The Process Induced by Slope Components of α-Regression Quantile

Jana Jurečková

Abstract We consider the linear regression model, along with the process induced by its α-regression quantile, $0 < \alpha < 1$. While only the intercept component of the α-regression quantile estimates the quantile $F^{-1}(\alpha)$ of the model errors, the α also affects the slope components, whose dispersion infinitely increases as $\alpha \to 0, 1$, in the same rate as the variance of the sample α-quantile. The process of the slope components of α-regression quantile over $\alpha \in (0, 1)$ is asymptotically equivalent to the process of R-estimates of the slope parameters in the linear model, generated by the Hájek rank scores. Both processes converge to the vector of independent Brownian bridges under exponentially tailed parent distribution F, after standardization by $f(F^{-1}(\alpha))$.

1 Introduction

We consider the linear regression model:

$$Y_{ni} = \beta_0 + \mathbf{x}_{ni}^\top \boldsymbol{\beta} + e_{ni}, \quad i = 1, \ldots, n \tag{1.1}$$

with observations $Y_{n1}, \ldots, Y_{nn}$, independent errors $e_{n1}, \ldots, e_{nn}$, identically distributed according to an unknown distribution function F. The parameter $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)^\top$ is of interest, β_0 is a nuisance intercept parameter, and $\mathbf{x}_{ni} = (x_{i1}, \ldots, x_{in})^\top$, $i = 1, \ldots, n$, is the vector of covariates.

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The regression α-quantiles, $0 \leq \alpha \leq 1$, introduced in [12], are an important research tool, mainly in economics, where the quantile regression became almost a technical term. Remind that the regression α-quantile of model (1.1) is defined as the solution of the minimization:

$$(\hat{\beta}_0(\alpha), \widehat{\boldsymbol{\beta}}(\alpha))^\top = \arg\min \Big\{\alpha \sum_{i=1}^{n} (Y_i - b_0 - \mathbf{x}_i^\top \mathbf{b})^+ + (1-\alpha) \sum_{i=1}^{n} (Y_i - b_0 - \mathbf{x}_i^\top \mathbf{b})^-,$$
$$b_0 \in \mathbb{R}_1,\ \mathbf{b} \in \mathbb{R}_p,\ 0 < \alpha < 1\Big\}. \tag{1.2}$$

The population counterpart of (1.2) is

$$\Big(\beta_0 + F^{-1}(\alpha), \beta_1, \beta_2, \ldots, \beta_p\Big)^\top,\ 0 < \alpha < 1.$$

Hence, only the intercept component of the α-regression quantile reflects the quantile $F^{-1}(\alpha)$ of the probability distribution F, while $\widehat{\boldsymbol{\beta}}(\alpha)$ only reflects the slopes. If α runs over the interval $(0, 1)$, we get the regression quantile process. Its trajectories are step functions, whose number of breakpoints increases with the number n of observations. There is a rich literature devoted to the concepts connected with regression quantile, its processes, and applications. We recommend Koenker's book [11] as well *Handbook of Quantile Regression* [13] as excellent reviews.

The choice of index $\alpha \in (0, 1)$ is an important decision, e.g., in the situation when Y_i measures the loss and α reflects the balance between underestimating and overestimating the risk of the loss. Moreover, important in the applications are also the shape of trajectories of the limiting process and the shape of various functionals of the regression quantile, which can characterize the economic market. Alternatively, the two-step regression α-quantile, proposed in [8], estimates separately the slope components $\boldsymbol{\beta}$ with the aid of rank estimator $\widetilde{\boldsymbol{\beta}}_{nR}$, and then estimates the intercept β_0 as the α-quantile of the residuals $Y_i - \mathbf{x}_i^\top \widetilde{\boldsymbol{\beta}}_{nR},\ i = 1, \ldots, n$. The resulting two-step regression quantile process is asymptotically equivalent to the ordinary regression quantile process (see Theorem 2 in [8], also Theorem 4.1 in [9]). Only the number of breakpoints of its trajectories is different, being exactly equal to n. The empirical processes corresponding to the regression quantiles and to their inversions are numerically illustrated in [9].

