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SELF REFERRED EQUATIONS WITH AN INTEGRAL BOUNDARY CONDITION

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ABSTRACT. In this note, we study three differential problems with a dynamic, which are be represented by a self referred equation and a boundary condition, which are expressed as an integral constraint. We prove that under certain assumptions, there exists at least one solution of for all of these problems by using Schauder's fixed point theorem. In the end, we propose briefly some open problems.

1. Introduction

The hereditary phenomena were widely studied in the past (see, e.g., [2, 3, 11]) due to their impact in applied sciences, for example, in engineering and biology. Hence, the so called *self-referred and hereditary equations* were proposed in order to write a model to describe this type of events (see, e.g., [1, 5–7] and [10]). Formally we may represent this class of equations as follows: let us consider X a space of functions, $A\colon X \to \mathbb{R}$, $B\colon X \to \mathbb{R}$ two functional operators. Then a self referred equation may be written as

$$Au(x,t) = u(Bu(x,t),t).$$

In our problems, the boundary condition will be expressed in the form

$$\int_0^\beta \dot{y}^2(t)\mathrm{d}t = \delta\beta > 0$$

or

$$\int_0^\beta y^2(x)\mathrm{d}x = \delta\beta > 0,$$

where $0 < \beta \leq 1$ and $0 < \delta < 1$. This constraint may represent, roughly speaking, a quantity which is preserved in an isolated system (e.g., the energy). This condition was implemented in

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an ODE problem in [4] with a dynamic in the form $\ddot{y}(t) = f(t, y(t), \dot{y}(t))$, with f was a positive, bounded and globally Lipschitz function.

The goal of this paper is to study a differential problem which mix self referred equations and integral boundary constraints. We study the differential problem

$$\begin{cases} \ddot{y}(t) = \beta y(y(t)), & t \in [0,1], \\ y(0) = \alpha, \\ \int_0^\beta \dot{y}^2(t)\mathrm{d}t = \delta\beta > 0, \end{cases} \tag{1.1}$$

where $0 < \delta < 1$, $0 \leq \alpha \leq 1$ and $0 < \beta \leq 1$. We may also study a variation of (1.1), where we do not know the initial state of our system. Hence, the second problem which will be studied is

$$\begin{cases} \dot{y}(t) = \beta y\big(\int_0^t y(s)\mathrm{d}s\big), & t \in [0,1], \\ \int_0^\beta y^2(t)\mathrm{d}t = \delta\beta, \end{cases} \tag{1.2}$$

where $0 < \delta < 1$ and $0 < \beta \leq 1$. In the last problem, we introduce also a space variable, i.e.,

$$\begin{cases} \frac{\partial}{\partial t} y(x,t) = \beta y\big(\int_0^t y(x,s)\mathrm{d}s, t\big), & (x,t) \in [0,1] \times [0,1], \\ \int_0^\beta y^2(x,t)\mathrm{d}t = \beta g(x), \end{cases} \tag{1.3}$$

where $0 < \beta \leq 1$ and $g\colon [0,1] \to [0,1]$ is smooth function.

The technique to find the existence of a solution follows these steps: we will study the problems heuristically, obtaining the operator which we want to study and the space, where we will work on. Then we will check that under certain conditions, the space of functions which we have found is bounded, closed and convex. Then we will prove that operator is continuous and compact and we conclude applying the Schauder's fixed point theorem.

The paper is organized as follows: in the Section 2, we recall briefly some preliminaries of functional analysis, such as Ascoli-Arzelà theorem and Schauder's fixed point theorem; in the Section 3, we prove that the problem (1.1) has at least one solution; in the Section 4, we prove a similar result for the problem (1.2); in the Section 5, we study the problem (1.3) and in the Section 6, we state briefly some open problems.

2. Preliminaries

Let us recall some classical definitions and results of functional analysis. For further details, we refer to [8].

Definition 1. Let us consider a sequence $\{f_n\}_{n\in\mathbb{N}}$ of continuous functions on an interval $I \subset \mathbb{R}$.

- The sequence $\{f_n\}_{n\in\mathbb{N}}$ is *equibounded* if there exists a real number $M > 0$ such that it holds true

$$|f_n(x)| \leq M$$

  for all $n \in \mathbb{N}$ and for every $x \in I$.

- The sequence $\{f_n\}_{n\in\mathbb{N}}$ is *equicontinuous* if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f_n(x) - f_n(y)| < \varepsilon$$

whenever $|x - y| < \delta$ and for all $n \in \mathbb{N}$.

Definition 2. Let us consider two normed spaces X, Y and an operator $T\colon X \to Y$. Then the operator T is *compact* if, for every bounded subsequence $\{x_n\}_{n\in\mathbb{N}} \subset X$, it is possible to extract a convergent subsequence of the sequence $\{Tx_n\}_{n\in\mathbb{N}} \subset Y$.

These definitions are necessary in order to state the following classical results of functional analysis. These theorems are crucial to prove the existence of a fixed point for a functional and then the existence of a solution for our differential problem.

Theorem 2.1 (Ascoli-Arzelà theorem). *Let us consider a sequence of real-valued continuous functions $\{f_n\}_{n\in\mathbb{N}}$ defined on a closed and bounded interval $[a, b]$ of the real line. If $\{f_n\}_{n\in\mathbb{N}}$ is equibounded and equicontinuous, then there exists a subsequence $\{f_{n_k}\}_{n_k\in\mathbb{N}}$ that converges uniformly.*

Theorem 2.2 (Fixed point Schauder's theorem). *Let us consider a bounded, closed and convex Banach spaces X and a continuous and compact operator $T\colon X \to X$. Thus the operator will have a fixed point.*

3. First problem

As stated in the Introduction, we will divide our proof in smaller steps. We will start finding by heuristic methods the operator which we want to study (namely T) and the space, where we will work on (namely $\mathbb{X}$). We will prove then that under certain conditions, $\mathbb{X}$ is bounded, closed and convex. Then we will prove that T is continuous and compact, and in the end, we conclude applying the Schauder's theorem.

3.1. The definition of the functional T

In this section, we will study heuristically the ODE problem stated in (1.1).

