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Pathwise Duality of Interacting Particle Systems

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Author's signature

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Abstract: In the study of Markov processes, duality is an important tool used to prove various types of long-time behavior. Nowadays, there exist two predominant approaches to Markov process duality: the algebraic one and the pathwise one. This thesis utilizes the pathwise approach in order to identify new dualities of interacting particle systems and to present previously known dualities within a unified framework. Three classes of pathwise dualities are identified by equipping the state space of an interacting particle system with the additional structure of a monoid, a module over a semiring, and a partially ordered set, respectively. This additional structure then induces a pathwise duality for each interacting particle system that preserves this structure in the sense that its generator can be written using only structure-preserving local maps.

Keywords: pathwise duality, interacting particle systems, monotone Markov process, monoid, module

Contents

Notation

Throughout this thesis, the following conventions are used:

- The symbol $\mathbb{N}$ denotes the set of positive integers so that $0 \notin \mathbb{N}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
- The notation $A \subset B$ means that each element in the set A is also in the set B so that A and B may coincide.
- A countable set is a set which admits a bijection onto an *infinite* subset of $\mathbb{N}$.
- In the context of continuous or measurable functions, each finite or countable set is always assumed to carry the discrete topology and the discrete σ-algebra.

Introduction

Interacting particle systems are used to model systems comprising numerous identical components, typically situated in a spatial relationship to one another. Examples may include populations of animals or plants, cells within an organism, or a material composed of molecules. For each such component, an interacting particle system tracks one or more traits, which are updated depending on the values of the traits at finitely many other components - usually those in its vicinity. For instance, binary traits are often used to model the presence of an individual at a given component, with trait updates corresponding to reproduction and death. Mathematically, an interacting particle system will be defined as a system of interacting Markov processes. Each such Markov process represents one component, and its jump rates depend on the values of finitely many other Markov processes (in its vicinity).

It is customary in the field of interacting particle systems to idealize such systems. Instead of studying systems of many, but finitely many components, as one may encounter in the real world, one studies infinite systems. This has several mathematical advantages. For instance, imagine a population confined to a finite space that reproduces (in some way) at exponentially distributed times, while the individuals die independently at exponentially distributed times. Mathematically, such a population is almost sure to die out at some finite time, even under the most favorable conditions. In contrast, on an infinite space, this does not have to be the case. The underlying philosophy of this idealization is that phenomena observed in the infinite system happen for the finite system with high probability on appropriate time scales.

The study of interacting particle systems, in the sense of this thesis, originated with the work of Spitzer [1969, 1970] and Dobrushin [1971a,b], with numerous other authors subsequently contributing to the field. Its roots lie in the study of statistical mechanics, a much older field of study. For example, the Ising model, an interacting particle system describing a ferromagnet, has (without time evolution) already been studied for more than a century [Lenz, 1920, Ising, 1925]. Over time, the books by Liggett [1985, 1999] have established themselves as the standard references in the field. The interested reader may consult them for an in-depth analysis of some of the most frequently studied interacting particle systems, as well as a detailed introduction to the most important techniques and tools utilized to study interacting particle systems.

Duality is one of the most powerful and most commonly used of these tools. It describes a relationship between two Markov processes: one process that is of primary interest and a dual process that is (hopefully) easier to study. The two processes are related through a duality function, which can be interpreted as an observable of the two processes. If an interacting particle system is the process of primary interest, duality typically yields a relationship to a process with only finitely many "active" components. Thus, duality (typically) allows one to study an interacting particle system with infinitely many "active" components through a process with only finitely many ones. This reduction to a "finite" process is of considerable help if one tries to study the (time-) invariant laws of an interacting particle system. Examples of this application of duality are provided in this thesis.

Initially, many useful duality relations of Markov processes were discovered with concrete processes in mind [Lindley, 1952, Lévy, 1954, Karlin and McGregor, 1957]. Over the following decades the theory has been formalized and generalized [Siegmund, 1976, Holley and Stroock, 1979, Ethier and Kurtz, 1986]. Despite numerous applications of duality to Markov processes, the question regarding sufficient and necessary conditions a Markov process has to satisfy in order to have a dual process has not yet been answered to full satisfaction. The task of finding a dual process has in [Etheridge, 2006] even been compared to "black art".

For several concrete interacting particle systems the existence of dual processes has been known since the early 1970s, shortly after the field of study was established as a distinct area of research. By the end of the 1970s a duality theory for two important classes of interacting particle systems had been established [Harris, 1976, 1978, Griffeath, 1979]. However, a truly systematic treatment of duality of interacting particle systems remains to be developed.

Over the past three decades, there has been a notable advancement in this direction. In [Lloyd and Sudbury, 1995, 1997, Sudbury, 2000] a systematic treatment of dualities for nearest-neighbor interacting particle systems is given, where the duality function is of a special "local" form. Recently, the so called "algebraic approach to duality" [Giardinà et al., 2007, 2009], which relies on a deep connection between duality and representations of Lie algebras, has advanced the systematic study of dualities considerably.

In addition to the algebraic approach, there also exists the "pathwise approach to duality" as coined (based on the notion of pathwise duality from [Jansen and Kurt, 2014]) in [Sturm et al., 2020]. Probably the most important recent publication following this approach is [Sturm and Swart, 2018]. The pathwise approach is based on the graphical representation of an interacting particle system, introduced by Harris [1978]. It has the advantage that the construction of a duality is graphic and easy to comprehend. Moreover, it yields a (slightly) stronger notion of duality than the algebraic approach. On the other hand, it has the disadvantage that the pathwise approach does not detect all dualities that may be found by the algebraic one. In [Sturm et al., 2020] more detailed remarks pointing out differences and similarities between the two approaches can be found.

This thesis uses the pathwise approach to duality in order to identify new dualities of interacting particle systems and to present previously known dualities within a unified framework. The reader of this thesis will observe that also the pathwise approach to duality relies on algebra, but only on elementary algebra.

The present thesis is based on the publications [Latz and Swart, 2023a,b] and the preprint [Latz and Swart, 2023c], all of which are joint work with Jan M. Swart. It combines the three articles into a single, coherent work, employing a unified notation. In comparison with the articles, several parts were reformulated, restructured or added. The most important changes are highlighted in the outline below.

Outline

Chapter 1 serves as a mathematically rigorous introduction to the construction of a pathwise duality of an interacting particle system. To do so, in Chapter 1.1

the construction of an interacting particle system via a graphical representation is presented. Chapter 1.2 presents a construction of an interacting particle system started in a "finite" configuration as a continuous-time Markov chain on a countable state space. This construction will be needed for the dual processes of interacting particle systems. The concept of duality and pathwise duality is introduced in Chapter 1.3, while in Chapter 1.4 a first "basic" pathwise duality is considered. This initial duality function will be the starting point of the construction of the more interesting dualities in the subsequent chapters. Chapter 1.5 introduces the notion of (weak) informativeness of a duality function and proves a number of key results regarding these notions.

Chapter 1 is mainly based on [Swart, 2022], while the exact formulation of (pathwise) duality is inspired by [Jansen and Kurt, 2014]. Theorem 1.6 is based on [Latz and Swart, 2023c, Lemma 1], but the conditions were modified. In particular, the summability condition (1.18) is newly introduced in this thesis. The notions of informativeness and weak informativeness were coined in [Latz and Swart, 2023a], and also Proposition 1.10 and Lemma 1.12 already appeared there.

Chapter 2 presents a duality theory for interacting particle systems based on monoids. Chapter 2.1 prepares the reader by studying monoid homomorphisms. In Chapter 2.2 a duality theory for monoids is presented. This theory is in Chapter 2.3 extended to monoids that carry a topology. Using the construction of Chapter 1, this notion of duality between monoids that carry a topology then induces a pathwise duality of interacting particle systems. Chapter 2.4 shows that additive and cancellative duality, two of the most used dualities of interacting particle systems, are of this form. Afterwards, in Chapter 2.5, it is shown how to identify dualities between monoids using a computer. Tables containing all dualities between monoids with up to four elements are included. In Chapter 2.6 the identified duality functions are divided into the group of those that are weakly informative and those that are not. Finally, in Chapter 2.7 an application of monoid duality is presented. There, monoid duality is used to compute all shift- and time-invariant measures of a variant of the well-known contact process.

Chapter 2 unifies results from [Latz and Swart, 2023b] and [Latz and Swart, 2023a]. Chapter 2.1, Chapter 2.2 and Chapter 2.5 are based on the content of [Latz and Swart, 2023b, Section 2 & Section 5]. The content of [Latz and Swart, 2023a, Section 2] is presented in Chapter 2.3. The two last results of Chapter 2.3 are newly added: Lemma 2.12 is inspired by the proof of [Latz and Swart, 2023c, Proposition 13] and Proposition 2.13 was mentioned in [Latz and Swart, 2023a] without a proof. The main ideas behind Chapter 2.4 are taken from [Latz and Swart, 2023b, Section 4]. Chapter 2.6 partially follows [Latz and Swart, 2023a, Section 3], but most of the computation in its second half were newly added. Finally, Chapter 2.7 is based on [Latz and Swart, 2023a, Sections 1,4,5 & the appendix].

Chapter 3 presents a duality theory for interacting particle systems based on modules over a semirings. As indicated by the titles of its subchapters, its structure follows the one of Chapter 2 closely.

The idea behind Chapter 3 is based on [Latz and Swart, 2023b, Section 3 & Section 5.2]. Most results in it are newly added. They are, however, mainly based on very similar results in Chapter 2.

Chapter 4 presents a duality theory for interacting particle systems based on partially ordered sets. In contrast to the two chapters prior, the dual process of this construction does not have the form of an interacting particle system. In Chapter 4.1 its state space is defined and equipped with both a topology and a partial order. The process itself is constructed in Chapter 4.2. Chapter 4.3 compares the construction of Chapter 4.2 to a previously known one [Gray, 1986, Sturm and Swart, 2018]. In Chapter 4.4 it is shown that the duality function used in Chapter 4 is (in multiple senses) informative. Chapter 4.5 connects the notions of the non-triviality of an upper invariant law and the survival of its dual process by means of monotone duality, the type of duality investigated in Chapter 4. Finally, Chapter 4.6 shows that additive duality is also a special case of monotone duality.

Chapter 4 is based on [Latz and Swart, 2023c]. The main result, Theorem 4.9, uses the new summability condition (1.18) as an assumption. The original result [Latz and Swart, 2023c, Theorem 5] was assuming both the summability condition (1.7) and a second one for the dual process. It only gave the second statement of Theorem 4.9. The usage of (1.18) has also made it possible to weaken the assumptions in some of the results presented in Chapter 4.5 and Chapter 4.6.1, compared to [Latz and Swart, 2023c]. In particular, the assumptions in Chapter 4.6.1 now better align with those in Chapter 2.

Appendix A collects examples that rule out the weak informativeness of some of the duality function identified in Chapter 2.5 and Chapter 3.4. The examples were computed newly for this thesis.

1. Constructing a pathwise duality

In this first chapter the general idea of how to construct a pathwise dual process of an interacting particle system is introduced. Let, throughout this thesis, S be a finite set with $|S| \geq 2$, and let Λ be a countable set. We will call S the *local state space* and Λ the *grid*.[1] Let S^Λ be the set of functions $x : \Lambda \to S$, equipped with the product topology. The space S^Λ will serve as the state space of an interacting particle system. We will call it sometimes the *global state space*. For $a \in S$ we will denote by $\underline{a} \in S^\Lambda$ the configuration that is constantly a, i.e.,

$$\underline{a}(i) := a \qquad (i \in \Lambda). \tag{1.1}$$

By Tychonoff's theorem (see, e.g., [Bredon, 1993, Theorem I.8.9]) S^Λ is compact. The product topology is metrizable. As the grid Λ is countable, we can find a bijection $\gamma : \Lambda \to \mathbb{N}$. Using this bijection we define

$$a_i := \frac{1}{3^{\gamma(i)}} \qquad (i \in \Lambda)$$

and define a metric d on S^Λ as

$$d(x, y) := \sum_{i \in \Lambda} a_i (1 - \delta_{x(i)y(i)}) \qquad (x, y \in S^\Lambda), \tag{1.2}$$

where δ_{ab} $(a, b \in S)$ denotes the Kronecker delta. It is well-known that the metric d generates the product topology. We are going to use this concrete metric in Chapter 4.

1.1 Interacting particle systems and graphical representations

We start by specifying some conventions for Markov processes that we will use throughout this thesis. Let E be a Polish space, i.e., a separable completely metrizable topological space, equipped with the Borel σ-algebra $\mathcal{B}(E)$. A *Markov semigroup* is a collection of probability kernels $(P_t)_{t \geq 0}$ on E such that

$$P_0 = 1 \qquad \text{and} \qquad P_s P_t = P_{s+t} \quad (s, t \geq 0),$$

where $P_s P_t$ denotes the concatenation of P_s and P_t and 1 denotes the identity kernel defined as $1(x, \cdot\,) := \delta_x$, the Dirac measure on $x \in E$. For our purposes a *Markov process* with semigroup $(P_t)_{t \geq 0}$ is a stochastic process $X = (X_t)_{t \geq 0}$ with values in E such that

$$\mathbb{P}[X_t \in \cdot \mid (X_u)_{0 \leq u \leq s}] = P_{t-s}(X_s, \cdot\,) \quad \text{a.s.} \qquad (0 \leq s \leq t),$$

[1] This is often called the *lattice* but we reserve the latter term for its order-theoretic meaning.

where, in the left-hand side above, we condition on the σ-algebra generated by $(X_u)_{0\leq u\leq s}$.[2] A Markov process with values in a finite or countable state space we call a *continuous-time Markov chain*.

Let E now be a compact metrizable space and let $\mathcal{M}_1(E)$ denote the space of probability measures on E, equipped with the topology of weak convergence. A *Feller semigroup* on E is a Markov semigroup $(P_t)_{t\geq 0}$ with the additional property that

$$(x,t)\mapsto P_t(x,\,\cdot\,)\text{ is a continuous map from } E\times[0,\infty)\text{ to }\mathcal{M}_1(E).$$

A *Feller process* is a Markov process whose semigroup is a Feller semigroup.

Before we construct an interacting particle system, we first note some properties of the compact metrizable space S^Λ. For each $\Delta\subset\Lambda$ and $x\in S^\Lambda$, we let $x_\Delta=(x(i))_{i\in\Delta}$ denote the restriction of x to Δ. Let T be a finite set. For any function $f:S^\Lambda\to T$ we call

$$\mathcal{R}(f):=\Big\{i\in\Lambda:\exists x,x'\in S^\Lambda\text{ s.t. }f(x)\neq f(x')\text{ and }x_{\Lambda\setminus\{i\}}=x'_{\Lambda\setminus\{i\}}\Big\}. \tag{1.3}$$

the set of *f-relevant sites*. Thus, changing the value of $x(i)$ for $i\in\mathcal{R}(f)$ may change $f(x)$. We cite the following result [Sturm and Swart, 2015, Lemma 24].

Lemma 1.1 (Continuous maps)**.** *A map $f:S^\Lambda\to T$ is continuous if and only if the following two conditions hold:*

(i) *$\mathcal{R}(f)$ is finite.*

(ii) *If $x,x'\in S^\Lambda$ satisfy $x(j)=x'(j)$ for all $j\in\mathcal{R}(f)$, then $f(x)=f(x')$.*

One can express the content of Lemma 1.1 in words by saying that a function $f:S^\Lambda\to T$ is continuous if and only if it depends on finitely many coordinates in the sense that there exists a finite set $\Delta\subset\Lambda$ and a function $f':S^\Delta\to T$ such that $f(x)=f'(x_\Delta)$ $(x\in S^\Lambda)$. The smallest possible set for which one can find a function $f':S^\Delta\to T$ such that $f(x)=f'(x_\Delta)$ $(x\in S^\Lambda)$ is then $\Delta:=\mathcal{R}(f)$.

For any map $m:S^\Lambda\to S^\Lambda$ and $i\in\Lambda$ we define $m[i]:S^\Lambda\to S$ via $m[i](x):=m(x)(i)$ $(x\in S^\Lambda)$. Note that m is continuous if and only if $m[i]$ is continuous for all $i\in\Lambda$. We also define

$$\mathcal{D}(m):=\Big\{i\in\Lambda:\exists x\in S^\Lambda\text{ s.t. }m(x)(i)\neq x(i)\Big\}, \tag{1.4}$$

the set of sites whose local states m may change.

By definition, a *local map* is a continuous map $m:S^\Lambda\to S^\Lambda$ for which $\mathcal{D}(m)$ is finite. It is not hard to prove that, in parallel to Lemma 1.1, a map $m:S^\Lambda\to S^\Lambda$ is local if and only if there exists a finite set $\Delta\subset\Lambda$ and a map $m':S^\Delta\to S^\Delta$ such that

$$m(x)(i)=\begin{cases}m'(x_\Delta)(i) & \text{if } i\in\Delta,\\ x(i) & \text{else,}\end{cases}\qquad (x\in S^\Lambda,\ i\in\Lambda). \tag{1.5}$$

[2]In the literature it is often required that a Markov process has càdlàg (i.e., right-continuous with left limits) sample paths. In this thesis, however, we also want to call processes with càglàd (i.e., left-continuous with right limits) sample paths Markov. The reader that does not like this convention may replace every “Markov process with càglàd sample paths” that appears in this thesis by “the left-continuous modification of a Markov process”.

Indeed, choosing $\Delta := \mathcal{D}(m) \cup \bigcup_{i \in \mathcal{D}(m)} \mathcal{R}(m[i])$ yields the smallest set satisfying (1.5). For any map $m : S^\Lambda \to S^\Lambda$ we set

$$\mathcal{R}_i^\downarrow(m) := \begin{cases} \mathcal{R}(m[i]) & \text{if } i \in \mathcal{D}(m), \\ \emptyset & \text{else,} \end{cases} \qquad (i \in \Lambda). \tag{1.6}$$

Let $\mathcal{G}$ be a countable collection of local maps and let $(r_m)_{m \in \mathcal{G}}$ be non-negative real constants that satisfy the summability condition

$$\sup_{i \in \Lambda} \sum_{m \in \mathcal{G}} r_m \Big(\mathbb{1}_{\mathcal{D}(m)}(i) + |\mathcal{R}_i^\downarrow(m)| \Big) < \infty, \tag{1.7}$$

where $\mathbb{1}_A$ denotes the indicator function of a set A. For each continuous real function f on S^Λ that depends on finitely many coordinates, we define

$$Gf(x) := \sum_{m \in \mathcal{G}} r_m \Big\{ f(m(x)) - f(x) \Big\} \qquad (x \in S^\Lambda). \tag{1.8}$$

Then it is known [Swart, 2022, Theorem 4.30] that G is closable and its closure generates a Feller semigroup $(P_t)_{t \geq 0}$. General theory [Kallenberg, 1997, Theorem 17.15] then says that for each initial law on S^Λ there exists a unique (in law) Feller process $X = (X_t)_{t \geq 0}$ with values in S^Λ and càdlàg sample paths such that the transition probabilities of X are given by $(P_t)_{t \geq 0}$. We call X the *interacting particle system* with generator G.[3]

The representation of G in (1.8) is called a *random mapping representation*. Random mapping representations are far from unique and different ones may lead to different dualities. Hence, it is an important question to decide for an interacting particle system if there exists a "special" random mapping representation, where $\mathcal{G}$ consists only of one class of functions. In this thesis we will skip this question and assume in our main results that such a special random mapping representation is given. Additional comments regarding this question of "representability" can be found throughout [Sturm and Swart, 2018, Section 2].

For any initial distribution $\nu \in \mathcal{M}_1(S^\Lambda)$ we denote by $\mathbb{P}^\nu$ the law of X started in ν. For the special case that X is started in a deterministic state $x \in S^\Lambda$ we write $\mathbb{P}^x$ as a shorthand for $\mathbb{P}^{\delta_x}$. Accordingly, we denote expectation with respect to $\mathbb{P}^\nu$ by $\mathbb{E}^\nu$ $(\nu \in \mathcal{M}_1(S^\Lambda))$ and expectation with respect to $\mathbb{P}^x$ by $\mathbb{E}^x$ $(x \in S^\Lambda)$.

It is known that X can be constructed from a *graphical representation* ω as follows. Let ρ be the measure on $\mathcal{G} \times \mathbb{R}$ defined by $\rho(\{m\} \times [s,t]) := r_m(t-s)$ $(m \in \mathcal{G},\ s \leq t)$ and let ω be a Poisson point set with intensity measure ρ. For each $s \leq u$, we set

$$\omega_{s,u} := \Big\{ (m,t) \in \omega : s < t \leq u \Big\}. \tag{1.9}$$

Each finite $\tilde{\omega} \subset \omega_{s,u}$ we can (a.s.) write as $\tilde{\omega} = \{(m_1,t_1), \ldots, (m_n,t_n)\}$ with $t_1 < \cdots < t_n$. For each such $\tilde{\omega}$, we define

$$\mathbf{X}_{s,u}^{\tilde{\omega}} := m_n \circ \cdots \circ m_1, \tag{1.10}$$

[3]Following the general theory for Feller processes, one should say more precisely that the closure of G is the generator of X. Since the corresponding theory for interacting particle systems is well-known (compare, e.g., [Liggett, 1985, Chapter I.2], [Swart, 2022, Chapter 4.2]), we will not distinguish between G and its closure in this thesis and usually write that a generator of an interacting particle system is "defined by (1.8)" or "given (as) in (1.8)".

i.e., $\mathbf{X}^{\tilde{\omega}}_{s,u}$ is the concatenation of the maps from $\tilde{\omega}$ in the time order in which they occur. The following result follows from [Swart, 2022, Lemma 4.24] and the proof of [Swart, 2022, Theorem 4.19]. Compare also [Swart, 2022, Theorem 6.16].

Lemma 1.2 (Finitely many relevant local maps). *Assume the summability condition (1.7). Then, almost surely, for each $s \leq u$ and $i \in \Lambda$, there exists a finite set $\omega_{s,u}(i) \subset \omega_{s,u}$ such that*

$$\mathbf{X}^{\tilde{\omega}}_{s,u}[i] = \mathbf{X}^{\omega_{s,u}(i)}_{s,u}[i] \quad \textit{for all finite } \tilde{\omega} \textit{ with } \omega_{s,u}(i) \subset \tilde{\omega} \subset \omega_{s,u}. \tag{1.11}$$

These finite sets can be chosen such that $\omega_{t,u}(i) = \omega_{s,u}(i) \cap \omega_{t,u}$ for all $s \leq t \leq u$ and $i \in \Lambda$.

