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RESEARCH PAPER

Global weight optimization of frame structures with polynomial programming

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Abstract

Weight optimization of frame structures with continuous cross-section parametrization is a challenging non-convex problem that has traditionally been solved by local optimization techniques. Here, we exploit its inherent semi-algebraic structure and adopt the Lasserre hierarchy of relaxations to compute the global minimizers. While this hierarchy generates a natural sequence of lower bounds, we show, under mild assumptions, how to project the relaxed solutions onto the feasible set of the original problem and thus construct feasible upper bounds. Based on these bounds, we develop a simple sufficient condition of global ε-optimality. Finally, we prove that the optimality gap converges to zero in the limit if the set of global minimizers is convex. We demonstrate these results by means of two academic illustrations.

Keywords Topology optimization · Frame structures · Semidefinite programming · Polynomial optimization · Global optimality

1 Introduction

Finding cross-section parameters that minimize the weight of frame structures for given performance constraints constitutes a fundamental problem of structural design (Bendsøe and Sigmund 2004; Saka and Geem 2013). This problem naturally arises, e.g., in civil (Thevendran et al. 1992; Mosquera and Gargoum 2014), automotive (Zuo et al. 2016), or machine (Tyburec et al. 2019) industries.

In contrast to optimization of trusses, for which several convex formulations were established thanks to the linear dependence of the stiffness matrix on the cross-section areas (Stolpe 2017; Kočvara 2017; Bendsøe and Sigmund 2004), the stiffness of frame elements is non-linear due to the non-linear coupling of cross-section areas with moments of inertia. Therefore, the emergent optimization problems are non-convex in general and very challenging to be solved globally (Yamada and Kanno 2015; Tyburec et al. 2021; Toragay et al. 2022).

Due to the non-convexity, majority of methods for optimizing frame structures are either local, or (meta-)heuristic (Saka and Geem 2013), thus converging to structures of unknown quality with respect to the global minimizers. For example, Saka (1980) optimized the weight of frame structures while accounting for stress and displacement constraints using sequential linear programming, and Wang and Arora (2006) improved over its solution efficiency by adopting sequential quadratic programming instead. For the same setting, Khan (1984) and Chan et al. (1995) developed optimality criteria methods. Further, Yamada and Kanno (2015) optimized the weight of frame structures for a prescribed fundamental free-vibration eigenfrequency lower bound using a sequence of semidefinite programming relaxations, concluding that good-quality local optima are attained.

Several optimization methods have also been developed for the discrete setting of the frame optimization problem, i.e., considering a catalog of available cross-sections. The associated optimization methods are naturally enumerative, with the global optimizers reachable by branch-and-bound-type methods. For example, Kureta and Kanno (2013); Hirota and Kanno (2015) and Mellaert et al. (2017) formulated

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mixed-integer linear programs, and Kanno (2016) developed a mixed-integer second-order conic programming formulation. Another related approach was introduced by Wang et al. (2021), who solved a single semidefinite programming relaxation whose solution served as an input to a (meta-heuristic) neighborhood search of a differential evolution algorithm. Many other heuristic and meta-heuristic approaches were presented, we refer the reader to (Saka and Geem 2013) for an extensive review.

Returning to the continuous case, which is the purpose of this work, only three global approaches have been presented to the best of our knowledge. First, Toragay et al. (2022) tackled the displacement-constrained weight minimization problem of frame structures by casting it into a mixed integer quadratically constrained program. This formulation further involved constraints preventing virtually intersecting frame elements, bounds on the cross-section areas, and a set of additional cut constraints to reduce the feasible design space. The emergent optimization problems assumed parametrization of the moments of inertia by degree-two polynomials of the cross-section areas, and were solved globally using a branch-and-bound method.

The remaining two methods are based on a fairly different concept: they provide hierarchies of relaxations of increasing size, hence avoid the need for enumeration. Using the sum-of-squares (SOS) hierarchy of specific semidefinite programming relaxations (Kojima 2003), (Murota et al. (2010), Section 5.3) optimized the weight of frame structures while bounding the fundamental free-vibration eigenvalue from below. Despite theoretically guaranteed convergence of the objective function values (Kojima and Muramatsu 2006), the associated optimal solutions may remain unknown.

Adopting a dual approach to Murota et al. (2010)—the moment-sum-of-squares (MSOS) hierarchy (Lasserre 2001; Henrion and Lasserre 2006)— Tyburec et al. (2021) considered compliance optimization of volume-constrained frame and shell structures while accounting for multiple loading scenarios and self-weight. Similarly to the SOS hierarchy, the MSOS hierarchy guaranteed a convergence of the objective function values, but further maintained a simple procedure for extracting the global minimizers at the convergence (Henrion and Lasserre 2006). Furthermore, Tyburec et al. (2021) provided a simple method for projecting the relaxed solutions onto the feasible set of the original problem, thereby providing both lower- and feasible upper-bounds. These bounds naturally assess the quality of the relaxations and guarantee performance gap of the upper-bound designs with respect to the global minimizers. Finally, they showed that the optimality gap approaches zero if the global minimizer is unique.

1.1 Aims and novelty

In this contribution, we investigate global weight optimization of frame structures with bounded compliance of multiple loading scenarios. This is achieved by extending our original results for global compliance optimization of frame structures via the MSOS hierarchy (Tyburec et al. 2021).

In the MSOS hierarchy, exactness of relaxations follows from a rank condition. If not satisfied, the relaxation is not exact and its quality is generally uncertain. The goal of this paper is thus to estimate the quality of inexact relaxations for the weight optimization problem by providing a method for constructing feasible upper bounds from the relaxed solutions. These ingredients also settle a simple sufficient condition of global ε-optimality, vanishing in the limit for problems with the global minimizers forming a convex set.

In contrast to (Toragay et al. 2022), our work avoids the need for an enumerative exploration of the feasible space. Moreover, our setup naturally accommodates degree-three (and possibly higher-degree) polynomials, which are necessary, e.g., for the height optimization of rectangular cross-sections, or for thickness optimization of plates and shells (Tyburec et al. 2021, Section 3.5).

This contribution is organized as follows. We start by a brief introduction to the moment-sum-of-squares hierarchy in Sect. 2 and formalizing the investigated optimization problem in Sect. 3.1. Section 3.2 provides basic mathematical properties of the compliance function. In particular, we show that compliance is monotonic w.r.t. the scaling of the cross-section areas and also derive the bounds for the compliance function. Sect. 3.3 is devoted to construction of upper bounds to the optimization problem. Its idea relies on scalarizing the optimization problem and showing that the bounds to the compliance function remain the same in most cases, rendering the scalarized compliance constraint feasible if the original problem admits a solution. Finally, Sect. 3.4 presents an optimization problem formulation that satisfies convergence assumptions of the moment-sum-of-squares hierarchy, constructs feasible upper bounds from the relaxations, and develops a certificate of global ε-optimality. We illustrate these theoretical findings numerically in Sect. 4, and summarize our contribution in Sect. 5.

2 Moment-sum-of-squares hierarchy

We start with a brief introduction to the moment-sum-of-squares hierarchy. We refer the reader to (Lasserre 2001, 2015; Henrion and Lasserre 2006) for a more thorough treatment.