The R-estimate of slope components $\boldsymbol{\beta}$ is defined as the minimizer $\widetilde{\boldsymbol{\beta}}_{nR}$ of the Jaeckel [5] measure of the rank dispersion:

$$\mathcal{D}_n(\mathbf{b}) = \sum_{i=1}^{n} (Y_{ni} - \mathbf{x}_{ni}^\top \mathbf{b}) \Big(a_n(R_{ni}(Y_i - \mathbf{x}_{ni}^\top \mathbf{b}) - \bar{a}_n\Big) \tag{1.3}$$
$$= \sum_{i=1}^{n} \Big[(Y_{ni} - \bar{Y}_n) - (\mathbf{x}_{ni} - \bar{\mathbf{x}}_n)^\top \mathbf{b})\Big] a_n(R_{ni}(Y_i - \mathbf{x}_{ni}^\top \mathbf{b}))F$$

with respect to $\mathbf{b} \in \mathbb{R}_p$, where $R_{ni}(Y_{ni} - \mathbf{x}_{ni}^\top \mathbf{b})$ is the rank of residual $Y_{ni} - \mathbf{x}_{ni}^\top \mathbf{b}$ among $Y_{n1} - \mathbf{x}_{n1}^\top \mathbf{b}, \ldots, Y_{nn} - \mathbf{x}_{nn}^\top \mathbf{b}$. Moreover, $\bar{Y}_n = \frac{1}{n}\sum_{i=1}^n Y_{ni}, \bar{\mathbf{x}}_n = \frac{1}{n}\sum_{i=1}^n \mathbf{x}_{ni}$ and $a_n(i)$ are the rank scores, $\bar{a}_n = \frac{1}{n}\sum_{i=1}^n a_n(i)$. Notice that $\widetilde{\boldsymbol{\beta}}_{nR}$ is invariant to the shift in location; hence, it is independent of β_0.

The scores $a_n(i), \;\; i = 1, \ldots, n$ are typically generated by a function $\varphi(u) : (0, 1) \mapsto \mathbb{R}_1$, nondecreasing and square integrable on (0,1), such that

$$\lim_{n\to\infty} \int_0^1 \Big(a_n(1 + [nu]) - \varphi(u)\Big)^2 du = 0.$$

For instance, $a_n(i) = \varphi\left(\frac{i}{n+1}\right), \;\; i = 1, \ldots, n.$

Particularly, we shall consider the family of score functions $\Big\{\varphi_\alpha(u), \;\; 0 \le \alpha \le 1, \;\; 0 \le u \le 1\Big\}$:

$$\varphi_\alpha(u) = \begin{cases} 0 \ldots 0 \le u \le \alpha \le 1 \\ 1 \ldots 1 \ge u > \alpha \ge 0. \end{cases}$$

As $n \to \infty$, the function $\varphi_\alpha(u)$ generates the following scores:

$$a_n(i, \alpha) = \begin{cases} 0 & \ldots i \le n\alpha \\ i - n\alpha & \ldots n\alpha \le i \le n\alpha + 1 \\ 1 & \ldots i \ge n\alpha + 1 \end{cases} \tag{1.4}$$

$i = 1, \ldots, n$. Notice that $a_n(i, \alpha)$ is continuous in $\alpha \in (0, 1)$. The scores $a_n(i, \alpha), \;\; i = 1, \ldots, n$ are known as Hájek's rank scores (see Hájek [3] and Hájek and Šidák [4]).

