Estimation of Conditional Value-at-Risk in Linear Model

Devoted to the Memory of Pranab Kumar Sen

Jana Jurečková[1], Jan Picek[2], and Jan Kalina[3]

[1] The Czech Academy of Sciences, Institute of Information Theory and Automation, CZ-182 00 Prague 8, Czech Republic, `jureckova@utia.cas.cz`, http://www.utia.cas.cz

[2] Technical University in Liberec, Faculty of Science, Humanities, and Education, Liberec, Czech Republic, `jan.picek@tul.cz`

[3] Charles University, Faculty of Mathematics and Physics, Prague, Czech Republic & The Czech Academy of Sciences, Institute of Computer Science, Prague, Czech Republic, `kalina@cs.cas.cz`

Abstract. The conditional value-at-risk (CVaR) represents a popular risk measure often exploited e.g. within portfolio optimization. The situation with a nuisance linear regression is considered here; in other words, we do not observe directly the loss Z of interest, but only $Y = \beta_0 + \mathbf{X}\boldsymbol{\beta} + Z$, where the covariates are not under our control. We propose a novel estimator of $\mathsf{CVaR(Z)}$ based on the *averaged two-step regression quantile* combined with an R-estimate of regression parameters.

Keywords: linear regression, risk measure, conditional value-at-risk, nuisance regression, averaged regression quantile, two-step regression quantile

1 Introduction

We follow the variables $\mathbf{Z}_n = (Z_1, \ldots, Z_n)^\top$ measuring the loss Z of an asset or of a portfolio at times 1, 2, . . . , n. They are assumed to be independent and identically distributed (*i.i.d.*) with a distribution function F, satisfying

$$\int z dF(z) = 0, \quad \int z^2 dF(z) < \infty, \tag{1}$$

otherwise unknown. We are interested in the possible risk of the loss Z in a given period and with a particular confidence level $\alpha \in (0,1)$. The popular risk measure is the *Conditional Value-at-Risk (*CVaR*)*, equal to

$$\mathsf{CVaR}_\alpha(\mathsf{Z}) = \mathbf{E}\{\mathsf{Z}|\mathsf{Z} > \mathsf{F}^{-1}(\alpha)\} = (1-\alpha)^{-1}\int_\alpha^1 \mathsf{F}^{-1}(\mathsf{t})\mathsf{dt} = (1-\alpha)^{-1}\int_{\mathsf{F}^{-1}(\alpha)}^\infty \mathsf{zdF(z)}. \tag{2}$$

It has obtained applications in many areas immediately after its introduction; let us mention the management of water supplies, risk management of the social security fund, the cash flow risk measurement for non-life insurance industry, the financial risk in the industrial areas, operational risk in the banks, and others. In the finance is CVaR taken as a popular a risk measure in portfolio optimization or management of extreme risk (cf. [5]). There is a rich bibliography on the subject, both from theoretical and applications points. Innovative ideas on the subject and estimation methods were presented e.g. in [1, 20, 21].

A review of estimation methods for CVaR, based on data, is collected in [18]. It describes the parametric, nonparametric and semiparametric estimation methods, supplemented with the computer software, partially available from the R package (*R* Development Core Team, 2012). In the parametric setup, [18] admits the skew and asymmetric probability distributions. The computing methods based on characteristic functions and Fourier analysis through the empirical data are studied in [22]. There is also compared the Value-at-Risk (VaR) and CVaR with the new *expectile based risk measure* (ERM); the ERM is found as the only risk measure that is both coherent and elicitable. Recently, [12] exploited the concept of pseudo-capacities, elaborated in [9] and [3], in estimating the CVaR in nonparametric and measurement error situations.

The risk of a loss is often affected by a nuisance regression caused by exogenous market variables as interest rates, overall market sentiment, liquidity shocks, etc. Effects of nuisance regression in testing predictive models were recently considered either in [17] for assessing value-at-risk or in [4] for combining two forecasts for assessing CVaR. Joint fitting VaR and CVaR in regression models by a two-step procedure has been recently considered in [8].