Let us consider optimization problems of the form

$$\min_{\mathbf{x}} \ f(\mathbf{x}) \tag{1a}$$

$$\text{s.t. } \mathbf{G}(\mathbf{x}) \succeq 0, \tag{1b}$$

where $f(\mathbf{x}) : \mathbb{R}^n \mapsto \mathbb{R}$ and $\mathbf{G}(\mathbf{x}) : \mathbb{R}^n \mapsto \mathbb{S}^m$ are real polynomial mappings and $\mathbb{S}^m$ stands for the space of $m \times m$ real symmetric square matrices. Further, the notation $\bullet \succ 0$ ($\bullet \succeq 0$) denotes positive definiteness (semi-definiteness) of $\bullet$ and $\mathcal{K}(\mathbf{G}(\mathbf{x}))$ represents the feasible set of (1b).

Let now $\mathbf{x} \mapsto \mathbf{b}_k(\mathbf{x})$ be the polynomial space basis of polynomials in $\mathbb{R}^n$ of degree at most k

$$\mathbf{b}_k(\mathbf{x}) = \begin{pmatrix} 1 & x_1 & \dots & x_n & x_1^2 & x_1x_2 & \dots & x_n^2 & \dots & x_n^k \end{pmatrix}. \tag{2}$$

Then, using a coefficient vector $\mathbf{q} \in \mathbb{R}^{|\mathbf{b}_k(\mathbf{x})|}$, we can write any polynomial $p(\mathbf{x}) : \mathbb{R}^n \mapsto \mathbb{R}$ of degree at most k as a linear combination of the monomial entries in the basis $\mathbf{b}_k(\mathbf{x})$, i.e., $p(\mathbf{x}) = \mathbf{q}^{\mathrm{T}}\mathbf{b}_k(\mathbf{x})$.

Also, let $\left\{\boldsymbol{\alpha} \in \mathbb{N}^n : \mathbf{1}^{\mathrm{T}}\boldsymbol{\alpha} \leq k, \prod_{i=1}^n x_i^{\alpha_i} \in \mathbf{b}_k(\mathbf{x})\right\}$ be a multi-index and $\mathbf{y} \in \mathbb{R}^{|\mathbf{b}_k(\mathbf{x})|}$ the moments of probability measures supported on $\mathcal{K}(\mathbf{G}(\mathbf{x}))$. In this work, we label the moment vector entries associated with the monomials $\prod_{i=1}^n x_i^{\alpha_i} \in \mathbf{b}_k(\mathbf{x})$ as $y_{\boldsymbol{\alpha}} = y_{\prod_{i=1}^n x_i^{\alpha_i}}$.

Furthermore, we need to set a formal definition of the (matrix) sum-of-squares decomposition:

Definition 1 The matrix $\boldsymbol{\Sigma}(\mathbf{x}) : \mathbb{R}^n \mapsto \mathbb{S}^m$ is a (matrix) sum-of-squares function if there exists a matrix $\mathbf{H}(\mathbf{x}) : \mathbb{R}^n \mapsto \mathbb{R}^{m \times o}$ such that $\forall \mathbf{x} : \boldsymbol{\Sigma}(\mathbf{x}) = \mathbf{H}(\mathbf{x}) \left[\mathbf{H}(\mathbf{x})\right]^{\mathrm{T}}$.

Notice that when $m = 1$, Definition 1 reduces to the case of scalar sum-of-squares polynomials.

Using this definition and $\langle \mathbf{X}, \mathbf{Y} \rangle = \mathrm{Tr}(\mathbf{X}\mathbf{Y}^{\mathrm{T}})$ to denote the inner product on $\mathbb{S}^m$, we provide the (Archimedean) assumption of algebraic compactness:

Assumption 1 (Henrion and Lasserre 2006) There exist sum-of-squares polynomials $p_0(\mathbf{x}) : \mathbb{R}^n \mapsto \mathbb{S}^1$ and $\mathbf{R}(\mathbf{x}) : \mathbb{R}^n \mapsto \mathbb{S}^m$ such that the superlevel set $\left\{\mathbf{x} \in \mathbb{R}^n : p_0(\mathbf{x}) + \langle \mathbf{R}(\mathbf{x}), \mathbf{G}(\mathbf{x}) \rangle \geq 0\right\}$ is compact.

If Assumption 1 holds, (1) is equivalent to an infinite-dimensional linear semidefinite (and hence convex) program

$$f^{(r)} = \min_{\mathbf{y}} \ \mathbf{q}_0^{\mathrm{T}}\mathbf{y} \tag{3a}$$

$$\text{s.t. } \mathbf{M}_{2r}(\mathbf{y}) \succeq 0 \tag{3b}$$

$$\mathbf{M}_{2r-d}(\mathbf{G}\mathbf{y}) \succeq 0, \tag{3c}$$

with the relaxation degree $r \to \infty$. For $r \in \mathbb{N}$ and finite, (3) then provides a finite-dimensional truncation of (1). In (3), d denotes the maximum degree of a polynomial in $\mathbf{G}(\mathbf{x})$, and $\mathbf{M}_{2r}(\mathbf{y})$ with $\mathbf{M}_{2r-d}(\mathbf{G}\mathbf{y})$ are the (truncated) moment and localizing matrices associated with the moments $\mathbf{y}$ and $\mathbf{G}\mathbf{y}$, respectively. We refer the reader to (Henrion and Lasserre 2006, Section D) for more details.

With increased relaxation degree r, larger portions of the infinite-dimensional program are incorporated, so that a convergence to the optimum value f^* of f is obtained in the limit.

Theorem 1 (Henrion and Lasserre 2006) *Let Assumption 1 be satisfied. Then,* $f^{(r)} \nearrow f^*$ *as* $r \to \infty$.

However, the convergence is generically finite (Nie 2013) and usually occurs at a low r. In addition to the convergence of the objective function value, the global optimality can be recognized and the corresponding minimizers extracted using the flat extension theorem of Curto and Fialkow (1996). We again refer an interested reader to (Henrion and Lasserre 2006; Lasserre 2015) for more information.

3 Methods

This section introduces the main theoretical results of this article. In particular, we first formalize the optimization problem in Sect. 3.1. Section 3.2 develops bounds for the compliance function and shows that the compliance function is monotonic w.r.t. scaling of the cross-section areas. In Sect. 3.2, we rely on the assumption of a statically admissible design. We show that scaling the corresponding cross-section areas preserves the bounds of the compliance function under mild assumptions, which allows us to construct feasible upper bounds to the original optimization problem. Finally, Sect. 3.4 modifies the original optimization problem formulation for the moment-sum-of-squares hierarchy, presents a sequence of lower and upper bounds, and develops a simple sufficiency condition of global ε-optimality.