More precisely, if $R_{n1}, \ldots, R_{nn}$ are the ranks of random variables $Z_1, \ldots, Z_n$, then the vector $(a(R_{n1}, \alpha), \ldots, a(R_{nn}, \alpha))$ is a solution of the linear programming:

$$\begin{aligned} &\sum_{i=1}^n Z_i \; a_n(R_{ni}, \alpha) = \max \\ \text{under} \quad &\sum_{i=1}^n a_n(R_{ni}, \alpha) = n(1 - \alpha) \\ &0 \le a_n(R_{ni}, \alpha) \le 1, \quad i = 1, \ldots, n \end{aligned} \tag{1.5}$$

(cf. also [1, 2]). As $n \to \infty$, the Jaeckel criterion (1.3) asymptotically simplifies to

$$\mathcal{D}_{n\alpha}(\mathbf{b}) = \\ = \sum_{i=1}^n \Big[(Y_{ni} - \bar{Y}_n) - (\mathbf{x}_{ni} - \bar{\mathbf{x}}_n)^\top \mathbf{b})\Big] \Big(I[R_{ni}(Y_i - \mathbf{x}_{ni}^\top \mathbf{b}) \ge n\alpha]$$

$$+(R_{ni}(Y_i - \mathbf{x}_{ni}^\top \mathbf{b}) - n\alpha)I[n\alpha \le R_{ni}(Y_i - \mathbf{x}_{ni}^\top \mathbf{b}) \le n\alpha + 1]\Big) \tag{1.6}$$

$$\approx \sum_{i=1}^{n} \Big[(Y_{ni} - \bar{Y}_n) - (\mathbf{x}_{ni} - \bar{\mathbf{x}}_n)^\top \mathbf{b})\Big] I[R_{ni}(Y_i - \mathbf{x}_{ni}^\top \mathbf{b}) \ge n\alpha].$$

Jaeckel [5] proves that $\mathcal{D}_{n\alpha}(\mathbf{b})$ is continuous, convex, and piecewise linear function of $\mathbf{b} \in \mathbb{R}_p$, thus differentiable with gradient

$$\frac{\partial \mathcal{D}_{n\alpha}(\mathbf{b})}{\partial \mathbf{b}}\Big|_{\mathbf{b}_0} = -\sum_{i=1}^{n} (\mathbf{x}_{ni} - \bar{\mathbf{x}}_n) I\left[R_{ni}(Y_i - \mathbf{x}_{ni}^\top \mathbf{b}_0) \ge n\alpha\right] \tag{1.7}$$

at any point $\mathbf{b}_0 \in \mathbb{R}_p$ of differentiability. Notice that the gradients of the Jaeckel measure coincide with the Hájek rank scores. Using the uniform asymptotic linearity of the Hájek scores (see Proposition 1), we can approximate the Jaeckel measure by a quadratic function.

Our aim is to investigate the limiting behavior of the process of R-estimators $\{\widetilde{\boldsymbol{\beta}}_{n\alpha},\ 0 < \alpha < 1\}$, generated by the Hájek scores (1.4). It coincides with the limiting behavior of the process of the slope components of the regression quantile. Indeed, notice that, because of the asymptotic equivalence of the regression quantile and of the two-step regression quantile, the R-estimator of the slope components is asymptotically equivalent to the slope component vector of the regression quantile. The relation of the extreme regression quantile and of an R-estimator is studied in [7].

The intercept component of the α-regression quantile explicitly reflects the population quantile $F^{-1}(\alpha)$. However, α affects also the process of slopes; namely its dispersion depends on α similarly as the variance of the α-quantile.

Hájek and Šidák [4] proved the weak convergence of the process of Hájek's rank scores (1.4), (1.5) to the Brownian bridge, under the i.i.d. observations as well as under contiguous (Pitman) alternatives. Under the conditions on the tails of distribution of model errors, following those imposed in [2], we can prove the weak convergence of the process $\{f(F^{-1}(\alpha))(\widetilde{\boldsymbol{\beta}}_{n\alpha} - \boldsymbol{\beta})\}$ to the vector of independent Brownian bridges over the compact subsets of [0, 1].