Let us start observing that the dynamic of the system is easily integrable by using standard technique of ODEs theory. By multiplying both sides with $\dot{y}(t)$, we get immediately

$$\ddot{y}(\tau)\dot{y}(\tau) = \beta y(y(\tau))\dot{y}(\tau) \implies \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\dot{y}^2(\tau) = \beta\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^{y(\tau)} y(s)\mathrm{d}s.$$

Then, by integrating on the interval $[0, t]$ both sides of the equation, it is straightforward to get

$$\begin{aligned}
&\frac{1}{2}\int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau}\dot{y}^2(\tau)\mathrm{d}\tau = \beta\int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^{y(\tau)} y(s)\mathrm{d}s\mathrm{d}\tau \\
\implies \quad &\frac{1}{2}\dot{y}^2(t) = \frac{1}{2}\dot{y}^2(0) + \beta\left[\int_0^{y(t)} y(s)\mathrm{d}s - \int_0^{\alpha} y(s)\mathrm{d}s\right],
\end{aligned}$$

which implies

$$\dot{y}^2(t) = \dot{y}^2(0) + 2\beta \int_{\alpha}^{y(t)} y(s)\mathrm{d}s. \tag{3.1}$$

Thus, by applying the condition (3.1) on our integrodifferential condition, we obtain

$$\delta\beta = \int_0^{\beta} \dot{y}^2(t)\mathrm{d}t = \dot{y}^2(0)\beta + 2\beta \int_0^{\beta}\int_{\alpha}^{y(t)} y(s)\mathrm{d}s\mathrm{d}t$$

from which we deduce

$$\dot{y}(0) = \pm\sqrt{\delta - 2\int_0^{\beta}\int_{\alpha}^{y(t)} y(s)\mathrm{d}s\mathrm{d}t}. \tag{3.2}$$

The existence condition on (3.2) leads us naturally to the definition of the operator T and of the space of function $\mathbb{X}$.

Definition 3. Let us consider the space of continuous functions $C([0,1],[0,1])$. Then we define the space of function $\mathbb{X} \subset C([0,1],[0,1])$ as

$$\mathbb{X} := \left\{ y \;\middle|\; y \in C([0,1],[0,1]) \text{ and } \delta - 2\int_0^{\beta}\int_{\alpha}^{y(t)} y(s)\mathrm{d}s\mathrm{d}t \geq 0 \right\}, \tag{3.3}$$

and the operator T as

$$Ty(t) = \alpha + t\sqrt{\delta - 2\int_0^{\beta}\int_{\alpha}^{y(t)} y(s)\mathrm{d}s\mathrm{d}t} + \beta\int_0^{t}\left(\int_0^{\tau} y(y(s))\mathrm{d}s\right)\mathrm{d}\tau, \qquad t \in [0,1]. \tag{3.4}$$

Let us remark some immediate properties related to the boundedness of our operator.

Remark 1. Since $y \in \mathbb{X}$, then

$$Ty(t) \geq \alpha$$

for all $y \in \mathbb{X}$ and for all $t \in [0,1]$.

Remark 2. Let us recall that for all $y \in \mathbb{X}$ and for all $t \in [0,1]$, it holds true that $|y(t)| \leq 1$. Hence, $\beta\int_0^t\int_0^{\tau} y(y(s))\mathrm{d}s\mathrm{d}\tau \leq \frac{\beta}{2}$. Therefore, we obtain

$$\begin{aligned} Ty(t) \leq \alpha + \sqrt{\delta - 2\int_0^{\beta}\int_{\alpha}^{y(t)} y(s)\mathrm{d}s\mathrm{d}t} + \frac{\beta}{2} &\leq \alpha + \sqrt{\delta + 2\int_{\{t\in[0,\beta] \mid y(t)<\alpha\}}\int_{y(t)}^{\alpha} y(s)\mathrm{d}s\mathrm{d}t} + \frac{\beta}{2} \\ &\leq \alpha + \sqrt{\delta + 2\int_{\{t\in[0,\beta] \mid y(t)<\alpha\}}\int_{0}^{\alpha} y(s)\mathrm{d}s\mathrm{d}t} + \frac{\beta}{2}. \end{aligned}$$

Thus we can write

$$Ty(t) \leq \alpha + \sqrt{\delta + 2\beta\alpha} + \frac{\beta}{2}.$$

Assuming that $\alpha + \sqrt{\delta + 2\beta\alpha} + \frac{\beta}{2} \leq 1$, we get that $\alpha \leq Ty(t) \leq 1$ for all $y \in \mathbb{X}$ and for all $t \in [0,1]$.

Remark 3. We estimate

$$\begin{aligned}\delta - 2\int_0^\beta \int_\alpha^{Ty(t)} Ty(s)\mathrm{d}s\mathrm{d}t &= \delta - 2\int_{\{t\in[0,\beta]\,|\,\alpha<Ty(t)\}} \int_\alpha^{Ty(t)} Ty(s)\mathrm{d}s\mathrm{d}t \\ &\quad + 2\int_{\{t\in[0,\beta]\,|\,\alpha\geq Ty(t)\}} \int_{Ty(t)}^{\alpha} Ty(s)\mathrm{d}s\mathrm{d}t \\ &\geq \delta - 2\int_{\{t\in[0,\beta]\,|\,\alpha<Ty(t)\}} \int_\alpha^{Ty(t)} Ty(s)\mathrm{d}s\mathrm{d}t \\ &\geq \delta - 2\int_{\{t\in[0,\beta]\,|\,\alpha<1\}} \int_\alpha^{1} Ty(s)\mathrm{d}s\mathrm{d}t \\ &\geq \delta - 2\beta(1-\alpha).\end{aligned}$$

Hence, if we consider $\delta - 2\beta(1-\alpha) \geq 0$, we have also $Ty \in \mathbb{X}$ for all $y \in \mathbb{X}$.

Remark 4. The Remarks 1, 2, 3 suggest us the following assumptions:

$$\begin{cases} \alpha + \sqrt{\delta + 2\beta\alpha} + \dfrac{\beta}{2} \leq 1, \\ \delta - 2\beta(1-\alpha) \geq 0. \end{cases} \tag{3.5}$$

Let us show that the system (3.5) has at least one solution. Let us consider $\alpha = 0$. Then

$$\begin{cases} \sqrt{\delta} \leq \dfrac{2-\beta}{2} \\ 2\beta \leq \delta \end{cases} \quad\Longrightarrow\quad 2\beta \leq \delta \leq \left(\frac{2-\beta}{2}\right)^2.$$

This system of inequalities is well defined since it is equivalent to $8\beta \leq 4 - 4\beta + \beta^2$, which implies that $0 < \beta \leq 6 - 4\sqrt{2}$. Hence, it is possible to find a triple (α, β, δ) which solves our system of inequalities.

Proposition 3.1. *Under the assumption* (3.5)*, the operator* $T\colon \mathbb{X} \to \mathbb{X}$ *is well defined.*

Proof. The proof is an immediate consequence of the Remarks 1, 2, 3 and 4. □

3.2. The properties of the space $\mathbb{X}$

Let us focus now on the topological properties on space $\mathbb{X}$. In order to apply our fixed point theorem, we have to check that the space $\mathbb{X}$ is closed and convex.