Lemma 1.2 implies that under (1.7), almost surely, we can define for all $s \leq u$ a random map $\mathbf{X}_{s,u} : S^\Lambda \to S^\Lambda$ via

$$\mathbf{X}_{s,u}(x) := \lim_{\tilde{\omega}_n \uparrow \omega_{s,u}} \mathbf{X}^{\tilde{\omega}_n}_{s,u}(x) \qquad (s \leq u,\ x \in S^\Lambda), \tag{1.12}$$

where $(\tilde{\omega}_n)_{n\in\mathbb{N}}$ is an arbitrary sequence of finite subsets of ω increasing to $\omega_{s,u}$. These random maps form a *stochastic flow*, in the sense that $\mathbf{X}_{s,s}$ is the identity map for all $s \in \mathbb{R}$, and $\mathbf{X}_{t,u} \circ \mathbf{X}_{s,t} = \mathbf{X}_{s,u}$ $(s \leq t \leq u)$.

From this stochastic flow we can construct the interacting particle system X as the following theorem shows. It follows from [Swart, 2022, Theorem 4.20 & Theorem 4.30].

Theorem 1.3 (Poisson construction). *Assume the summability condition (1.7). If X_0 is a random variable with values in S^Λ that is independent of the graphical representation ω, then setting*

$$X_t := \mathbf{X}_{s,s+t}(X_0) \qquad (t \geq 0) \tag{1.13}$$

defines (for fixed $s \in \mathbb{R}$) a Feller process $X = (X_t)_{t\geq 0}$ with càdlàg sample paths. The generator of this Feller process is the one given in (1.8).

For later use we note two key properties of the stochastic flow $(\mathbf{X}_{s,u})_{s\leq u}$, defined via (1.12). Considering $\mathbf{X}_{s,u}[i]$ for fixed $s \leq u$ and $i \in \Lambda$ both statements follow immediately from Lemma 1.2.

Lemma 1.4 (Flow properties). *Assume the summability condition (1.7). Then, almost surely, the maps $\mathbf{X}_{s,u} : S^\Lambda \to S^\Lambda$ $(s \leq u)$, defined in (1.12), are continuous. Moreover, assume that there exists an $x \in S^\Lambda$ such that $m(x) = x$ for all $m \in \mathcal{G}$. Then, almost surely, $\mathbf{X}_{s,u}(x) = x$ for all $s \leq u$.*

The stochastic flow $(\mathbf{X}_{s,u})_{s\leq u}$ can also be characterized via an evolution equation. Assuming (1.7), for each $s \in \mathbb{R}$ and $x \in S^\Lambda$, there almost surely exists a unique càdlàg function $[s,\infty) \ni t \mapsto X_t \in S^\Lambda$ that solves the evolution equation

$$X_s = x \quad \text{and} \quad X_t = \begin{cases} m(X_{t-}) & \text{if } (m,t) \in \omega, \\ X_{t-} & \text{else,} \end{cases} \qquad (t > s), \tag{1.14}$$

where $X_{t-} = \lim_{t'\uparrow t} X_{t'}$ denotes the state of the process just before time t. The unique function is given via $X_t := \mathbf{X}_{s,t}(x)$ $(t \geq s)$. This fact follows from [Swart, 2022, Theorem 4.19 and its proof].

1.2 Staying finite

In the upcoming chapters we will always encounter the situation that there exists a special element $0 \in S$ that has the property that $m(\underline{0}) = \underline{0}$ for all $m \in \mathcal{G}$, where $\underline{0}$ is defined by (1.1). The element $\underline{0}$ then becomes a trap for the interacting particle system X defined in Chapter 1.1. If such an element $0 \in S$ exists, we set $\mathrm{supp}(x) := \{i \in \Lambda : x(i) \neq 0\}$ $(x \in S^\Lambda)$ and call $\mathrm{supp}(x)$ the *support* of x. Moreover, we set

$$S^\Lambda_{\mathrm{fin}} := \Big\{x \in S^\Lambda : |\mathrm{supp}(x)| < \infty\Big\}. \tag{1.15}$$

For any $a \in S$ and $i \in \Lambda$ we define $\delta^a_i \in S^\Lambda_{\mathrm{fin}}$ as

$$\delta^a_i(k) := \begin{cases} a & \text{if } k = i, \\ 0 & \text{else,} \end{cases} \qquad (k \in \Lambda). \tag{1.16}$$

In the special case that $S = \{0,1\}$, we set $\delta_i := \delta^1_i$ $(i \in \Lambda)$. The reader is advised to not confuse the notion from (1.16) with the one of a Dirac measure: $\delta_i \in \{0,1\}^\Lambda$ $(i \in \Lambda)$ denotes a single configuration while, e.g., $\delta_{\underline{0}} \in \mathcal{M}_1(\{0,1\}^\Lambda)$ denotes the Dirac measure on the "all 0 configuration". We will caution the reader throughout this thesis at points where confusion may arise.

Let $X = (X_t)_{t\geq 0}$ be the interacting particle system defined in Chapter 1.1 and assume that there exists an element $0 \in S$ such that $m(\underline{0}) = \underline{0}$ for all $m \in \mathcal{G}$. We say that X *survives* if there exists an $x \in S^\Lambda_{\mathrm{fin}}$ such that

$$\mathbb{P}^x[\exists t \geq 0 : X_t = \underline{0}] < 1.$$

Otherwise we say that X *dies out*. Moreover, we say that a probability measure $\mu \in \mathcal{M}_1(S^\Lambda)$ is X*-non-trivial* if it is concentrated on those $x \in S^\Lambda$ that are not traps for X. Hence, in the setup of this subchapter, any X-non-trivial probability measure μ has to satisfy $\mu(\{\underline{0}\}) = 0$. We will discuss the survival of interacting particle systems shortly in Chapter 2.7.2 and in more detail in Chapter 4.5. Distributions that are X-non-trivial will play an important role in Chapter 2.7.5.

Note that S^Λ_{fin} is a countable set and that a local map $m : S^\Lambda \to S^\Lambda$ maps S^Λ_{fin} into itself. It is an interesting question under which conditions we can define a stochastic flow on S^Λ_{fin} as in (1.12). In parallel to (1.6), we set for any map $m : S^\Lambda \to S^\Lambda$

$$\mathcal{R}^\uparrow_i(m) := \Big\{j \in \mathcal{D}(m) : i \in \mathcal{R}(m[j])\Big\}. \tag{1.17}$$

It will turn out that assuming

$$\sup_{i\in\Lambda} \sum_{m\in\mathcal{G}} r_m\Big(1_{\mathcal{D}(m)}(i) + |\mathcal{R}^\uparrow_i(m)|\Big) < \infty \tag{1.18}$$

instead of (1.7) will allow us to define a stochastic flow on S^Λ_{fin} as in (1.12). To see this, we first consider continuous-time Markov chains on general state spaces.

Let $\mathcal{X}$ be a countable set and let $\mathcal{X}_\infty := \mathcal{X} \cup \{\infty\}$ denote the one-point compactification of $\mathcal{X}$. Recall that a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathcal{X}$ then converges to ∞ if for all finite $\mathcal{X}' \subset \mathcal{X}$ there exists an $N = N(\mathcal{X}') \in \mathbb{N}$ such that $x_n \notin \mathcal{X}'$ for all $n \geq N$.

Let $\mathcal{G}$ now be a countable collection of maps mapping from $\mathcal{X}$ to itself and extend all $m \in \mathcal{G}$ to $\mathcal{X}_\infty$ by setting $m(\infty) := \infty$. Let $(r_m)_{m\in\mathcal{G}}$ be non-negative real constants and define a Poisson point set ω as in Chapter 1.1. Then define $\omega_{s,u}$ $(s \leq u)$ as in (1.9) and $\mathbf{X}^{\tilde{\omega}}_{s,u}$ for each finite $\tilde{\omega} \subset \omega_{s,u}$ as in (1.10). We are going to need that

$$\sum_{\substack{m\in\mathcal{G}:\\ m(x)\neq x}} r_m < \infty \quad \text{for all } x \in \mathcal{X}. \tag{1.19}$$

The proof of the following result is inspired by the one of [Swart, 2022, Proposition 2.8].

Proposition 1.5 (Stochastic flow on countable space)**.** *Let $\mathcal{X}$ be countable and assume the summability condition (1.19). Then, almost surely, setting*

$$\mathbf{X}_{s,u}(x) := \lim_{\tilde{\omega}_n\uparrow\omega_{s,u}} \mathbf{X}^{\tilde{\omega}_n}_{s,u}(x) \qquad (s \leq u,\ x \in \mathcal{X}_\infty), \tag{1.20}$$

where $(\tilde{\omega}_n)_{n\in\mathbb{N}}$ is an arbitrary sequence of finite subsets of ω increasing to $\omega_{s,u}$, yields a well-defined random map $\mathbf{X}_{s,u} : \mathcal{X}_\infty \to \mathcal{X}_\infty$ for all $s \leq u$.

Proof. Since $\mathcal{X}$ is countable, it suffices to show that $\mathbf{X}_{s,u}(x)$ is almost surely well-defined for all $s \leq u$ but *fixed* $x \in \mathcal{X}_\infty$. In fact, as by definition $\omega_{s,s} = \emptyset$ $(s \in \mathbb{R})$ and hence $\mathbf{X}_{s,s}(x) = x$ $(s \in \mathbb{R})$, it suffices to show that $\mathbf{X}_{s,u}(x)$ is almost surely well-defined for all $s < u$ and fixed $x \in \mathcal{X}_\infty$.

Let $x \in \mathcal{X}_\infty$ be fixed. If the sum in (1.19) is 0, then there does not exist a map $m \in \mathcal{G}$ with $m(x) \neq x$ and $r_m > 0$. Hence, almost surely, $\mathbf{X}^{\tilde{\omega}}_{s,u}(x) = x$ for any finite subset $\tilde{\omega} \subset \omega_{s,u}$ $(s < u)$, implying that $\mathbf{X}_{s,u}(x)$ is (a.s.) well-defined as $\mathbf{X}_{s,u}(x) = x$ for all $s < u$. Note that this is in particular the case for $x = \infty$.

Otherwise, if the sum in (1.19) equals a positive real number, there almost surely exists a set $\{s_k : k \in \mathbb{Z}\} \subset \mathbb{R}$ with $s_{k-1} < s_k < s_{k+1}$ $(k \in \mathbb{Z})$ such that

$$I(x) := \Big\{s \in \mathbb{R} : \exists (m, s) \in \omega \text{ with } m(x) \neq x\Big\} = \Big\{s_k : k \in \mathbb{Z}\Big\}. \tag{1.21}$$

We will first show that $\mathbf{X}_{s_k,u}(x)$ is almost surely well-defined for all $k \in \mathbb{Z}$ and $u \in (s_k, \infty)$. From this it will then follow that $\mathbf{X}_{s,u}(x)$ is almost surely well-defined for all $s < u$. As $I(x)$ is (a.s.) countable we may fix $k \in \mathbb{Z}$.

For fixed $k \in \mathbb{Z}$, due to (1.19), there almost surely exists a sequence $(u_n)_{n\in\mathbb{N}} \subset (s_k, \infty]$ such that $u_1 = s_{k+1}$ and

$$u_n = \inf \Big\{(u_{n-1}, \infty) \cap I(m_{n-1} \circ \cdots \circ m_1(x))\Big\} \qquad (n \in \mathbb{N} \setminus \{1\}),$$

where the infimum over the empty set is ∞. Here, for $u_n < \infty$, m_n denote the corresponding map such that $(m_n, u_n) \in \omega$. Let $\tau := \lim_{n\to\infty} u_n \in (s_k, \infty]$. If $\tau = \infty$, then for all $u \in (s_k, \infty)$ we can find an $n \in \mathbb{N}$ such that $u_n \leq u < u_{n+1}$. Let $\tilde{\omega} = \{(m_1, u_1), \ldots, (m_n, u_n)\}$ so that $\mathbf{X}^{\tilde{\omega}}_{s_k,u}(x) = m_n \circ \cdots \circ m_1(x)$. As $\omega_{s_k,u}$ is almost surely at most countable, for any sequence $(\tilde{\omega}_n)_{n\in\mathbb{N}}$ that increases to $\omega_{s_k,u}$ there (a.s.) has to exist an $n_0 \in \mathbb{N}$ such that $\tilde{\omega} \subset \tilde{\omega}_n$ for all $n \geq n_0$. It follows that $\mathbf{X}_{s_k,u}(x)$ is (a.s.) well-defined as

$$\mathbf{X}_{s_k,u}(x) = \mathbf{X}^{\tilde{\omega}}_{s_k,u}(x) = m_n \circ \cdots \circ m_1(x).$$

On the other hand, if $\tau < \infty$, we claim that $\mathbf{X}_{s_k,u}(x)$ is almost surely well-defined as $\mathbf{X}_{s_k,u}(x) = \infty$ for all $u \geq \tau$ while $\mathbf{X}_{s_k,u}(x)$ is defined for $u < \tau$ as above. Indeed, let $\mathcal{X}' \subset \mathcal{X}$ be finite. Then, almost surely,

$$[s_k, \tau) \cap \bigcup_{x' \in \mathcal{X}'} I(x')$$

is finite. It follows that there has to exist an $N = N(\mathcal{X}') \in \mathbb{N}$ such that for all $n > N$ one has $m_n(x') = x'$ for all $x' \in \mathcal{X}'$. Let $\tilde{\omega} := \{(m_1, u_1), \ldots, (m_N, u_N)\}$. Then, one has that $\mathbf{X}^{\tilde{\omega}}_{s_k,\tau}(x) \notin \mathcal{X}'$. Indeed, $\mathbf{X}^{\tilde{\omega}}_{s_k,\tau}(x) = x' \in \mathcal{X}'$ would imply that $m_{N+1}(x') \neq x'$, contradicting the definition of N. As above, for any sequence $(\tilde{\omega}_n)_{n\in\mathbb{N}}$ that increases to $\omega_{s_k,\tau}$ there has to exist an $n_0 \in \mathbb{N}$ such that $\tilde{\omega} \subset \tilde{\omega}_n$ for all $n \geq n_0$. It follows that $\mathbf{X}^{\tilde{\omega}_n}_{s_k,\tau}(x) \notin \mathcal{X}'$ for all $n \geq n_0$. As there are only countably many finite subsets of $\mathcal{X}$, we conclude from the definition of convergence to ∞ that, almost surely, $\mathbf{X}_{s_k,\tau}(x)$ is well-defined as $\mathbf{X}_{s_k,\tau}(x) = \infty$. It follows from the way we extended the maps in $\mathcal{G}$ to $\mathcal{X}_\infty$ that (a.s.) also $\mathbf{X}_{s_k,u}(x) = \infty$ for all $u > \tau$. We conclude that $\mathbf{X}_{s_k,u}(x)$ is almost surely well-defined for all $s_k < u$ and $k \in \mathbb{Z}$.

Finally, we extend this result to arbitrary $s \in \mathbb{R}$. Due to the structure of $I(x)$ in (1.21), for all $s \in \mathbb{R}$ we can almost surely find a $k \in \mathbb{Z}$ such that $s_k \leq s < s_{k+1}$ and $\mathbf{X}_{s,u}(x) = \mathbf{X}_{s_k,u}(x)$ for $u \in (s, \infty)$. Hence, $\mathbf{X}_{s,u}(x)$ is almost surely well-defined for all $s < u$. This implies the statement of the proposition as outlined at the beginning of the proof. □

Let X_0 be a $\mathcal{X}_\infty$-valued random variable, independent of ω. Setting for fixed $s \in \mathbb{R}$

$$X_t := \mathbf{X}_{s,s+t}(X_0) \qquad (t \geq 0)$$

as in (1.13) yields by [Swart, 2022, Proposition 2.9] a continuous-time Markov process $X = (X_t)_{t\geq 0}$ with state space $\mathcal{X}_\infty$. We refer to it as the continuous-time Markov chain with càdlàg sample paths and generator G, defined as in (1.8) but for $x \in \mathcal{X}$ and bounded $f : \mathcal{X} \to \mathbb{R}$.[4] We call $\tau := \inf_{t\geq 0}\{X_t = \infty\}$ the *explosion time* of X and say that X is *non-explosive* if $\tau = \infty$ almost surely. We now consider the case that $\mathcal{X} = S^\Lambda_{\text{fin}}$.

Theorem 1.6 (Finite initial states). *Assume that there exists an element* $0 \in S$ *such that*

$$m(\underline{0}) = m(\underline{0}) \qquad (m \in \mathcal{G}), \tag{1.22}$$

and that instead of (1.7), the rates satisfy (1.18). Then, almost surely, one can define a random map $\mathbf{X}_{s,u} : S^\Lambda_{\text{fin}} \to S^\Lambda_{\text{fin}}$ *via (1.20) for all* $s \leq u$.

Proof. In the first step we want to apply Proposition 1.5. To do so, we verify that (1.19) holds. Let $x \in S^\Lambda_{\text{fin}}$ and $m \in \mathcal{G}$. Assume that $m(x) \neq x$. Then, either m changes a value on $\text{supp}(x)$, i.e., there exists a $j \in \Lambda$ such that $j \in$

[4] Usually, the generator of a continuous-time Markov chain is viewed as an infinite matrix, the so called Q-matrix, and not as a linear operator (compare, e.g., [Liggett, 2010]). It is not hard to confirm that also the the form of the generator G presented here uniquely determines the transition probabilities of X. Compare the comment below [Swart, 2022, Proposition 2.9].

$\mathcal{D}(m) \cap \operatorname{supp}(x)$, or a value outside of $\operatorname{supp}(x)$, i.e., there exists a $j \in \Lambda$ such that $j \in \mathcal{D}(m) \cap \operatorname{supp}(x)^{\mathrm{c}}$. But in the latter case, due to the assumption that $m(\underline{0}) = \underline{0}$, there has to exist an $i \in \mathcal{R}(m[j]) \cap \operatorname{supp}(x)$. Hence,

$$\sum_{\substack{m\in\mathcal{G}:\\ m(x)\neq x}} r_m \leq \sum_{j\in\operatorname{supp}(x)} \sum_{m\in\mathcal{G}} r_m \mathbb{1}_{\mathcal{D}(m)}(j) + \sum_{i\in\operatorname{supp}(x)} \sum_{m\in\mathcal{G}} r_m |\mathcal{R}_i^{\uparrow}(m)|$$

and the finiteness of $\operatorname{supp}(x)$ implies together with (1.18) that (1.19) holds.

Hence, due to Proposition 1.5, we can almost surely define $\mathbf{X}_{s,u}(x)$ via (1.20) for all $s \leq u$ and $x \in S^\Lambda_{\text{fin}}$. We are left to show that (a.s.) $\mathbf{X}_{s,u}(x) \in S^\Lambda_{\text{fin}}$ for all $s \leq u$ and $x \in S^\Lambda_{\text{fin}}$. To do so, we are going to use a variant of the "paths of potential influence" defined in [Swart, 2022, Display (4.20)]. The same variant was also used in the proof of [Swart, 2022, Lemma 4.22].

For $i, j \in \Lambda$ and $s \leq u$ we write $(i, s) \rightsquigarrow (j, u)$ if there exists a càdlàg function $\xi : [s, u] \to \Lambda$ with $\xi(s) = i$, $\xi(u) = j$ and the property that

- if $\xi(t-) \neq \xi(t)$ for some $t \in (s, u]$, then there exists a map $m \in \mathcal{G}$ such that $(m, t) \in \omega$, $\xi(t) \in \mathcal{D}(m)$ and $\xi(t-) \in \mathcal{R}(m[\xi(t)])$.

We set

$$\zeta_{s,u}(I) := \Big\{ j \in \Lambda : (i, s) \rightsquigarrow (j, u) \text{ for some } i \in I \Big\} \qquad (s \leq u,\ I \subset \Lambda).$$

To show that $\mathbf{X}_{s,u}(x) \neq \infty$, the "infinity element" of the one-point compactification of S^Λ_{fin}, it suffices to show that $|\zeta_{s,u}(\operatorname{supp}(x))|$ stays finite ($x \in S^\Lambda_{\text{fin}}$, $s \leq u$). Indeed, if, for some $x \in S^\Lambda_{\text{fin}}$ and $s \leq u$, $\mathbf{X}_{s,u}(x) = \infty$, then, by definition, for all finite $\mathcal{S} \subset S^\Lambda_{\text{fin}}$ there has to exist a $t_0 \in (s, u)$ such that $\mathbf{X}_{s,t}(x) \notin \mathcal{S}$ for all $t \in [t_0, u]$. This, in particular, holds for

$$\mathcal{S} := \Big\{ x' \in S^\Lambda : \operatorname{supp}(x') \subset \Delta \Big\},$$

where $\Delta \subset \Lambda$ is some finite set. Together with the observation at the beginning of the proof this implies that if $|\zeta_{s,u}(\operatorname{supp}(x))|$ is finite, then $\mathbf{X}_{s,u}(x) \in S^\Lambda_{\text{fin}}$ ($x \in S^\Lambda_{\text{fin}}$, $s \leq u$).