3.1 Optimization problem formulation

This paper deals with a global solution of the weight minimization (4a) problem with bounded compliances of the n_{lc} load cases (4c) under linear-elastic equilibrium (4b) and non-negativity of the design variables (4d):

$$\min_{\mathbf{a},\mathbf{u}} \ \sum_{e=1}^{n_{\mathrm{e}}} \rho_e \ell_e a_e \tag{4a}$$

$$\text{s.t. } \mathbf{K}_j(\mathbf{a})\mathbf{u}_j = \mathbf{f}_j, \quad \forall j \in \{1, \dots, n_{\mathrm{lc}}\}, \tag{4b}$$

$$\overline{c}_j - \mathbf{f}_j^{\mathrm{T}}\mathbf{u}_j \geq 0, \quad \forall j \in \{1, \dots, n_{\mathrm{lc}}\}, \tag{4c}$$

$$\mathbf{a} \geq \mathbf{0}. \tag{4d}$$

In (4), $\mathbf{a} \in \mathbb{R}^{n_e}_{\geq 0}$ is the vector of the design variables such as the cross-section areas of frames, n_e denotes the number of elements, $\boldsymbol{\ell} \in \mathbb{R}^{n_e}_{>0}$ stands for a vector of volume multipliers so that the volume of the e-th element amounts to $\ell_e a_e$, and $\boldsymbol{\rho} \in \mathbb{R}^{n_e}_{>0}$ are the element densities. Further, $\overline{\mathbf{c}} \in \mathbb{R}^{n_{lc}}_{>0}$ are upper bounds for the compliance of the n_{lc} load cases, and $\mathbf{u}_j \in \mathbb{R}^{n_{dof,j}}$ with $\mathbf{f}_j \in \mathbb{R}^{n_{dof,j}}$ stand respectively for the generalized displacement and force vectors of the j-th load case, with $n_{dof,j}$ being the number of degrees of freedom. Without loss of generality, we assume that $\forall j \in \{1,\dots,n_{lc}\}\} : \mathbf{f}_j \neq \mathbf{0}$.

For bending-resistant structures, such as frames and flat shells, the structural stiffness matrices $\mathbf{K}_j(\mathbf{a})$ follow from the assembly

$$\mathbf{K}_j(\mathbf{a}) = \mathbf{K}_{j,0} + \sum_{e=1}^{n_e} \left[a_e \mathbf{K}^{(1)}_{j,e} + a_e^2 \mathbf{K}^{(2)}_{j,e} + a_e^3 \mathbf{K}^{(3)}_{j,e} \right], \tag{5}$$

in which $\mathbf{K}_{j,0} \in \mathbb{S}^{n_{dof,j}}_{\succeq 0}$ constitutes a design-independent stiffness matrix, where the notation $\mathbb{S}^{n_{dof,j}}_{\succeq 0}$ denotes the space of $n_{dof,j} \times n_{dof,j}$ symmetric positive semidefinite matrices, and $\forall i \in \{1,2,3\} : \mathbf{K}^{(i)}_{j,e} \in \mathbb{S}^{n_{dof,j}}_{\succeq 0}$ are portions of the e-th element stiffness matrix that depend on the monomials a_e^i linearly. In the case of frame structures, this dependence assumes that the moment of inertia I_e can be expressed in the form

$$I_e(a_e) = c_{II} a_e^2 + c_{III} a_e^3. \tag{6}$$

For the optimization problem (4), it is natural to assume solvability of the equilibrium system (4b) and forbid rigid body motions if all optimized elements are present:

Assumption 2 $\forall \mathbf{a} > \mathbf{0}, \forall j \in \{1,\dots,n_{lc}\} : \mathbf{K}_j(\mathbf{a}) \succ 0$.

Then, we can reformulate (4) equivalently to a non-linear semidefinite program, see, e.g., (Achtziger and Kočvara 2008; Kanno 2011; Tyburec et al. 2021),

$$\min_{\mathbf{a}} \sum_{e=1}^{n_e} \rho_e \ell_e a_e \tag{7a}$$

$$\text{s.t.} \begin{pmatrix} \overline{c}_j & -\mathbf{f}_j^{\mathrm{T}} \\ -\mathbf{f}_j & \mathbf{K}_j(\mathbf{a}) \end{pmatrix} \succeq 0, \quad \forall j \in \{1,\dots,n_{lc}\}, \tag{7b}$$

$$\mathbf{a} \geq \mathbf{0}. \tag{7c}$$

The problem (7) is in general non-convex due to the polynomial nature of $\mathbf{K}_j(\mathbf{a})$, recall Eq. (5). Nevertheless, (7b) maintains a special structure that is described next.

3.2 Properties of the compliance function

Based on variational principles, we can express the compliance function $c_j(\mathbf{a})$ of the j-th load case as

$$c_j : \left\{\mathbf{a} \mid \mathbf{a} \geq \mathbf{0}, \mathbf{f}_j \in \mathrm{Im}(\mathbf{K}_j(\mathbf{a}))\right\} \mapsto \mathbb{R},$$
$$c_j(\mathbf{a}) := \max_{\mathbf{u}_j} \left(2\mathbf{f}_j^{\mathrm{T}} \mathbf{u}_j - \mathbf{u}_j^{\mathrm{T}} \mathbf{K}_j(\mathbf{a}) \mathbf{u}_j \right) \tag{8}$$

Remark 1 Stationarity condition of the maximum imply that the maximizer $\mathbf{u}_j^*$ of the concave function solves the equilibrium equation $\mathbf{K}_j(\mathbf{a})\mathbf{u}_j^* = \mathbf{f}_j$, and thus based on (Tyburec et al. 2021, Lemma 1), $\mathbf{u}_j^* = \mathbf{K}_j(\mathbf{a})^\dagger \mathbf{f}_j$ with $\bullet^\dagger$ denoting the Moore-Penrose pseudo-inverse of $\bullet$. Therefore, the actual value of the maximum evaluates as $\mathbf{f}_j^{\mathrm{T}} \mathbf{K}_j(\mathbf{a})^\dagger \mathbf{f}_j$, rendering the above definition equivalent to

$$c_j : \left\{\mathbf{a} \mid \mathbf{a} \geq \mathbf{0}, \mathbf{f}_j \in \mathrm{Im}(\mathbf{K}_j(\mathbf{a}))\right\} \mapsto \mathbb{R},$$
$$c_j(\mathbf{a}) := \mathbf{f}_j^{\mathrm{T}} \mathbf{K}_j(\mathbf{a})^\dagger \mathbf{f}_j. \tag{9}$$

Next, we state basic properties of c_j.

Proposition 1 (Monotonicity of scalarized compliance) *For a statically admissible design* $\left\{\tilde{\mathbf{a}} \mid \tilde{\mathbf{a}} \geq \mathbf{0}, \mathbf{f}_j \in \mathrm{Im}(\mathbf{K}_j(\tilde{\mathbf{a}}))\right\}$ *and* $\delta_2 \geq \delta_1 > 0$, *it holds that* $c_j(\delta_1 \tilde{\mathbf{a}}) \geq c_j(\delta_2 \tilde{\mathbf{a}})$.