Proposition 3.2. *Under the assumptions* (3.5)*, the space* $\mathbb{X}$ *as defined in* (3.3) *is closed.*

Proof. Let us consider a Cauchy sequence $\{y_n\}_{n\in\mathbb{N}}$ in $\mathbb{X} \subset C([0,1],[0,1])$. Let us recall that $C([0,1],[0,1])$ is complete. Hence, there will exists $y_\infty \in C([0,1],[0,1])$ such that $y_n \to y_\infty$ uniformly in [0,1]. We obtain, of course, that $y_\infty \in C([0,1],[0,1])$. We have to prove that y_∞ satisfies the conditions of the space $\mathbb{X}$. To remark it, let us observe that

$$\left|\int_\alpha^{y_n(t)} y_n(s)\mathrm{d}s - \int_\alpha^{y_\infty(t)} y_\infty(s)\mathrm{d}s\right| \leq \left|\int_\alpha^{y_n(t)} y_n(s)\mathrm{d}s - \int_\alpha^{y_\infty(t)} y_n(s)\mathrm{d}s\right| + \left|\int_\alpha^{y_\infty(t)} y_n(s)\mathrm{d}s - \int_\alpha^{y_\infty(t)} y_\infty(s)\mathrm{d}s\right|$$

$$\leq \left| \int_{y_\infty(t)}^{y_n(t)} y_n(s)\mathrm{d}s \right| + \left| \int_{\alpha}^{y_\infty(t)} |y_n(s) - y_{\infty(s)|}\mathrm{d}s \right|$$
$$\leq 2\|y_n - y_\infty\|_\infty,$$

where $\| \|_\infty$ is the standard L^∞ norm. Hence, it is straightforward to get

$$\lim_{n\to\infty} \int_{\alpha}^{y_n(t)} y_n(t) = \int_{\alpha}^{y_\infty(t)} y_\infty(s)\mathrm{d}s \tag{3.6}$$

and, by using the standard limit theorem for integrals, we obtain that since for all $n \in \mathbb{N}$ it holds $y_n \in \mathbb{X}$, then $y_\infty \in \mathbb{X}$. □

Proposition 3.3. *Under the assumptions given in* (3.5)*, the space* $\mathbb{X}$ *is convex.*

P r o o f. Let us consider y_1, $y_2 \in \mathbb{X}$ and μ, $\lambda \in [0,1]$ such that $\lambda + \mu = 1$. It is immediate to observe that $\lambda y_1 + \mu y_2 \in C([0,1],[0,1])$. Now our goal is to prove that $\lambda y_1 + \mu y_2 \in \mathbb{X}$, which is immediate remarking

$$\begin{aligned}
&\delta - 2\int_0^\beta \int_\alpha^{\lambda y_1+\mu y_2} (\lambda y_1 + \mu y_2)\mathrm{d}s\mathrm{d}t \\
= \quad & \delta - 2 \int_{\{t\in[0,\beta] \mid \alpha\leq \lambda y_1(t)+\mu y_2(t)\}} \int_\alpha^{\lambda y_1(t)+\mu y_2(t)} (\lambda y_1(s) + \mu y_2(s))\mathrm{d}s\mathrm{d}t \\
& + 2 \int_{\{t\in[0,\beta] \mid \alpha> \lambda y_1(t)+\mu y_2(t)\}} \int_{\lambda y_1(t)+\mu y_2(t)}^{\alpha} (\lambda y_1(s) + \mu y_2(s))\mathrm{d}s\mathrm{d}t \\
&\geq \delta - 2\beta(1-\alpha) \geq 0.
\end{aligned}$$

Hence, the space $\mathbb{X}$ is convex. □

Hence, we may conclude stating the following theorem which sum up all our results.

Theorem 3.4. *Let us consider the operator* $T\colon \mathbb{X} \to \mathbb{X}$ *as defined in* Definition 3. *Under the assumptions* (3.5)*, the operator* $T\colon \mathbb{X} \to \mathbb{X}$ *is well defined and the space* $\mathbb{X}$ *is closed, bounded and convex.*

P r o o f. The statement is a direct consequence of Propositions 3.1, 3.2 and 3.3. □

3.3. The continuity and compactness of T

Let us study now the properties of the operator T. In the following proposition, we show that T is continuous.

Proposition 3.5. *Under the assumptions given in* (3.5)*, the operator* $T\colon \mathbb{X} \to \mathbb{X}$ *as defined in* Definition 3 *is continuous.*

Proof. Let us consider the sequence of functions $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{X}$ and such that $y_n \to y_\infty$ uniformly in the set $\mathbb{X}$. Let us consider the sequence $\{Ty_n\}_{n\in\mathbb{N}}$ and Ty_∞. Let us remark we may deduce from (3.6) that

$$\lim_{n\to\infty} \int_0^\beta \int_\alpha^{y_n(t)} y_n(s)\mathrm{d}s\mathrm{d}t = \int_0^\beta \int_\alpha^{y_\infty(t)} y_\infty(s)\mathrm{d}s\mathrm{d}t.$$

Then it is obvious to obtain by continuity that

$$\lim_{n\to\infty} \sqrt{\delta - 2\int_0^\beta \int_\alpha^{y_n(t)} y_n(s)\mathrm{d}s\mathrm{d}t} = \sqrt{\delta - 2\int_0^\beta \int_\alpha^{y_\infty(t)} y_\infty(s)\mathrm{d}s\mathrm{d}t}.$$

Let us recall that $y_\infty \in C([0,1],[0,1])$ is uniformly continuous in the set $[0,1]$. Then we have that

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \text{ such that } \xi, \eta \in [0,1] \ |\xi - \eta| < \delta \Rightarrow |y_\infty(\xi) - y_\infty(\eta)| < \varepsilon.$$

We estimate

$$\begin{aligned} |y_n(y_n(t)) - y_\infty(y_\infty(t))| &\le |y_n(y_n(t)) - y_\infty(y_n(t))| + |y_\infty(y_n(t)) - y_\infty(y_\infty(t))| \\ &\le \|y_n - y_\infty\| + |y_\infty(y_n(t)) - y_\infty(y_\infty(t))| \\ &\le \|y_n - y_\infty\| + \varepsilon \end{aligned} \tag{3.7}$$

for a sufficiently large n. The inequality (3.7) implies that for all $\varepsilon > 0$ and for all $t \in [0,1]$,

$$\lim_{n\to\infty} |y_n(y_n(t)) - y_\infty(y_\infty(t))| \le \varepsilon$$

and, as consequence,

$$\lim_{n\to\infty} |y_n(y_n(t)) - y_\infty(y_\infty(t))| = 0.$$

We deduce immediately that

$$\lim_{n\to\infty} \int_0^t \int_0^\tau y_n(y_n(s))\mathrm{d}s\mathrm{d}\tau = \int_0^t \int_0^\tau y_\infty(y_\infty(s))\mathrm{d}s\mathrm{d}\tau,$$

and then we conclude that T is continuous. □

In the following proposition, we prove that the operator T is compact.