In the present paper, we consider estimating CVaR in the situation that the loss of interest is unobservable, being affected by the covariates within a nuisance linear regression model. Moreover, the situation is nonparameric, with an unknown probability distribution of the loss. The literature is almost void in estimation of CVaR for models with a nuisance regression. Mathematical solution of estimating CVaR under a nuisance linear regression was considered by Trindade et al. [21]; they proposed to estimate the nuisance regression parameters with an M-estimate, minimizing a convex criterion under the restriction that the resulting CVaR does not exceed a given $\eta > 0$. However, the M-estimator and the resulting CVaR are not scale equivariant, while the scale equivariance is a desirable property of measures in applications. Comparing with the proposal in [21], our novel estimator of CVaR proposed here is equal to the *averaged regression α-quantile* of residuals with respect to the R-estimate of the regression parameters. As such, the estimate of CVaR is scale-equivariant and its asymptotics follows from the asymptotics for R-estimators and averaged regression quantile.

Section 2 of this paper overviews regression quantiles together with averaged two-step regression quantiles, which are exploited in Section 3 to define a novel CVaR estimate for a nuisance regression.

2 Regression quantile and its two-step version

2.1 Motivation and notation

The averaged two-step regression α-quantile, introduced in [11], approximates the quantile function $F^{-1}(\alpha)$ of the errors asymptotically in probability for $n \to \infty$, up to the standardization with the regression parameters (cf. [14]). Because its number of breakpoints equals exactly to n, while in the case of ordinary regression quantile it is much larger, and it is nondecreasing in $\alpha \in (0,1)$, the averaged two-step regression quantile also facilitates joint fitting the VaR and CVaR in the regression model.

Even if distribution function of Z is unknown, the estimate of $\mathsf{CVaR}_\alpha(\mathsf{Z})$ in (2) can be obtained from the empirical quantile function of independent observations $Z_1, Z_2, \ldots, Z_n$. The resulting estimate would be

$$\widehat{\mathsf{CVaR}}_\alpha(Z) = \lfloor n(1-\alpha)\rfloor^{-1} \sum_{i=\lfloor n\alpha\rfloor}^{n} Z_{n:i} = \lfloor n(1-\alpha)\rfloor^{-1} \sum_{i=1}^{n} Z_i \, \mathcal{I}[Z_i \geq Z_{n:[n\alpha]}]. \tag{3}$$

However, the financial returns are often regressed on some covariates, and we can observe only the variables $\mathbf{Y}_n = (Y_{n1}, ..., Y_{nn})^\top$, what are the $\mathbf{Z}_n$ affected by covariates $\mathbf{X}_n$ with unknown intensities, measured by regression coefficients $\beta_1, \ldots, \beta_n$. Taking this into account, we work with the linear regression model

$$\mathbf{Y}_n = \beta_0 \mathbf{1}_n + \mathbf{X}_n \boldsymbol{\beta} + \mathbf{Z}_n \tag{4}$$

with observations $\mathbf{Y}_n = (Y_{n1}, \ldots, Y_{nn})^\top$, unknown parameters β_0 (intercept), $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_p)^\top$ (scales), $\mathbf{1}_n = (1, \ldots, 1)^\top \in \mathbb{R}_n$ and the $n \times p$ matrix $\mathbf{X} = \mathbf{X}_n$ of covariates. It is a known matrix with the rows $\mathbf{x}_i^\top = (x_{i1}, \ldots, x_{ip}),\ i = 1, \ldots, n$.

Because $Z_1, \ldots, Z_n$ are not available, every inference on the loss Z and on F^{-1}, hence also estimating $\mathsf{CVaR}_\alpha(Z)$, are possible only by means of observations of Y. Thus we have to look for an alternative explicit estimating of $\mathsf{CVaR}_\alpha(\mathsf{Z})$. Pioneering ideas in estimating the CVaR were presented in [1,20,21]. Compared to them, we take as a main tool for estimating CVaR the two-step α-regression quantile of the model (4), originated and illustrated by the present authors in [11,13,14].