Proof Using triangular inequality, we get

$$\begin{aligned} & c_j(\delta_2 \tilde{\mathbf{a}}) - c_j(\delta_1 \tilde{\mathbf{a}}) \\ & \leq \max_{\mathbf{u}_j} \left[-\mathbf{u}_j^{\mathrm{T}} \mathbf{K}_j(\delta_2 \tilde{\mathbf{a}}) \mathbf{u}_j + \mathbf{u}_j^{\mathrm{T}} \mathbf{K}_j(\delta_1 \tilde{\mathbf{a}}) \mathbf{u}_j \right] \\ & = \max_{\mathbf{u}_j} \left[\mathbf{u}_j^{\mathrm{T}} \left(\sum_{e=1}^{n_e} \sum_{i=1}^{3} (\delta_1^i - \delta_2^i) \mathbf{K}^{(i)}_{j,e} \right) \mathbf{u}_j \right] \leq 0 \end{aligned} \tag{10}$$

as required because $\mathbf{K}^{(i)}_{j,e} \succeq 0$ and $0 \leq \delta_1 \leq \delta_2$. □

Further, we investigate the range of c_j, for which it suffices to find $\inf_{\mathbf{a}} c_j(\mathbf{a})$ and $\sup_{\mathbf{a}} c_j(\mathbf{a})$. To this goal and similarly to (Tyburec et al. 2019, Appendix A), we partition $\mathbf{K}_j(\mathbf{a})$ and $\mathbf{f}_j$ in (7b) according to the dependence on the design variables $\mathbf{a}$ as follows. Let $\mathbf{U}_{N,j}$ be orthonormal bases belonging to $\mathrm{Ker}\left(\sum_{e=1}^{n_e} \sum_{i=1}^{3} a_e^i \mathbf{K}^{(i)}_{j,e}\right)$ and let $\mathbf{U}_{R,j}$ be the bases of $\mathrm{Im}\left(\sum_{e=1}^{n_e} \sum_{i=1}^{3} a_e^i \mathbf{K}^{(i)}_{j,e}\right)$. We wish to emphasize here the omitted term $\mathbf{K}_{j,0}$ in the calculation of the bases. Consequently, $\mathbf{U}_{R,j}$ may not span the whole space even if $\mathbf{a} > \mathbf{0}$, see, e.g., the reinforcement problem in Sect. 4.2 After projecting $\mathbf{K}_j(\mathbf{a})$ via these bases, we receive the partitioning

$$\begin{pmatrix} \mathbf{U}_{R,j}^{\mathrm{T}} \\ \mathbf{U}_{N,j}^{\mathrm{T}} \end{pmatrix} \mathbf{K}_j(\mathbf{a}) \begin{pmatrix} \mathbf{U}_{R,j} & \mathbf{U}_{N,j} \end{pmatrix} = \begin{pmatrix} \mathbf{K}_{A,j}(\mathbf{a}) & \mathbf{K}_{AB,j}^{\mathrm{T}} \\ \mathbf{K}_{AB,j} & \mathbf{K}_{B,j} \end{pmatrix} \tag{11}$$

in which

$$\forall \mathbf{a} > \mathbf{0} : \mathbf{U}_{R,j}^{\mathrm{T}} \left(\sum_{e=1}^{n_e} \sum_{i=1}^{3} a_e^i \mathbf{K}^{(i)}_{j,e} \right) \mathbf{U}_{R,j} \succ 0. \tag{12}$$

Moreover, $\mathbf{K}_{A,j}(\mathbf{a})$ is the only part that depends on $\mathbf{a}$.

Similarly, we define

$$\begin{pmatrix}\mathbf{f}_{\mathrm{A},j}\\ \mathbf{f}_{\mathrm{B},j}\end{pmatrix} = \begin{pmatrix}\mathbf{U}_{\mathrm{R},j}^{\mathrm{T}}\\ \mathbf{U}_{\mathrm{N},j}^{\mathrm{T}}\end{pmatrix}\mathbf{f}_j. \tag{13}$$

Then, (7b) can be equivalently rewritten[1] as

$$\begin{pmatrix} c_j & \mathbf{f}_{\mathrm{A},j}^{\mathrm{T}} & \mathbf{f}_{\mathrm{B},j}^{\mathrm{T}}\\ \mathbf{f}_{\mathrm{A},j} & \mathbf{K}_{\mathrm{A},j}(\mathbf{a}) & \mathbf{K}_{\mathrm{AB},j}^{\mathrm{T}}\\ \mathbf{f}_{\mathrm{B},j} & \mathbf{K}_{\mathrm{AB},j} & \mathbf{K}_{\mathrm{B},j}\end{pmatrix} \succeq 0. \tag{14}$$

Because $\mathbf{K}_{\mathrm{B},j} \succ 0$ due to Assumption 2, then, using the Schur complement lemma (Haynsworth 1968), (14) is equivalent to

$$\begin{pmatrix} c_{\mathrm{sch},j} & -\mathbf{f}_{\mathrm{sch},j}^{\mathrm{T}}\\ -\mathbf{f}_{\mathrm{sch},j} & \mathbf{K}_{\mathrm{sch},j}(\mathbf{a})\end{pmatrix} \succeq 0. \tag{15}$$

with

$$\mathbf{f}_{\mathrm{sch},j} = \mathbf{f}_{\mathrm{A},j} - \mathbf{K}_{\mathrm{AB},j}^{\mathrm{T}}\mathbf{K}_{\mathrm{B},j}^{-1}\mathbf{f}_{\mathrm{B},j} \tag{16a}$$

$$\mathbf{K}_{\mathrm{sch},j}(\mathbf{a}) = \mathbf{K}_{\mathrm{A},j}(\mathbf{a}) - \mathbf{K}_{\mathrm{AB},j}^{\mathrm{T}}\mathbf{K}_{\mathrm{B},j}^{-1}\mathbf{K}_{\mathrm{AB},j} \tag{16b}$$

$$c_{\mathrm{sch},j}(\mathbf{a}) = c_j(\mathbf{a}) - \mathbf{f}_{\mathrm{B},j}^{\mathrm{T}}\mathbf{K}_{\mathrm{B},j}^{-1}\mathbf{f}_{\mathrm{B},j} \tag{16c}$$

being the condensed force vector, stiffness matrix, and compliance, respectively. Then, we are ready to prove the following proposition.

Proposition 2 *For the partitioning in Eq. (14), it holds that*

$$\inf_{\mathbf{a}} c_j(\mathbf{a}) = \mathbf{f}_{\mathrm{B},j}^{\mathrm{T}}\mathbf{K}_{\mathrm{B},j}^{-1}\mathbf{f}_{\mathrm{B},j}. \tag{17}$$

Proof Based on (15), we have $c_{\mathrm{sch},j}(\mathbf{a}) \geq 0$. Hence, $c_j(\mathbf{a}) \geq \mathbf{f}_{\mathrm{B},j}^{\mathrm{T}}\mathbf{K}_{\mathrm{B},j}^{-1}\mathbf{f}_{\mathrm{B},j}$ due to (16c). Finally, it suffices to show that $c_j(\mathbf{a}) \to \mathbf{f}_{\mathrm{B},j}^{\mathrm{T}}\mathbf{K}_{\mathrm{B},j}^{-1}\mathbf{f}_{\mathrm{B},j}$ for $\mathbf{a} \to \infty$ component-wise. Because $\forall i, e : \mathbf{K}_{j,e}^{(i)} \succeq 0$ and (12), the eigenvalues of $\mathbf{K}_{\mathrm{A},j}(\mathbf{a})$ approach infinity as $\mathbf{a} \to \infty$. Hence,

$$c_{\mathrm{sch},j}(\mathbf{a}) = \mathbf{f}_{\mathrm{sch},j}^{\mathrm{T}}\mathbf{K}_{\mathrm{sch},j}(\mathbf{a})^{-1}\mathbf{f}_{\mathrm{sch},j} \to 0 \text{ as } \mathbf{a} \to \infty \tag{18}$$

and, therefore, $c_j(\mathbf{a}) \to \mathbf{f}_{\mathrm{B},j}^{\mathrm{T}}\mathbf{K}_{\mathrm{B},j}^{-1}\mathbf{f}_{\mathrm{B},j}$ based on (16c). □

Remark 2 For the case of $\mathbf{K}_{\mathrm{B},j} \in \mathbb{S}^0$, we have $\inf_{\mathbf{a}} c_j(\mathbf{a}) \to 0$.