Proposition 3.6. *Under the assumption* (3.5)*, the operator* $T\colon \mathbb{X} \to \mathbb{X}$ *as defined in* Definition 3 *is compact.*

Proof. Let us remark that for all $y \in \mathbb{X}$, it holds true

$$0 \le Ty(t) \le \alpha + \frac{\beta}{2} + \sqrt{\delta + 2\beta\alpha} \le 1.$$

Thus if we consider an equibounded sequence $\{y_n\}_{n\in\mathbb{N}}$, then we also get that $\{Ty_n\}_{n\in\mathbb{N}}$ is equibounded. Moreover, from the definition of T, we remark that for every $y \in \mathbb{X}$, there exists the derivative $\frac{\mathrm{d}}{\mathrm{d}t}Ty(t)$. In particular, we get

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}Ty(t)\right| \le \alpha + \frac{\beta}{2} + \sqrt{\delta + 2\beta\alpha}.$$

Then the sequence $\{Ty_n\}_{n\in\mathbb{N}}$ is equicontinuous. Thus, by Ascoli-Arzelà theorem, it is possible to extract a convergent subsequence, which implies that T is a compact operator. □

We conclude this section proving the theorem which allows us to state that there is at least one solution of the differential problem (1.1).

Theorem 3.7. *Under the assumptions* (3.5)*, the differential problem*

$$\begin{cases} \ddot{y}(t) = \beta y(y(t)), \qquad t \in [0,1], \\ y(0) = \alpha, \\ \int\limits_0^\beta \dot{y}^2(t)\mathrm{d}t = \delta\beta > 0 \end{cases}$$

has at least one solution.

P r o o f. Let us recall that by Theorem 3.4, the space $\mathbb{X}$ is bounded, closed and convex. Furthermore, the operator $T\colon \mathbb{X} \to \mathbb{X}$ is continuous and compact. Then, by Schauder's fixed point theorem, we obtain immediately that there exists a $y \in \mathbb{X}$ such that $T(y) = y$. □

***Example* 1.** By Theorem 3.7, we get that the integrodifferential problem

$$\begin{cases} \ddot{y}(t) = \beta y(y(t)), \quad t \in [0,1], \\ y(0) = 0, \\ \int\limits_0^{\frac{1}{10}} \dot{y}^2(t)\mathrm{d}t = \frac{1}{50} \end{cases} \tag{3.8}$$

has at least one solution.

4. Second problem

We prove now that the problem (1.2) has at least one solution. As in the previous case, we will find an operator T_1, a space of function $\mathbb{X}_1 \subset C([0,1],[0,1])$ and some assumptions which will allow us to apply the Schauder's fixed point theorem and recover the existence of a solution. Also in this case, we have to show that $\mathbb{X}_1$ is bounded, closed and convex and our operator T_1 is continuous and compact.

4.1. The definition of the operator T_1

We start studying heuristically the differential problem (1.2).

Let us remark that multiplying both sides our ODE by $y(t)$ and applying standard techniques, we deduce

$$\dot{y}(\tau)y(\tau) = \beta y\bigg(\int\limits_0^\tau y(s)\mathrm{d}s\bigg)y(\tau) \implies \frac{\mathrm{d}}{\mathrm{d}\tau}\int\limits_0^t \frac{y^2(\tau)}{2}\mathrm{d}\tau = \beta\int\limits_0^t \frac{\mathrm{d}}{\mathrm{d}\tau}\int\limits_0^{\int_0^\tau y(s)\mathrm{d}s} y(s)\mathrm{d}s.$$

This implies

$$y^2(t) = y^2(0) + 2\beta \int\limits_0^{\int_0^t y(\tau)\mathrm{d}\tau} y(s)\mathrm{d}s.$$

Hence, by using the boundary condition of the problem (1.2), we obtain

$$\delta\beta = \int_0^\beta y^2(t)\mathrm{d}t = \beta y^2(0) + 2\beta \int_0^\beta \Bigg(\int_0^{\int_0^t y(\tau)\mathrm{d}\tau} y(s)\mathrm{d}s \Bigg)\mathrm{d}t.$$

Hence, we may rewrite

$$y(0) = \pm\sqrt{\delta - 2\int_0^\beta \Bigg(\int_0^{\int_0^t y(\tau)\mathrm{d}\tau} y(s)\mathrm{d}s \Bigg)\mathrm{d}t}.$$

We define the space of functions $\mathbb{X}_1$ and the operator T_1 which will be studied in order to find the existence of a solution of (1.2).

Definition 4. Let us consider the space of continuous functions $C([0,1],[0,1])$. We define the space of functions $\mathbb{X}_1 \subset C([0,1],[0,1])$ as

$$\mathbb{X}_1 = \left\{ y \in C([0,1],[0,1]) \;\middle|\; \delta - 2\int_0^\beta \Bigg(\int_0^{\int_0^t y(\tau)\mathrm{d}\tau} y(s)\mathrm{d}s \Bigg)\mathrm{d}t \geq 0 \right\} \tag{4.1}$$

and the operator T_1 as

$$T_1 y(t) = \sqrt{\delta - 2\int_0^\beta \Bigg(\int_0^{\int_0^t y(\tau)\mathrm{d}\tau} y(s)\mathrm{d}s \Bigg)\mathrm{d}t} + \beta \int_0^t \Bigg(y\Bigg(\int_0^\tau y(s)\mathrm{d}s \Bigg)\Bigg)\mathrm{d}\tau.$$

Remark 5. Let us remark that the function $F(t) = \int_0^t y(\tau)\mathrm{d}\tau$ is increasing in t since $y \in C([0,1],[0,1])$ and that, since $|y(t)| \leq 1$ for all $t \in [0,1]$, it holds true that $0 \leq F(t) \leq t$.

Remark 6. If $\delta - 2\beta \geq 0$, then it is immediate to get

$$T_1 y(t) \geq \sqrt{\delta - 2\beta} \geq 0$$

for all $y \in \mathbb{X}_1$ and $t \in [0,1]$.