2 Process of R-Estimates of Slopes and Its Asymptotics

Let $\widetilde{\boldsymbol{\beta}}_{n\alpha}$ be the R-estimator of $\boldsymbol{\beta}$, based on the Hájek rank scores, i.e., the minimizer of (1.6). Following the steps of [2], we shall first study the order of $\widetilde{\boldsymbol{\beta}}_{n\alpha}$ over $(\alpha_n^*, 1-\alpha_n^*)$ and show that the process of R-estimators converges to the vector of independent Brownian bridges for some $\alpha_n^* \downarrow 0$ as $n \to \infty$. This, in turn, will lead to the convergence over $\alpha \in (\alpha_0, 1-\alpha_0)$ with any $0 < \alpha_0 < 1/2$ fixed.

Consider the process of the Hájek rank scores:

$$\left\{\mathcal{A}_{n\alpha}(n^{-1/2}\mathbf{b}) = n^{-1/2}\sum_{i=1}^{n}(\mathbf{x}_{ni} - \bar{\mathbf{x}}_n)a_{n\alpha}(R_{ni}(Y_i - n^{-1/2}\mathbf{x}_{ni}^\top\mathbf{b}), \alpha) : \ 0 \le \alpha \le 1\right\} \tag{2.1}$$

for $\mathbf{b} \in \mathbb{R}_p$. The R-estimator $\widetilde{\boldsymbol{\beta}}_{n\alpha}$ is the minimizer of $\mathcal{D}_{n\alpha}(n^{-1/2}\mathbf{b})$ with respect to $\mathbf{b} \in \mathbb{R}_p$, and $\mathcal{A}_{n\alpha}(n^{-1/2}\mathbf{b})$ is its gradient, due to (1.6) and (1.7). It follows from [2] and [6] that the process (2.1) is uniformly asymptotically linear in $\mathbf{b}$, what enables to approximate $\mathcal{D}_{n\alpha}(n^{-1/2}\mathbf{b})$ by a quadratic function and then to approximate $\widetilde{\boldsymbol{\beta}}_{n\alpha}$ by its minimizer.

In order to realize these approximations, we impose the following conditions on the distribution of the model errors and on the triangular array of covariates $\mathbf{x}_{n1}, \dots, \mathbf{x}_{nn}$. These conditions are only sufficient and apparently can be weakened.

(F1) The density $f(x) = F'(x)$ is absolutely continuous and bounded with bounded derivative f' for $A < x < B$, where $-\infty \le A = \sup\{x : F(x) = 0\}$ and $+\infty \ge B = \inf\{x : F(x) = 1\}$.
(F2) The density $f(x) = F'(x)$ is increasing (decreasing) on an interval to the right of A (to the left of B), and $|f'(x)/f(x)| \le c|x|$ for $x \ge K(\ge 0), \ c > 0$.
(F3) $|F^{-1}(\alpha)| \le c(\alpha(1-\alpha))^{-a}$ and similarly, $1/f(F^{-1}(\alpha)) \le c(\alpha(1-\alpha))^{-a-1}$ for $0 < \alpha \le \alpha_0$ and $1-\alpha_0 \le \alpha < 1$ where $0 < a < \frac{1}{4}-\varepsilon$, $\varepsilon > 0, \ 0 < \alpha_0 \le 1/2$.