Next, we consider the supremum part.

[1] Notice, however, that the solution $\tilde{\mathbf{u}}_j$ to the transformed system $\left(\mathbf{U}_{\mathrm{R},j}^{\mathrm{T}}\ \mathbf{U}_{\mathrm{N},j}^{\mathrm{T}}\right)^{\mathrm{T}}\mathbf{K}_j(\mathbf{a})\left(\mathbf{U}_{\mathrm{R},j}\ \mathbf{U}_{\mathrm{N},j}\right)\tilde{\mathbf{u}}_j = \left(\mathbf{U}_{\mathrm{R},j}^{\mathrm{T}}\ \mathbf{U}_{\mathrm{N},j}^{\mathrm{T}}\right)^{\mathrm{T}}\mathbf{f}_j$ differs from $\mathbf{u}_j$ in (4b). The original vector field $\mathbf{u}_j$ can be recovered by another transformation as $\mathbf{u}_j = \left(\mathbf{U}_{\mathrm{R},j}\ \mathbf{U}_{\mathrm{N},j}\right)\tilde{\mathbf{u}}_j$.

Proposition 3 *For the partitioning in Eq. (14), it holds that*

1. $\sup_{\mathbf{a}} c_j(\mathbf{a}) = \mathbf{f}_{\mathrm{sch},j}^{\mathrm{T}}\left(\mathbf{K}_{\mathrm{A},j,0} - \mathbf{K}_{\mathrm{AB},j}^{\mathrm{T}}\mathbf{K}_{\mathrm{B},j}^{-1}\mathbf{K}_{\mathrm{AB},j}\right)^{-1}\mathbf{f}_{\mathrm{sch},j}$ *if* $\mathbf{f}_{\mathrm{sch},j} \in \mathrm{Im}(\mathbf{K}_{\mathrm{A},j,0} - \mathbf{K}_{\mathrm{AB},j}^{\mathrm{T}}\mathbf{K}_{\mathrm{B},j}^{-1}\mathbf{K}_{\mathrm{AB},j})$
2. $\sup_{\mathbf{a}} c_j(\mathbf{a}) = \infty$ *otherwise.*

Proof The first part follows from (15) and corresponds to the setting when fixed elements are able to transmit prescribed loading to supports. For the second part, setting $\mathbf{a} \to \mathbf{0}$ renders the displacement field arbitrarily large, and thus the compliance infinite. □

3.3 Upper bounds to program (7) by scalarization

Using these compliance function properties, this section develops a method for obtaining feasible upper bounds to (7) under mild assumptions.

Let $\tilde{\mathbf{a}} \in \mathbb{R}_{\geq 0}^{n_{\mathrm{e}}}$ be a vector of fixed ratios of the cross-section areas such that $\forall j \in \{1, \dots, n_{\mathrm{lc}}\} : \mathbf{f}_j \in \mathrm{Im}\left(\mathbf{K}_j(\tilde{\mathbf{a}})\right)$. Further, define a scaling parametrization of the cross-section areas via a parameter $\delta > 0$, i.e., $\mathbf{a}(\delta) = \delta\tilde{\mathbf{a}}$. In what follows, we state the conditions under which the values in Propositions 2 and 3 remain valid even though we replace $c_j(\mathbf{a})$ with $c_j(\mathbf{a}(\delta))$.

Proposition 4 *If* $\mathbf{f}_{\mathrm{A},j} \in \mathrm{Im}\left(\mathbf{U}_{\mathrm{R},j}^{\mathrm{T}}\left(\sum_{e=1}^{n_{\mathrm{e}}}\sum_{i=1}^{3}\tilde{a}_e^i\mathbf{K}_{j,e}^{(i)}\right)\mathbf{U}_{\mathrm{R},j}\right)$ holds, then, $\inf_{\mathbf{a}} c_j(\mathbf{a}) = \inf_\delta c_j(\mathbf{a}(\delta))$.

Proof The proof follows from Proposition 2. □

Remark 3 A statically admissible design may violate the condition in Proposition 4 only if $\mathbf{K}_{j,0} \neq \mathbf{0}$ and $\exists e : \tilde{a}_e = 0$ at the same time, i.e., when optimizing topology of a reinforcement. From the mechanical point of view, such situation corresponds to the case of carrying loads through elements with prescribed stiffness, although the optimized domain would allow load transfer through elements that are eliminated with $\tilde{a}_e = 0$. On the other hand, the infima are the same for standard topology optimization and for sizing optimization of reinforcement problems.

Remark 4 If there is no fixed structural stiffness $\mathbf{K}_{j,0} = \mathbf{0}$, then the condition in Proposition 4 simplifies to statical admissibility of the design, i.e., $\mathbf{f}_j \in \mathrm{Im}(\mathbf{K}_j(\tilde{\mathbf{a}}))$.

For the case of upper bounds in Proposition 3, the situation is considerably easier—they remain the same regardless of $\tilde{\mathbf{a}}$.

Finally, we can state the procedure for constructing feasible upper bounds.

Proposition 5 *Let* $\forall j \in \{1, \dots, n_{\mathrm{lc}}\} : \inf_\delta c_j(\mathbf{a}(\delta)) < \overline{c}_j$ *hold and let the feasible set of (4) have a non-empty interior. Then, there exists* $\delta > 0$ *such that* $\delta\tilde{\mathbf{a}}$ *is feasible to*

(7). Furthermore, δ follows from a solution to the univariate optimization problem

$$\min_{\delta} \ \delta \tag{19a}$$

$$\text{s.t. } \mathbf{f}_j^{\mathrm{T}} \mathbf{K}_j(\mathbf{a}(\delta))^{\dagger} \mathbf{f}_j \leq \overline{c}_j, \quad \forall j \in \{1, \dots, n_{\mathrm{lc}}\}, \tag{19b}$$

$$\delta > 0. \tag{19c}$$

Proof Due to the assumption on the infimum, (19) is solvable whenever (7) is, i.e., when $\inf_{\mathbf{a}} c_j(\mathbf{a}) \leq \inf_{\delta} c_j(\mathbf{a}(\delta)) < \overline{c}_j$. Moreover, since $\sup_{\delta} c_j(\mathbf{a}(\delta))$ does not depend on $\tilde{\mathbf{a}}$, Proposition 3, and (19b) is a monotonic function due to Proposition 1, the equality sign in (19b) can always be satisfied for at least one load case if $\overline{c}_j \leq \sup_{\mathbf{a}} c_j(\mathbf{a}) = \inf_{\delta} c_j(\mathbf{a}(\delta))$. □

From the numerical perspective, the value of $\inf_{\delta} c_j(\mathbf{a}(\delta))$ is obtained via Proposition 2 for a partitioning in (14) that follows from $\mathbf{a}(\delta) = \tilde{\mathbf{a}}\delta$, with $\tilde{a}_e = 0$ considered as an empty contribution to $\mathbf{K}_{j,0}$.