By definition, we just require jumps of the càdlàg function ξ to correspond to $(m, t) \in \omega$, while ξ may ignore some $(m, t) \in \omega$ "on its way". Hence, $I \subset \zeta_{s,u}(I)$ and

$$\zeta_{s,u}(I) \subset \zeta_{\lfloor s\rfloor,\lceil u\rceil}(I) \qquad (s \leq u,\ I \subset \Lambda),$$

where $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ denote the floor and the ceiling of a real number, respectively. Recall that S^Λ_{fin} is countable. Hence, in order to show that

$$|\zeta_{s,u}(\operatorname{supp}(x))|$$

is almost surely finite *for all* $x \in S^\Lambda_{\text{fin}}$ and $s \leq u$, it suffices to show that

$$|\zeta_{s,u}(\operatorname{supp}(x))|$$

is almost surely finite for *fixed* $x \in S^\Lambda_{\text{fin}}$ and $s \leq u$ with $s, u \in \mathbb{Z}$. To do so, we use a standard generator computation. Let $(\Delta_n)_{n\in\mathbb{N}}$ be a sequence of finite subsets of Λ with $\Delta_n \nearrow \Lambda$ as $n \to \infty$. Let $\mathcal{P}(\Delta_n)$ denote the power set of Δ_n ($n \in \mathbb{N}$). We define

$$\zeta^n_{s,u}(I) := \Big\{ j \in \Lambda : (i,s) \rightsquigarrow_n (j,u) \text{ for some } i \in I \Big\} \qquad (n \in \mathbb{N},\ s \le u,\ I \subset \Delta_n),$$

where $(i,s) \rightsquigarrow_n (j,u)$ is defined as $(i,s) \rightsquigarrow (j,u)$ above, but with the càdlàg function ξ being required to map to Δ_n instead of the whole Λ. We claim that $(\zeta^n_{s,u}(\mathrm{supp}(x)))_{u \ge s}$ is a Markov process with finite state space $\mathcal{P}(\Delta_n)$ and generator G_n of the form

$$G_n f(I) := \sum_{m \in \mathcal{G}} r_m \Big\{ f(m_n(I)) - f(I) \Big\} \quad (n \in \mathbb{N},\ I \subset \Delta_n,\ f : \mathcal{P}(\Delta_n) \to \mathcal{P}(\Delta_n)),$$

where for all $m \in \mathcal{G}$ and $n \in \mathbb{N}$ we define $m_n : \mathcal{P}(\Delta_n) \to \mathcal{P}(\Delta_n)$ by

$$m_n(I) := I \cup \Big[\Big\{ j \in \mathcal{D}(m) : \exists i \in I \cap \mathcal{R}(m[j]) \Big\} \cap \Delta_n \Big] \qquad (n \in \mathbb{N},\ I \subset \Delta_n).$$

Indeed, this follows from standard theory (compare [Swart, 2022, Proposition 2.7]) and the fact that

$$\sum_{\substack{m \in \mathcal{G}: \\ m_n(I) \neq I}} r_m \le \sum_{i \in \Delta_n} \sum_{m \in \mathcal{G}} r_m |\mathcal{R}^\uparrow_i(m)|,$$

which, due to (1.18), says that the total rate of Poisson events that can change the state of the process is finite in any state $I \in \mathcal{P}(\Delta_n)$. Choosing f to be the function computing the cardinality of a set, i.e., $f(I) := |I|$, one has that

$$G_n f(I) \le \sum_{i \in I} \sum_{m \in \mathcal{G}} r_m |\mathcal{R}^\uparrow_i(m)| \le K f(I) \qquad (n \in \mathbb{N},\ I \subset \Delta_n), \tag{1.23}$$

where $K < \infty$ is the supremum in (1.18).[5] Standard theory (compare the proof of [Swart, 2022, Lemma 4.21]) then implies that

$$\mathbb{E}\Big[|\zeta^n_{s,u}(\mathrm{supp}(x))| \Big] \le |\mathrm{supp}(x)| e^{K(u-s)} < \infty \qquad (n \in \mathbb{N},\ u \in \mathbb{Z} : u \ge s).$$

Letting $n \to \infty$ and using monotone convergence, it follows that $|\zeta_{s,u}(\mathrm{supp}(x))|$ is almost surely finite. □

Hence, Theorem 1.6 implies that under the summability condition (1.18), setting $X_t := \mathbf{X}_{s,s+t}(X_0)$ ($t \ge 0$) for fixed $s \in \mathbb{R}$ as in (1.13) yields a continuous-time Markov chain $X = (X_t)_{t \ge 0}$ with state space S^Λ_{fin}, càdlàg sample paths and generator G, defined as in (1.8) but for $x \in S^\Lambda_{\mathrm{fin}}$ and bounded $f : S^\Lambda_{\mathrm{fin}} \to \mathbb{R}$.

Moreover, following [Swart, 2022, Proposition 2.8], it can be shown that, assuming (1.18) instead of (1.7), for each $s \in \mathbb{R}$ and $x \in S^\Lambda_{\mathrm{fin}}$ there almost surely exists a unique càdlàg function $[s,\infty) \ni t \mapsto X_t \in S^\Lambda_{\mathrm{fin}}$ that solves (1.14), given via $X_t := \mathbf{X}_{s,t}(x)$ ($t \ge s$).

To conclude this subchapter, we want to stress that the summability conditions (1.7) and (1.18) do not imply each other. Indeed, let $S = \{0,1\}$ and $\Lambda = \mathbb{N}$. Let $m_k : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ ($k \in \mathbb{N}$) be defined by

$$m_k(x)(\ell) := \begin{cases} x(1) & \text{if } \ell = k, \\ x(\ell) & \text{else,} \end{cases} \qquad (x \in \{0,1\}^{\mathbb{N}},\ \ell \in \mathbb{N}).$$

[5] Any f that satisfies (1.23) is sometimes called *Lyapunov function* for the Markov process with generator G_n.

One has for $\ell \in \mathbb{N}$ and $k \in \mathbb{N} \setminus \{1\}$ that

$$\mathcal{R}^{\downarrow}_{\ell}(m_k) = \begin{cases} \{1\} & \text{if } \ell = k, \\ \emptyset & \text{else,} \end{cases} \quad \text{and} \quad \mathcal{R}^{\uparrow}_{\ell}(m_k) = \begin{cases} \{k\} & \text{if } \ell = 1, \\ \emptyset & \text{else.} \end{cases}$$

Setting now $\mathcal{G} := \{m_k : k \in \mathbb{N}\}$ and $r_m := \varepsilon > 0$ for all $m \in \mathcal{G}$ yields

$$\sup_{k\in\mathbb{N}} \sum_{m\in\mathcal{G}} r_m \Big(\mathbb{1}_{\mathcal{D}(m)}(k) + |\mathcal{R}^{\downarrow}_k(m)| \Big) = 2\varepsilon,$$
$$\sup_{k\in\mathbb{N}} \sum_{m\in\mathcal{G}} r_m \Big(\mathbb{1}_{\mathcal{D}(m)}(k) + |\mathcal{R}^{\uparrow}_k(m)| \Big) = \sum_{n=2}^{\infty} \varepsilon = \infty.$$

Thus, (1.7) holds, but (1.18) does not.

On the other hand, let $n_k : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ $(k \in \mathbb{N})$ be defined by

$$n_k(x)(\ell) := \begin{cases} \max\left\{ x\left(\frac{k(k-1)}{2}+1\right), \dots, x\left(\frac{k(k+1)}{2}\right)\right\} & \text{if } \ell = k, \\ x(\ell) & \text{else,} \end{cases}$$

where $x \in \{0,1\}^{\mathbb{N}}$ and $\ell \in \mathbb{N}$. Thus, $n_1(x)(1) = x(1)$, $n_2(x)(2) = \max\{x(2), x(3)\}$, $n_3(x)(3) = \max\{x(4), x(5), x(6)\}$, and so on. One has for $\ell \in \mathbb{N}$ and $k \in \mathbb{N} \setminus \{1\}$ that

$$\mathcal{R}^{\downarrow}_{\ell}(n_k) = \begin{cases} \left\{\frac{k(k-1)}{2}+1, \dots, \frac{k(k+1)}{2}\right\} & \text{if } \ell = k, \\ \emptyset & \text{else,} \end{cases}$$

and

$$\mathcal{R}^{\uparrow}_{\ell}(n_k) = \begin{cases} \{k\} & \text{if } \frac{k(k-1)}{2}+1 \le \ell \le \frac{k(k+1)}{2}, \\ \emptyset & \text{else.} \end{cases}$$

Setting now $\mathcal{G} := \{n_k : k \in \mathbb{N}\}$ and $r_m := \varepsilon > 0$ for all $m \in \mathcal{G}$ yields

$$\sup_{k\in\mathbb{N}} \sum_{m\in\mathcal{G}} r_m \Big(\mathbb{1}_{\mathcal{D}(m)}(k) + |\mathcal{R}^{\downarrow}_k(m)| \Big) = \varepsilon \sup_{k\in\mathbb{N}\setminus\{1\}} (k+1) = \infty,$$
$$\sup_{k\in\mathbb{N}} \sum_{m\in\mathcal{G}} r_m \Big(\mathbb{1}_{\mathcal{D}(m)}(k) + |\mathcal{R}^{\uparrow}_k(m)| \Big) = 2\varepsilon.$$

Thus, (1.18) holds, but (1.7) does not.

1.3 Duality

Let $\mathcal{Y}$ be a Polish space equipped with the Borel σ-algebra and let $Y = (Y_t)_{t\ge0}$ be a Markov process on $\mathcal{Y}$. Similar to our conventions for the interacting particle system $X = (X_t)_{t\ge0}$, for any initial distribution $\mu \in \mathcal{M}_1(\mathcal{Y})$ we denote by $\mathbb{P}_\mu$ the law of Y started in μ. For the special case that Y is started in a deterministic state $y \in \mathcal{Y}$ we write $\mathbb{P}_y$ as a shorthand for $\mathbb{P}_{\delta_y}$. Accordingly, we denote expectation with respect to $\mathbb{P}_\mu$ by $\mathbb{E}_\mu$ $(\mu \in \mathcal{M}_1(\mathcal{Y}))$ and expectation with respect to $\mathbb{P}_y$ by $\mathbb{E}_y$ $(y \in \mathcal{Y})$.

Let $\boldsymbol{\psi} : S^\Lambda \times \mathcal{Y} \to \mathbb{V}$ be a bounded measurable function, where $\mathbb{V}$ is some finite-dimensional normed vector space. We say that the interacting particle system X, constructed in Chapter 1.1, is *dual to Y with respect to the duality function $\boldsymbol{\psi}$* if

$$\mathbb{E}^x[\boldsymbol{\psi}(X_t, y)] = \mathbb{E}_y[\boldsymbol{\psi}(x, Y_t)] \qquad (x \in S^\Lambda,\ y \in \mathcal{Y}). \tag{1.24}$$

We will sometimes call S^Λ the *primal space* and $\mathcal{Y}$ the *dual space*.

The main goal of the present thesis is to identify sufficient conditions for the existence of a "useful" duality relation. The adjective useful was added to stress that we are not trying to just find any duality relation - in this case one could just choose $\boldsymbol{\psi}$ or Y to be constant. In Chapter 1.5 below we will specify what makes a duality useful.

A slightly stronger notion than duality is *pathwise duality*. We say that the interacting particle system X is *pathwise dual to Y with respect to $\boldsymbol{\psi}$* if, for each $t \geq 0$, we can construct families $\{X^x = (X^x_s)_{s\geq 0} : x \in S^\Lambda\}$ and $\{Y^y = (Y^y_s)_{s\geq 0} : y \in \mathcal{Y}\}$ on one common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that X^x has law $\mathbb{P}^x$, Y^y has law $\mathbb{P}_y$ and

$$[0,t] \ni s \mapsto \boldsymbol{\psi}(X^x_s, Y^y_{t-s}) \text{ is } \mathbb{P}\text{-a.s. constant} \qquad (x \in S^\Lambda,\ y \in \mathcal{Y}). \tag{1.25}$$

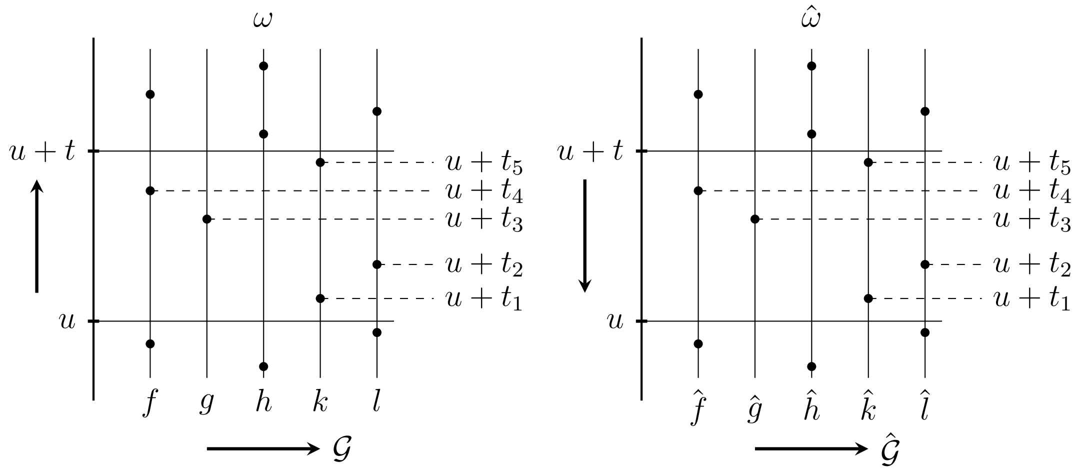
Clearly, considering $s = 0$ and $s = t$, pathwise duality implies duality. It has the additional advantage that the definition still makes sense if we allow $\mathbb{V}$ to be a more general space in which it might not be clear how to compute expectations. In fact, for now we will replace the finite-dimensional normed vector space $\mathbb{V}$ by a finite set T. The idea is to later identify T with elements of $\mathbb{V}$ in order to be able to take expectation and conclude (1.24) from (1.25). This step is mainly discussed in Chapter 2.6 and Chapter 3.5. In Chapter 4 we will work with $T = \{0, 1\}$, which we naturally embed in $\mathbb{V} = \mathbb{R}$.

Pathwise dualities for interacting particle systems are usually constructed using their graphical representation. To do so, one has to introduce the concept of a *dual map*. We say that the maps $m : S^\Lambda \to S^\Lambda$ and $\hat{m} : \mathcal{Y} \to \mathcal{Y}$ are *dual with respect to $\boldsymbol{\psi} : S^\Lambda \times \mathcal{Y} \to T$* if

$$\boldsymbol{\psi}(m(x), y) = \boldsymbol{\psi}(x, \hat{m}(y)) \qquad (x \in S^\Lambda,\ y \in \mathcal{Y}). \tag{1.26}$$

If every map $m \in \mathcal{G}$ now possesses a unique dual map $\hat{m}$, the idea is to replace each m that appears in the graphical representation ω by its dual map $\hat{m}$ in order to construct a new graphical representation $\hat{\omega} := \{(\hat{m}, t) : (m, t) \in \omega\}$. Using $\hat{\omega}$ and "changing the direction of time" we wish to define a *backward stochastic flow* $(\mathbf{Y}_{u,s})_{u\geq s}$ in the sense that $\mathbf{Y}_{s,s}$ is the identity for all $s \in \mathbb{R}$ and $\mathbf{Y}_{t,s} \circ \mathbf{Y}_{u,t} = \mathbf{Y}_{u,s}$ $(u \geq t \geq s)$. Let X_0 be a random variable with values in S^Λ that is independent of ω. Recall from Theorem 1.3 that under (1.7), for any fixed $s \in \mathbb{R}$, setting $X_t := \mathbf{X}_{s,s+t}(X_0)$ $(t \geq 0)$ based on (1.12) yields the interacting particle system X. Let Y_0 be a random variable with values in $\mathcal{Y}$ that is also independent of ω. The properties of the backward stochastic flow then should imply that for any fixed $u \in \mathbb{R}$ setting $Y_t := \mathbf{Y}_{u,u-t}(Y_0)$ $(t \geq 0)$ yields Y. As both stochastic flows were constructed from the same underlying Poisson point set, the duality between the maps then should imply a duality between the flows in the sense that, almost surely,

$$\boldsymbol{\psi}(\mathbf{X}_{s,u}(x), y) = \boldsymbol{\psi}(x, \mathbf{Y}_{u,s}(y)) \tag{1.27}$$



$$\mathbf{X}_{u,u+t} = k \circ f \circ g \circ l \circ k \qquad \mathbf{Y}_{u+t,u} = \hat{k} \circ \hat{l} \circ \hat{g} \circ \hat{f} \circ \hat{k}$$

Figure 1.1: An illustration how to construct a pathwise duality for an interacting particle system if $\mathcal{G}$ were finite. Time is depicted vertically.

holds almost surely for all $s \le u$, $x \in S^\Lambda$ and $y \in \mathcal{Y}$ simultaneously.

Let $t \ge 0$ be fixed and choose an arbitrary $u \in \mathbb{R}$. Assume that the construction of the previous paragraph works. Then we define the families $\{X^x = (X_s^x)_{s\ge 0} : x \in S^\Lambda\}$ and $\{Y^y = (Y_s^y)_{s\ge 0} : y \in \mathcal{Y}\}$ on the probability space of the underlying Poisson point process by setting $X_s^x := \mathbf{X}_{u,u+s}(x)$ for $x \in S^\Lambda$ and $Y_s^y := \mathbf{Y}_{u+t,u+t-s}(y)$ for $y \in \mathcal{Y}$ ($s \ge 0$). The duality between the flows in (1.27) then implies (1.25). Indeed, let $x \in S^\Lambda$ and $y \in \mathcal{Y}$. Then, almost surely for all $s_1, s_2 \in [0,t]$ with $s_1 \le s_2$,

$$\begin{aligned}
\boldsymbol{\psi}(X_{s_2}^x, Y_{t-s_2}^y) &= \boldsymbol{\psi}(\mathbf{X}_{u,u+s_2}(x), \mathbf{Y}_{u+t,u+s_2}(y)) \\
&= \boldsymbol{\psi}(\mathbf{X}_{u+s_1,u+s_2} \circ \mathbf{X}_{u,u+s_1}(x), \mathbf{Y}_{u+t,u+s_2}(y)) \\
&= \boldsymbol{\psi}(\mathbf{X}_{u,u+s_1}(x), \mathbf{Y}_{u+s_2,u+s_1} \circ \mathbf{Y}_{u+t,u+s_2}(y)) \\
&= \boldsymbol{\psi}(\mathbf{X}_{u,u+s_1}(x), \mathbf{Y}_{u+t,u+s_1}(y)) \\
&= \boldsymbol{\psi}(X_{s_1}^x, Y_{t-s_1}^y),
\end{aligned}$$

where we used (1.27) in the third equality and the properties of the two stochastic flows in the second and fourth equality. The construction is illustrated in Figure 1.1.

To summarize, to construct a pathwise duality in the outlined way, we first have to identify a Polish space $\mathcal{Y}$, a finite set T and a measurable function $\boldsymbol{\psi} : S^\Lambda \times \mathcal{Y} \to T$ for which every $m \in \mathcal{G}$ possesses a unique dual map $\hat{m}$. Then we have to confirm that

- using $\hat{\omega}$, almost surely we can construct for every $u \ge s$ a well-defined random map $\mathbf{Y}_{u,s} : \mathcal{Y} \to \mathcal{Y}$ such that $(\mathbf{Y}_{u,s})_{u\ge s}$ forms a backward stochastic flow,
- for any $u \in \mathbb{R}$, setting $Y_t = \mathbf{Y}_{u,u-t}(Y_0)$, where Y_0 is a $\mathcal{Y}$-valued random variable independent of ω, yields the same Markov process $Y = (Y_t)_{t\ge 0}$ (in law),

- (1.27) holds almost surely for all $s \le u$, $x \in S^\Lambda$ and $y \in \mathcal{Y}$ simultaneously.

As outlined above, the first important step is that every $m \in \mathcal{G}$ possesses a unique dual map with respect to $\boldsymbol{\psi}$. We will prove a useful characterization of this being the case. Let T be again a finite set and let $\mathcal{F}(S^\Lambda, T)$ denote the collection of all functions from S^Λ to T.[6] Let $\mathcal{H} \subset \mathcal{F}(S^\Lambda, T)$. We say that $m : S^\Lambda \to S^\Lambda$ *preserves* $\mathcal{H}$ if

$$f \circ m \in \mathcal{H} \quad \text{whenever} \quad f \in \mathcal{H}. \tag{1.28}$$

Let $\mathcal{Y}$ be a Polish space and let $\boldsymbol{\psi} : S^\Lambda \times \mathcal{Y} \to T$ be a measurable function. Let $\mathcal{H}_{\boldsymbol{\psi}} \subset \mathcal{F}(S^\Lambda, T)$ be defined as

$$\mathcal{H}_{\boldsymbol{\psi}} := \Big\{ \boldsymbol{\psi}(\,\cdot\,, y) : y \in \mathcal{Y} \Big\}. \tag{1.29}$$

We have the following characterization.

Lemma 1.7 (Maps with a dual I)**.** *A map $m : S^\Lambda \to S^\Lambda$ has a dual map $\hat{m} : \mathcal{Y} \to \mathcal{Y}$ with respect to $\boldsymbol{\psi}$ if and only if m preserves $\mathcal{H}_{\boldsymbol{\psi}}$. Moreover, if*

$$\boldsymbol{\psi}(x, y) = \boldsymbol{\psi}(x, y') \text{ for all } x \in S^\Lambda \text{ implies } y = y' \qquad (y, y' \in \mathcal{Y}), \tag{1.30}$$

the dual map $\hat{m}$, if it exists, is unique.

Proof. If m preserves $\mathcal{H}_{\boldsymbol{\psi}}$, then one has that $\boldsymbol{\psi}(m(\,\cdot\,), y) \in \mathcal{H}_{\boldsymbol{\psi}}$. Hence, by definition, there exists an element $\hat{m}(y) \in \mathcal{Y}$ such that $\boldsymbol{\psi}(m(\,\cdot\,), y) = \boldsymbol{\psi}(\,\cdot\,, \hat{m}(y))$. If (1.30) holds, the element $\hat{m}(y) \in \mathcal{Y}$ is unique. This implies that m has a unique dual map if it preserves $\mathcal{H}_{\boldsymbol{\psi}}$. That m preserves $\mathcal{H}_{\boldsymbol{\psi}}$ if it has a dual map follows by reversing the above arguments. $\square$

In Chapter 2 and Chapter 3 we are going to work with countable $\mathcal{Y}$. We are going to show that in this case, under mild assumptions on $\boldsymbol{\psi}$, the three bullet points in the list above hold true as long as we assume the summability condition (1.7).

Let $\mathcal{Y}$ be countable and let $\mathcal{Y}_\infty := \mathcal{Y} \cup \{\infty\}$ denote the one-point compactification of $\mathcal{Y}$. Assume that $\boldsymbol{\psi} : S^\Lambda \times \mathcal{Y} \to T$ is a measurable function such that every $m \in \mathcal{G}$ possesses a unique dual map with respect to $\boldsymbol{\psi}$. We define

$$\widehat{\mathcal{G}} := \Big\{ \hat{m} : m \in \mathcal{G} \Big\}$$

and extend all $\hat{m} \in \widehat{\mathcal{G}}$ to $\mathcal{Y}_\infty$ by setting $\hat{m}(\infty) := \infty$. Recall the definition of $\omega_{s,u}$ ($s \le u$) from (1.9). For each finite $\tilde{\omega} \subset \omega_{s,u}$ given as $\tilde{\omega} = \{(m_1, t_1), \dots, (m_n, t_n)\}$ with $t_1 < \dots < t_n$, we define

$$\mathbf{Y}^{\tilde{\omega}}_{u,s} := \hat{m}_1 \circ \dots \circ \hat{m}_n.$$

The following assertion can be proved in a similar way as Proposition 1.5, with trivial modifications to the proof.

[6] Note that $\mathcal{F}(S^\Lambda, T)$ is nothing else than $T^{(S^\Lambda)}$. However, we are going to reserve the latter notation for cases where we want to stress the interpretation of the elements of such a space as vectors or configurations. In all other cases we will use the notation with $\mathcal{F}$.