2.2 Regression quantile

The model (4) can be rewritten as the model

$$Y_{ni} = \beta_0 + \mathbf{x}_{ni}^\top \boldsymbol{\beta} + Z_{ni},\ i = 1, \ldots, n \tag{5}$$

with covariates $\mathbf{x}_{n1}, \ldots, \mathbf{x}_{nn}$, each element of $\mathbb{R}_p$. For the sake of brevity, we also use the notation $\mathbf{x}_{ni}^* = (1, x_{i1}, \ldots, x_{ip})^\top,\ i = 1, \ldots, n$. Let $\widehat{\boldsymbol{\beta}}_n(\alpha) \in R_{p+1}$

be the α-regression quantile of model (4), $0 < \alpha < 1$, i.e. the solution of the minimization

$$\sum_{i=1}^{n} \rho_\alpha(Y_i - b_0 - \mathbf{x}_i^\top \mathbf{b}) = \min,\ b_0 \in \mathbb{R}_1,\ \mathbf{b} \in \mathbb{R}_p. \tag{6}$$

If derivative f of F exists and is positive in a neighborhood of the quantile $F^{-1}(\alpha)$, and if the matrix

$$\mathbf{Q}_n = n^{-1} \left(\mathbf{1}_n, \mathbf{X}_n\right)^\top \left(\mathbf{1}_n, \mathbf{X}_n\right)$$

is positively definite starting with some n, then $n^{\frac{1}{2}}(\widehat{\boldsymbol{\beta}}_n(\alpha) - \check{\boldsymbol{\beta}}(\alpha))$ admits the asymptotic representation (see e.g. [15])

$$n^{\frac{1}{2}}(\widehat{\boldsymbol{\beta}}_n(\alpha) - \check{\boldsymbol{\beta}}(\alpha)) = n^{-\frac{1}{2}}(f(F^{-1}(\alpha))^{-1}\mathbf{Q}_n^{-1} \sum_{i=1}^{n} \mathbf{x}_i^*(\alpha - I[Z_i < F^{-1}(\alpha)]) + o_p(1) \tag{7}$$

as $n \to \infty$, where $\check{\boldsymbol{\beta}}(\alpha) = (F^{-1}(\alpha) + \beta_0, \beta_1, \dots, \beta_p)^\top$ is the population counterpart of the regression quantile. The intercept part of the representation (7) is rewritten as

$$\begin{aligned}\hat{\beta}_{n0}(\alpha) - \beta_0 - F^{-1}(\alpha) &= (nf(F^{-1}(\alpha)))^{-1} \sum_{i=1}^{n}(\alpha - I[Z_i < F^{-1}(\alpha)]) + o_p(n^{-\frac{1}{2}}) \\ &= Z_{n:[n\alpha]} - F^{-1}(\alpha) + o_p(n^{-\frac{1}{2}}) \end{aligned} \tag{8}$$

as $n \to \infty$, where the first equality follows from (7), while the second equality follows from the Bahadur representation of sample quantile. $\hat{\beta}_{n1}(\alpha), \dots, \hat{\beta}_{np}(\alpha)$ are consistent estimates of the slope parameters $\beta_1 \dots, \beta_p$. The slope components of regression quantile are asymptotically independent of the intercept component $\hat{\beta}_{n0}(\alpha)$. The solution of (6) minimizes the $(\alpha, 1-\alpha)$ convex combination of residuals $(Y_i - \mathbf{x}_i^{*\top}\mathbf{b})$ over $\mathbf{b} \in \mathbf{R}^{p+1}$, where the choice of α depends on the balance between underestimating and overestimating the respective losses Y_i. The increasing $\alpha \nearrow 1$ reflects a greater concern about underestimating losses Y, comparing to overestimating. A useful functional of the regression quantile is the *averaged regression α-quantile*, the weighted mean of components of $\widehat{\boldsymbol{\beta}}_n(\alpha),\ 0 \leq \alpha \leq 1$:

$$\bar{B}_n(\alpha) = \overline{\mathbf{x}}_n^{*\top}\widehat{\boldsymbol{\beta}}_n(\alpha) = \widehat{\beta}_{n0}(\alpha) + \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{p} x_{ij}\widehat{\beta}_j(\alpha), \quad \overline{\mathbf{x}}_n^* = \frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i^* \tag{9}$$

As shown in [11], the $\bar{B}_n(\alpha) - \beta_0 - \bar{\mathbf{x}}_n^\top\boldsymbol{\beta}$ is asymptotically equivalent to the $[n\alpha]$-quantile $e_{n:[n\alpha]}$ of the model errors, if they are identically distributed.