Remark 7. Since $y \in C([0,1],[0,1])$, we observe that by Remark 5, we obtain

$$\int_0^\beta \Bigg(\int_0^{\int_0^t y(\tau)\mathrm{d}\tau} y(s)\mathrm{d}s \Bigg)\mathrm{d}t \leq \beta \qquad \text{and} \qquad \beta \int_0^t \Bigg(y\Bigg(\int_0^\tau y(s)\mathrm{d}s \Bigg)\Bigg)\mathrm{d}\tau \leq \beta t$$

and, if $\sqrt{\delta} + \beta \leq 1$, then

$$\begin{aligned} T_1 y(t) &= \sqrt{\delta - 2\int_0^\beta \Bigg(\int_0^{\int_0^t y(\tau)\mathrm{d}\tau} y(s)\mathrm{d}s \Bigg)\mathrm{d}t} + \beta \int_0^t \Bigg(y\Bigg(\int_0^\tau y(s)\mathrm{d}s \Bigg)\Bigg)\mathrm{d}\tau \\ &\leq \sqrt{\delta} + \beta t \leq \sqrt{\delta} + \beta \leq 1. \end{aligned}$$

Remark 8. The conditions for which our operator is well defined are

$$\begin{cases} \sqrt{\delta} + \beta \leq 1, \\ \delta - 2\beta \geq 0. \end{cases} \tag{4.2}$$

The system (4.2) has at least one solution. In fact it may be solved, for instance, by the couple $(\delta, \beta) = (\frac{1}{4}, \frac{1}{8})$.

Hence, we may conclude stating the following proposition.

Proposition 4.1. *Under the conditions* (4.2)*, the operator* $T_1 \colon \mathbb{X}_1 \to \mathbb{X}_1$ *as in* Definition 4 *is well defined.*

P r o o f. It follows immediately from the Remarks 6, 7, 8. □

4.2. The properties of the space $\mathbb{X}_1$

In this section, we will prove that $\mathbb{X}_1$ is closed and convex, as did previously with the space $\mathbb{X}$.

Proposition 4.2. *Under the assumptions* (4.2)*, the space* $\mathbb{X}_1$ *is closed.*

P r o o f. Let us consider a Cauchy sequence $\{y_n\}_{n\in\mathbb{N}}$ in $\mathbb{X}_1 \subset C([0,1],[0,1])$. Then, by the completeness of this space, there will exist a $y_\infty \in C([0,1],[0,1])$ such that $y_n \to y_\infty$ uniformly in $[0,1]$. We get immediately that $y_\infty \in C([0,1],[0,1])$. Furthermore, we remark that

$$\begin{aligned} &\left| \int_0^{\int_0^t y_n(\tau)} y_n(s)\mathrm{d}s - \int_0^{\int_0^t y_\infty(\tau)\mathrm{d}\tau} y_\infty(s)\mathrm{d}s \right| \\ &\leq \left| \int_0^{\int_0^t y_n(s)\mathrm{d}s} y_n(\tau)\mathrm{d}\tau - \int_0^{\int_0^t y_\infty(s)\mathrm{d}s} y_n(\tau)\mathrm{d}\tau \right| \\ &\qquad + \left| \int_0^{\int_0^t y_\infty(s)\mathrm{d}s} y_n(\tau)\mathrm{d}\tau - \int_0^{\int_0^t y_\infty(s)\mathrm{d}s} y_\infty(\tau)\mathrm{d}\tau \right| \\ &\leq \left| \int_0^t y_n(\tau)\mathrm{d}\tau - \int_0^t y_\infty(\tau)\mathrm{d}\tau \right| + \left| \int_0^{\int_0^t y_\infty(s)\mathrm{d}s} (y_n(\tau) - y_\infty(\tau))\mathrm{d}\tau \right| \end{aligned} \tag{4.3}$$

which will converge to zero as $n \to \infty$. Hence,

$$\lim_{n\to\infty} \int_0^{\int_0^t y_n(\tau)} y_n(s)\mathrm{d}s = \int_0^{\int_0^t y_\infty(\tau)\mathrm{d}\tau} y_\infty(s)\mathrm{d}s \tag{4.4}$$

which concludes our proof. □

Proposition 4.3. *Under the assumptions* (4.2)*, the space* $\mathbb{X}_1$ *is convex.*

Proof. Let us consider $y_1, y_2 \in \mathbb{X}_1$ and $\lambda, \mu \in [0,1]$ and $\lambda + \mu = 1$. It is immediate to deduce that $\lambda y_1 + \mu y_2 \in C([0,1],[0,1])$. To prove that $\lambda y_1 + \mu y_2 \in \mathbb{X}_1$ is sufficient to remark that

$$\delta - 2\int_0^\beta \Bigg(\int_0^{\int_0^t \lambda y_1(\tau) + \mu y_2(\tau) \mathrm{d}\tau} \lambda y_1(s) + \mu y_2(s) \mathrm{d}s \Bigg) \mathrm{d}t$$

$$\geq \delta - 2\int_0^\beta \Bigg(\int_0^{\int_0^t (\lambda+\mu) \mathrm{d}\tau} (\lambda + \mu) \mathrm{d}s \Bigg) \geq \delta - 2\beta \geq 0. \qquad \square$$

Hence, we deduce the following theorem which sum up all our results about the space $\mathbb{X}_1$.

Theorem 4.4. *Under the assumptions* (4.2)*, the operator* $T_1 \colon \mathbb{X}_1 \to \mathbb{X}_1$ *is well defined and the space* $\mathbb{X}_1$ *is bounded, closed and convex.*

Proof. It is an immediate consequence of the Propositions 4.2 and 4.3. $\square$

4.3. The continuity and compactness of T_1

Our goal now is to check that our operator T_1 is continuous and compact. This is a necessary step in order to apply the Schauder's theorem.

Proposition 4.5. *Under the assumptions* (4.2)*, the operator* $T_1 \colon \mathbb{X}_1 \to \mathbb{X}_1$ *as defined in* Definition 4 *is continuous.*

Proof. We have obtained from (4.3) that if $\{y_n\}_{n\in\mathbb{N}}$ converges uniformly to y_∞, then

$$\lim_{n\to\infty} \int_0^\beta \int_0^{\int_0^t y_n(\tau)} y_n(s) \mathrm{d}s = \int_0^\beta \int_0^{\int_0^t y_\infty(\tau)\mathrm{d}\tau} y_\infty(s) \mathrm{d}s \tag{4.5}$$

as $n \to \infty$. Hence, by standard theorems of convergence and continuity, it is straightforward to remark that

$$\lim_{n\to\infty} \sqrt{\delta - 2\int_0^\beta \Bigg(\int_0^{\int_0^t y_n(\tau)\mathrm{d}\tau} y_n(s)\mathrm{d}s \Bigg)\mathrm{d}t} = \sqrt{\delta - 2\int_0^\beta \Bigg(\int_0^{\int_0^t y_\infty(\tau)\mathrm{d}\tau} y_\infty(s)\mathrm{d}s \Bigg)\mathrm{d}t}.$$

Let us recall that $y_\infty \in C([0,1],[0,1])$ is uniformly continuous in the set $[0,1]$. Then we have that

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \text{ such that } \forall \xi, \eta \in [0,1] \quad |\xi - \eta| < \delta \Rightarrow |y_\infty(\xi) - y_\infty(\eta)| < \varepsilon.$$