Proposition 1.8 (Dual flow on countable spaces)**.** *Let $\mathcal{Y}$ be countable and assume that*

$$\sum_{\substack{m\in\mathcal{G}:\\ \hat{m}(y)\neq y}} r_m < \infty \quad \textit{for all } y \in \mathcal{Y}. \tag{1.31}$$

Then, almost surely, setting

$$\mathbf{Y}_{u,s}(y) := \lim_{\tilde{\omega}_n \uparrow \omega_{s,u}} \mathbf{Y}^{\tilde{\omega}_n}_{u,s}(y) \qquad (u \geq s,\ y \in \mathcal{Y}_\infty), \tag{1.32}$$

where $(\tilde{\omega}_n)_{n\in\mathbb{N}}$ is an arbitrary sequence of finite subsets of ω increasing to $\omega_{s,u}$, yields a well-defined random map $\mathbf{Y}_{u,s} : \mathcal{Y}_\infty \to \mathcal{Y}_\infty$ for all $u \geq s$.

It follows readily that $(\mathbf{Y}_{u,s})_{u\geq s}$ is a backward stochastic flow. In parallel to (1.14) it can be characterized by an evolution equation. Almost surely, for each $s \in \mathbb{R}$ and $y \in \mathcal{Y}$, there exists a unique càdlàg function $(-\infty, s] \ni t \mapsto Y_t \in \mathcal{Y}$ that solves the evolution equation

$$Y_s = y \quad \text{and} \quad Y_{t-} = \begin{cases} \hat{m}(Y_t) & \text{if } (m,t) \in \omega, \\ Y_t & \text{else,} \end{cases} \qquad (t \leq s),$$

and this function is given via $Y_t := \mathbf{Y}_{s,t}(y)$ $(t \leq s)$. This fact is proved in parallel to [Swart, 2022, Proposition 2.8].

Analogous to the construction in Chapter 1.2, let Y_0 be a $\mathcal{Y}_\infty$-valued random variable, independent of ω. Setting for fixed $u \in \mathbb{R}$

$$Y_s := \mathbf{Y}_{u,u-s}(Y_0) \qquad (s \geq 0) \tag{1.33}$$

yields, again by [Swart, 2022, Proposition 2.9], a continuous-time Markov chain $Y = (Y_s)_{s\geq 0}$ on $\mathcal{Y}_\infty$, now with *càglàd* sample paths and generator $\widehat{G}$, defined as

$$\widehat{G}f(y) = \sum_{m\in\mathcal{G}} r_m \Big\{ f(\hat{m}(y)) - f(y) \Big\} \qquad (y \in \mathcal{Y}) \tag{1.34}$$

for bounded $f : \mathcal{Y} \to \mathbb{R}$. The following result is cited from [Swart, 2022, Theorem 6.20].

Theorem 1.9 (Duality with countable dual space)**.** *Let $\mathcal{Y}$ be countable and assume the summability condition (1.7). Moreover, assume (1.30), that all $m \in \mathcal{G}$ preserve $\mathcal{H}_\psi$ and that*

$$\boldsymbol{\psi} : S^\Lambda \times \mathcal{Y} \to T \textit{ is continuous with respect to the product topology on } S^\Lambda \times \mathcal{Y}. \tag{1.35}$$

Then also the summability condition (1.31) holds and the continuous-time Markov chain from (1.33), with initial initial distribution Y_0 concentrated on $\mathcal{Y}$, is non-explosive. Moreover, (1.27) holds almost surely for all $s \leq u$, $x \in S^\Lambda$ and $y \in \mathcal{Y}$ simultaneously.

In Chapter 4 we are going to work with an uncountable dual space $\mathcal{Y}$, so Proposition 1.8 and Theorem 1.9 are not going to be applicable. We are going to resolve this issue in Section 4.2.

1.4 A first pathwise duality

After having defined duality in Chapter 1.3 and having shown how a duality between maps can induce a pathwise duality between processes, we now aim to identify useful dualities for the maps in $\mathcal{G}$. The starting point is the following observation. Let T be again a finite set and recall that $\mathcal{F}(S^\Lambda, T)$ denotes the collection of all functions from S^Λ to T. We define $\boldsymbol{\psi}_{\text{basic}} : S^\Lambda \times \mathcal{F}(S^\Lambda, T) \to T$ as

$$\boldsymbol{\psi}_{\text{basic}}(x, f) := f(x) \qquad (x \in S^\Lambda,\ f \in \mathcal{F}(S^\Lambda, T)). \tag{1.36}$$

Then, as $\mathcal{H}_{\boldsymbol{\psi}_{\text{basic}}} = \mathcal{F}(S^\Lambda, T)$ and (1.30) clearly holds, Lemma 1.7 implies that *every* map $m : S^\Lambda \to S^\Lambda$ has a unique dual map $\hat{m} : \mathcal{F}(S^\Lambda, T) \to \mathcal{F}(S^\Lambda, T)$ with respect to $\boldsymbol{\psi}_{\text{basic}}$. The unique dual map of $m : S^\Lambda \to S^\Lambda$ is given by

$$\hat{m}(f) := f \circ m \qquad (f \in \mathcal{F}(S^\Lambda, T)). \tag{1.37}$$

Indeed, for $x \in S^\Lambda$ and $f \in \mathcal{F}(S^\Lambda, T)$,

$$\boldsymbol{\psi}_{\text{basic}}(m(x), f) = f(m(x)) = (f \circ m)(x) = \boldsymbol{\psi}_{\text{basic}}(x, \hat{m}(f)).$$

In particular, for $T = \{0, 1\}$ we can identify $\mathcal{F}(S^\Lambda, \{0, 1\})$ with $\mathcal{P}(S^\Lambda)$, the power set of S^Λ, via the bijection $f \mapsto \{x \in S^\Lambda : f(x) = 1\}$. In this case we can write $\boldsymbol{\psi}_{\text{basic}} : S^\Lambda \times \mathcal{P}(S^\Lambda) \to \{0, 1\}$ as

$$\boldsymbol{\psi}_{\text{basic}}(x, A) := \mathbb{1}_A(x) \qquad (x \in S^\Lambda,\ A \in \mathcal{P}(S^\Lambda)). \tag{1.38}$$

Under this identification the dual map $\hat{m}$ from (1.37) becomes the preimage map m^{-1}.

The collection $\mathcal{F}(S^\Lambda, T)$, however, is too large to work with. Let $\mathcal{C}(S^\Lambda, T)$ denote the collection of all *continuous* functions from S^Λ to T. As, by Lemma 1.1, any $f \in \mathcal{C}(S^\Lambda, T)$ depends only on finitely many coordinates, it is easy to see that $\mathcal{C}(S^\Lambda, T)$ is countable. We equip $\mathcal{C}(S^\Lambda, T)$ as usual with the discrete topology and the discrete σ-algebra. We now restrict $\boldsymbol{\psi}_{\text{basic}}$ from (1.36) in the second coordinate to $\mathcal{C}(S^\Lambda, T)$. Then (1.30) still holds and (1.35) is also satisfied. Moreover, now $\mathcal{H}_{\boldsymbol{\psi}_{\text{basic}}} = \mathcal{C}(S^\Lambda, T)$ and, as each $m \in \mathcal{G}$ is local and thus, by definition, also continuous, any $m \in \mathcal{G}$ preserves $\mathcal{C}(S^\Lambda, T) = \mathcal{H}_{\boldsymbol{\psi}_{\text{basic}}}$. Hence, by Proposition 1.8 and Theorem 1.9, for *every* interacting particle system $X = (X_t)_{t \geq 0}$ that satisfies the summability condition (1.7), we can construct a pathwise duality to a continuous-time Markov chain with state space $\mathcal{C}(S^\Lambda, T)$ and càglàd sample paths.

Although $\mathcal{C}(S^\Lambda, T)$ is only countable, the resulting dual process is rather abstract and not easy to work with. In Chapter 2 and Chapter 3 we are going to study dualities based on $\boldsymbol{\psi}_{\text{basic}}$ that are associated with preserved subspaces of $\mathcal{C}(S^\Lambda, T)$. In Chapter 4 we are going to study a duality based on $\boldsymbol{\psi}_{\text{basic}}$ in the formulation of (1.38) that is associated with a preserved subspace of $\mathcal{P}(S^\Lambda)$. In all cases the preserved subspaces are constructed by equipping S^Λ with additional structure (in Chapter 2 S^Λ will become a monoid, in Chapter 3 a module over a semiring, and in Chapter 4 a partially ordered set). The preserved subspaces of $\mathcal{F}(S^\Lambda, T)$ consist then of those functions, that preserve the structure of either a monoid, a module, or a partially ordered set (where we equip $T = \{0, 1\}$ with the natural partial order $0 \leq 1$). This leaves room for further research to either generalize this method or to equip S^Λ with some other structure in order to find additional dualities.

1.5 Informativeness

In this subchapter we specify what kind of dualities are useful. As we use duality mainly as a tool to study an interacting particle system $X = (X_t)_{t\geq 0}$, we would like the duality to determine the law of X_t for fixed $t \geq 0$. As in Section 1.3 let $\mathcal{Y}$ be a Polish space (equipped with the Borel σ-algebra), let T be a finite set and assume that an interacting particle system X on S^Λ and a Markov process $Y = (Y_t)_{t\geq 0}$ on $\mathcal{Y}$ are pathwise dual with respect to $\boldsymbol{\psi} : S^\Lambda \times \mathcal{Y} \to T$. A necessary condition for the duality function $\boldsymbol{\psi}$ to be able to determine the law of X_t for fixed $t \geq 0$ is that it separates the points in its first coordinate in the sense that

$$\boldsymbol{\psi}(x, y) = \boldsymbol{\psi}(x', y) \text{ for all } y \in \mathcal{Y} \text{ implies } x = x' \qquad (x, x' \in S^\Lambda). \tag{1.39}$$

Since measures on S^Λ are characterized by their finite-dimensional marginals, property (1.39) implies that for $x \in S^\Lambda$ and $t \geq 0$ the law $\mathbb{P}^x[X_t \in \cdot\,]$ is uniquely determined by all probabilities of the form

$$\mathbb{P}[\boldsymbol{\psi}(X_t^x, y_1) = z_1, \ldots, \boldsymbol{\psi}(X_t^x, y_n) = z_n] = \mathbb{P}[\boldsymbol{\psi}(x, Y_t^{y_1}) = z_1, \ldots, \boldsymbol{\psi}(x, Y_t^{y_n}) = z_n] \\ (n \in \mathbb{N},\ y_1, \ldots, y_n \in \mathcal{Y},\ z_1, \ldots, z_n \in T), \tag{1.40}$$

where $\mathbb{P}$ denotes the probability measure of the common probability space on which the families $\{X^x = (X_s^x)_{s\geq 0} : x \in S^\Lambda\}$ and $\{Y^y = (Y_s^y)_{s\geq 0} : y \in \mathcal{Y}\}$ are defined. Many duality functions that will appear in this thesis possess a stronger property regarding the characterization of the law of the interacting particle system. To introduce it, we first investigate families of functions defined on S^Λ in greater detail.

Let $\mathbb{V}$ be a finite-dimensional normed vector space and let V be a measurable space. For an arbitrary index set I we call a collection $(f_i)_{i\in I}$ of bounded measurable functions $f_i : S^\Lambda \to \mathbb{V}$ *distribution determining* if, for two random variables X and X' on S^Λ,

$$\mathbb{E}[f_i(X)] = \mathbb{E}[f_i(X')] \quad \forall i \in I \qquad \text{implies} \qquad X \stackrel{d}{=} X',$$

where $\stackrel{d}{=}$ denotes equality in distribution. Similarly, we call a collection $(g_i)_{i\in I}$ of measurable functions $g_i : S^\Lambda \to V$ *weakly distribution determining* if

$$g_i(X) \stackrel{d}{=} g_i(X') \quad \forall i \in I \qquad \text{implies} \qquad X \stackrel{d}{=} X'.$$

A family $(f_i)_{i\in I}$ of functions $f_i : S^\Lambda \to \mathbb{V}$ that is distribution determining is clearly also weakly distribution determining. The reverse implication is not true in general, but holds in the following special case. Recall that $v_1, \ldots, v_n \in \mathbb{V}$ are called *affinely independent* if

$$\sum_{k=1}^n \lambda_k v_k = 0 \text{ with scalars } \lambda_1, \ldots, \lambda_n \text{ s.t. } \sum_{k=1}^n \lambda_k = 0 \quad \text{implies} \quad \lambda_1 = \ldots = \lambda_n = 0.$$

Proposition 1.10 (Equality of notions)**.** *Let $(f_i)_{i\in I}$ be a family of functions $f_i : S^\Lambda \to \{v_1, \ldots, v_n\} \subset \mathbb{V}$. If $v_1, \ldots, v_n$ are affinely independent, then $(f_i)_{i\in I}$ is distribution determining if and only if it is weakly distribution determining.*

Proof. Comparing the definitions, it suffices to show that for fixed $i \in I$, under the assumption of the proposition, $\mathbb{E}[f_i(X)] = \mathbb{E}[f_i(X')]$ implies $f_i(X) \overset{d}{=} f_i(X')$. As the set $\{v_1, \ldots, v_n\}$ is finite, the condition $\mathbb{E}[f_i(X)] = \mathbb{E}[f_i(X')]$ is equivalent to writing

$$\sum_{k=1}^{n} v_k \Big(\mathbb{P}[f_i(X) = v_k] - \mathbb{P}[f_i(X') = v_k] \Big) = 0.$$

But as $v_1, \ldots, v_n$ are affinely independent, then also

$$\mathbb{P}[f_i(X) = v_k] - \mathbb{P}[f_i(X') = v_k] = 0 \qquad (k = 1, \ldots, n),$$

i.e., $f_i(X)$ and $f_i(X')$ are equal in distribution. □

We introduce a further notion that highlights the strength of a family of functions that is distribution determining. For an arbitrary index set I we call a family $(f_i)_{i \in I}$ of bounded continuous functions $f_i : S^\Lambda \to \mathbb{V}$ *convergence determining* if for probability measures $(\mu_n)_{n \in \mathbb{N}}, \mu$ on S^Λ,

$$\int f(x)\, \mathrm{d}\mu_n(x) \underset{n\to\infty}{\longrightarrow} \int f(x)\, \mathrm{d}\mu(x)\ \forall i \in I \quad \text{implies} \quad \mu_n \underset{n\to\infty}{\Longrightarrow} \mu,$$

where $\Rightarrow$ denotes weak convergence. The following statement follows from [Swart, 2022, Lemma 4.38].[7]

Lemma 1.11 (Convergence determining family)**.** *Let $\mathbb{V}$ be a finite-dimensional normed vector space and let $(f_i)_{i \in I}$ be a family of bounded continuous functions $f_i : S^\Lambda \to \mathbb{V}$. If $(f_i)_{i \in I}$ is distribution determining, then it is also convergence determining.*

We are going to use a Stone-Weierstrass argument in order to identify collections of functions that are distribution determining. We recall a couple of standard definitions. Let $\mathcal{X}$ and $\mathcal{Z}$ be arbitrary spaces. Recall that we denote by $\mathcal{F}(\mathcal{X}, \mathcal{Z})$ the collection of functions from $\mathcal{X}$ to $\mathcal{Z}$. We say that $\mathcal{H} \subset \mathcal{F}(\mathcal{X}, \mathcal{Z})$ *separates points* if for $x, x' \in \mathcal{X}$ with $x \neq x'$ there exists a function $f \in \mathcal{H}$ such that $f(x) \neq f(x')$. Moreover, we say that $\mathcal{H} \subset \mathcal{F}(\mathcal{X}, \mathbb{C})$ is *self-adjoint* if $f \in \mathcal{H}$ implies $\overline{f} \in \mathcal{H}$, where $\overline{f}(x) := \overline{f(x)}$ $(x \in \mathcal{X})$, the complex conjugate of $f(x)$.

Lemma 1.12 (Application of Stone-Weierstrass)**.** *Let E be a compact metrizable space. Assume that $\mathcal{H} \subset \mathcal{C}(E, \mathbb{C})$ separates points and is closed under products. Then $\mathcal{H}$ is distribution determining.*

Proof. The statement with $\mathbb{C}$ replaced by $\mathbb{R}$ is already proved in [Swart, 2022, Lemma 4.37]. Note that

$$\mathbb{E}[f(X)] = \mathbb{E}[f(X')] \quad \text{implies} \quad \mathbb{E}[\overline{f}(X)] = \mathbb{E}[\overline{f}(X')] \qquad (f \in \mathcal{H}), \tag{1.41}$$

as $\mathbb{E}[\overline{f}(X)] = \overline{\mathbb{E}[f(X)]}$, where X and X' are random variables on E. We can enlarge $\mathcal{H}$ with the constant function 1, take linear combinations and complex conjugates and receive an algebra $\mathcal{A} \supset \mathcal{H}$ that is closed under products, self-adjoint and separates points. If $\mathbb{E}[f(X)] = \mathbb{E}[f(X')]$ for all $f \in \mathcal{H}$ then also $\mathbb{E}[f(X)] = \mathbb{E}[f(X')]$ for all $f \in \mathcal{A}$ by the linearity of the integral and (1.41). We then can apply the complex version of the Stone-Weierstrass theorem and continue as in the proof of [Swart, 2022, Lemma 4.37]. □

[7][Swart, 2022, Lemma 4.38] is formulated for real-valued function, but the proof does not change if one also allows more general functions.

After this excursion we again consider the function $\boldsymbol{\psi} : S^\Lambda \times \mathcal{Y} \to T$, where $\mathcal{Y}$ is a Polish space and T is a finite set. We say that $\boldsymbol{\psi}$ is *weakly informative* if

$$(\boldsymbol{\psi}(\,\cdot\,, y))_{y \in \mathcal{Y}} \tag{1.42}$$

is weakly distribution determining. If T is a subset of a finite-dimensional normed vector space $\mathbb{V}$, we say that $\boldsymbol{\psi}$ is *informative* if the functions in (1.42) are distribution determining. Informativeness will be an important property in the applications of duality in Chapter 2.7.4 and Chapter 4.5.

By Lemma 1.12, if T is a subset of $\mathbb{C}$, then there exists a strategy how to prove informativeness: It suffices to show that $\boldsymbol{\psi}$ separates points in its first coordinate in the sense of (1.39) and is closed under products in the second coordinate in the sense that

$$\boldsymbol{\psi}(x, y_1)\boldsymbol{\psi}(x, y_2) = \boldsymbol{\psi}(x, y_3) \text{ for some } y_3 \in \mathcal{Y} \qquad (x \in S^\Lambda,\ y_1, y_2 \in \mathcal{Y}). \tag{1.43}$$

If T is not a subset of $\mathbb{C}$, it becomes much harder to say anything. In Chapter 2.6 we will develop an iterative procedure that can detect the absence of weak informativeness for duality functions that do not map to a subset of $\mathbb{C}$.

Both in Chapter 2 and in Chapter 3 we are going to encounter duality functions that satisfy (1.39) but are not weakly informative. As they still determine the law $\mathbb{P}^x[X_t \in \,\cdot\,]$ for fixed $x \in S^\Lambda$ and $t \geq 0$ via (1.40), the question becomes to which extend they are useful considering we can not work with just the expectation of the duality function as in Chapter 2.7.4 and Chapter 4.5. Moreover, we can ask (somewhat vaguely) what is the "minimal" information about the probabilities in (1.39) that is needed to uniquely determine the law $\mathbb{P}^x[X_t \in \,\cdot\,]$ for fixed $x \in S^\Lambda$ and $t \geq 0$. We leave this as an open problem for future research.

2. Monoid duality

As explained at the end of Chapter 1.4, the idea is to equip S^Λ with an additional structure in order to find new dualities and to explain the existence of known ones. In this chapter this additional structure will be the one of a monoid. Recall that, by definition, a *semigroup* is a pair $\mathbf{M} = (M, +)$ where M is a set and $+$ is an associative binary operator on M, i.e.,

$$(a_1 + a_2) + a_3 = a_1 + (a_2 + a_3) \qquad (a_1, a_2, a_3 \in M).$$

A semigroup is *commutative* if moreover

$$a_1 + a_2 = a_2 + a_1 \qquad (a_1, a_2 \in M).$$

A *neutral element* of a semigroup $\mathbf{M}$ is an element $0 \in M$ such that

$$a + 0 = a = 0 + a \qquad (a \in M).$$

It is easy to see that the neutral element, if it exists, is unique. By definition, a *monoid* is a semigroup $\mathbf{M} = (M, +)$ that is equipped with a neutral element 0. For a subset $M' \subset M$ that contains 0 and is closed under application of the operator $+$, $\mathbf{M'} = (M', +)$ is called a *submonoid* of $\mathbf{M}$. Then $\mathbf{M'}$ is itself a monoid with neutral element 0.

If we can equip the local state space S from Chapter 1 with an associative binary operator $+$ so that $\mathbf{S} = (S, +)$ is a semigroup, then we can also define an associative binary operator $+$ on the global state space S^Λ by setting

$$(x + x')(i) := x(i) + x'(i) \qquad (x, x' \in S^\Lambda,\ i \in \Lambda). \tag{2.1}$$

Then $\mathbf{S}^\Lambda := (S^\Lambda, +)$ becomes a semigroup as well. If $\mathbf{S}$ is commutative, then so is $\mathbf{S}^\Lambda$, and if $\mathbf{S}$ has a neutral element 0, then $\underline{0}$, defined as in (1.1), is the neutral element of $\mathbf{S}^\Lambda$.

The idea for the construction of the (pathwise) duality of this chapter is the following. Let T be a finite set. We start with the duality function $\boldsymbol{\psi}_{\text{basic}} : S^\Lambda \times \mathcal{C}(S^\Lambda, T) \to T$ from (1.36). If both S and T are equipped with the structure of a monoid, we can restrict $\boldsymbol{\psi}_{\text{basic}}$ in the second coordinate to the collection of all continuous monoid homomorphisms. By Lemma 1.7, any $m : S^\Lambda \to S^\Lambda$, that is a local monoid homomorphism, has $\hat{m}$ from (1.37) as its unique dual map with respect to this restriction of $\boldsymbol{\psi}_{\text{basic}}$. Consequently, by Proposition 1.8 and Theorem 1.9, any interacting particle system that has a generator G represented as in (1.8) with $\mathcal{G}$ consisting only of local monoid homomorphisms from S^Λ to itself, has a continuous-time Markov chain taking values in the collection of all continuous monoid homomorphism from S^Λ to T as a pathwise dual process. While this construction works for any pair of monoids the finite sets S and T might be equipped with, the resulting dual process is still quite abstract. The main contribution of this chapter consists of systematically finding cases where the collection of all continuous monoid homomorphism from S^Λ to T can be identified with a set of the form R^Λ_{fin}, where R is a finite set.