2.3 Two-step regression quantile

The two-step regression quantile was introduced in [13] and later studied e.g. in [14], where it was shown that it is asymptotically equivalent to the ordinary

α-regression quantile. The two-step regression α-quantile combines the rank-estimator (R-estimator) $\widetilde{\boldsymbol{\beta}}_{nR}$ of the slope components $\boldsymbol{\beta}$ with the $[n\alpha]$ order statistics of the residuals $Y_i - \mathbf{x}_i^\top \widetilde{\boldsymbol{\beta}}_{nR},\ i = 1, \ldots, n$. The two-step regression quantile first estimates the slope components $\boldsymbol{\beta}$ by means of an R-estimate $\widetilde{\boldsymbol{\beta}}_{nR}(\lambda) \in \mathbb{R}^p$, defined as a minimizer of the Jaeckel's measure of the rank dispersion [10] with a fixed $\lambda \in (0,1)$:

$$\sum_{i=1}^{n}(Y_i - \mathbf{x}_i^\top \mathbf{b})[a_i(\lambda, \mathbf{b}) - \bar{a}_n(\lambda)] = \min \tag{10}$$

with respect to $\mathbf{b} = (b_1, b_2, \ldots, b_p)^\top \in \mathbb{R}^p$. The notation in (10) means:

$$a_i(\lambda, \mathbf{b}) = \begin{cases} 0 & \ldots \quad R_{ni}(Y_i - \mathbf{x}_i^\top \mathbf{b}) < n\lambda \\ R_i - n\lambda & \ldots \quad n\lambda \le R_{ni}(Y_i - \mathbf{x}_i^\top \mathbf{b}) < n\lambda + 1 \\ 1 & \ldots \quad n\lambda + 1 \le R_{ni}(Y_i - \mathbf{x}_i^\top \mathbf{b}), \end{cases} .$$

Here $R_{ni}(Y_i - \mathbf{x}_i^\top \mathbf{b}), i = 1, \ldots, n$ are the ranks of the residuals, and $a_i(\lambda, \mathbf{b})$ are known as *Hájek's rank scores* (see [7]). Note that $\bar{a}_n(\lambda) = \frac{1}{n}\sum_{i=1}^n a_i(\lambda, \mathbf{b})$ is constant in $\mathbf{b}$, as an average of the rank scores. The minimization (10) can be rewritten as

$$\sum_{i=1}^{n} \left(Y_i - \bar{Y}_n - (\mathbf{x}_i - \bar{\mathbf{x}}_n)^\top \mathbf{b}\right) a_i(\lambda, \mathbf{b}) = \min. \tag{11}$$

It implies that the solution of (10) is invariant to the intercept, which is a nuisance component. The solution of (10) and (11) is the R-estimator $\widetilde{\boldsymbol{\beta}}_{nR}(\lambda)$ of $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)^\top$, generated by the following score function $\varphi_\lambda : (0,1) \mapsto \mathbb{R}^1$:

$$\varphi_\lambda(u) + (1 - \lambda) = \begin{cases} 0 & \ldots \quad 0 \le u < \lambda \\ 1 & \ldots \quad \lambda \le u \le 1. \end{cases} . \tag{12}$$

Generally, as the score function we can use another nondecreasing square integrable function on $(0,1)$. By [15], $\widetilde{\boldsymbol{\beta}}_{nR}(\lambda)$ consistently estimates $\boldsymbol{\beta}$ under the following conditions on F and on $\mathbf{X}_n$:

(F1) The distribution function F has a continuous density f with a positive an finite Fisher information $\mathcal{I}(f)$.

(X1) Assume that, as $n \to \infty$,

$$n^{-1}\mathbf{V}_n = O_p(1) \quad \text{where} \quad \mathbf{V}_n = \sum_{i=1}^{n}(\mathbf{x}_{ni} - \bar{\mathbf{x}}_n)(\mathbf{x}_{ni} - \bar{\mathbf{x}}_n)^\top, \tag{13}$$

$$\max_{1 \le i \le n} \|\mathbf{x}_{ni} - \bar{\mathbf{x}}_n\| = o(n^{1/4}), \quad \bar{\mathbf{x}}_n = n^{-1}\sum_{i=1}^{n}\mathbf{x}_{ni}.$$