Due to the monotonicity of (19b) in δ, the optimal scaling factor δ, and thus an (upper-bound) feasible solution to (7), can be found by a bisection-type algorithm.

3.4 Moment-sum-of-squares hierarchy

In this section, we modify (7) to be practically solvable to global optimality by the moment-sum-of-squares hierarchy, develop a sequence of feasible upper bounds, and settle a simple sufficiency condition of global ε-optimality in the spirit of (Tyburec et al. 2021).

3.4.1 Polynomial programming reformulation

For convergence guarantees of the moment-sum-of-squares hierarchy, we need to certify algebraic compactness of the feasible set, recall Assumption 1 and Theorem 1. This can be secured by bounding the design variables through quadratic constraints (Tyburec et al. 2021, Proposition 4).

To set these constraints, we first notice that while the lower bounds for $\mathbf{a}$ come directly from the problem formulation, recall (7c), the upper bounds can be established by exploiting the results in Sect. 3.3. In particular, for any fixed $\tilde{\mathbf{a}} > \mathbf{0}$, the condition in Proposition 4 is satisfied, allowing us to compute optimal scaling δ^* through the program (19), and thus construct a feasible upper-bound to (7). An upper-bound structural weight then amounts to

$$\overline{w} = \delta^* \sum_{e=1}^{n_{\mathrm{e}}} \rho_e \ell_e \tilde{a}_e. \tag{20}$$

Since (20) bounds the weight from above, none of the structural elements can exceed the weight $\overline{w}$ at the optimum. Therefore, the individual variables a_e can be bounded as

$$0 \leq a_e \leq \frac{\overline{w}}{\rho_e \ell_e} \tag{21}$$

which is then equivalent to

$$a_e \left(\frac{\overline{w}}{\rho_e \ell_e} - a_e \right) \geq 0. \tag{22}$$

From the numerical perspective, it is further advantageous to scale the design variables, i.e., solve the optimization problem in terms of $\forall e \in \{1, \dots, n_{\mathrm{e}}\} : a_{\mathrm{s},e} \in [-1, 1]$ rather than in $\forall e \in \{1, \dots, n_{\mathrm{e}}\} : a_e \in [0, \overline{w}/(\ell_e \rho_e)]$, which is achieved by inserting

$$a_e = \frac{a_{\mathrm{s},e} + 1}{2} \frac{\overline{w}}{\rho_e \ell_e}. \tag{23}$$

After these modifications, the final formulation reads as

$$\min_{\mathbf{a}_{\mathrm{s}}} \ 0.5\overline{w} \left(n_{\mathrm{e}} + \mathbf{1}^{\mathrm{T}} \mathbf{a}_{\mathrm{s}} \right) \tag{24a}$$

$$\text{s.t. } \begin{pmatrix} \overline{c}_j & -\mathbf{f}_j^{\mathrm{T}} \\ -\mathbf{f}_j & \mathbf{K}_j(\mathbf{a}_{\mathrm{s}}) \end{pmatrix} \succeq 0, \quad \forall j \in \{1, \dots, n_{\mathrm{lc}}\}, \tag{24b}$$

$$a_{\mathrm{s},e}^2 \leq 1, \quad \forall j \in \{1, \dots, n_{\mathrm{e}}\}. \tag{24c}$$

3.4.2 Recovering feasible upper bounds and sufficient condition of global ε-optimality

In order to solve (24) globally, we generate a hierarchy of convex outer approximations of the feasible set $\mathcal{K}(\mathbf{G}(\mathbf{x}))$, recall Sect. 2. The feasible set of these relaxations is described in terms of the moments $\mathbf{y}$ that are indexed in the polynomial space basis $\mathbf{b}_{2r}(\mathbf{a}_{\mathrm{sc}}) = \{1, a_{\mathrm{s},1}, \dots, a_{\mathrm{s},n_{\mathrm{e}}}, \dots\}$. Because the emerging relaxations are linear in $\mathbf{y}$, recall (3), we solve a sequence of convex linear semidefinite programming problems.

Let now $\mathbf{y}_{\mathbf{a}_{\mathrm{s}}^1}^{(r)}$ be the optimal first-order moments associated with degree-1 polynomials in $\mathbf{b}_{2r}(\mathbf{a}_{\mathrm{sc}})$ of the r-th degree relaxation. Unscaling these first-order moments provides us with an estimate on the optimal scaling factors $\tilde{\mathbf{a}}$, i.e.,

$$\tilde{a}_e = \frac{y_{a_{\mathrm{s},e}^1}^{(r)} + 1}{2} \frac{\overline{w}}{\rho_e \ell_e}, \forall e \in \{1, \dots, n_{\mathrm{e}}\}. \tag{25}$$

For $\tilde{\mathbf{a}}$, it holds that $\mathbf{f}_j \in \mathrm{Im}\left(\mathbf{K}_j(\tilde{\mathbf{a}})\right)$ by (Tyburec et al. 2021, Proposition 6). Consequently, we show how to construct feasible upper bounds next.

Theorem 2 *Let $\mathbf{y}_{\mathbf{a}_{\mathrm{s}}^1}^{(r)}$ be the optimal first-order moments associated with the r-th relaxation. If $\inf_{\delta} c_j(\mathbf{a}(\delta)) < \overline{c}_j$ holds for all $j \in \{1 \dots n_{\mathrm{lc}}\}$ and the problem (4) is solvable, then $\delta^* \tilde{\mathbf{a}}$,* *with $\tilde{\mathbf{a}}$ set as (25) and δ^* computed based on Proposition 5, is a feasible (upper-bound) solution to (4).*

Proof Same as in Proposition 5. □

Finally, we also show that $\varepsilon \to 0$ for problems with minimizers forming a convex set.

Remark 5 Due to $\mathbf{f}_j \in \operatorname{Im}\left(\mathbf{K}_j(\tilde{\mathbf{a}})\right)$ being always satisfied, the assumptions of Theorem 2 may be violated only when $\inf_\delta c_j(\mathbf{a}(\delta)) > \overline{c}_j$, which can happen only if $\mathbf{K}_{j,0} \neq \mathbf{0}$ and $\exists e : \tilde{a}_e = 0$ at the same time, recall Remark 3. For $\mathbf{K}_{j,0} = \mathbf{0}$, the upper bounds can always be constructed.