2.1 Monoid homomorphisms

We begin the just outlined investigation by studying monoid homomorphisms, with a particular emphasis on product spaces. Recall that if $\mathbf{M} = (M, +)$ and $\mathbf{T} = (T, \otimes)$ are monoids with neutral elements 0 and 1, respectively, then a *monoid homomorphism* from $\mathbf{M}$ to $\mathbf{T}$ is a function $h : M \to T$ such that

(i) $h(a + a') = h(a) \otimes h(a') \qquad (a, a' \in M)$,

(ii) $h(0) = 1$.

We denote the collection of all monoid homomorphisms from $\mathbf{M}$ to $\mathbf{T}$ by $\mathcal{H}(\mathbf{M}, \mathbf{T})$. If $h \in \mathcal{H}(\mathbf{M}, \mathbf{T})$ is a bijection, then it is easy to see that $h^{-1} \in \mathcal{H}(\mathbf{T}, \mathbf{M})$. In this case, h is called a *monoid isomorphism*. The following simple lemma shows that if $\mathbf{T}$ is commutative, then $\mathcal{H}(\mathbf{M}, \mathbf{T})$ naturally has the structure of a commutative monoid. We call $(\mathcal{H}(\mathbf{M}, \mathbf{T}), \otimes)$ the $\mathbf{T}$-*adjoint* of the monoid $\mathbf{M}$.

Lemma 2.1 (Adjoint of a monoid)**.** *Let $\mathbf{M} = (M, +)$ and $\mathbf{T} = (T, \otimes)$ be monoids with neutral elements 0 and 1, respectively, and assume that $\mathbf{T}$ is commutative. Then $(\mathcal{H}(\mathbf{M}, \mathbf{T}), \otimes)$ is a submonoid of $(\mathcal{F}(M, T), \otimes)$, where $\otimes$ is defined as in (2.1).*

Proof. The argument that $(\mathcal{F}(M, T), \otimes)$ is a monoid with neutral element $\underline{1}$, the function that is constantly $1 \in T$, is the same as for $\mathbf{S}^\Lambda$ above.

It is easy to see that $\underline{1} \in \mathcal{H}(\mathbf{M}, \mathbf{T}), \otimes)$, so it remains to show that $f \otimes g \in \mathcal{H}(\mathbf{M}, \mathbf{T})$ for all $f, g \in \mathcal{H}(\mathbf{M}, \mathbf{T})$. Indeed, for each $a, a' \in M$ and $f, g \in \mathcal{H}(\mathbf{M}, \mathbf{T})$,

$$\begin{aligned}(f \otimes g)(a + a') &= f(a + a') \otimes g(a + a') = (f(a) \otimes f(a')) \otimes (g(a) \otimes g(a')) \\ &= f(a) \otimes g(a) \otimes f(a') \otimes g(a') = (f \otimes g)(a) \otimes (f \otimes g)(a'),\end{aligned}$$

where we have used the commutativity of $\mathbf{T}$ in the third step. Since moreover $(f \otimes g)(0) = f(0) \otimes g(0) = 1 \otimes 1 = 1$, this shows that $f \otimes g \in \mathcal{H}(\mathbf{M}, \mathbf{T})$. □

Let $\mathbf{M}$ and $\mathbf{T}$ be monoids and assume that $\mathbf{T}$ is commutative. We claim that there exists a natural monoid homomorphism from $\mathbf{M}$ to the $\mathbf{T}$-adjoint of the $\mathbf{T}$-adjoint of $\mathbf{M}$. To see this, for each $a \in M$, we define $L_a : \mathcal{H}(\mathbf{M}, \mathbf{T}) \to T$ by

$$L_a(h) := h(a) \qquad (a \in M,\ h \in \mathcal{H}(\mathbf{M}, \mathbf{T})). \tag{2.2}$$

With this definition, the following lemma holds.

Lemma 2.2 (Adjoint of the adjoint)**.** *Let $\mathbf{M}$ and $\mathbf{T}$ be monoids and assume that $\mathbf{T}$ is commutative. Let $\mathbf{M}''$ denote the $\mathbf{T}$-adjoint of the $\mathbf{T}$-adjoint of $\mathbf{M}$. Then the map $a \mapsto L_a$ is a monoid homomorphism from $\mathbf{M}$ to $\mathbf{M}''$.*

Proof. As above assume that $\mathbf{M}$ and $\mathbf{T}$ can be written as $\mathbf{M} = (M, +)$ and $\mathbf{T} = (T, \otimes)$, and that their neutral elements are denoted by $0 \in M$ and $1 \in T$. Let $\underline{1}$ again denote the function that is constantly $1 \in T$. Since, for each $a \in M$,

$$\begin{aligned}&L_a(f \otimes g) = (f \otimes g)(a) = f(a) \otimes g(a) = L_a(f) \otimes L_a(g) \qquad (f, g \in \mathcal{H}(\mathbf{M}, \mathbf{T})), \\ &L_a(\underline{1}) = \underline{1}(a) = 1,\end{aligned}$$

we see that L_a is a monoid homomorphism from $(\mathcal{H}(\mathbf{M},\mathbf{T}),\otimes)$ to $\mathbf{T}$, i.e., L_a is an element of $\mathbf{M}''$. Since, for each $f \in \mathcal{H}(\mathbf{M},\mathbf{T})$,

$$\begin{aligned}&L_{a+a'}(f) = f(a+a') = f(a)\otimes f(a') = L_a(f)\otimes L_{a'}(f) \qquad (a,a'\in M),\\&L_0(f) = f(0) = 1,\end{aligned}$$

it follows that $a \mapsto L_a$ is a monoid homomorphism from $\mathbf{M}$ to $\mathbf{M}''$. □

Due to the idea to equip the global state space S^Λ with a monoid structure, we are especially interested in product monoids. We have already seen that if $\mathbf{M} = (M,+)$ is a monoid with neutral element 0 and Γ is a set, then also M^Γ has the structure of a monoid with operator $+$, defined as in (2.1), and neutral element $\underline{0}$, defined analogously to (1.1). Analogously, if $\mathbf{M}_1 = (M_1,+_1),\dots,\mathbf{M}_n = (M_n,+_n)$ are monoids, we can equip $M_1\times\cdots\times M_n$ with the structure of a monoid that we are going to denote by $\mathbf{M}_1\times\cdots\times\mathbf{M}_n$.

We claim that if $\mathbf{M}_1 = (M_1,+_1),\dots,\mathbf{M}_n = (M_n,+_n)$ and $\mathbf{T} = (T,\otimes)$ are monoids and $\mathbf{T}$ is commutative, then there exists a natural monoid isomorphism between $\mathcal{H}(\mathbf{M}_1,\mathbf{T})\times\cdots\times\mathcal{H}(\mathbf{M}_n,\mathbf{T})$ and $\mathcal{H}(\mathbf{M}_1\times\cdots\times\mathbf{M}_n,\mathbf{T})$. To see this, for each $\mathbf{f} = (\mathbf{f}_1,\dots,\mathbf{f}_n) \in \mathcal{H}(\mathbf{M}_1,\mathbf{T})\times\cdots\times\mathcal{H}(\mathbf{M}_n,\mathbf{T})$, we define a function $F_\mathbf{f} : M_1\times\cdots\times M_n \to T$ by

$$F_\mathbf{f}(x) := \bigotimes_{i=1}^n \mathbf{f}_i(x(i)) \qquad (x = (x(i))_{i\in\{1,\dots,n\}} \in M_1\times\cdots\times M_n). \tag{2.3}$$

Due to the commutativity of $\mathbf{T}$, the function $F_\mathbf{f}$ is invariant under permutations of the order of $M_1,\dots,M_n$.

Lemma 2.3 (Adjoints of product spaces)**.** *Let $\mathbf{M}_1,\dots,\mathbf{M}_n$ and $\mathbf{T}$ be monoids and assume that $\mathbf{T}$ is commutative. Then the map*

$$\mathcal{H}(\mathbf{M}_1,\mathbf{T})\times\cdots\times\mathcal{H}(\mathbf{M}_n,\mathbf{T}) \ni \mathbf{f} \longmapsto F_\mathbf{f} \in \mathcal{H}(\mathbf{M}_1\times\cdots\times\mathbf{M}_n,\mathbf{T})$$

is a monoid isomorphism from the product of the $\mathbf{T}$-adjoints of $\mathbf{M}_1,\dots,\mathbf{M}_n$ to the $\mathbf{T}$-adjoint of $\mathbf{M}_1\times\cdots\times\mathbf{M}_n$.

Proof. Assume that $\mathbf{M}_1,\dots,\mathbf{M}_n$ and $\mathbf{T}$ are given as $\mathbf{M}_1 = (M_1,+_1),\dots,\mathbf{M}_n = (M_n,+_n)$ and $\mathbf{T} = (T,\otimes)$, and let their neutral elements be given as $0_1,\dots,0_n$ and 1, respectively. Moreover, let $\mathbf{0} := (0_1,\dots,0_n)$ denote the neutral element of $\mathbf{M}_1\times\cdots\times\mathbf{M}_n$ and let $+$ denote the binary operator of $\mathbf{M}_1\times\cdots\times\mathbf{M}_n$, i.e., for $x,x' \in M_1\times\cdots\times M_n$ one has

$$(x+x')(i) := x(i) +_i x'(i) \qquad (i\in\{1,\dots,n\}).$$

We first check that $F_\mathbf{f} \in \mathcal{H}(\mathbf{M}_1\times\cdots\times\mathbf{M}_n,\mathbf{T})$ for all $\mathbf{f} \in \mathcal{H}(\mathbf{M}_1,\mathbf{T})\times\cdots\times\mathcal{H}(\mathbf{M}_n,\mathbf{T})$. Indeed, using the commutativity of $\mathbf{T}$, we see that

$$\begin{aligned}F_\mathbf{f}(x+x') &= \bigotimes_{i=1}^n \mathbf{f}_i((x+x')(i)) = \bigotimes_{i=1}^n \mathbf{f}_i(x(i)+_i x'(i)) = \bigotimes_{i=1}^n \Big(\mathbf{f}_i(x(i))\otimes\mathbf{f}_i(x'(i))\Big)\\&= \Big(\bigotimes_{i=1}^n \mathbf{f}_i(x(i))\Big)\otimes\Big(\bigotimes_{i=1}^n \mathbf{f}_i(x'(i))\Big) = F_\mathbf{f}(x)\otimes F_\mathbf{f}(x')\\&\qquad\qquad\qquad\qquad\qquad\qquad (x,x'\in M_1\times\cdots\times M_n)\end{aligned}$$

and

$$F_{\mathbf{f}}(\mathbf{0}) = \bigotimes_{i=1}^{n} \mathbf{f}_i(\mathbf{0}(i)) = \bigotimes_{i=1}^{n} \mathbf{f}_i(0_i) = \bigotimes_{i=1}^{n} 1 = 1.$$

Next we check that $\mathbf{f} \mapsto F_{\mathbf{f}}$ is a bijection. We first show that it is injective. For each $i \in \{1, \ldots, n\}$ and $a \in M_i$, let us define $a^i \in M_1 \times \cdots \times M_n$ by

$$a^i(j) := \begin{cases} a & \text{if } j = i, \\ 0_j & \text{else,} \end{cases} \qquad (j \in \{1, \ldots, n\}). \tag{2.4}$$

Then $\mathbf{f} \neq \mathbf{g}$ implies that $\mathbf{f}_i \neq \mathbf{g}_i$ for some $i \in \{1, \ldots, n\}$ and hence there exists an $a \in M_i$ such that $\mathbf{f}_i(a) \neq \mathbf{g}_i(a)$. Now $F_{\mathbf{f}}(a^i) = \mathbf{f}_i(a) \neq \mathbf{g}_i(a) = F_{\mathbf{g}}(a^i)$, which shows that $F_{\mathbf{f}} \neq F_{\mathbf{g}}$. It remains to show that $\mathbf{f} \mapsto F_{\mathbf{f}}$ is surjective. For each $F \in \mathcal{H}(\mathbf{M}_1 \times \cdots \times \mathbf{M}_n, \mathbf{T})$, we define $\mathbf{f} \in \mathcal{H}(\mathbf{M}_1, \mathbf{T}) \times \cdots \times \mathcal{H}(\mathbf{M}_n, \mathbf{T})$ by $\mathbf{f}_i(a) := F(a^i)$ $(i \in \{1, \ldots, n\},\ a \in M_i)$. Then, for each $x \in M_1 \times \cdots \times M_n$, we have

$$F(x) = F\Big(\sum_{i=1}^{n} (x(i))^i\Big) = \bigotimes_{i=1}^{n} F\Big((x(i))^i\Big) = \bigotimes_{i=1}^{n} \mathbf{f}_i(x(i)) = F_{\mathbf{f}}(x),$$

which shows that $F = F_{\mathbf{f}}$.

To complete the proof, we must show that $\mathbf{f} \mapsto F_{\mathbf{f}}$ is a monoid homomorphism. For each $i \in \{1, \ldots, n\}$, we denote the neutral element of $(\mathcal{H}(\mathbf{M}_i, \mathbf{T}), \otimes)$ by o_i and the neutral element of $(\mathcal{H}(\mathbf{M}_1, \mathbf{T}) \times \cdots \times \mathcal{H}(\mathbf{M}_n, \mathbf{T}), \otimes)$ by $\mathbf{o}$. Then

$$\begin{aligned} F_{\mathbf{f} \otimes \mathbf{g}}(x) &= \bigotimes_{i=1}^{n} (\mathbf{f} \otimes \mathbf{g})_i(x(i)) = \bigotimes_{i=1}^{n} (\mathbf{f}_i \otimes \mathbf{g}_i)(x(i)) = \bigotimes_{i=1}^{n} \Big(\mathbf{f}_i(x(i)) \otimes \mathbf{g}_i(x(i))\Big) \\ &= \Big(\bigotimes_{i=1}^{n} \mathbf{f}_i(x(i))\Big) \otimes \Big(\bigotimes_{i=1}^{n} \mathbf{f}_i(x(i))\Big) = F_{\mathbf{f}}(x) \otimes F_{\mathbf{g}}(x) \qquad (x \in M_1 \times \cdots \times M_n) \end{aligned}$$

and

$$F_{\mathbf{o}}(x) = \bigotimes_{i=1}^{n} \mathbf{o}_i(x(i)) = \bigotimes_{i=1}^{n} o_i(x(i)) = \bigotimes_{i=1}^{n} 1 = 1 \qquad (x \in M_1 \times \cdots \times M_n),$$

i.e., $F_{\mathbf{o}}$ is constantly 1 and thus the neutral element of the $\mathbf{T}$-adjoint of $\mathbf{M}_1 \times \cdots \times \mathbf{M}_n$. □

As an application of Lemma 2.3, we obtain a characterization of $\mathcal{H}(\mathbf{M}^\Gamma, \mathbf{M}^\Gamma)$ for a finite set Γ, or, somewhat more generally, of the set of homomorphisms between two product monoids $\mathbf{M}^\Gamma$ and $\mathbf{N}^{\Gamma'}$.[1]

Lemma 2.4 (Homomorphisms between product spaces)**.** *Let $\mathbf{M} = (M, +)$, $\mathbf{N} = (N, \oplus)$ be monoids and assume that that $\mathbf{N}$ is commutative. Let Γ, Γ' be finite sets, and let $m : M^\Gamma \to N^{\Gamma'}$ be a map, with $m(x) = (m_i(x))_{i \in \Gamma'}$. Then one has $m \in \mathcal{H}(\mathbf{M}^\Gamma, \mathbf{N}^{\Gamma'})$ if and only if there exists a matrix $(M_{ij})_{i \in \Gamma', j \in \Gamma}$ with $M_{ij} \in \mathcal{H}(\mathbf{M}, \mathbf{N})$ for each $i \in \Gamma'$ and $j \in \Gamma$, such that*

$$m_i(x) = \bigoplus_{j \in \Gamma} M_{ij}(x(j)) \qquad (x \in M^\Gamma,\ i \in \Gamma'). \tag{2.5}$$

[1]Compared to [Latz and Swart, 2023a] and [Latz and Swart, 2023b] we have changed the notation slightly so that (2.5) resembles the formula for the usual matrix-vector multiplication. The same change was made in Lemma 2.10 (leading also to a change in (2.13)).

Proof. This follows from applying Lemma 2.3 to the maps m_i for each $i \in \Gamma'$. $\square$

Of course, the grid Λ is infinite, so the previous two lemmas cannot be applied directly to $\mathbf{S}^\Lambda$, constructed at the beginning of the chapter. Nevertheless, the two lemmas above will be important intermediate results used in the upcoming subchapters.

2.2 Duality of monoids

As explained above Chapter 2.1, the aim is to identify pairs of monoids $\mathbf{S}$ and $\mathbf{T}$ such that $\mathcal{H}(\mathbf{S}^\Lambda, \mathbf{T}) \cap \mathcal{C}(\mathbf{S}^\Lambda, \mathbf{T})$ can be identified with a less abstract space, where, as in Chapter 1.4, $\mathcal{C}(\mathbf{S}^\Lambda, \mathbf{T})$ denotes the collection of continuous function from S^Λ to T. This aim is formalized by defining a notion of duality of monoids. As we first want to equip S with the structure of a monoid, which then induces a monoid structure also on S^Λ as explained above, it is natural to first consider the local state space S instead of the global state space S^Λ.

Let $\mathbf{M} = (M, +)$, $\mathbf{N} = (N, \oplus)$, and $\mathbf{T} = (T, \otimes)$ be monoids and let $\psi : M \times N \to T$ be a function. We say that $\mathbf{M}$ is $\mathbf{T}$-*dual* to $\mathbf{N}$ *with respect to* ψ if ψ has the following properties:

(i) $\psi(a, b) = \psi(a', b)$ for all $b \in N$ implies $a = a'$ $(a, a' \in M)$,

(ii) $\mathcal{H}(\mathbf{M}, \mathbf{T}) = \big\{\psi(\,\cdot\,, b) : b \in N\big\}$,

(iii) $\psi(a, b) = \psi(a, b')$ for all $a \in M$ implies $b = b'$ $(b, b' \in N)$,

(iv) $\mathcal{H}(\mathbf{N}, \mathbf{T}) = \big\{\psi(a, \,\cdot\,) : a \in M\big\}$.

In words, the four properties above say that fixing for ψ either an $a \in M$ or a $b \in N$ yields a monoid homomorphism, and all monoid homomorphisms from $\mathbf{M}$ or $\mathbf{N}$ to $\mathbf{T}$ are of this form. Moreover, ψ separates the points in both coordinates. The definition of duality of monoids implies the following result.

Proposition 2.5 (Maps with a dual II). *Let* $\mathbf{M} = (M, +)$, $\mathbf{N} = (N, \oplus)$, *and* $\mathbf{T} = (T, \otimes)$ *be monoids such that* $\mathbf{M}$ *is* $\mathbf{T}$*-dual to* $\mathbf{N}$ *with respect to* $\psi : M \times N \to T$. *Then a map* $m : M \to M$ *has a dual map* $\hat{m} : N \to N$ *with respect to* ψ *if and only if* $m \in \mathcal{H}(\mathbf{M}, \mathbf{M})$. *The dual map* $\hat{m}$, *if it exists, is unique and satisfies* $\hat{m} \in \mathcal{H}(\mathbf{N}, \mathbf{N})$.

Proof. Recall from (1.28) when a map m preserves a subspace and let $\mathcal{H}_\psi$ be defined analogously to (1.29). Due to property (ii) of the definition of duality one has $\mathcal{H}_\psi = \mathcal{H}(\mathbf{M}, \mathbf{T})$. Each $m \in \mathcal{H}(\mathbf{M}, \mathbf{M})$ clearly preserves $\mathcal{H}(\mathbf{M}, \mathbf{T})$. Hence, as in the proof of Lemma 1.7, we conclude that each such m has a unique dual map $\hat{m} : N \to N$ with respect to the duality function ψ. Note that (the analogue of) (1.30) is satisfied by property (iii) of the definition of duality.

Assume, conversely, that $m : M \to M$ has a dual map $\hat{m} : N \to N$ with respect to ψ. Then $\psi(m(a+a'), b) = \psi(a+a', \hat{m}(b)) = \psi(a, \hat{m}(b)) \otimes \psi(a', \hat{m}(b)) = \psi(m(a), b) \otimes \psi(m(a'), b) = \psi(m(a) + m(a'), b)$ for all $a, a' \in M$ and $b \in N$, so using property (i) of the definition of duality we see that $m(a+a') = m(a)+m(a')$ for all $a, a' \in M$. Since moreover $\psi(m(0), b) = \psi(0, \hat{m}(b)) = 1$ for all $b \in N$, this proves that also $m(0) = 0$. Hence $m \in \mathcal{H}(\mathbf{M}, \mathbf{M})$.

This completes the proof that a map $m : M \to M$ has a dual map $\hat{m} : N \to N$ with respect to ψ if and only if $m \in \mathcal{H}(\mathbf{M}, \mathbf{M})$, and moreover shows that such a dual map is unique. Since $\hat{m}$ has a dual with respect to the duality function $\psi^\dagger(b, a) := \psi(a, b)$ $(b \in N,\ a \in M)$, namely, the map $m : S \to S$, by what we have already proved, we must have $\hat{m} \in \mathcal{H}(\mathbf{N}, \mathbf{N})$. □

Together with (the proof of) Lemma 1.7 we conclude that $m : M \to M$ preserves $\mathcal{H}(\mathbf{M}, \mathbf{T})$ if and only if $m \in \mathcal{H}(\mathbf{M}, \mathbf{M})$. Note that if $\mathbf{M}$ is not commutative, then properties (i) and (ii) of the definition of duality imply that $\mathbf{T}$ cannot be commutative either. Analogously, if $\mathbf{N}$ is not commutative, then properties (iii) and (iv) of the definition of duality imply that $\mathbf{T}$ cannot be commutative. The commutativity of $\mathbf{T}$, however, will be crucial in the statements to follow. Hence, we will only consider commutative monoids from now on. The following proposition says that any duality of commutative monoids can be "lifted" to a duality between product spaces.