Moreover, we assume that $\mathbf{V}_n$ satisfies

$$\lim_{n\to\infty} \max_{1 \le i \le n} (\mathbf{x}_{ni} - \bar{\mathbf{x}}_n)^\top \mathbf{V}_n^{-1}(\mathbf{x}_{ni} - \bar{\mathbf{x}}_n) = 0. \tag{14}$$

Under conditions (F1) and (X1), the R-estimator $\widetilde{\boldsymbol{\beta}}_{nR} = \widetilde{\boldsymbol{\beta}}_{nR}(\lambda)$ admits the following asymptotic representation, as $n \to \infty$ (see e.g. [15] for the proof):

$$\widetilde{\boldsymbol{\beta}}_{nR}-\boldsymbol{\beta} = (f(F^{-1}(\lambda))^{-1}\mathbf{V}_n^{-1}\sum_{i=1}^{n}(\mathbf{x}_{ni}-\bar{\mathbf{x}}_n)\Big(I[Z_{ni} > F^{-1}(\lambda)]-(1-\lambda)\Big)+o_p(n^{-1/2}), \tag{15}$$

hence $\|n^{1/2}(\widetilde{\boldsymbol{\beta}}_{nR} - \boldsymbol{\beta})\| = O_p(1)$. The intercept component of the two-step regression
α-quantile is defined as the $[n\alpha]$-quantile of the residuals $Y_i - \mathbf{x}_i^\top \widetilde{\boldsymbol{\beta}}_{nR}(\lambda),\ i = 1,\dots,n$. Denote it as $\widetilde{\beta}_{nR,0}(\alpha)$, hence

$$\widetilde{\beta}_{nR,0}(\alpha) = \Big(Y_i - \mathbf{x}_i^\top \widetilde{\boldsymbol{\beta}}_{nR}(\lambda)\Big)_{n:[n\alpha]}$$

and we define the two-step α-regression quantile as the vector in $\mathbb{R}_{p+1}$

$$\widetilde{\boldsymbol{\beta}}_n(\alpha) = \Big(\widetilde{\beta}_{nR,0}(\alpha), (\widetilde{\boldsymbol{\beta}}_{nR}(\lambda))^\top\Big)^\top. \tag{16}$$

Hence, the averaged two-step regression α-quantile equals to

$$\widetilde{B}_{n\alpha} = \widetilde{\beta}_{nR,0}(\alpha) + \bar{\mathbf{x}}_n^\top \widetilde{\boldsymbol{\beta}}_{nR}(\lambda) = \Big(Y_i - (\mathbf{x}_i - \bar{\mathbf{x}}_n)^\top \widetilde{\boldsymbol{\beta}}_{nR}(\lambda)\Big)_{n:[n\alpha]} \tag{17}$$

It has been introduced in [14], where it is proven that

$$\widetilde{B}_{n\alpha} - \beta_0 - \bar{\mathbf{x}}_n^\top \boldsymbol{\beta} = Z_{n:[n\alpha]} + o_p(n^{-1/2}) = F^{-1}(\alpha) + o_p(n^{-1/2}) \ \text{ as } \ n \to \infty \tag{18}$$

uniformly for $\alpha \in (\varepsilon, 1-\varepsilon),\ 0 < \varepsilon \le 1/2$, and for any fixed $\lambda \in 0, 1$.

3 Estimation of CVaR_α

If there are available independent observations $Z_1, Z_2, \dots, Z_n$, then $\mathsf{CVaR}_\alpha(\mathsf{Z})$ in (2) can be estimated with the aid of their empirical quantile function, even if distribution function of Z is unknown. The corresponding estimate of the conditional value-at-risk of Z would be

$$\widehat{\mathsf{CVaR}}_{n\alpha}(Z) = \lfloor n(1-\alpha)\rfloor^{-1} \sum_{i=\lfloor n\alpha\rfloor}^{n} Z_{n:i} = \lfloor n(1-\alpha)\rfloor^{-1} \sum_{i=1}^{n} Z_i\ I[Z_i \ge Z_{n:[n\alpha]}]. \tag{19}$$