Having the sequence of upper bounds in Theorem 2, and a natural sequence of lower bounds from the relaxations, we arrive at a simple condition of global ε-optimality.

Proposition 6 *Let $\delta^* \tilde{\mathbf{a}}$ be a feasible (upper-bound) solution to (4) constructed based on Theorem 2. Then,*

$$\left(\delta^* - 1\right) 0.5\overline{w}(n_\mathrm{e} + \mathbf{1}^\mathrm{T}\mathbf{y}^{(r)}_{\mathbf{a}^1_\mathrm{s}}) \leq \varepsilon \tag{26}$$

is a sufficient condition of global ε-optimality.

Proof The lower bound for the objective function amounts to $0.5\overline{w}(n_\mathrm{e} + \mathbf{1}^\mathrm{T}\mathbf{y}^{(r)}_{\mathbf{a}^1_\mathrm{s}})$, recall (24a) and (Henrion and Lasserre 2006, Section D). Similarly, the upper bound evaluates as

$$\delta^* \sum_{e=1}^{n_\mathrm{e}} \frac{y^{(r)}_{a^1_{\mathrm{s},e}} + 1}{2} \frac{\overline{w}}{\rho_e \ell_e} \rho_e \ell_e = 0.5\delta^* \overline{w}(n_\mathrm{e} + \mathbf{1}^\mathrm{T}\mathbf{y}^{(r)}_{\mathbf{a}^1_\mathrm{s}})$$

based on Eq. (25) and Theorem 2. Subtracting the lower bound from the upper bound provides the worst-case estimate for the optimality gap ε, Eq. (26). □

Theorem 3 *Let $\delta^* \tilde{\mathbf{a}}$ be a feasible (upper-bound) solution to (4) constructed based on Theorem 2. If the set of global minimizers is convex, then, as $r \to \infty$,*

$$\left(\delta^* - 1\right) 0.5\overline{w}(n_\mathrm{e} + \mathbf{1}^\mathrm{T}\mathbf{y}^{(r)}_{\mathbf{a}^1_\mathrm{s}}) = 0. \tag{27}$$

Proof Because of Theorem 1 and satisfied Assumption 1, for $r \to \infty$, optimization over a set $\mathcal{K}$ is equivalent to optimization over its convex hull $\mathrm{Conv}(\mathcal{K})$ (Tyburec et al. 2021, Proposition 7). By Assumption 1, $\mathcal{K}$ is compact, and, thus, $\mathrm{Conv}(\mathcal{K})$ is too. Therefore, we can express the convex hull using its limits points $\mathbf{d}_1, \mathbf{d}_2, \dots,$

$$\mathrm{Conv}\,(\mathcal{K}) = \mathrm{Conv}\left(\cup_{i=1}^{\infty} \{\mathbf{d}_i\}\right). \tag{28}$$

Having assumed that the set of global minimizers is convex, there must exist a convex set $\mathrm{Conv}\left(\cup_{i=1}^{\infty}\left\{\mathbf{d}_i^*\right\}\right) \subseteq \mathrm{Conv}\,(\mathcal{K})$ with points $\mathbf{d}_i^*$ that are associated with the minimum. □

Table 1 24-element frame structure optimization. LB abbreviates lower bound, UB stands for feasible upper bounds, and r is the relaxation number. Further, $n_\mathrm{c} \times m$ denotes the number of n_c semidefinite constraints of the size m, and n is the number of variables

r	LB	UB	Time [s]	$n_\mathrm{c} \times m$	n
1	0.046	0.160	0.10	$10, 9 \times 1, 37$	54
2	0.103	0.125	16.65	$55, 9 \times 10, 370$	714
3	0.118	0.118	12 732.90	$220, 9 \times 55, 2\,035$	5 004

4 Examples

In this section, we demonstrate the capabilities of the presented method by means of two illustrations: a modular three-story structure, and a part design. All computations were performed on a personal laptop with 24 GB of RAM and Intel® Core™ i5-8350U CPU. For optimization, we relied on the MOSEK (MOSEK 2019) solver.

4.1 24-element modular frame structure

As the first illustration we investigate a modular frame structure containing 24 Euler-Bernoulli finite elements and 36 degrees of freedom, see Fig. 1a. For simplicity, we assume the dimensionless Young modulus $E = 1.0$, density $\rho = 1.0$, as well as the dimensionless structural dimensions.

The frame structure is clamped at the bottom nodes ⓐ and ⓑ, and subjected to horizontal loads at nodes ⓒ, ⓔ and ⓖ, and to vertical forces acting at ⓘ, ⓙ and ⓚ.

We split the structural elements into several groups: first, we use a different cross-section parametrization within the structural elements, Fig. 1b. In addition, we also specify groups of elements that must have the same cross-section size to maintain structural symmetry: while for the columns, we set $a_1 = a_2, a_3 = a_4, a_5 = a_6$, we require $a_7 = a_8, a_9 = a_{10}$ and $a_{11} = a_{12}$ for the horizontal beams. Finally, we enforce equal cross-section sizes within the circular tubes in a single story, i.e., $a_{13} = a_{14} = a_{15} = a_{16}, a_{17} = a_{18} = a_{19} = a_{20}$, and $a_{21} = a_{22} = a_{23} = a_{24}$. Hence, we have nine independent cross-section areas in total.

Because all structural elements are being optimized, the term $\mathbf{K}_0$ in (5) is empty. Based on Remark 2, we thus have $\inf_\mathbf{a} c(\mathbf{a}) = 0$. As any positive compliance can thus be attained, we set the compliance upper-bound to $\overline{c} = 5\,000$. Starting with a uniform distribution of cross-section areas, $\tilde{\mathbf{a}} = \mathbf{1}$, the optimization problem (19) yields an upper-bound weight of $\overline{w} = 0.150$.

Moreover, for any feasible first-order moments, Remark 5 assures us that feasible upper bounds can always be constructed. In the lowest, first-degree relaxation, we receive the lower-bound weight of 0.046. Using the cross-section area distribution provided by the optimal first-order moments

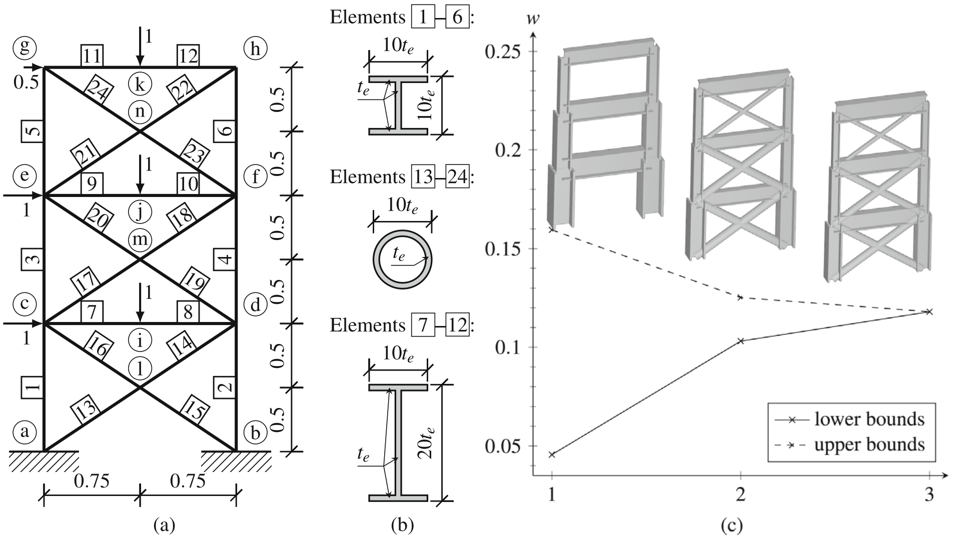

Fig. 1 24-element frame structure: (a) boundary conditions, (b) cross-section parametrization, and (c) convergence of the proposed relaxation-based approach with visualized feasible upper-bound designs