Proposition 2.6 (Duality of product monoids)**.** *Let* $\mathbf{M}_1 = (M_1, +_1), \dots, \mathbf{M}_n = (M_n, +_n)$*,* $\mathbf{N}_1 = (N_1, \oplus_1), \dots, \mathbf{N}_n = (N_n, \oplus_n)$*, and* $\mathbf{T} = (T, \otimes)$ *be commutative monoids such that* $\mathbf{M}_i$ *is* $\mathbf{T}$*-dual to* $\mathbf{N}_i$ *with respect to* ψ_i *(*$i \in \{1, \dots, n\}$*). Then* $\mathbf{M}_1 \times \cdots \times \mathbf{M}_n$ *is* $\mathbf{T}$*-dual to* $\mathbf{N}_1 \times \cdots \times \mathbf{N}_n$ *with respect to* $\boldsymbol{\psi} : (M_1 \times \cdots \times M_n) \times (N_1 \times \cdots \times N_n) \to T$*, defined as*

$$\boldsymbol{\psi}(x, y) := \bigotimes_{i=1}^{n} \psi_i(x(i), y(i)) \qquad (x \in M_1 \times \cdots \times M_n,\ y \in N_1 \times \cdots \times N_n). \tag{2.6}$$

In particular, if Γ *is a finite set and* $\mathbf{M}$ *is* $\mathbf{T}$*-dual to* $\mathbf{N}$*, then* $\mathbf{M}^\Gamma$ *is* $\mathbf{T}$*-dual to* $\mathbf{N}^\Gamma$ *with respect to* $\boldsymbol{\psi}(x, y) : M^\Gamma \times N^\Gamma \to T$*, defined as*

$$\boldsymbol{\psi}(x, y) := \bigotimes_{i \in \Gamma} \psi(x(i), y(i)) \qquad (x \in M^\Gamma,\ y \in N^\Gamma). \tag{2.7}$$

Proof. We need to check that $\boldsymbol{\psi}$ from (2.6) satisfies properties (i)–(iv) of the definition of duality. By the symmetry between the M_i's and N_i's, it suffices to check properties (i) and (ii). For each $i \in \{1, \dots, n\}$ and $b \in N_i$, let $b^i \in N_1 \times \cdots \times N_n$ be defined as in (2.4). Then, assuming that $\boldsymbol{\psi}(x, y) = \boldsymbol{\psi}(x', y)$ $(x, x' \in M_1 \times \cdots \times M_n)$ for all $y \in N_1 \times \cdots \times N_n$, in particular implies for $i \in \{1, \dots, n\}$ that

$$\psi_i(x(i), b) = \boldsymbol{\psi}(x, b^i) = \boldsymbol{\psi}(x', b^i) = \psi_i(x'(i), b)$$

for all $b \in N_i$. Hence, property (i) of the duality of $\mathbf{M}_i$ and $\mathbf{N}_i$ implies that $x(i) = x'(i)$ for all $i \in \{1, \dots, n\}$ and thus $x = x'$.

To prove also property (ii), we must show that

$$\mathcal{H}(\mathbf{M}_1 \times \cdots \times \mathbf{M}_n, \mathbf{T}) = \Big\{\boldsymbol{\psi}(\,\cdot\,, y) : y \in N_1 \times \cdots \times N_n\Big\}. \tag{2.8}$$

By Lemma 2.3, each $F \in \mathcal{H}(\mathbf{M}_1 \times \cdots \times \mathbf{M}_n, \mathbf{T})$ is of the form $F(x) = \bigotimes_{i=1}^n \mathbf{f}_i(x(i))$ for some $\mathbf{f}_i \in \mathcal{H}(\mathbf{M}_i, \mathbf{T})$ $(i \in \{1, \dots, n\})$. Since, for all $i \in \{1, \dots, n\}$, the monoid $\mathbf{M}_i$ is $\mathbf{T}$-dual to $\mathbf{N}_i$ with respect to ψ_i, property (ii) of the definition of duality implies that there exist $b_i \in N_i$ $(i \in \{1, \dots, n\})$ such that $\mathbf{f}_i = \psi_i(\,\cdot\,, b_i)$. Defining $y \in N_1 \times \cdots \times N_n$ via $y(i) := b_i$ $(i \in \{1, \dots, n\})$ then yields $F = \boldsymbol{\psi}(\,\cdot\,, y)$, proving the inclusion $\subset$ in (2.8).

The inclusion $\supset$ in (2.8) follows from property (ii) of the dualities of $\mathbf{M}_i$ and $\mathbf{N}_i$ $(i \in \{1, \dots, n\})$ and the definition of $\boldsymbol{\psi}$. □

Let $\mathbf{M} = (M, +)$ be a commutative monoid and let Γ be a finite set. Then, by Proposition 2.5, a map $m : M^\Gamma \to M^\Gamma$ has a unique dual map with respect to the function $\boldsymbol{\psi}$ defined in (2.7) if and only if $m \in \mathcal{H}(\mathbf{M}^\Gamma, \mathbf{M}^\Gamma)$. By Lemma 2.4, maps $m \in \mathcal{H}(\mathbf{M}^\Gamma, \mathbf{M}^\Gamma)$ are uniquely characterized by a matrix with values in $\mathcal{H}(\mathbf{M}, \mathbf{M})$. The next step is to prove a similar statement when the finite set Γ is replaced by a countable one, so that this statement can then be applied to S^Λ.

2.3 Duality of topological monoids

Recall that S^Λ is equipped with the product topology. Hence, if we equip S with a binary operator $+$ so that $\mathbf{S} = (S, +)$ becomes a monoid, then $\mathbf{S}^\Lambda$ becomes a monoid that additionally carries a topology. In general, we say that a monoid $\mathbf{M} = (M, +)$ is a *topological monoid* if it is equipped with a topology so that the map $M \times M \ni (x, x') \mapsto x + x' \in M$ is continuous, where $M \times M$ is equipped with the product topology. Before focusing on $\mathbf{S}^\Lambda$, we introduce a notion of duality of topological monoids.

Let $\mathbf{M} = (M, +)$, $\mathbf{N} = (N, \oplus)$ and $\mathbf{T} = (T, \otimes)$ be topological monoids. We say that $\mathbf{M}$ is $\mathbf{T}$-*dual* to $\mathbf{N}$ with respect to $\boldsymbol{\psi} : M \times N \to T$ if $\boldsymbol{\psi}$ has the following properties:

(i) $\boldsymbol{\psi}(x, y) = \boldsymbol{\psi}(x', y)$ for all $y \in N$ implies $x = x'$ $(x, x' \in M)$,

(ii) $\mathcal{H}(\mathbf{M}, \mathbf{T}) \cap \mathcal{C}(\mathbf{M}, \mathbf{T}) = \big\{\boldsymbol{\psi}(\,\cdot\,, y) : y \in N\big\}$,

(iii) $\boldsymbol{\psi}(x, y) = \boldsymbol{\psi}(x, y')$ for all $x \in M$ implies $y = y'$ $(y, y' \in N)$,

(iv) $\mathcal{H}(\mathbf{N}, \mathbf{T}) \cap \mathcal{C}(\mathbf{N}, \mathbf{T}) = \big\{\boldsymbol{\psi}(x, \,\cdot\,) : x \in M\big\}$,

where, as in Chapter 1.4, $\mathcal{C}(\mathbf{M}, \mathbf{T})$ and $\mathcal{C}(\mathbf{N}, \mathbf{T})$ denote the collection of continuous function from M to T and from N to T, respectively. Thus, duality of topological monoids is defined as duality of "usual" monoids, with the sole distinction being that monoid homomorphisms are replaced by continuous monoid homomorphisms.

Due to our convention to equip all finite and countable sets with the discrete topology, if $\mathbf{M}$, $\mathbf{N}$ and $\mathbf{T}$ are finite monoids, then $\mathcal{C}(\mathbf{M}, \mathbf{T}) = \mathcal{F}(\mathbf{M}, \mathbf{T})$ and $\mathcal{C}(\mathbf{N}, \mathbf{T}) = \mathcal{F}(\mathbf{N}, \mathbf{T})$ so that $\mathbf{M}$ is $\mathbf{T}$-dual to $\mathbf{N}$ in the sense of topological monoids if and only $\mathbf{M}$ is $\mathbf{T}$-dual to $\mathbf{N}$ in the sense of "usual" monoids, i.e., in the sense of Chapter 2.2. The following proposition is the analogue of Proposition 2.5. Recall the notion of a map preserving a subspace from (1.28).

Proposition 2.7 (Maps with a dual III)**.** *Let* $\mathbf{M} = (M, +)$, $\mathbf{N} = (N, \oplus)$ *and* $\mathbf{T} = (T, \otimes)$ *be commutative topological monoids such that* $\mathbf{M}$ *is* $\mathbf{T}$*-dual to* $\mathbf{N}$ *with respect to* $\boldsymbol{\psi}$*. Then a map* $m : M \to M$ *has a dual map* $\hat{m} : N \to N$ *with respect to* $\boldsymbol{\psi}$ *if and only if* m *preserves* $\mathcal{H}(\mathbf{M}, \mathbf{T}) \cap \mathcal{C}(\mathbf{M}, \mathbf{T})$*. The dual map* $\hat{m}$*, if it exists, is unique and preserves* $\mathcal{H}(\mathbf{N}, \mathbf{T}) \cap \mathcal{C}(\mathbf{N}, \mathbf{T})$*.*

Proof. As in the proof of Proposition 2.5, property (ii) of the definition of duality of topological monoids implies that $\mathcal{H}_{\boldsymbol{\psi}} = \mathcal{H}(\mathbf{M}, \mathbf{T}) \cap \mathcal{C}(\mathbf{M}, \mathbf{T})$ with $\mathcal{H}_{\boldsymbol{\psi}}$ being defined as in (1.29). Hence, the first assertion follows as in the proof of Lemma 1.7.

Again, (the analogue of) (1.30) is satisfied by property (iii) of the definition of duality of (topological) monoids.

For the second assertion, if $\hat{m}$ exists, then it has m as a dual map with respect to $\boldsymbol{\psi}^\dagger : N \times M \to T$ defined as $\boldsymbol{\psi}^\dagger(y,x) := \boldsymbol{\psi}(x,y)$ $(y \in N,\ x \in M)$, and the previously proved statement implies that $\hat{m}$ has to preserve $\mathcal{H}(\mathbf{N},\mathbf{T}) \cap \mathcal{C}(\mathbf{N},\mathbf{T})$. $\square$

Clearly, any $m \in \mathcal{H}(\mathbf{M},\mathbf{M}) \cap \mathcal{C}(\mathbf{M},\mathbf{M})$ preserves $\mathcal{H}(\mathbf{M},\mathbf{T}) \cap \mathcal{C}(\mathbf{M},\mathbf{T})$. Conversely, if the assumptions on $\mathbf{M}$, $\mathbf{N}$ and $\mathbf{T}$ from Proposition 2.7 are satisfied and $m : M \to M$ preserves $\mathcal{H}(\mathbf{M},\mathbf{T})$, then the proof of Proposition 2.5 shows that m has to be a monoid homomorphism. However, while property (ii) of the definition of duality implies that $\boldsymbol{\psi}(m(\,\cdot\,),y)$ is continuous for all $y \in N$, we do not know if m itself always has to be continuous. For the duality we are about to introduce, Proposition 2.13 below implies that there exist no discontinuous maps that preserve $\mathcal{H}(\mathbf{M},\mathbf{T}) \cap \mathcal{C}(\mathbf{M},\mathbf{T})$.

We are now ready to study $\mathbf{S}^\Lambda$, the state space of the interacting particle system $X = (X_t)_{t\geq 0}$ if its local state space S is equipped with the structure of a monoid. Clearly, $\mathbf{S}^\Lambda$ equipped with the product topology, is a topological monoid. Recall from the beginning of this chapter that the aim is to construct a pathwise duality for an interacting particle system that has a generator G represented as in (1.8) with $\mathcal{G}$ consisting only of local monoid homomorphisms from $\mathbf{S}^\Lambda$ to itself. Moreover, recall that, by definition, any monoid homomorphism $m \in \mathcal{H}(\mathbf{S}^\Lambda,\mathbf{S}^\Lambda)$ satisfies $m(\underline{0}) = \underline{0}$, where $0 \in S$ denotes the neutral element of $\mathbf{S}$. Hence, we are in the setup of Chapter 1.2 and can repeat the notions of $\mathrm{supp}(x)$, the support of $x \in S^\Lambda$, S^Λ_{fin}, and $\delta_i^a \in S^\Lambda_{\mathrm{fin}}$ $(i \in \Lambda,\ a \in S)$ based on 0. In particular, $\mathbf{S}^\Lambda_{\mathrm{fin}} = (S^\Lambda_{\mathrm{fin}},+)$ becomes a countable submonoid of $\mathbf{S}^\Lambda$. We equip it according to our conventions with the discrete topology to make it a topological monoid as well.

Let $\mathbf{S} = (S,+)$, $\mathbf{R} = (R,\oplus)$ and $\mathbf{T} = (T,\otimes)$ be finite commutative monoids. We denote the neutral elements of $\mathbf{S}$, $\mathbf{R}$ and $\mathbf{T}$ by 0, $\mathbf{0}$ and 1, respectively. If $\mathbf{S}$ is $\mathbf{T}$-dual to $\mathbf{R}$ with respect to $\psi : S \times R \to T$, then we define $\boldsymbol{\psi} : S^\Lambda \times R^\Lambda_{\mathrm{fin}} \to T$ by

$$\boldsymbol{\psi}(x,y) := \bigotimes_{i\in\Lambda} \psi(x(i),y(i)) \qquad (x \in S^\Lambda,\ y \in R^\Lambda_{\mathrm{fin}}). \tag{2.9}$$

Note that $\boldsymbol{\psi}$ is well-defined as for all but finitely many $i \in \Lambda$ one has $y(i) = \mathbf{0}$, and $\psi(\,\cdot\,,\mathbf{0}) = o$ due to property (iv) of the definition of duality, where $o : S \to T$ is the function that is constantly 1. Using Lemma 1.1 we can prove the following analogue of Proposition 2.6.

Proposition 2.8 (Duality of infinite product monoids)**.** *Let* $\mathbf{S}$, $\mathbf{R}$ *and* $\mathbf{T}$ *be finite commutative monoids. If* $\mathbf{S}$ *is* $\mathbf{T}$*-dual to* $\mathbf{R}$ *with respect to* ψ*, then the topological monoid* $\mathbf{S}^\Lambda$ *is* $\mathbf{T}$*-dual to the topological monoid* $\mathbf{R}^\Lambda_{\mathrm{fin}}$ *with respect to* $\boldsymbol{\psi}$ *from (2.9).*

Proof. The properties (i) and (iii) of the definition of duality of topological monoids follow directly from the corresponding properties of the duality of $\mathbf{S}$ and $\mathbf{R}$ exactly in the same way as in the proof of Proposition 2.6.

The fact that $\boldsymbol{\psi}(\,\cdot\,,y)$ and $\boldsymbol{\psi}(x,\,\cdot\,)$ are monoid homomorphisms for all $y \in R^\Lambda_{\mathrm{fin}}$ and for all $x \in S^\Lambda$, respectively, are implied by properties (ii) and (iv) of the duality of $\mathbf{S}$ and $\mathbf{R}$ and the definition of $\boldsymbol{\psi}$. Since $\mathbf{R}^\Lambda_{\mathrm{fin}}$ is countable, this implies that $\boldsymbol{\psi}(x,\cdot\,) \in \mathcal{H}(\mathbf{R}_{\text{fin}}^{\Lambda},\mathbf{T}) \cap \mathcal{C}(\mathbf{R}_{\text{fin}}^{\Lambda},\mathbf{T})$. Recall the definition of $\mathcal{R}(f)$ from (1.3). For $y \in R_{\text{fin}}^{\Lambda}$ we have that $\mathcal{R}(\boldsymbol{\psi}(\,\cdot\,,y)) = \{j \in \Lambda : y(j) \neq \mathbf{0}\}$, so $\boldsymbol{\psi}(\,\cdot\,,y)$ satisfies the conditions of Lemma 1.1 and hence also $\boldsymbol{\psi}(\,\cdot\,,y) \in \mathcal{H}(\mathbf{S}^{\Lambda},\mathbf{T}) \cap \mathcal{C}(\mathbf{S}^{\Lambda},\mathbf{T})$. Thus, we have proved the implication $\supset$ in properties (ii) and (iv) of the definition of duality of topological monoids.

To prove the implication $\subset$ in property (iv), assume that $g \in \mathcal{H}(\mathbf{R}_{\text{fin}}^{\Lambda},\mathbf{T}) \cap \mathcal{C}(\mathbf{R}_{\text{fin}}^{\Lambda},\mathbf{T})$. Then, with $\delta_i^b \in R_{\text{fin}}^{\Lambda}$ $(i \in \Lambda,\ b \in R)$ being defined analogously to (1.16), we define $g_i : R \to T$ as $g_i(b) := g(\delta_i^b)$ $(i \in \Lambda,\ b \in R)$. The fact that $g \in \mathcal{H}(\mathbf{R}_{\text{fin}}^{\Lambda},\mathbf{T})$ implies that $g_i \in \mathcal{H}(\mathbf{R},\mathbf{T})$, and property (iv) of the duality of $\mathbf{S}$ and $\mathbf{R}$ implies that there exists an $x_i \in S$ such that $g_i = \psi(x_i,\cdot\,)$. Defining $x \in S^{\Lambda}$ by $x(i) := x_i$, one has for $y \in R_{\text{fin}}^{\Lambda}$ that

$$\begin{aligned} g(y) = g\Big(\bigoplus\nolimits_{i\in\text{supp}(y)} \delta_i^{y(i)}\Big) &= \bigotimes_{i\in\text{supp}(y)} g\big(\delta_i^{y(i)}\big) = \bigotimes_{i\in\text{supp}(y)} g_i(y(i)) \\ &= \bigotimes_{i\in\text{supp}(y)} \psi(x_i, y(i)) = \boldsymbol{\psi}(x,y), \end{aligned}$$

which finishes the proof of property (iv) of the definition of duality.

Lastly, we prove the implication $\subset$ in property (ii) of the definition of duality of topological monoids. Assume that $f \in \mathcal{H}(\mathbf{S}^{\Lambda},\mathbf{T}) \cap \mathcal{C}(\mathbf{S}^{\Lambda},\mathbf{T})$. Then, as written below Lemma 1.1, there exists a finite set $\Delta \subset \Lambda$ such that f only depends on the coordinates in Δ. We define for general $\Gamma \subset \Lambda$ the element $x\lfloor_{\Gamma} \in S^{\Lambda}$ as

$$x\lfloor_{\Gamma}(i) := \begin{cases} x(i) & \text{if } i \in \Gamma, \\ 0 & \text{else,} \end{cases} \qquad (i \in \Lambda). \tag{2.10}$$

Then

$$f(x) = f(x\lfloor_{\Delta^c} + x\lfloor_{\Delta}) = f(x\lfloor_{\Delta^c}) \otimes \bigotimes_{i\in\Delta} f\big(\delta_i^{x(i)}\big).$$

But, as f does not depend on Δ^c, we conclude that

$$f(x\lfloor_{\Delta^c}) = f(\underline{0}\lfloor_{\Delta^c}) = f(\underline{0}) = 1.$$

Analogously to above, we can now define $y \in R_{\text{fin}}^{\Lambda}$ by $y(i) := y_i$ for $i \in \Delta$ and $y(i) := 0$ for $i \in \Delta^c$, where $y_i \in R$ satisfies $f(\delta_i^{x(i)}) = \psi(x(i), y_i)$ for all values of $x(i)$. Then $f = \boldsymbol{\psi}(\,\cdot\,,y)$, which finishes the proof of property (ii) of the definition of duality. Thus, the proof is complete. □

With this we have completed the objective outlined above Chapter 2.1: By property (ii) of the definition of duality, if $\mathbf{S}$ is $\mathbf{T}$-dual to $\mathbf{R}$, then one can identify $\mathcal{H}(\mathbf{S}^{\Lambda},\mathbf{T}) \cap \mathcal{C}(\mathbf{S}^{\Lambda},\mathbf{T})$ with R_{fin}^{Λ}. Similar to how we argued for $\boldsymbol{\psi}_{\text{basic}} : S^{\Lambda} \times \mathcal{C}(\mathbf{S}^{\Lambda},\mathbf{T}) \to T$ in Chapter 1.4, we conclude the following.

Theorem 2.9 (Pathwise monoid duality)**.** *Let there exist an associative binary operator $+$ on the local state space S such that $\mathbf{S} = (S,+)$ is a commutative monoid. Assume that $\mathbf{R} = (R,\oplus)$ and $\mathbf{T} = (T,\otimes)$ are further finite commutative monoids so that $\mathbf{S}$ is $\mathbf{T}$-dual to $\mathbf{R}$ with respect to $\psi : S \times R \to T$. Let G and $\widehat{G}$ be the generators from (1.8) and (1.34) defined via $\mathcal{G} \subset \mathcal{H}(\mathbf{S}^{\Lambda},\mathbf{S}^{\Lambda})$, a countable collection of local monoid homomorphisms. Assuming, as usual, that G satisfies*

(1.7), there exists a continuous-time Markov chain $Y = (Y_t)_{t\geq 0}$ with generator $\widehat{G}$, state space R^{Λ}_{fin} and càglàd sample paths such that X, the interacting particle system defined in Chapter 1.1, is pathwise dual to Y with respect to $\boldsymbol{\psi}$, the duality function defined in (2.9).

Proof. As R^{Λ}_{fin} is countable the statement follows readily from Proposition 1.8 and Theorem 1.9. In particular, (1.30) is property (iii) of the definition of duality (of topological monoids). As $\mathcal{H}_{\boldsymbol{\psi}} = \mathcal{H}(\mathbf{S}^{\Lambda}, \mathbf{T}) \cap \mathcal{C}(\mathbf{S}^{\Lambda}, \mathbf{T})$ by property (ii) of the definition of duality, $\mathcal{G} \subset \mathcal{H}(\mathbf{S}^{\Lambda}, \mathbf{S}^{\Lambda}) \cap \mathcal{C}(\mathbf{S}^{\Lambda}, \mathbf{S}^{\Lambda})$ implies that every $m \in \mathcal{G}$ preserves $\mathcal{H}_{\boldsymbol{\psi}}$. Lastly, (1.35) follows from property (ii) of the definition of duality and the fact that we have equipped R^{Λ}_{fin} with the discrete topology. □

Apart from the abstract definition in (1.37), the only information about the dual maps we have obtained until now is their existence and their uniqueness. For the practical use of the above theorem we need to know how to compute the dual maps efficiently. The first step is to better understand local monoid homomorphisms. The following lemma generalizes Lemma 2.4.