(X1) The matrix

$$\mathbf{Q}_n = \sum_{i=1}^{n}(\mathbf{x}_{ni} - \bar{\mathbf{x}}_n)(\mathbf{x}_{ni} - \bar{\mathbf{x}}_n)^\top, \quad \bar{\mathbf{x}}_n = \frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_{ni}$$

has the rank p and $n^{-1}\mathbf{Q}_n \to \mathbf{C}$ as $n \to \infty$, where $\mathbf{C}$ is a positively definite $p \times p$ matrix. Moreover, we assume:

$$\lim_{n\to\infty}\max_{1\le i\le n}(\mathbf{x}_{ni} - \bar{\mathbf{x}}_n)^\top\mathbf{Q}_n^{-1}(\mathbf{x}_{ni} - \bar{\mathbf{x}}_n) = 0 \quad (\textit{Noether condition}). \tag{2.2}$$

(X2) $n^{-1}\sum_{i=1}^{n}\|\mathbf{x}_{ni}\|^4 = O(1)$ as $n \to \infty$, and $\max_{1\le i\le n}\|\mathbf{x}_{ni}\| = O\left(n^{(2(b-a)-\delta)/(1+4b)}\right)$ as $n \to \infty$ for some $b > 0, \ \delta > 0$ such that $0 < b - a < \frac{\varepsilon}{2}$ (hence $0 < b < \frac{1}{4} - \frac{\varepsilon}{2}$).

As a consequence of Section V.3.5 in [4], we get the following weak convergence in the Prokhorov topology under $\mathbf{b} = \mathbf{0}$:

$$\left\{n^{1/2}\mathbf{Q}_n^{-1/2}\mathcal{A}_{n\alpha}(\mathbf{0}) : 0 \le \alpha \le 1\right\} \stackrel{\mathcal{D}}{\to} \mathbf{W}_p^* \tag{2.3}$$

as $n \to \infty$, where $\mathbf{W}_p^*$ is the vector of p independent Brownian bridges (see [4] and [1]). Furthermore, under a sequence of contiguous alternatives, when $Y_{ni} = Y_{ni}^0 + n^{-1/2}\mathbf{x}_{ni}^\top \mathbf{b},\ i = 1, \ldots, n$ with Y_{ni}^0 *i.i.d.* with distribution function F, there also takes place the following convergence to the vector of p independent Brownian bridges:

$$\left\{ n^{1/2}\mathbf{Q}_n^{-1/2}\mathcal{A}_n(\alpha, n^{-1/2}\mathbf{b}) - n^{-1/2}\mathbf{Q}_n^{1/2}\mathbf{b} f(F^{-1}(\alpha)) \ :\ 0 \le \alpha \le 1 \right\} \xrightarrow{\mathcal{D}} \mathbf{W}_p^* \tag{2.4}$$

as $n \to \infty$ (see [4], Theorem VI.3.2). The first important property is the uniform asymptotic linearity of $A_n(\alpha, n^{-1/2}\mathbf{b})$ in $\mathbf{b}$, proven in [6].

Denote:

$$\sigma_\alpha = \frac{(\alpha(1-\alpha))^{1/2}}{f(F^{-1}(\alpha))},\ 0 < \alpha < 1 \quad \text{and} \quad \alpha_n^* = 1/n^{1+4b} \quad \text{with } b \text{ given in } (X2). \tag{2.5}$$

Proposition 1 *Assume that F and $\mathbf{X}_n$ satisfy (F1)–(F3) and (X1)–(X2). Then*

$$\sup_{\|\mathbf{b}\| \le K,\ \alpha_n^* \le \alpha \le 1-\alpha_n^*} \left\{ \frac{\left| \mathcal{A}_n(\alpha, n^{-1/2}\sigma_\alpha \mathbf{b}) - \mathcal{A}_n(\alpha, \mathbf{0}) + n^{-1}\mathbf{Q}_n \mathbf{b} \right|}{(\alpha(1-\alpha))^{1/2}} \right\} \xrightarrow{p} 0 \tag{2.6}$$

and

$$\sup_{\|\mathbf{b}\| \le K,\ 0 \le \alpha \le 1} \left\{ \left| \mathcal{A}_n(\alpha, n^{-1/2}\mathbf{b}) - \mathcal{A}_n(\alpha, \mathbf{0}) + f(F^{-1}(\alpha)) n^{-1}\mathbf{Q}_n \mathbf{b} \right| \right\} \xrightarrow{p} 0 \tag{2.7}$$

as $n \to \infty$, for any fixed $K,\ 0 < K < \infty$.