Thus we estimate

$$\begin{aligned} &\bigg| y_n\Big(\int_0^\tau y_n(s)\mathrm{d}s\Big) - y_\infty\Big(\int_0^\tau y_\infty(s)\mathrm{d}s\Big)\bigg| \\ &\leq \bigg| y_n\Big(\int_0^\tau y_n(s)\mathrm{d}s\Big) - y_\infty\Big(\int_0^\tau y_n(s)\mathrm{d}s\Big)\bigg| + \bigg| y_\infty\Big(\int_0^\tau y_n(s)\mathrm{d}s\Big) - y_\infty\Big(\int_0^\tau y_\infty(s)\mathrm{d}s\Big)\bigg| \\ &\leq \|y_n - y_\infty\|_\infty + \varepsilon, \end{aligned}$$

where $\| \|_\infty$ is the standard L^∞ norm. Thus it is immediate to get the continuity. □

Proposition 4.6. *Under the assumptions* (4.2), *the operator* $T_1 \colon \mathbb{X}_1 \to \mathbb{X}_1$ *as defined in* Definition 4 *is compact.*

P r o o f. Let us consider the sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{X}_1$. Then it is straightforward to see that also the sequence $\{T_1 y_n\}_{n\in\mathbb{N}}$ will be equibounded. Let us remark that

$$\frac{\mathrm{d}}{\mathrm{d}t} T_1 y_n(t) = \beta y_n\biggl(\int_0^t y_n(\tau)\mathrm{d}\tau\biggr) y_n(\tau),$$

and then, $\left|\frac{\mathrm{d}}{\mathrm{d}t} T_1 y_n(t)\right| \le \beta$. Hence, the sequence $\{T y_n\}_{n\in\mathbb{N}}$ is equicontinuous. Hence, by the Ascoli-Arzelà theorem it is possible to extract a subsequence. Thus T is compact. □

We conclude this section showing the existence of a solution of the problem (1.2).

Theorem 4.7. *Under the assumptions* (4.2)*, the integrodifferential problem*

$$\begin{cases} \dot{y}(t) = \beta y\bigl(\int_0^t y(s)\mathrm{d}s\bigr), \quad t \in [0,1], \\ \int_0^\beta y^2(t)\mathrm{d}t = \delta\beta \end{cases}$$

with $0 < \beta \le 1$ *has at least one solution.*

P r o o f. Let us recall that by Theorem 4.4, the space $\mathbb{X}_1$ is bounded, closed and convex. Furthermore, the operator $T_1 \colon \mathbb{X}_1 \to \mathbb{X}_1$ is continuous and compact. Then, by Schauder's theorem, we obtain immediately that there exists a $y \in \mathbb{X}_1$ such that $T_1(y) = y$. □

***Example* 2.** By Theorem 4.7, we get that the integrodifferential problem

$$\begin{cases} \dot{y}(t) = \frac{1}{8} y\bigl(\int_0^t y(s)\mathrm{d}s\bigr), \quad t \in [0,1], \\ \int_0^{\frac{1}{8}} y^2(t)\mathrm{d}t = \dfrac{1}{32} \end{cases}$$

has at least one solution.

5. Third problem

5.1. The definition of the operator T_L

Finally, let us study the problem (1.3).

Let us start remarking that by multiplying both sides of the integrodifferential equation of (1.3) by $y(x,t)$, we get

$$\frac{\partial}{\partial t} y^2(x,t) = 2\beta y(x,t) y\biggl(\int_0^t y(x,s)\mathrm{d}s, t\biggr)$$

which implies, by integrating both sides w.r.t. the time variable between 0 and t

$$y^2(x,t) = y^2(x,0) + 2\beta \int_0^t y(x,s)y\Big(\int_0^s y(x,\tau)\mathrm{d}\tau, s\Big)\mathrm{d}s. \tag{5.1}$$

Integrating one more time w.r.t. the time variable both the sides of the integrodifferential equation (5.1) and by using the boundary condition of (1.3), we get

$$\beta g(x) = \beta y^2(x,0) + 2\beta \int_0^\beta \int_0^t y(x,\tau)y\Big(\int_0^\tau y(x,s)\mathrm{d}s, \tau\Big)\mathrm{d}\tau\mathrm{d}t.$$

Then it is straightforward to deduce

$$y^2(x,0) = g(x) - 2 \int_0^\beta \int_0^t y(x,\tau)y\Big(\int_0^\tau y(x,s)\mathrm{d}s, \tau\Big)\mathrm{d}\tau\mathrm{d}t.$$

Now we define the space of functions and the operator that we need to prove the existence of a solution of (1.3).

Definition 5. Let us consider the space of continuous functions $C([0,1]\times[0,1],[0,1])$. We define the space $\mathbb{X}_L \subset C([0,1]\times[0,1],[0,1])$ as

$$\mathbb{X}_L = \Big\{ y \in C\big([0,1]\times[0,1],[0,1]\big) \,\Big|\, g(x) - 2\int_0^\beta \int_0^t y(x,\tau)y\Big(\int_0^\tau y(x,s)\mathrm{d}s,\tau\Big)\mathrm{d}\tau\mathrm{d}t \geq \varepsilon_0$$
$$\text{with } |y(x_2,t) - y(x_1,t)| \leq L|x_2 - x_1| \text{ for a } L>0 \text{ and for all } x_1,x_2\in[0,1] \text{ and } t\in[0,1]\Big\}$$

with $\varepsilon_0 > 0$. We define the functional T_L as

$$T_L y(x,t) = \sqrt{g(x) - 2\int_0^\beta \int_0^t y(x,\tau)y\Big(\int_0^\tau y(x,s)\mathrm{d}s,\tau\Big)\mathrm{d}\tau\mathrm{d}t} + \beta\int_0^t y\Big(\int_0^\tau y(x,s)\mathrm{d}s,\tau\Big)\mathrm{d}\tau.$$

Remark 9. Let us remark that for all $y \in \mathbb{X}_L$, we get

$$T_L y(x,t) \geq \sqrt{\varepsilon_0} > 0 \qquad \text{and} \qquad T_L y(x,t) \leq \sqrt{g(x)} + \beta.$$

If we assume as condition $\sqrt{g(x)} + \beta \leq 1$ for all $x \in [0,1]$, then it is straightfoward to deduce that $0 < \sqrt{\varepsilon_0} \leq T_L y(x,t) \leq 1$.