Lemma 2.10 (Local monoid homomorphisms)**.** *Let $\mathbf{S} = (S, +)$ be a finite commutative monoid. Let $(M_{ij})_{i,j\in\Lambda}$ be an infinite matrix with values in $\mathcal{H}(\mathbf{S}, \mathbf{S})$ such that the set*

$$\Delta := \Big\{(i,j) \in \Lambda^2 : i \neq j,\ M_{ij} \neq o\Big\} \cup \Big\{(i,i) \in \Lambda^2 : M_{ii} \neq \text{id}\Big\} \tag{2.11}$$

is finite, where $o \in \mathcal{H}(\mathbf{S}, \mathbf{S})$ denotes the function constantly equal to the neutral element of $\mathbf{S}$ and $\text{id} \in \mathcal{H}(\mathbf{S}, \mathbf{S})$ denotes the identity. Then setting

$$m[i](x) := \sum_{j\in\Lambda} M_{ij}(x(j)) \qquad (i \in \Lambda,\ x \in S^{\Lambda}) \tag{2.12}$$

defines a local map $m \in \mathcal{H}(\mathbf{S}^{\Lambda}, \mathbf{S}^{\Lambda})$. Conversely, each local map $m \in \mathcal{H}(\mathbf{S}^{\Lambda}, \mathbf{S}^{\Lambda})$ is of this form.

Proof. First assume that m is of the form (2.12). Then m is well-defined as Δ from (2.11) is finite. Since $(M_{ij})_{i,j\in\Lambda}$ takes values in $\mathcal{H}(\mathbf{S}, \mathbf{S})$, it follows readily that $m[i] \in \mathcal{H}(\mathbf{S}^{\Lambda}, \mathbf{S})$ for all $i \in \Lambda$, thus $m \in \mathcal{H}(\mathbf{S}^{\Lambda}, \mathbf{S}^{\Lambda})$. Let $i \in \Lambda$. One sees that

$$\mathcal{R}(m[i]) = \begin{cases} \{j \in \Lambda \setminus \{i\} : (i,j) \in \Delta\} \cup \{i\} & \text{if } M_{ii} \neq o, \\ \{j \in \Lambda \setminus \{i\} : (i,j) \in \Delta\} & \text{if } M_{ii} = o. \end{cases}$$

In both cases $\mathcal{R}(m[i])$ satisfies the conditions of Lemma 1.1. Additionally,

$$\mathcal{D}(m) = \Big\{i \in \Lambda : \exists j \in \Lambda : (i,j) \in \Delta\Big\}$$

is finite and it follows that m is local.

Now assume that $m \in \mathcal{H}(\mathbf{S}^{\Lambda}, \mathbf{S}^{\Lambda})$ is local. Then, one has in particular that $m[i] : S^{\Lambda} \to S$ is continuous for all $i \in \Lambda$ by the properties of the product topology. Moreover, $\mathcal{D}(m) \subset \Lambda$ is finite and, by definition, for $i \in \mathcal{D}(m)^{\text{c}}$ one has $m[j](x) = x(j)$ for all $x \in S^{\Lambda}$. Due to Lemma 1.1, for each $i \in \mathcal{D}(m)$ the set $\mathcal{R}(m[i])$ is finite and we can identify $m[i]$ with a map $m[i]' : S^{\mathcal{R}(m[i])} \to S$. By

Lemma 2.4 there exists a vector $M^i = (M^i_j)_{j\in\mathcal{R}(m[i])}$ with coordinates in $\mathcal{H}(\mathbf{S},\mathbf{S})$ such that

$$m[i]'(x) = \sum_{j\in\mathcal{R}(m[i])} M^i_j(x(j)) \qquad (i \in \Lambda,\ x \in S^{\mathcal{R}(\mathtt{m}[i])}).$$

Defining now $(M_{ij})_{i,j\in\Lambda}$ as

$$M_{ij} := \begin{cases} M^i_j & \text{if } i \in \mathcal{D}(m),\ j \in \mathcal{R}(m[i]), \\ \mathrm{id} & \text{if } i \notin \mathcal{D}(m),\ j = i, \\ o & \text{else,} \end{cases} \qquad (i,j \in \Lambda),$$

gives a representation of $m[i]$ for all $i \in \Lambda$ as in (2.12) with the property that the set Δ from (2.11) is finite. This completes the proof. □

With the help of the above lemma we can compute the dual map of each local $m \in \mathcal{H}(\mathbf{S}^\Lambda,\mathbf{S}^\Lambda)$ with respect to $\boldsymbol{\psi}$ from (2.9). Let, as in Lemma 2.10, for any monoid $\mathbf{M}$ the elements $o, \mathrm{id} \in \mathcal{H}(\mathbf{M},\mathbf{M})$ denote the function constantly equal to the neutral element and the identity, respectively.

Proposition 2.11 (Dual local homomorphisms I)**.** *Let $\mathbf{S} = (S,+)$, $\mathbf{R} = (R,\oplus)$ and $\mathbf{T} = (T,\otimes)$ be finite commutative monoids so that $\mathbf{S}$ is $\mathbf{T}$-dual to $\mathbf{R}$ with respect to $\psi : S \times R \to T$. For each local $m \in \mathcal{H}(\mathbf{S}^\Lambda,\mathbf{S}^\Lambda)$ there exists a local map $\hat{m} \in \mathcal{H}(\mathbf{R}^\Lambda,\mathbf{R}^\Lambda)$ so that the restriction of $\hat{m}$ to R^Λ_{fin} is the unique dual map of m with respect to $\boldsymbol{\psi}$ from (2.9). If $(M_{ij})_{i,j\in\Lambda}$ denotes the matrix from Lemma 2.10 such that (2.12) holds, then $\hat{m}$ is given via*

$$\hat{m}[i](y) = \bigoplus_{j\in\Lambda} \widehat{M}_{ji}(y(j)) \qquad (i \in \Lambda,\ y \in R^\Lambda), \tag{2.13}$$

where, for $i,j \in \Lambda$, $\widehat{M}_{ij} \in \mathcal{H}(\mathbf{R},\mathbf{R})$ is the (unique) dual map of $M_{ij} \in \mathcal{H}(\mathbf{S},\mathbf{S})$ with respect to ψ.

Proof. Let $x \in S^\Lambda$, $y \in R^\Lambda_{\mathrm{fin}}$ and let $\hat{m}$ be defined via (2.13). Note that $\hat{m}$ indeed maps R^Λ_{fin} into itself as Δ from (2.11) is finite for m and the (unique) dual maps of $o, \mathrm{id} \in \mathcal{H}(\mathbf{S},\mathbf{S})$ with respect to ψ are $o \in \mathcal{H}(\mathbf{R},\mathbf{R})$ and $\mathrm{id} \in \mathcal{H}(\mathbf{R},\mathbf{R})$, respectively. Moreover, Lemma 2.10 implies that $\hat{m} \in \mathcal{H}(\mathbf{R}^\Lambda,\mathbf{R}^\Lambda)$ and that it is local. We compute that

$$\begin{aligned} \boldsymbol{\psi}(m(x),y) &= \bigotimes_{i\in\Lambda} \psi\left(\textstyle\sum_{j\in\Lambda} M_{ij}(x(j)),\, y(i)\right) = \bigotimes_{i,j\in\Lambda} \psi\Big(M_{ij}(x(j)),\, y(i)\Big) \\ &= \bigotimes_{i,j\in\Lambda} \psi\Big(x(j),\, \widehat{M}_{ij}(y(i))\Big) = \bigotimes_{j\in\Lambda} \psi\left(x(j),\, \textstyle\bigoplus_{i\in\Lambda} \widehat{M}_{ij}(y(i))\right) \\ &= \boldsymbol{\psi}(x,\hat{m}(y)). \end{aligned}$$

This completes the proof. □

We reiterate the previous two results in words: Each local monoid homomorphism $m : S^\Lambda \to S^\Lambda$ can be characterized by an infinite matrix $(M_{ij})_{i,j\in\Lambda}$ via (2.12). Its unique dual map with respect to $\boldsymbol{\psi}$ from (2.9), that exists due to Proposition 2.7, is then characterized by transposing the matrix $(M_{ij})_{i,j\in\Lambda}$ and replacing all its entries by their unique dual maps with respect to ψ. Note that Proposition 2.11 in particular implies that the dual process from Theorem 2.9 has $\underline{\mathbf{0}}$ as a trap (where $\mathbf{0}$ denotes the neutral element of $\mathbf{R}$).

The dual process found via Theorem 2.9 is basically an "interacting particle system" with càglàd sample paths started in a finite configuration.[2] Applying an analogue of Theorem 1.3, if $\widehat{G}$ from (1.34) also satisfies (1.7), i.e., if

$$\sup_{i\in\Lambda}\sum_{m\in\mathcal{G}} r_m\Big(\mathbb{1}_{\mathcal{D}(\hat{m})}(i)+|\mathcal{R}_i^{\downarrow}(\hat{m})|\Big)<\infty, \tag{2.14}$$

then we can start the dual process Y also in an infinite initial state. More concretely, $\mathbf{Y}_{u,s}(y)$ $(u\geq s)$, defined as in (1.32), is then almost surely well-defined for *all* $y\in R^{\Lambda}$.

If $\widehat{G}$ satisfies (2.14), then, by symmetry and Theorem 2.9, we can use $\mathcal{G}$ to define a continuous-time Markov chain on S^{Λ}_{fin}. Recall that, by Theorem 1.6, this is also implied by (1.18). We note the following.

Lemma 2.12 (Summability conditions)**.** *Under the conditions of Theorem 2.9 the summability condition (1.18) implies the summability condition (2.14).*

Proof. Let $\mathbf{S}$ be a finite commutative monoid, fix a local map $m\in\mathcal{H}(\mathbf{S}^{\Lambda},\mathbf{S}^{\Lambda})$ and let $(M_{ij})_{i,j\in\Lambda}$ be the corresponding infinite matrix from Lemma 2.10. Due to (2.13) one has that

$$\begin{aligned}\mathcal{D}(\hat{m})&=\Big\{i\in\Lambda:\widehat{M}_{ii}\neq\mathrm{id}\Big\}\cup\Big\{i\in\Lambda:\exists j\in\Lambda\setminus\{i\}:\widehat{M}_{ji}\neq o\Big\}\\&=\Big\{i\in\Lambda:M_{ii}\neq\mathrm{id}\Big\}\cup\Big\{i\in\Lambda:\exists j\in\Lambda\setminus\{i\}:M_{ji}\neq o\Big\},\end{aligned}$$

from which one concludes with (2.12) that

$$\mathcal{D}(\hat{m})\subset\mathcal{D}(m)\cup\bigcup_{j\in\mathcal{D}(m)}\mathcal{R}(m[j]).$$

Moreover,

$$\mathcal{R}(\hat{m}[i])=\Big\{j\in\Lambda:\widehat{M}_{ji}\neq o\Big\}=\Big\{j\in\Lambda:M_{ji}\neq o\Big\}=\Big\{j\in\Lambda:i\in\mathcal{R}(m[j])\Big\}\qquad(i\in\Lambda).$$

Hence, recalling the definition of $\mathcal{R}_i^{\uparrow}(m)$ from (1.17) and noting that $j\notin\mathcal{D}(m)$ implies $\mathcal{R}(m[j])=\{j\}$, it follows that

$$|\mathcal{R}(\hat{m}[i])|=|\mathcal{R}_i^{\uparrow}(m)|+\mathbb{1}_{\mathcal{D}(m)^{\mathrm{c}}}(i)\qquad(i\in\Lambda)$$

and we conclude that

$$|\mathcal{R}_i^{\downarrow}(\hat{m})|\leq\begin{cases}|\mathcal{R}_i^{\uparrow}(m)|+\mathbb{1}_{\mathcal{D}(m)^{\mathrm{c}}}(i) & \text{if } i\in\mathcal{D}(m)\cup\bigcup_{j\in\Lambda}\mathcal{R}_j^{\downarrow}(m),\\ 0 & \text{else,}\end{cases}\qquad(i\in\Lambda).$$

[2]The quotation marks were added since we have defined an interacting particle system to have càdlàg sample paths.

If follows that

$$\begin{aligned}
&\sum_{m\in\mathcal{G}} r_m\Big(\mathbb{1}_{\mathcal{D}(\hat{m})}(i) + |\mathcal{R}_i^{\downarrow}(\hat{m})|\Big)\\
&= \sum_{m\in\mathcal{G}} r_m \mathbb{1}_{\mathcal{D}(\hat{m})}(i)\Big(1 + |\mathcal{R}_i^{\downarrow}(\hat{m})|\Big)\\
&\leq \sum_{m\in\mathcal{G}} r_m \mathbb{1}_{\mathcal{D}(m)\cup\bigcup_{j\in\Lambda}\mathcal{R}_j^{\downarrow}(m)}(i)\Big(1 + |\mathcal{R}_i^{\uparrow}(m)| + \mathbb{1}_{\mathcal{D}(m)^{\mathrm{c}}}(i)\Big)\\
&\leq \sum_{m\in\mathcal{G}} r_m\Big(\mathbb{1}_{\mathcal{D}(m)}(i) + |\mathcal{R}_i^{\uparrow}(m)| + |\mathcal{R}_i^{\uparrow}(m)| + \mathbb{1}_{\mathcal{D}(m)^{\mathrm{c}}\cap\bigcup_{j\in\Lambda}\mathcal{R}_j^{\downarrow}(m)}(i)\Big)\\
&\leq \sum_{m\in\mathcal{G}} r_m\Big(\mathbb{1}_{\mathcal{D}(m)}(i) + 3|\mathcal{R}_i^{\uparrow}(m)|\Big)
\end{aligned}$$

for $i \in \Lambda$. Hence,

$$\begin{aligned}
\sup_{i\in\Lambda}\sum_{m\in\mathcal{G}} r_m\Big(\mathbb{1}_{\mathcal{D}(\hat{m})}(i) + |\mathcal{R}_i^{\downarrow}(\hat{m})|\Big) &\leq \sup_{i\in\Lambda}\sum_{m\in\mathcal{G}} r_m\mathbb{1}_{\mathcal{D}(m)}(i) + 3\sup_{i\in\Lambda}\sum_{m\in\mathcal{G}} r_m|\mathcal{R}_i^{\uparrow}(m)|\\
&\leq 4\sup_{i\in\Lambda}\sum_{m\in\mathcal{G}} r_m\Big(\mathbb{1}_{\mathcal{D}(m)}(i) + |\mathcal{R}_i^{\uparrow}(m)|\Big),
\end{aligned}$$

implying the statement of the lemma. □

It is important to note that, for the duality function $\boldsymbol{\psi}$ from (2.9), $\boldsymbol{\psi}(x, y)$ does not have to be defined for all $(x, y) \in S^\Lambda \times R^\Lambda$. Hence, in general one cannot hope for a pathwise duality between two "interacting particle systems". However, there exist exceptions as Proposition 2.16 below shows.

To conclude this subsection, we take a step back from our focus on interacting particle systems. From the viewpoint of monoid duality, Theorem 2.9 may not be formulated in its strongest form. As written in the remark below Proposition 2.7, any $m \in \mathcal{H}(\mathbf{S}^\Lambda, \mathbf{S}^\Lambda) \cap \mathcal{C}(\mathbf{S}^\Lambda, \mathbf{S}^\Lambda)$ preserves $\mathcal{H}(\mathbf{S}^\Lambda, \mathbf{T}) \cap \mathcal{C}(\mathbf{S}^\Lambda, \mathbf{T})$ and hence has a unique dual map with respect to $\boldsymbol{\psi}$ from (2.9). So one could try to construct a process on S^Λ based on a countable collection of general continuous monoid homomorphisms instead of only local ones. Theorem 2.9 then still applies if one can guarantee that both processes are well-defined. This, however, is outside the scope of the present thesis.

A further generalization, which would also allow the use of discontinuous monoid homomorphisms, is not possible. Using the properties of the product topology we can show that $m : S^\Lambda \to S^\Lambda$ preserves $\mathcal{H}(\mathbf{S}^\Lambda, \mathbf{T}) \cap \mathcal{C}(\mathbf{S}^\Lambda, \mathbf{T})$ *if and only if* $m \in \mathcal{H}(\mathbf{S}^\Lambda, \mathbf{S}^\Lambda) \cap \mathcal{C}(\mathbf{S}^\Lambda, \mathbf{S}^\Lambda)$.

Proposition 2.13 (No discontinuous maps have duals)**.** *Let* $\mathbf{S}$*,* $\mathbf{R}$ *and* $\mathbf{T}$ *be finite commutative monoids such that* $\mathbf{S}$ *is* $\mathbf{T}$*-dual to* $\mathbf{R}$ *with respect to* ψ*. Let* $\boldsymbol{\psi}$ *be as in (2.9). Then a map* $m : S^\Lambda \to S^\Lambda$ *has a dual map* $\hat{m} : R^\Lambda_{\mathrm{fin}} \to R^\Lambda_{\mathrm{fin}}$ *with respect to* $\boldsymbol{\psi}$ *if and only if* $m \in \mathcal{H}(\mathbf{S}^\Lambda, \mathbf{S}^\Lambda) \cap \mathcal{C}(\mathbf{S}^\Lambda, \mathbf{S}^\Lambda)$*. The dual map* $\hat{m}$*, if it exists, is unique and satisfies* $\hat{m} \in \mathcal{H}(\mathbf{R}^\Lambda_{\mathrm{fin}}, \mathbf{R}^\Lambda_{\mathrm{fin}})$*.*

Proof. Let $x, x_1, x_2, \ldots \in S^\Lambda$. Assume that $m : S^\Lambda \to S^\Lambda$ preserves $\mathcal{H}(\mathbf{S}^\Lambda, \mathbf{T}) \cap \mathcal{C}(\mathbf{S}^\Lambda, \mathbf{T})$ and that $x_n \to x$ in the product topology. Fix an $i \in \Lambda$. As m preserves $\mathcal{H}(\mathbf{S}^\Lambda, \mathbf{T}) \cap \mathcal{C}(\mathbf{S}^\Lambda, \mathbf{T}) = \{\boldsymbol{\psi}(\,\cdot\,, y) : y \in R^\Lambda_{\mathrm{fin}}\}$, one has for all $b \in R$ that

$$\psi(m(x_n)(i), b) = \boldsymbol{\psi}(m(x_n), \delta_i^b) \underset{n\to\infty}{\longrightarrow} \boldsymbol{\psi}(m(x), \delta_i^b) = \psi(m(x)(i), b)$$

in the discrete topology on T, where $\delta_i^b \in R_{\rm fin}^\Lambda$ is defined as in (1.16). This implies that there exists an $N = N(i,b) \in \mathbb{N}$ such that $\psi(m(x_n)(i),b) = \psi(m(x)(i),b)$ for all $n \geq N$. Using the finiteness of R, setting $N^* := \max\{N(i,b) : b \in R\}$, one has for $n \geq N^*$ that $\psi(m(x_n)(i),b) = \psi(m(x)(i),b)$ for all $b \in R$. Property (i) of the definition of duality then implies that $m(x_n)(i) = m(x)(i)$ for $n \geq N^*$, hence $m(x_n) \to m(x)$ in the product topology and $m \in \mathcal{C}(\mathbf{S}^\Lambda, \mathbf{S}^\Lambda)$. The claims now follow from Proposition 2.7 and the remark below it. □

2.4 Previously known special cases

While monoid duality was newly introduced in [Latz and Swart, 2023b], some special cases yield dualities that were already known in the literature.

2.4.1 Additive duality

A *lattice* is a partially ordered set $(L,\leq)$ with the property that for every $a, a' \in L$ there exist (necessarily unique) elements $a \vee a'$ (the *join*) and $a \wedge a'$ (the *meet*) such that $a \vee a'$ is the least upper bound of a and a', and $a \wedge a'$ is the greatest lower bound of a and a'. It is easy to see that each finite lattice $(L,\leq)$ has a least element 0 and a greatest element $\top$, i.e., $0 \leq a \leq \top$ for all $a \in L$. If $(L,\leq)$ is a lattice, then so is $(L^\Lambda,\leq)$ equipped with the product order $\leq$, where $x \vee x'$ and $x \wedge x'$ are the coordinatewise join and meet of $x, x' \in L^\Lambda$.

It is easy to see that if $(L,\leq)$ is a lattice with least element 0, then $\mathbf{L} := (L,\vee)$ is a monoid with neutral element 0. For two lattices $(L,\leq)$ and $(T,\leq)$ an *additive map* is, by definition, a map $m : L \to T$ that is a monoid homomorphism between $\mathbf{L}$ and $\mathbf{T}$, with $\mathbf{T}$ being defined analogously to $\mathbf{L}$.

Following [Sturm and Swart, 2018, Subsection 2.4], we say that a partially ordered set $(\widehat{L},\leq)$ is *dual* to $(L,\leq)$ if there exists a bijection $L \ni a \mapsto \hat{a} \in \widehat{L}$ such that

$$a_1 \leq a_2 \quad \text{if and only if} \quad \hat{a}_2 \leq \hat{a}_1 \qquad (a_1, a_2 \in L).$$

Clearly, each partially ordered set has a dual, and all duals of a partially ordered set are naturally isomorphic. Moreover, $(L,\leq)$ is naturally isomorphic to the dual of $(\widehat{L},\leq)$ when one sets $\hat{\hat{a}} := a$. Note that if L has a least element 0, then $\hat{0}$ is the greatest element of $\widehat{L}$ and vice versa. If $(L,\leq)$ is a finite partially ordered set and L^Λ is equipped with the product order, then we define $\hat{x}(i) := \widehat{x(i)}$ $(x \in L^\Lambda,\ i \in \Lambda)$ coordinatewise. Then naturally $(\widehat{L}^\Lambda,\leq)$ is dual to $(L^\Lambda,\leq)$.

The following lemma says that the monoids $(L,\vee)$ and $(\widehat{L},\vee)$ are dual in the sense defined in Chapter 2.2. In the definition of $\mathbf{T}$ below $\vee$ denotes the usual maximum on $\{0,1\}$.

