the conflict based on non-conflicting parts of belief functions (see Daniel (2014)) $Conf(m',m'') = 0.40$, see Daniel and Kratochvíl (2020).

X	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_1,\omega_2\}$	$\{\omega_1,\omega_3\}$	$\{\omega_2,\omega_3\}$	$\{\omega_1,\omega_2,\omega_3\}$	$\emptyset$
$m'(X)$	0.0	0.0	0.0	0.60	0.40	0.00	0.00	–
$m''(X) =$	0.0	0.0	0.0	0.00	0.00	1.00	0.00	–
$(m'\textcircled{\cap}m'')(X)$	0.00	0.60	0.40	0.00	0.00	0.00	0.00	0.00
$(m'\textcircled{\cap}m''\textcircled{\cap}m'\textcircled{\cap}m'')(X)$	0.00	0.36	0.16	0.00	0.00	0.00	0.00	0.48

Table 9: Belief masses in the Introductory Example.

Little Angel Example. Let us have two BFs Bel_1 and Bel_2 on $\Omega_5 = \{\omega_1,\omega_2,...,\omega_5\}$: $\mathcal{F}_1 = \{A,B,C\} = \{\{\omega_1,\omega_2,\omega_5\}, \{\omega_1,\omega_2,\omega_3, \omega_4\}, \{\omega_1,\omega_3,\omega_4,\omega_5\}\}$, $\mathcal{F}_2 = \{D\} = \{\{\omega_2,\omega_3, \omega_4,\omega_5\}\}$, i.e., $|\mathcal{F}_1| = 3$ while $|\mathcal{F}_2| = 1$. Respective structures can be seen in Fig. 2 where sets of focal elements of individual BFs m_1 $(3\times)$ and m_2 $(1\times)$ are depicted in its first row (ω_1 is on the top with ω_is clock-wise enumerated). Again, there is $(m_1\textcircled{\cap}m_2)(\emptyset) = 0$ (there is no empty intersection of any $X \in \mathcal{F}_1$ with $Y \in \mathcal{F}_2$). Moreover, $\textcircled{\cap}_1^2(m_1\textcircled{\cap}m_2)(\emptyset) = 0$ in this example. Finally, $(\textcircled{\cap}_1^3(m_1\textcircled{\cap}m_2))(\emptyset) > 0$. Thus there is a hidden conflict of the 2-nd degree. Following the second line of Figure 2, the empty set emerges as the intersection of focal elements drawn by red color, i.e., it appears already in $m_1\textcircled{\cap}m_1\textcircled{\cap}m_1\textcircled{\cap}m_2$.

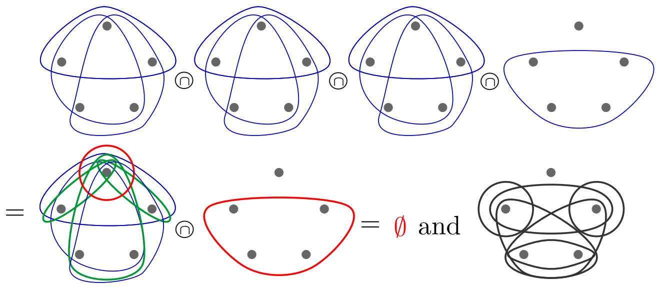


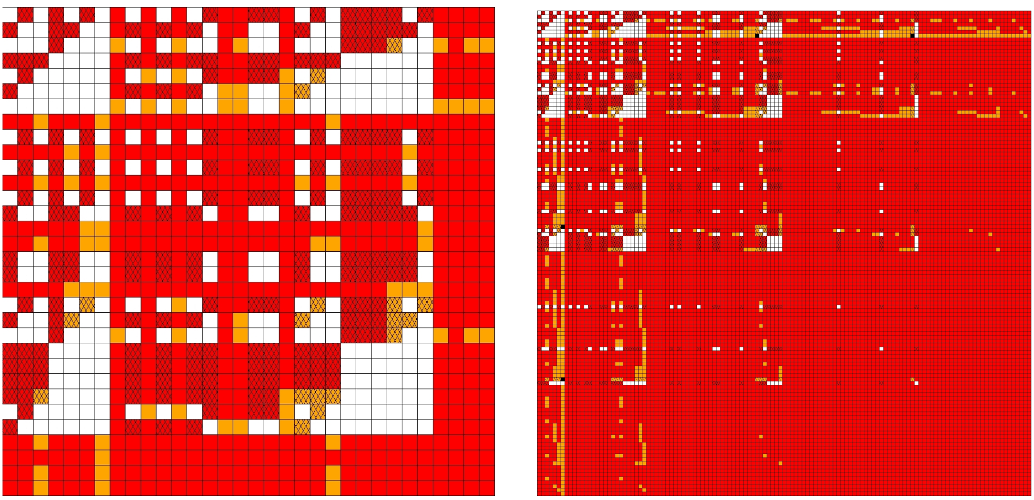
Figure 2: Little Angel example: focal elements of m_1, m_1, m_1, m_2; $\circledcap_1^3 m_1, m_2$; $(\circledcap_1^3 m_1) \circledcap m_2$

Consistency of both m_1 and m_2 is underlined in Daniel and Kratochvíl (2020), nevertheless we already know, that non-consistency of their combination $m_1 \circledcap m_2$ is more important.

For numeric values, see the original instance of Little Angel example in Daniel and Kratochvíl (2020), where is $(\circledcap_1^3 (m_1 \circledcap m_1))(\emptyset) = 0.108 > 0$ and $Conf(m_1, m_2) = 0.1 > 0$.

Appendix 2: Bitmaps of Couples of Ω_3 Belief Structures

Analogously to Table 1 in Section 4 for Ω_2, we can present situation for Ω_3 by a bitmap presented in Figure 3.



(a) Zoom of left upper part (32 × 32)

(b) Full 127 × 127 bitmap

Figure 3: Hiddeness degree bitmap of conjunctive conflict of belief structures on Ω_3: White – full non-conflict (degree 3), Black – HC_2, Orange – HC_1, Red – conflict – HC_0 (degree 0).