to construct a feasible upper bound, recall Theorem 2, we receive the weight 0.160. In the second relaxation, we obtain the lower bound objective 0.103, from which we recover an upper-bound weight 0.125. Final, third relaxation yields a lower-bound weight of 0.118, and the projected upper-bound design of weight 0.118, see Table 1. Hence, the global optimality of the design is certified based on Proposition 6. Similarly, the hierarchy also converged based on the flat extension theorem of Curto and Fialkow (1996), allowing for extracting the unique global minimizer. The upper-bound and the optimal design appear visualized in Fig. 1c.

4.2 Part design

Second, we consider the problem of optimizing a structural part. Such problems appear, e.g., in stiffening and reinforcing structure design or in structural component optimizations.

Here, we assume the problem shown in Fig. 2a, consisting of 12 nodes that are interconnected with 20 Euler-Bernoulli beam elements. All these elements have square cross-sections. While the elements drawn in Fig 2a with a solid line are subject of optimization, the elements denoted with dashed lines share the cross-section area of 0.01. Further, we have again the dimensionless Young modulus $E = 1.0$ and the density $\rho = 1.0$.

The structure is fully clamped at the nodes (a) and (h), whereas symmetric boundary conditions are assumed along the (d)–(k) axis. The structure is loaded by unitary vertical forces at the nodes (i) and (j), and by a halved vertical force at the node (k).

Because of the fixed elements [1]–[4], [12], [13], and [18], the term $\mathbf{K}_0$ is now present in (5). Using Proposition 2, we thus receive a positive value of the compliance infimum, $\inf_{\mathbf{a}} c(\mathbf{a}) = 578.9$, which can be approached by the optimized elements only in the limit. Thus, we set $\overline{c} = 1\,000$.

After setting $\tilde{\mathbf{a}} = \mathbf{1}$, the optimization problem (19) provides us with the upper-bound solution of the weight $\overline{w} = 0.285$. Using this bound to make the design space compact and solving the emerging optimization problem (24) via the MSOS hierarchy, we receive the lower bound weight of 0.133

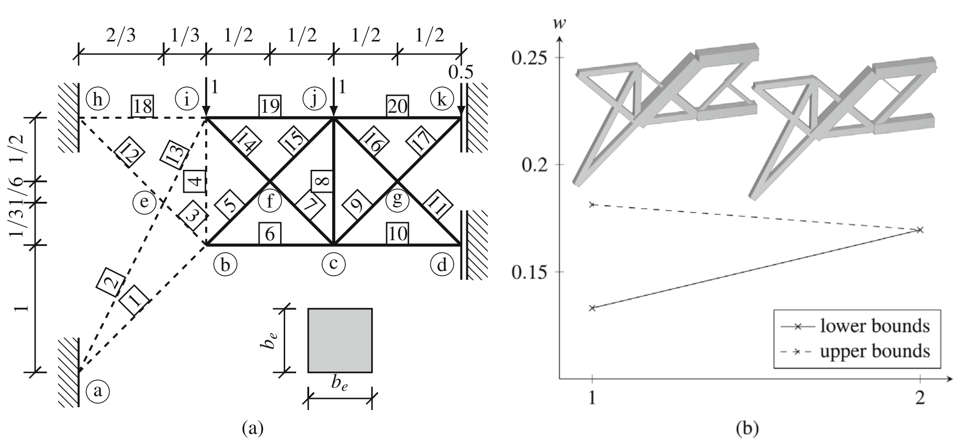

Fig. 2 Frame structure reinforcement problem. (a) Discretization, boundary conditions and cross-section parametrization, and (b) convergence of the proposed relaxation-based approach with visualized feasible upper-bound designs

Table 2 Part design optimization. LB abbreviates lower bound, UB stands for feasible upper bounds, and r is the relaxation number. Further, $n_c \times m$ denotes the number of n_c semidefinite constraints of the size m, and n is the number of variables

r	LB	UB	Time [s]	$n_c \times m$	n
1	0.133	0.181	0.04	14, 13 × 1, 19	104
2	0.170	0.170	29.52	105, 13 × 14, 266	2 379

in the first relaxation and the associated feasible upper-bound design of the weight 0.181. The second relaxation makes the hierarchy converge with respect to both the optimality gap and the flatness of the moment matrices ranks (Curto and Fialkow 1996). The optimal design, shown in Fig. 2, weights 0.170. In all the relaxations, setting $\tilde{\mathbf{a}}$ based on the first-order moments produced the same value of $\inf_\delta c(\mathbf{a}(\delta)) = 578.9$.

The hierarchy convergence appear summarized in Table 2. We note here that because of the partitioning in (14) and five constant rows/columns, we adopted the Schur complement lemma to reduce the problem size and accelerate its solution.

5 Results and discussion

In this contribution, we have extended our previous results for global compliance optimization of bending resistant structures (Tyburec et al. 2021) to the weight minimization setting. To this goal, we have first exploited monotonicity of the scalarized compliance function, and developed a univariate problem (19) for computing feasible upper bounds to the weight optimization problem (4) when the ratio of cross-section areas is fixed. We proposed to solve this problem by a bisection-type algorithm.

Based on such constructed upper bound, it is possible to bound the design variables from above, and thus show that the assumption of algebraic compactness, which is needed for the convergence of the Lasserre hierarchy, is satisfied. Developing and solving an efficient polynomial programming formulation, we have shown that, under mild assumptions, the first-order moments from the relaxations may serve as the ratios of the cross-section areas, enabling a construction of feasible upper bounds in each relaxation. Finally, a comparison of the relaxations lower bounds with the constructed upper bounds establishes a simple sufficient condition of global ε-optimality, and this condition converges to zero in the limit in the case of a convex set of global minimizers.

We have illustrated these theoretical results on a set of two optimization problems. These problems revealed applicability of the approach to small-scale problems and a rapid convergence of the hierarchy.

We plan to extend our approach in several directions. First, we are interested in problems in structural dynamics such as eigenvalue problems (Achtziger and Kočvara 2008) and steady-state harmonic oscillations. Second, we aim to investigate methods for accelerating the optimization process, e.g., by exploiting structural (Zheng et al. 2021; Kočvara 2020) and term sparsity, or adopting the very recent results in optimization problems with tame structure (Aravanis et al. 2022).

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Declarations

Conflict of interest The authors declare no competing interests.

Replication of results Source codes are available at (Tyburec et al. 2022).

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