Proof The proposition is proven in [6].

The following theorem describes the asymptotic behavior of the R-estimator of slope parameter over the interval $[\alpha_n^*, 1-\alpha_n^*]$. The convergence of the process over interval $(\alpha_n^*, 1-\alpha_n^*)$ means the convergence of this process over every compact subinterval of $(\alpha_n^*, 1-\alpha_n^*)$.

Theorem 1 *Under the conditions of Proposition 1, as $n \to \infty$*

$$\sup_{\alpha_n^* \le \alpha \le 1-\alpha_n^*} \left\{ n^{1/2}\sigma_\alpha^{-1} \|\widetilde{\boldsymbol{\beta}}_{n\alpha} - \boldsymbol{\beta}\| \right\} = O_p(1) \tag{2.8}$$

and

$$\sup_{\alpha_n^* \le \alpha \le 1-\alpha_n^*} \left\{ \| n^{1/2}\sigma_\alpha^{-1}[\widetilde{\boldsymbol{\beta}}_{n\alpha} - \boldsymbol{\beta}] - (\alpha(1-\alpha))^{-1/2}\ n\mathbf{Q}_n^{-1}\mathcal{A}_{n\alpha}(\mathbf{0}) \| \right\} = o_p(1). \tag{2.9}$$

Moreover, the process

$$\left\{ f(F^{-1}(\alpha))\mathbf{Q}_n^{1/2}(\widetilde{\boldsymbol{\beta}}_{n\alpha} - \boldsymbol{\beta}) : \ \alpha_n^* \leq \alpha \leq 1 - \alpha_n^* \right\} \tag{2.10}$$

converges to the vector of independent Brownian bridges.

Proof The theorem is proven in Sect. 3. □

Corollary *Under the conditions of Theorem 1, the process of R-estimators of the slope components*

$$\left\{ f(F^{-1}(\alpha))\mathbf{Q}_n^{1/2}(\widetilde{\boldsymbol{\beta}}_{n\alpha} - \boldsymbol{\beta}) : \ 0 < \alpha < 1 \right\} \tag{2.11}$$

and the process of the slope components of the regression quantile of model (1.1)

$$\left\{ f(F^{-1}(\alpha))\mathbf{Q}_n^{1/2}(\widehat{\boldsymbol{\beta}}_{n\alpha} - \boldsymbol{\beta}) : \ 0 < \alpha < 1 \right\} \tag{2.12}$$

converge to the vector of independent Brownian bridges in $\mathcal{D}(0,1)^p$. The convergence over $0 < \alpha < 1$ is in the sense that the process converges over the interval $[\alpha_0, 1-\alpha_0]$ for any fixed $0 < \alpha_0 < 1/2$, i.e., converges over the compact subsets of (0,1). □

3 Proofs

3.1 Proof of Theorem 1

Notice that

$$\mathbf{t}_{n\alpha} = n^{1/2}\sigma_\alpha^{-1}(\widetilde{\boldsymbol{\beta}}_{n\alpha} - \boldsymbol{\beta}) \tag{3.1}$$

minimizes $[\mathcal{D}_{n\alpha}(n^{-1/2}\sigma_\alpha \mathbf{b}) - \mathcal{D}_{n\alpha}(\mathbf{0})]$. Proposition 1 leads to the following quadratic approximation of $D_{n\alpha}(\mathbf{b})$:

$$\sup \left\{ \left| (\alpha(1-\alpha))^{-1/2}\Big(\sigma_\alpha^{-1}[\mathcal{D}_{n\alpha}(n^{-1/2}\sigma_\alpha \mathbf{b}) - \mathcal{D}_n(\mathbf{0})] + \mathbf{b}^\top \mathcal{A}_{n\alpha}(\mathbf{0})\Big) - \frac{1}{2}n^{-1}\mathbf{b}^\top \mathbf{Q}_n \mathbf{b} \right| : \right.$$
$$\left. \|\mathbf{b}\| \leq K, \alpha_n^* \leq \alpha \leq 1-\alpha_n^* \right\} \xrightarrow{p} 0 \ \text{ as } n \to \infty \ \text{ for any fixed } K > 0. \tag{3.2}$$