Remark 10. Let us remark that for all $y \in \mathbb{X}_L$, there exists the derivative w.r.t. the time variable of the operator $T_L y$. In particular,

$$0 \leq \frac{\partial}{\partial t} T_L y(x,t) \leq \beta y\Big(\int_0^t y(x,s)\mathrm{d}s, t\Big) \leq \beta.$$

Remark 11. Since $g\colon [0,1] \to [0,1]$ is smooth, then we remark that there exists a constant $L_g > 0$ such that $|g(x_2) - g(x_1)| \le L_g |x_2 - x_1|$. Thus we may estimate the difference

$$\begin{aligned} T_L y(x_2,t) - T_L y(x_1,t) = {}& \sqrt{g(x_2) - 2\int_0^\beta \int_0^t y(x_2,\tau) y\bigg(\int_0^\tau y(x_2,s)\mathrm{d}s, \tau\bigg)\mathrm{d}\tau \mathrm{d}t} \\ & - \sqrt{g(x_1) - 2\int_0^\beta \int_0^t y(x_1,\tau) y\bigg(\int_0^\tau y(x_1,s)\mathrm{d}s, \tau\bigg)\mathrm{d}\tau \mathrm{d}t} \\ & + \beta \int_0^t \bigg[y\bigg(\int_0^\tau y(x_2,s)\mathrm{d}s, \tau\bigg) - y\bigg(\int_0^\tau y(x_1,s)\mathrm{d}s, \tau\bigg)\bigg]\mathrm{d}\tau. \end{aligned}$$

We remark, using the Lipschitz property of the function y, that

$$\begin{aligned} & \bigg|\int_0^t \bigg[y\bigg(\int_0^\tau y(x_2,s)\mathrm{d}s, \tau\bigg) - y\bigg(\int_0^\tau y(x_1,s)\mathrm{d}s, \tau\bigg)\bigg]\mathrm{d}\tau\bigg| \\ & \le L \int_0^t \int_0^\tau |y(x_2,s) - y(x_1,s)|\mathrm{d}s\mathrm{d}\tau \le \frac{L^2}{2}|x_2 - x_1| \end{aligned}$$

and, by considering standard algebraic computations, the Remark 9 and the Lipschitz condition of the functions g and y, we get

$$\begin{aligned} & \bigg| \sqrt{g(x_2) - 2\int_0^\beta \int_0^t y(x_2,\tau) y\bigg(\int_0^\tau y(x_2,s)\mathrm{d}s, \tau\bigg)\mathrm{d}\tau \mathrm{d}t} \\ & \quad - \sqrt{g(x_1) - 2\int_0^\beta \int_0^t y(x_1,\tau) y\bigg(\int_0^\tau y(x_1,s)\mathrm{d}s, \tau\bigg)\mathrm{d}\tau \mathrm{d}t}\bigg| \\ & \le \frac{1}{2\sqrt{\varepsilon_0}} \bigg[|g(x_2) - g(x_1)| \\ & \quad + \int_0^\beta \int_0^t \bigg| y(x_2,\tau) y\bigg(\int_0^\tau y(x_2,s)\mathrm{d}s, \tau\bigg) - y(x_1,\tau) y\bigg(\int_0^\tau y(x_1,s)\mathrm{d}s, \tau\bigg)\bigg|\mathrm{d}\tau\mathrm{d}t\bigg] \\ & \le \frac{1}{2\sqrt{\varepsilon_0}} \bigg[L_g |x_2 - x_1| + 2\int_0^\beta \int_0^t |y(x_2,\tau) - y(x_1,\tau)|\mathrm{d}\tau\mathrm{d}t \\ & \quad + 2\int_0^\beta \int_0^t \bigg| y\bigg(\int_0^\tau y(x_2,s)\mathrm{d}s, \tau\bigg) - y\bigg(\int_0^\tau y(x_1,s)\mathrm{d}s, \tau\bigg)\bigg|\mathrm{d}\tau\mathrm{d}t\bigg] \\ & \le \frac{1}{2\sqrt{\varepsilon_0}} \bigg[L_g + L\beta^2 + L^2 \frac{\beta^3}{3}\bigg] |x_2 - x_1|. \end{aligned}$$

If we consider $L > 0$ such that $\frac{1}{2\sqrt{\varepsilon_0}}\big[L_g + L\beta^2 + L^2\frac{\beta^3}{3}\big] + \beta\frac{L^2}{2} \le L$, then we have that $|T_Ly(x_2,t) - T_Ly(x_1,t)| \le L|x_2 - x_1|$ for all $y \in \mathbb{X}_L$.

Remark 12. Let us observe that if we assume that $g(x) - 2\beta^2 \ge \varepsilon_0$ for all $x \in [0,1]$, we observe

$$g(x) - 2\int_0^\beta \int_0^t T_Ly(x,\tau)T_Ly\Big(\int_0^\tau T_Ly(x,s)\mathrm{d}s, \tau\Big)\mathrm{d}\tau\mathrm{d}t \ge g(x) - 2\beta^2 \ge \varepsilon_0.$$

Remark 13. Remarks 9, 10, 11, 12 allow us to define as set of assumptions

$$\begin{cases} \sqrt{g(x)} + \beta \le 1, \\ |g(x_2) - g(x_1)| \le L_g|x_2 - x_1|, \\ \dfrac{1}{2\sqrt{\varepsilon_0}}\Big[L_g + L\beta^2 + L^2\dfrac{\beta^3}{3}\Big] + \beta\dfrac{L^2}{2} \le L, \\ g(x) - 2\beta^2 \ge \varepsilon_0. \end{cases} \tag{5.2}$$

By the previous remarks, we get immediately the following result.

Proposition 5.1. *Under the assumptions* (5.2)*, the operator* $T_L\colon \mathbb{X}_L \to \mathbb{X}_L$ *as in* Definition 5 *is well defined.*

Proof. It is an immediate consequence of the Remarks 9, 10, 11, 12. □

5.2. The properties of the space $\mathbb{X}_L$

As did for the two previous problems, we prove now that $\mathbb{X}_L$ is closed and convex.

Proposition 5.2. *Under the assumptions* (5.2)*, the space* $\mathbb{X}_L$ *is closed.*

Proof. Let us suppose that $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{X}_L$ and, due to the completeness of $C([0,1]\times[0,1],[0,1])$, $y_n \to y_\infty$ uniformly with $y_\infty \in C([0,1]\times[0,1],[0,1])$. It is immediate to remark that since it holds true that $|y_n(x_2,t) - y_n(x_1,t)| \le L|x_2 - x_1|$, then we may estimate y_∞ as $|y_\infty(x_2,t) - y_\infty(x_1,t)| \le L|x_2 - x_1|$.