If the observations $Z_1, Z_2, \dots, Z_n$, are not at disposal, we can profit from approximations (17) and (18) and estimate the CVaR_α by means of the averaged two-step regression quantile $\widetilde{B}_n(\alpha)$, which explicitly contains only observations $Y_1, \dots, Y_n$. The resulting estimator is

$$\widehat{\mathsf{CVaR}}_{n\alpha}(Y) = \lfloor n(1-\alpha)\rfloor^{-1} \sum_{\alpha \le \delta < 1} \widetilde{B}_n(\delta) - \beta_0 - \bar{\mathbf{x}}_n^\top \boldsymbol{\beta}. \tag{20}$$

Hence, the estimate is determined up to the standardization with nuisance $\beta_0 + \bar{\mathbf{x}}_n^\top \boldsymbol{\beta}$, what can be approximated by means of $Y_{n1}, \ldots, Y_{nn}$ under the conditions (F1) and (X1), namely by $\bar{\mathbf{x}}_n^\top \widetilde{\boldsymbol{\beta}}_{nR}(\lambda)$ and by $\bar{Y}_n$. Indeed, by (5) and (1),

$$\bar{Y}_n = \beta_0 + \bar{\mathbf{x}}_n^\top \boldsymbol{\beta} + \bar{Z}_n = \beta_0 + \bar{\mathbf{x}}_n^\top \boldsymbol{\beta} + o_p(n^{-1/2})$$

as $n \to \infty$.

Summarizing, our proposed estimate of the $\mathsf{CVaR}_\alpha(\mathsf{Z})$ in the linear model is described in the following theorem:

Theorem 1. *Assume that the distribution of Z satisfies condition (F1), let $\int z dF(z) = 0,\ 0 < \int z^2 dF(z) < \infty$, and let the matrix $\mathbf{X}_n$ satisfy (X1). Then*

$$\widehat{\mathsf{CVaR}}_{n\alpha}(Y) = \lfloor n(1-\alpha) \rfloor^{-1} \sum_{\delta:\alpha \le \delta < 1} \Big(Y_i - (\mathbf{x}_i - \bar{\mathbf{x}}_n)^\top \widetilde{\boldsymbol{\beta}}_{nR}(\lambda) \Big)_{n:[n\delta]} - \bar{Y}_n \quad (21)$$

is a $\sqrt{n}$-consistent estimate of $\mathsf{CVaR}_\alpha(\mathsf{Z})$ for any fixed $\lambda \in (0,1)$; more precisely

$$\widehat{\mathsf{CVaR}}_{n\alpha}(Y) = \mathsf{CVaR}_\alpha(Z) + o_p(n^{-1/2}) \quad \text{as} \quad n \to \infty. \quad (22)$$

for any $\alpha \in (0,1)$. Moreover,

$$\Big(Y_i - (\mathbf{x}_i - \bar{\mathbf{x}}_n)^\top \widetilde{\boldsymbol{\beta}}_{nR}(\lambda) \Big)_{n:[n\alpha]} - \bar{Y}_n = \mathsf{VaR}_\alpha(\mathsf{Z}) + \mathsf{o_p}(\mathsf{n}^{-1/2}) \quad \text{as} \quad \mathsf{n} \to \infty. \quad (23)$$

Conclusion

A novel nonparametric estimate of the Conditional Value-at-Risk of variable Z is proposed here in the situation that the observations are the responses under the presence of nuisance regression. The estimate is based on the averaged two-step regression α-quantile of the linear model, exploiting an R-estimator of the slope components. As such, the estimates are $\sqrt{n}$ consistent and scale equivariant estimates of the Conditional Value-at-Risk of variable Z, in spite of the presence of the nuisance regression.

The novel estimators may find applications not only in finance, but also in various other (non-financial) models, even in situations under the presence of nuisance covariates. This can be of interest e.g. in the risk assessment in metrology [2, 16] or within integrated energy systems for optimal scheduling the energy supply for large cities [19]). The novel tools come to hand, if the risk is influenced by a combination of more nuisance effects, including weather conditions (temperature, humidity, wind speed, precipitation, seasonal trends), energy price fluctuations, natural events (power outages), or industrial activity.

Acknowledgement

The authors thank the Editor and Referees for their attentive reading the text and for their comments. The research was supported by the grants 22-03636S (J. Jurečková, J. Picek) and 24-11146S (J. Kalina) of the Czech Science Foundation.