This further implies that as $n \to \infty$

$$\min_{\|\mathbf{b}\| \leq K} \left[(\alpha(1-\alpha))^{-1/2}\sigma_\alpha^{-1}[\mathcal{D}_{n\alpha}(n^{-1/2}\sigma_\alpha \mathbf{b}) - \mathcal{D}_{n\alpha}(\mathbf{0})] \right] \tag{3.3}$$
$$= \min_{\|\mathbf{b}\| \leq K} \left[\frac{1}{2}n^{-1}\mathbf{b}^\top \mathbf{Q}_n \mathbf{b} - (\alpha(1-\alpha))^{-1/2}\mathbf{b}^\top \mathcal{A}_{n\alpha}(\mathbf{0}) \right] + o_p(1)$$

for any $K,\ 0 < K < \infty$, uniformly for every compact subinterval of $(\alpha_n^*, 1-\alpha_n^*)$.
Moreover,

$$\min_{\mathbf{b}\in\mathbb{R}_p}\Big[\frac{1}{2}n^{-1}\mathbf{b}^\top\mathbf{Q}_n\mathbf{b}-(\alpha(1-\alpha))^{-1/2}\mathbf{b}^\top\mathcal{A}_{n\alpha}(\mathbf{0})\Big]$$
$$=-\frac{1}{2}(\alpha(1-\alpha))^{-1}\mathcal{A}_{n\alpha}^\top(\mathbf{0})\ n\mathbf{Q}_n^{-1}\mathcal{A}_{n\alpha}(\mathbf{0}) \tag{3.4}$$

and

$$\arg\min_{\mathbf{b}\in\mathbb{R}_p}\Big[\frac{1}{2}n^{-1}\mathbf{b}^\top\mathbf{Q}_n\mathbf{b}-(\alpha(1-\alpha))^{-1/2}\mathbf{b}^\top\mathcal{A}_{n\alpha}(\mathbf{0})\Big]$$
$$=(\alpha(1-\alpha))^{-1/2}\ n\mathbf{Q}_n^{-1}\mathcal{A}_{n\alpha}(\mathbf{0}) \tag{3.5}$$
$$=\mathbf{u}_{n\alpha}\quad \text{(SAY)}\ .$$

Notice that $\|\mathbf{u}_{n\alpha}\| = O_p(1)$ uniformly in $\alpha_n^* \le \alpha \le 1-\alpha_n^*$ by (2.1). Inserting $\mathbf{b}=\mathbf{u}_{n\alpha}$ in (3.2), we obtain:

$$\sup\Big\{\Big|(\alpha(1-\alpha))^{-1/2}\sigma_\alpha^{-1}[\mathcal{D}_{n\alpha}(n^{-1/2}\sigma_\alpha\mathbf{u}_{n\alpha})-\mathcal{D}_n(\mathbf{0})] \tag{3.6}$$
$$+\frac{1}{2}(\alpha(1-\alpha))^{-1}\mathcal{A}_{n\alpha}^\top(\mathbf{0})\ n\mathbf{Q}_n^{-1}\mathcal{A}_{n\alpha}(\mathbf{0})\Big|:\ \alpha_n^*\le\alpha\le 1-\alpha_n^*\Big\}=o_p(1)$$

Hence, using the convexity of $\mathcal{D}_n$, we apply the approach of Pollard in [14] and in [10] and conclude:

$$\sup\big\{\|\mathbf{t}_{n\alpha}-\mathbf{u}_{n\alpha}\|:\ \alpha_n^*\le\alpha\le 1-\alpha_n^*\big\}=o_p(1) \tag{3.7}$$

and remind the convergence (2.3) of $\mathbf{u}_{n\alpha}$.