Furthermore, let us remark

$$\begin{aligned} &\Big|\int_0^\beta\int_0^t y_n(x,t)y_n\Big(\int_0^\tau y_n(x,s)\mathrm{d}s,\tau\Big)\mathrm{d}\tau\mathrm{d}t - \int_0^\beta\int_0^t y_\infty(x,t)y_\infty\Big(\int_0^\tau y_\infty(x,s)\mathrm{d}s,\tau\Big)\mathrm{d}\tau\mathrm{d}t\Big| \\ &\le \int_0^\beta\int_0^t |y_n(x,\tau) - y_\infty(x,\tau)|\mathrm{d}\tau\mathrm{d}t + \int_0^\beta\int_0^t \Big|y_n\Big(\int_0^\tau y_n(x,s)\mathrm{d}s,\tau\Big) - y_\infty\Big(\int_0^\tau y_\infty(x,s)\mathrm{d}s,\tau\Big)\Big|\mathrm{d}\tau\mathrm{d}t \\ &\le \frac{\beta^2}{2}\|y_n - y_\infty\| + \int_0^\beta\int_0^t \Big|y_n\Big(\int_0^\tau y_n(x,s)\mathrm{d}s,\tau\Big) - y_n\Big(\int_0^\tau y_\infty(x,s)\mathrm{d}s,\tau\Big) \\ &\qquad + y_n\Big(\int_0^\tau y_\infty(x,s)\mathrm{d}s,\tau\Big) - y_\infty\Big(\int_0^\tau y_\infty(x,s)\mathrm{d}s,\tau\Big)\Big|\mathrm{d}\tau\mathrm{d}t \\ &\le \Big|\frac{\beta^2}{2} + L\frac{\beta^3}{3} + \frac{\beta^2}{2}\Big|\|y_n - y_\infty\|_\infty, \end{aligned} \tag{5.3}$$

where $\| \|_\infty$ is the standard L^∞ norm. Finally, remarking that

$$\varepsilon_0 \le \lim_{n\to\infty} \left[g(x) - 2 \int_0^\beta \int_0^t y_n(x,\tau) y_n\left(\int_0^\tau y_n(x,s)\mathrm{d}s, \tau \right) \mathrm{d}\tau \mathrm{d}t \right],$$

then it is immediate to conclude that $\mathbb{X}_L$ is closed. □

Proposition 5.3. *Under the assumptions* (5.2)*, the space* $\mathbb{X}_L$ *is convex.*

Proof. Let us consider $y_1, y_2 \in \mathbb{X}_L$ and $\lambda, \mu \in [0,1]$ and $\lambda + \mu = 1$. It is easy to observe that $\lambda y_1 + \mu y_2 \in C([0,1]\times[0,1],[0,1])$. To prove that $\lambda y_1 + \mu y_2 \in \mathbb{X}_L$ is sufficient to remark that

$$\begin{aligned} &g(x) - 2 \int_0^\beta \int_0^t (\lambda y_1 + \mu y_2)(x,\tau)(\lambda y_1 + \mu y_2)\left(\int_0^\tau (\lambda y_1 + \mu y_2)(x,s)\mathrm{d}s, \tau \right) \mathrm{d}\tau \mathrm{d}t \\ &\ge g(x) - 2\beta^2 \ge \varepsilon_0. \end{aligned}$$

□

Thus we may state the following theorem.

Theorem 5.4. *Under the assumptions* (5.2)*, the operator* $T_L \colon \mathbb{X}_L \to \mathbb{X}_L$ *as defined in* Definition 5 *is well defined and* $\mathbb{X}_L$ *is a bounded, closed and convex space.*

Proof. It is an immediate consequence of the Propositions 5.1, 5.2 and 5.3. □

5.3. The continuity and compactness of T_L

We show now that the operator T_L is continuous and compact.

Proposition 5.5. *Under the assumptions* (5.2)*, the operator* $T_L \colon \mathbb{X}_L \to \mathbb{X}_L$ *as defined in* Definition 5 *is continuous.*

Proof. From Proposition 5.2, we get

$$\lim_{n\to\infty} \int_0^t y_n\left(\int_0^\tau y_n(x,s)\mathrm{d}s, \tau \right) \mathrm{d}\tau = \int_0^t y_\infty\left(\int_0^\tau y_\infty(x,s)\mathrm{d}s, \tau \right) \mathrm{d}\tau.$$

From this estimate, it is immediate to deduce the continuity of our operator T_L. □

Proposition 5.6. *Under the assumptions* (5.2)*, the operator* $T_L \colon \mathbb{X}_L \to \mathbb{X}_L$ *defined as in* Definition 5 *is compact.*

Proof. Let us consider the sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{X}_L$. It is straightforward to remark that $\{T_L y_n\}_{n\in\mathbb{N}}$ is equibounded by Remark 9. Let us observe that also $\{T_L y_n\}_{n\in\mathbb{N}}$ is equicontinuous, since by Remarks 10 and 11, it holds true

$$|T_L y_n(x_2,t) - T_L y_n(x_1,t)| \le L|x_2 - x_1| \qquad \text{and} \qquad 0 \le \frac{\partial}{\partial t} T_L y_n(x,t) \le \beta \tag{5.4}$$

for all $n \in \mathbb{N}$. Hence, by Ascoli-Arzelà theorem, it is possible to extract a convergent subsequence of our sequence. Thus the operator T_L is compact. □

We conclude stating and proving that our differential problem (1.3) has at least one solution.

Theorem 5.7. *Under the assumptions* (5.2)*, the integrodifferential problem*

$$\begin{cases} \dfrac{\partial}{\partial t} y(x,t) = \beta y\Big(\int\limits_0^t y(x,s)\mathrm{d}s, t\Big), & (x,t) \in [0,1]\times[0,1], \\ \int\limits_0^\beta y^2(x,t) = \beta g(x) \end{cases}$$

with $0 < \beta \le 1$ *has at least one solution.*

Proof. Let us recall that by Theorem 5.4, the space $\mathbb{X}_L$ is bounded, closed and convex. Furthermore, the operator $T_L \colon \mathbb{X}_L \to \mathbb{X}_L$ is continuous and compact. Then, by Schauder's theorem, we obtain immediately that there exists a $y \in \mathbb{X}_L$ such that $T_L(y) = y$. □

6. Open problems

We propose some open problems which are a further generalization of (1.3):

(1)

$$\begin{cases} \dfrac{\partial}{\partial t} y(x,t) = \beta y\Big(\int\limits_0^x y(\xi,t)\mathrm{d}\xi, t\Big), & x \in [0,1],\ t \in [0,1], \\ \int\limits_0^\beta y^2(x,t)\mathrm{d}t = \beta g(x), \end{cases}$$

where $0 < \beta \le 1$ and $g \colon [0,1] \to [0,1]$ is smooth function;

(2)

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} y(x,t) = \beta y\Big(\int\limits_0^x y(\xi,s)\mathrm{d}\xi, t\Big), & x \in [0,1],\ t \in [0,1], \\ y(x,0) = f(x), \\ \int\limits_0^\beta \big[\frac{\partial}{\partial s} y(x,s)\big]^2 \mathrm{d}s = \beta g(x), \end{cases}$$

where $f \colon [0,1] \to [0,1]$ and $g \colon [0,1] \to [0,1]$ suitable smooth functions and $0 < \beta \le 1$.

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