References

1. Bassett, G.W., Koenker, R., Kordas, W. (2004). Pessimistic portfolio allocation and Choquet Expected Utility. *Journal Financial Economics* 2, 477–492.
2. Božić, D., Runje, B., Razumić, A. (2024). Risk assessment for linear regression models in metrology. *Applied Sciences* 14, 2605.
3. Buja, A. (1986). On the Huber-Strassen theorem. *Probab. Theory Rel.* 73, 149–152. *Annals of Operations Research* 313, 755–772.
4. Dimitriadis T., Liu, X., and Schnaitmann, J. (2023). Encompassing tests for value at risk and expected shortfall multistep forecasts based on inference on the boundary. *Journal of Financial Econometrics* 21, 412–444.
5. Dixit, V., Tiwari, M.K. (2020). Project portfolio selection and scheduling optimization based on risk measure: a conditional value at risk approach. *Annals of Operations Research* 285, 9–33.
6. Gneiting, T. (2011). Making and evaluating point forecasts. *Journal of the American Statistical Association*, 106, 746–62.
7. Hájek, J. (1965). Extension of the Kolmogorov-Smirnov test to regression alternatives. *Proc. of Bernoulli-Bayes-Laplace Seminar* (L. LeCam, ed.), pp. 45–60. Univ. of California Press.
8. He, X., Tan, K. M. and Zho, W.-X. (2023). Robust estimation and inference for expected shortfall regression with many regressors. *Journal of the Royal Statistical Society Series B (Statistical Methodology)*. DOI: 10.1093/jrsssb/qkad063.
9. Huber, P., Strassen, V. (1973). Minimax tests and the Neyman-Pearson lemma for capacities. *Ann. Statist.* 2, 251–273.
10. Jaeckel, L. A. (1972). Estimating regression coefficients by minimizing the dispersion of the residuals. *Ann. Math. Statist.* **43**, 1449–1459.
11. Jurečková, J. and Picek, J. (2014). Averaged regression quantiles. In: *Contemporary Developments in Statistical Theory* (Lahiri, S. et al., Editors), pp. 203–216 (Chapter 12). Springer, Switzerland.
12. Jurečková, J., Kalina, J. and Večeř, J. (2022). Estimation of expected shortfall under various experimental conditions. arXiv:2212.12419v1
13. Jurečková, J. and Picek, J. (2005). Two-step regression quantiles. *Sankhya* A 67, Part 2, 227–252.
14. Jurečková, J., Picek, J. and Schindler, M. (2020). Empirical regression quantile processes. *Aplications of Mathematics* 65, 257—269.
15. Jurečková, J., Sen, P.K. and Picek, J. (2013). *Methodology in Robust and Nonparametric Statistics.* CRC Press, Boca Raton, FL.
16. Kalina, J., Vidnerová, P., Soukup, L. (2023). Modern approaches to statistical estimation of measurements in the location model and regression. In: *Handbook of Metrology and Applications* (Aswal, D.K. et al., Editors), pp. 2355–2376. Springer, Singapore.
17. Khalaf, L., Leccadito, A., and Urga, G. (2022). Multilevel and tail risk management. *Journal of Financial Econometrics* 20, 839–874.
18. Nadarajah, S., Bo Zhang and Chan, S. (2013). Estimation methods for expected shortfall. *Quantitative Finance* 14/2, 271–291.
19. Padhan, H., Padhang, P.C., Tiwari, A.K., Ahmed, R., and Hammoudeh, S. (2020). Renewable energy consumption and robust globalization(s) in OECD countries: Do oil, carbon emissions and economic activity matter? *Energy Strategy Reviews* 32, 100535.

20. Pflug, G. C. (2000). Some Remarks on the Value-at-Risk and the Conditional Value-at-Risk. In: *Probabilistic Constrained Optimization Methodology and Applications* (Uryasev, S. P., Ed.). Springer, New York.
21. Trindade, A. A., Uryasev, S. P., Shapiro, A. and Zrazhevsky, G. (2007). Financial prediction with constrained tail risk. *Journal of Banking & Finance* 31, 3524–3538.
22. Zaevski, T. S. and Nedeltchev, D. C. (2023). From BASEL III to BASEL IV and beyond: Expected shortfall and expectile risk measures. *International Review of Financial Analysis* 87, 102645.