Moreover, if $1-\alpha \ge 1-\alpha_n^*$, then $R_{ni}(Y_i-\mathbf{x}_i^\top\mathbf{b}) \ge n(1-\alpha)$ only for the maximal residual $Y_i-\mathbf{x}_i^\top\mathbf{b}$, hence $R_{ni}(Y_i-\mathbf{x}_i^\top\mathbf{b}) = n$ for $\alpha \le \alpha_n^*$. Hence the estimator $\widetilde{\boldsymbol{\beta}}_{n(1-\alpha)}$ minimizes the maximal residual over $\mathbf{b}\in\mathbb{R}_p$. More precisely,

$$\widetilde{\boldsymbol{\beta}}_{n(1-\alpha)}=\arg\min_{\mathbf{b}\in\mathbb{R}_p}\Big\{[Y_{ni}-\bar{Y}_n-(\mathbf{x}_{ni}-\bar{\mathbf{x}}_n)^\top\mathbf{b}]_{n:n}\Big\}. \tag{3.8}$$

Moreover, notice that $\widetilde{\boldsymbol{\beta}}_{n(1-\alpha)}$ is then constant for $0<\alpha\le\alpha_n^*$, i.e.,

$$\widetilde{\boldsymbol{\beta}}_{n(1-\alpha)}=\widetilde{\boldsymbol{\beta}}_{n(1-\alpha_n^*)}\ \text{ for }\ 0<\alpha\le\alpha_n^*.$$

Denote as D_n the antirank of the maximal residual. Then

$$\Big[Y_{nD_n}-\bar{Y}_n-(\mathbf{x}_{nD_n}-\bar{\mathbf{x}}_n)^\top\widetilde{\boldsymbol{\beta}}_{n(1-\alpha)}\Big]\le[Y_{nD_n}-\bar{Y}_n-(\mathbf{x}_{nD_n}-\bar{\mathbf{x}}_n)^\top\mathbf{b}],$$

hence

$$(\mathbf{x}_{nD_n} - \bar{\mathbf{x}}_n)^\top \widetilde{\boldsymbol{\beta}}_{n(1-\alpha)} \geq (\mathbf{x}_{nD_n} - \bar{\mathbf{x}}_n)^\top \mathbf{b} \qquad (3.9)$$

for $0 < \alpha \leq \alpha_n^*$, any $\mathbf{b} \in \mathbb{R}_p$, including $\mathbf{0}$ and any other estimator of $\boldsymbol{\beta}$. Analogously, $\widetilde{\boldsymbol{\beta}}_{n\alpha_0} = \widetilde{\boldsymbol{\beta}}_{n\alpha_n^*}$ for $0 < \alpha_0 \leq \alpha_n^*$; hence, the convergence holds over $[\alpha_0, 1 - \alpha_0]$ for any $0 < \alpha_0 < 1/2$ and thus for the compact subintervals of $(0, 1)$. As a consequence, we conclude that the process $\{f(F^{-1}(\alpha))\mathbf{Q}_n^{1/2}(\widetilde{\boldsymbol{\beta}}_{n\alpha} - \boldsymbol{\beta})\}$ converges to the vector of Brownian bridges over the interval $[\alpha_0, 1 - \alpha_0]$ for any fixed $0 < \alpha_0 < 1/2$, i.e., over the compact subsets of (0,1). □

4 Concluding Remarks

We observe that the process of the slope components of the regression quantile converges to the vector of the independent Brownian bridges, standardized with the density quantile function $f(F^{-1}(\alpha))$, $0 < \alpha < 1$. This important function and its derivative typically appear in relations with the tails and shape of the probability distribution. Properties of the density quantile function, and their impact, are intensively studied by Staudte and Xia [15], among others.

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