Lemma 2.14 (Lattice duality)**.** *Let* $\mathbf{T} := (\{0,1\},\vee)$*. Let* $(L,\leq)$ *be a finite lattice and let* $(\widehat{L},\leq)$ *be its dual lattice. Then* $\mathbf{L} := (L,\vee)$ *is* $\mathbf{T}$*-dual to* $\widehat{\mathbf{L}} := (\widehat{L},\vee)$ *with respect to* $\psi_{\rm add} : L \times \widehat{L} \to \{0,1\}$ *defined as*

$$\psi_{\rm add}(a,\hat{b}) := \begin{cases} 1 & \text{if } a \not\leq b, \\ 0 & \text{else,} \end{cases} \qquad (a \in L,\ \hat{b} \in \widehat{L}). \tag{2.15}$$

Proof. We write $\psi_{\text{add}}(a,\hat{b}) = \mathbb{1}_{\{a\not\leq b\}}$, the indicator function of the set $\{a \not\leq b\} = \{\hat{b} \not\leq \hat{a}\} \subset L \times \widehat{L}$ $(a \in L,\, \hat{b} \in \widehat{L})$. We check properties (i)–(iv) of the definition of duality of commutative monoids from Chapter 2.2. By the symmetry of $\{a \not\leq b\} = \{\hat{b} \not\leq \hat{a}\}$, it suffices to check properties (i) and (ii). For property (i) note that $\psi_{\text{add}}(a,\hat{b}) = \psi_{\text{add}}(a',\hat{b})$ for all $\hat{b} \in \widehat{L}$ implies that $a \leq b$ if and only if $a' \leq b$ $(b \in L)$. Setting first $b = a$ and then $b = a'$ implies that $a \leq a' \leq a$ and hence $a = a'$. To check property (ii), we observe that

$$\psi_{\text{add}}(0,\hat{b}) = \mathbb{1}_{\{0\not\leq b\}} = 0 \qquad (\hat{b} \in \widehat{L})$$

and

$$\psi_{\text{add}}(a \vee a',\hat{b}) = \mathbb{1}_{\{a\vee a'\not\leq b\}} = \mathbb{1}_{\{a\not\leq b\}} \vee \mathbb{1}_{\{a'\not\leq b\}} = \psi_{\text{add}}(a,\hat{b}) \vee \psi_{\text{add}}(a',\hat{b})$$

for all $a, a' \in L$ and $\hat{b} \in \widehat{L}$. This shows that $\psi_{\text{add}}(\,\cdot\,,\hat{b}) \in \mathcal{H}(\mathbf{L},\mathbf{T})$ for all $\hat{b} \in \widehat{L}$.

Assume, conversely, that $h \in \mathcal{H}(\mathbf{L},\mathbf{T})$. To complete the proof, we must show that $h(a) = \mathbb{1}_{\{a\not\leq b\}}$ $(a \in L)$ for some $b \in L$. Since $h(0) = 0$, the set $\{a : h(a) = 0\}$ is non-empty, so using the finiteness of L we can define $b := \bigvee\{a : h(a) = 0\}$. We observe that $h(a) = 0 = h(a')$ implies

$$h(a \vee a') = h(a) \vee h(a') = 0 \vee 0 = 0.$$

It follows that $h(b) = 0$ and more generally

$$0 \leq h(a) \leq h(a) \vee h(b) = h(a \vee b) = h(b) = 0$$

for all $a \in S$ with $a \leq b$ and hence $h(a) = 0$. Conversely, $h(a) = 0$ implies that a is an element of $\{a' : h(a') = 0\}$ and hence $a \leq b$ by the definition of b. □

Note that setting $\mathbf{R} := (\{0,1\},\wedge)$ we could have equivalently proved that $\mathbf{L}$ is $\mathbf{R}$-dual to $\widehat{\mathbf{L}}$ with respect to $\tilde{\psi}_{\text{add}} : L \times \widehat{L} \to \{0,1\}$, defined as

$$\tilde{\psi}_{\text{add}}(a,\hat{b}) := \begin{cases} 1 & \text{if } a \leq b, \\ 0 & \text{else,} \end{cases} \qquad (a \in L,\ \hat{b} \in \widehat{L}). \tag{2.16}$$

If the local state space S is a lattice, the duality function $\boldsymbol{\psi}$ from (2.9) is, by Lemma 2.14, given as $\boldsymbol{\psi}_{\text{add}} : S^\Lambda \times \widehat{S}^\Lambda_{\text{fin}} \to \{0,1\}$ defined by

$$\boldsymbol{\psi}_{\text{add}}(x,\hat{y}) := \begin{cases} 1 & \text{if } x \not\leq y, \\ 0 & \text{else,} \end{cases} \qquad (x \in S^\Lambda,\ \hat{y} \in \widehat{S}^\Lambda_{\text{fin}}). \tag{2.17}$$

One concludes the following result from Theorem 2.9.

Theorem 2.15 (Additive duality)**.** *Let there exist a partial order on the local state space S such that $(S,\leq)$ is a lattice. Let G and $\widehat{G}$ be the generators from (1.8) and (1.34) defined via $\mathcal{G}$, a countable collection of local additive maps. Assuming, as usual, that G satisfies (1.7), there exists a continuous-time Markov chain $Y = (Y_t)_{t\geq 0}$ with generator $\widehat{G}$, state space $\widehat{S}^\Lambda_{\text{fin}}$ and càglàd sample paths such that $X = (X_t)_{t\geq 0}$, the interacting particle system defined in Chapter 1.1, is pathwise dual to Y with respect to $\boldsymbol{\psi}_{\text{add}}$, the duality function defined in (2.17).*

We call an interacting particle system that satisfies the conditions of Theorem 2.15 *additive*. Thus, we call an interacting particle system additive if it has a generator from (1.8) defined via $\mathcal{G}$, a countable collection of local additive maps.

Additive duality of interacting particle systems has been much studied and has found many applications since the foundational work of Harris [1976, 1978] and Griffeath [1979]. This foundational work has been concerned with the local state space $S = \{0, 1\}$. Some of the most studied interacting particle systems with local state space $\{0, 1\}$ are additive, making Theorem 2.15 applicable to them. Examples include the voter model and the contact process [Liggett, 1999]. Additive duality is one of the most important tools in their study.

To recover Harris and Griffeath's formulation of additive duality from ours, we have a closer look at the special case that $(S, \leq)$ is totally ordered, i.e., that $(S, \leq) = (\{0, \dots, n\}, \leq)$ for some $n \in \mathbb{N}$, where $\leq$ denotes the usual total order on $\mathbb{N}_0$. We set $\widehat{S} = \{0, \dots, n\}$, with the corresponding bijection being $\{0, \dots, n\} \ni b \mapsto n - b \in \{0, \dots, n\}$. Then $\hat{y} = \underline{n} - y$ for $y \in \{0, \dots, n\}^\Lambda$, and we can write $\boldsymbol{\psi}_{\text{add}}$ from (2.17) as

$$\boldsymbol{\psi}_{\text{add}}(x, \hat{y}) := \begin{cases} 1 & \text{if } \exists i \in \Lambda : x(i) + \hat{y}(i) > n, \\ 0 & \text{else,} \end{cases}$$

for $x \in \{0, \dots, n\}^\Lambda$ and $\hat{y} \in \{0, \dots, n\}^\Lambda_{\text{fin}}$. In the special case that $n = 1$ this yields

$$\boldsymbol{\psi}_{\text{add}}(x, \hat{y}) := \begin{cases} 1 & \text{if } \exists i \in \Lambda : x(i) = \hat{y}(i) = 1, \\ 0 & \text{else,} \end{cases} \qquad (x \in \{0, 1\}^\Lambda,\ \hat{y} \in \{0, 1\}^\Lambda_{\text{fin}}).$$

This is the formulation of the additive duality function also found in [Harris, 1976, Formula (1.1)] and [Griffeath, 1979, Formula (II.1.9)].

Pathwise dualities based on general dual lattices were studied in [Sturm and Swart, 2018]. In particular, [Sturm and Swart, 2018, Theorem 33] gives the statement of Theorem 2.15.[3] In fact, without using the terminology of lattice theory, additive duality for general finite lattices S was already studied by Foxall [2016].

The study of general additive duality is able to explain previously found dualities. In particular, as discussed in [Sturm and Swart, 2018, Section 3.3], the duality of the two-stage contact process discovered by Krone [1999] is based on an additive duality, where the local state space S is of the form $\{0, 1, 2\}$.

Additive duality has the advantage that $\boldsymbol{\psi}_{\text{add}}$ from (2.17) is always informative, as we will see in Chapter 2.6. In contrast to many other duality functions arising from monoid duality, $\boldsymbol{\psi}_{\text{add}}$ has the additional advantage that $\boldsymbol{\psi}_{\text{add}}(x, \hat{y})$ is well-defined for all $(x, \hat{y}) \in S^\Lambda \times \widehat{S}^\Lambda$. In fact, [Sturm and Swart, 2018, Theorem 33] implies that if, apart from (1.7), also (2.14) holds, then we actually get a pathwise duality between two "interacting particle systems" in the following sense.

[3] In their definition of the duality function $\boldsymbol{\psi}_{\text{add}}$ Sturm and Swart [2018] exchange the values of 0 and 1, see [Sturm and Swart, 2018, Formula (23)], i.e., they use the "local duality function" $\tilde{\psi}_{\text{add}}$ from (2.16). This, of course, does not change the assertion.

Proposition 2.16 (Infinite dual process)**.** *Assume that, additionally to the assumptions of Theorem 2.15, also (1.18) holds. Then,* $\widehat{G}$ *from (1.34) can be defined for function on* $\widehat{S}^\Lambda$ *as in Chapter 1.1, and there exists a Feller process* $Y = (Y_t)_{t\geq 0}$ *with this generator, state space* $\widehat{S}^\Lambda$ *and càglàd sample paths such that* $X = (X_t)_{t\geq 0}$*, the interacting particle system defined in Chapter 1.1, is pathwise dual to* Y *with respect to* $\boldsymbol{\psi}_{\text{add}} : S^\Lambda \times \widehat{S}^\Lambda \to \{0,1\}$*, defined as in (2.17) but allowing all* $\hat{y} \in \widehat{S}^\Lambda$ *in its second coordinate.*

Proof. This follows readily from Lemma 2.12, the comment below (2.14), [Sturm and Swart, 2018, Theorem 33] and the first half of Chapter 1.3. □

2.4.2 Cancellative duality

Let $n \in \mathbb{N}$. Denoting addition modulo n by $\oplus$, one has that $\mathbf{M} = (\{0,\dots,n-1\},\oplus)$ is a finite commutative monoid. Let multiplication modulo n be denoted by $\odot$. The following lemma says that $\mathbf{M}$ is $\mathbf{M}$-dual to itself.

Lemma 2.17 (Modulo duality)**.** *Let* $n \in \mathbb{N}$ *and* $\mathbf{M} = (\{0,\dots,n-1\},\oplus)$*. Then* $\mathbf{M}$ *is* $\mathbf{M}$*-dual to itself with respect to* $\psi_{\text{canc}} : \{0,\dots,n-1\} \times \{0,\dots,n-1\} \to \{0,\dots,n-1\}$ *defined as*

$$\psi_{\text{canc}}(a,b) := a \odot b \qquad (a,b \in \{0,\dots,n-1\}).$$

Proof. By symmetry, it again suffices to check properties (i) and (ii) of the definition of duality. If $n = 1$ the proof is trivial. Hence assume that $n \in \mathbb{N}\setminus\{1\}$. If $\psi_{\text{canc}}(a,b) = \psi_{\text{canc}}(a',b)$ for all $b \in \{0,\dots,n-1\}$, then in particular also

$$a = \psi_{\text{canc}}(a,1) = \psi_{\text{canc}}(a',1) = a' \qquad (a,a' \in \{0,\dots,n-1\}),$$

implying property (i).

To prove also property (ii), we note that $\psi_{\text{canc}}(0,b) = 0$ for all $b \in \{0,\dots,n-1\}$ and

$$\begin{aligned}\psi_{\text{canc}}(a\oplus a',b) &= (a\oplus a')\odot b = (a\odot b)\oplus(a'\odot b)\\ &= \psi_{\text{canc}}(a,b)\oplus\psi_{\text{canc}}(a',b)\end{aligned}$$

for all $a,a',b \in \{0,\dots,n-1\}$. This shows that $\psi_{\text{canc}}(\,\cdot\,,b) \in \mathcal{H}(\mathbf{M},\mathbf{M})$ for all $b \in \{0,\dots,n-1\}$.

Assume, conversely, that $h \in \mathcal{H}(\mathbf{M},\mathbf{M})$. Then, for each $k \in \{0,\dots,n-1\}$, one has that

$$\begin{aligned}h(k) &= h(1\oplus\cdots\oplus 1) = h(1)\oplus\cdots\oplus h(1)\\ &= \psi_{\text{canc}}(1,h(1))\oplus\cdots\oplus\psi_{\text{canc}}(1,h(1)) = \psi_{\text{canc}}(k,h(1)),\end{aligned}$$

where, in the three terms with the dots, there are exactly k summands. This proves property (ii) of the definition of duality and the proof is complete. □

For $S = \{0,\dots,n-1\}$, equipped with addition modulo n, the duality function $\boldsymbol{\psi}$ from (2.9) is, by Lemma 2.17, given as $\boldsymbol{\psi}_{\text{canc}} : \{0,\dots,n-1\}^\Lambda \times \{0,\dots,n-1\}^\Lambda_{\text{fin}} \to \{0,\dots,n-1\}$ defined by

$$\boldsymbol{\psi}_{\text{canc}}(x,y) := \bigoplus_{i\in\Lambda} x(i)\odot y(i) \qquad (x \in \{0,\dots,n-1\}^\Lambda,\ y \in \{0,\dots,n-1\}^\Lambda_{\text{fin}}). \tag{2.18}$$

In parallel to Chapter 2.4.1, we call a map $m : \{0, \dots, n-1\}^\Lambda \to \{0, \dots, n-1\}^\Lambda$ *cancellative* if $m \in \mathcal{H}(\mathbf{M}^\Lambda, \mathbf{M}^\Lambda)$, where the monoid $\mathbf{M}$ is defined as in Lemma 2.17. One concludes the following result from Theorem 2.9.

Theorem 2.18 (Cancellative duality)**.** *Assume that $S = \{0, \dots, n-1\}$ for some $n \in \mathbb{N}$. Let G and $\widehat{G}$ be the generators from (1.8) and (1.34) defined via $\mathcal{G}$, a countable collection of cancellative local maps $m : \{0, \dots, n-1\}^\Lambda \to \{0, \dots, n-1\}^\Lambda$. Assuming, as usual, that G satisfies (1.7), there exists a continuous-time Markov chain $Y = (Y_t)_{t \geq 0}$ with generator $\widehat{G}$, state space $\{0, \dots, n-1\}^\Lambda_{\text{fin}}$ and càglàd sample paths such that $X = (X_t)_{t \geq 0}$, the interacting particle system defined in Chapter 1.1, is pathwise dual to Y with respect to $\boldsymbol{\psi}_{\text{canc}}$, the duality function defined in (2.18).*

We call an interacting particle system that satisfies the conditions of Theorem 2.18 *cancellative*. Thus, we call an interacting particle system cancellative if it has a generator from (1.8) defined via $\mathcal{G}$, a countable collection of local cancellative maps.

For $n = 2$ one gets the classical cancellative duality relation from [Griffeath, 1979, Theorem III.1.5]. This classical form of cancellative duality has successfully been applied in the study of various nonlinear voter models [Cox and Durrett, 1991, Handjani, 1999, Sturm and Swart, 2008a] and annihilating branching processes [Bramson et al., 1991]. We will show in Chapter 2.6 that $\boldsymbol{\psi}_{\text{canc}}$ is for all $n \in \mathbb{N}$ weakly informative.

2.5 Computing monoid dualities

In Chapter 2.4 we saw that two of the most used dualities in the field of interacting particle systems are special cases of monoid dualities. The question becomes if we can find additional dualities that do not fit into one of the two classes of Section 2.4. Due to Proposition 2.8 we focus on dualities of finite commutative monoids that give rise to dualities of the corresponding product monoids. The aim is to list all such dualities only involving monoids up to a "reasonable size". The number of commutative monoids, up to isomorphisms, with $1, 2, 3, 4, 5, 6, 7, \dots$ elements is $1, 2, 5, 19, 78, 421, 2637, \dots$ (sequence A058131 in the OEIS [OEIS Foundation Inc., 2024]), so beyond cardinality four the sort of brute force approach we are going to apply below quickly becomes impractical. Therefore, we settled on listing all dualities of finite commutative monoids of cardinality up to four.

In order to compute all such dualities, the following proposition is crucial. It links duality in the sense of Chapter 2.2 to the concept of the $\mathbf{T}$-adjoint defined in Chapter 2.1. For monoids $\mathbf{M}$ and $\mathbf{T}$, we say that $\mathbf{M}$ is $\mathbf{T}$-*reflexive* if the map $a \mapsto L_a$ defined in (2.2) is a bijection (and hence a monoid isomorphism from $\mathbf{M}$ to $\mathbf{M}''$, the $\mathbf{T}$-adjoint of the $\mathbf{T}$-adjoint of $\mathbf{M}$).

Proposition 2.19 (Duality and reflexivity)**.** *Let $\mathbf{M} = (M, +)$, $\mathbf{N} = (N, \oplus)$, and $\mathbf{T} = (T, \otimes)$ be finite commutative monoids and let $\mathbf{M}'$ and $\mathbf{N}'$ be the $\mathbf{T}$-adjoints of $\mathbf{M}$ and $\mathbf{N}$, respectively. Then:*

(i) *If $\mathbf{M}$ is $\mathbf{T}$-dual to $\mathbf{N}$ with respect to $\psi : M \times N \to T$, then the map $b \mapsto \psi(\,\cdot\,, b)$ is a monoid isomorphism from $\mathbf{N}$ to $\mathbf{M}'$ and the map $a \mapsto \psi(a, \,\cdot\,)$ is a monoid isomorphism from $\mathbf{M}$ to $\mathbf{N}'$. Moreover, $\mathbf{M}$ and $\mathbf{N}$ are $\mathbf{T}$-reflexive.*

(ii) *If* $\mathbf{M}$ *is* $\mathbf{T}$*-reflexive, then* $\mathbf{M}$ *is* $\mathbf{T}$*-dual to* $\mathbf{M}'$ *with respect to* $\psi : M \times \mathcal{H}(\mathbf{M}, \mathbf{T}) \to T$*, defined as*

$$\psi(a, h) := h(x) \qquad (a \in M,\ h \in \mathcal{H}(\mathbf{M}, \mathbf{T})). \tag{2.19}$$

Proof. Let 0, $\mathbf{0}$ and 1 denote the neutral elements of $\mathbf{M}$, $\mathbf{N}$ and $\mathbf{T}$, respectively. Assume that $\mathbf{M}$ is $\mathbf{T}$-dual to $\mathbf{N}$ with respect to $\psi : M \times N \to T$. Property (iv) of the definition of duality implies that $\psi(a, b \otimes b') = \psi(a, b) \otimes \psi(a, b')$ and $\psi(a, \mathbf{0}) = 1$, so the map $b \mapsto \psi(\,\cdot\,, b)$ is a monoid homomorphism from $\mathbf{N}$ to $\mathbf{M}'$. By property (ii) of the definition of duality, the map $b \mapsto \psi(\,\cdot\,, b)$ is surjective and by property (i) of the definition of duality it is injective, so we conclude that it is a monoid isomorphism. Since $\mathbf{N}$ is $\mathbf{T}$-dual to $\mathbf{M}$ with respect to $\psi^\dagger(b, a) := \psi(a, b)$, the same argument shows that the map $a \mapsto \psi(a, \,\cdot\,)$ is a monoid isomorphism from $\mathbf{M}$ to $\mathbf{N}'$.

If we identify $\mathbf{N}$ with $\mathbf{M}'$ using the isomorphism $b \mapsto \psi(\,\cdot\,, b)$, then we can identify the function $L_a : \mathcal{H}(\mathbf{M}, \mathbf{T}) \to T$ defined in (2.2) with the function $L_a : N \to T$ defined as $L_a(b) := \psi(a, b)$ $(a \in M,\ b \in N)$. This means that the map $a \mapsto L_a$ from M to $\mathcal{H}(\mathbf{M}', \mathbf{T})$ corresponds to the map $a \mapsto \psi(a, \,\cdot\,)$ from M to $\mathcal{H}(\mathbf{N}, \mathbf{T})$, which we have just shown to be a monoid isomorphism. This proves that $\mathbf{M}$ is $\mathbf{T}$-reflexive, and by the symmetry between $\mathbf{M}$ and $\mathbf{N}$, the same is true for $\mathbf{N}$.

Assume, conversely, that $\mathbf{M}$ is $\mathbf{T}$-reflexive. To show that $\mathbf{M}$ is $\mathbf{T}$-dual to $\mathbf{M}'$ with respect to ψ defined in (2.19), we must show that:

(i) $\psi(a, h) = \psi(a, h')$ for all $a \in M$ implies $h = h'$,

(ii) $\mathcal{H}(\mathbf{M}, \mathbf{T}) = \big\{\psi(\,\cdot\,, h) : h \in \mathcal{H}(\mathbf{M}, \mathbf{T})\big\}$,

(iii) $\psi(a, h) = \psi(a', h)$ for all $h \in \mathcal{H}(\mathbf{M}, \mathbf{T})$ implies $a = a'$,

(iv) $\mathcal{H}(\mathbf{M}', \mathbf{T}) = \big\{\psi(a, \,\cdot\,) : a \in M\big\}$.

Properties (i) and (ii) are trivial consequences of the definition of ψ. Analogously, if we define $\psi' : \mathcal{H}(\mathbf{M}, \mathbf{T}) \times \mathcal{H}(\mathbf{M}', \mathbf{T}) \to T$ by $\psi'(h, L) := L(h)$ $(h \in \mathcal{H}(\mathbf{M}, \mathbf{T}),\ L \in \mathcal{H}(\mathbf{M}', \mathbf{T}))$, then

(i) $\psi'(h, L) = \psi'(h, L')$ for all $h \in \mathcal{H}(\mathbf{M}, \mathbf{T})$ implies $L = L'$,

(ii) $\mathcal{H}(\mathbf{M}', \mathbf{T}) = \big\{\psi'(\,\cdot\,, L) : L \in \mathcal{H}(\mathbf{M}', \mathbf{T})\big\}$.

Since by assumption, $\mathbf{M}$ is $\mathbf{T}$-reflexive, we may identify $\mathbf{M}$ with $\mathbf{M}''$ via $a \mapsto L_a$. As we have $\psi'(h, L_a) = L_a(h) = h(a) = \psi(a, h)$ properties (i) and (ii) of the duality function ψ' imply properties (iii) and (iv) of the duality function ψ. □

Due to Proposition 2.19, the strategy is to list all 26 commutative monoids of cardinality between two and four and then compute $\mathcal{H}(\mathbf{M}, \mathbf{T})$ for each pair $(\mathbf{M}, \mathbf{T}) = ((M, +), (T, \otimes))$ of monoids from this list. If $\mathcal{H}(\mathbf{M}, \mathbf{T})$ has at most four elements, then $\mathbf{M}' = (\mathcal{H}(\mathbf{M}, \mathbf{T}), \otimes)$, the $\mathbf{T}$-adjoint of $\mathbf{M}$, has to be isomorphic to one of the monoids from the original list, and using the previous computations we can check whether $\mathbf{M}''$ is isomorphic to $\mathbf{M}$, i.e., whether there *exists* a monoid isomorphism between $\mathbf{M}$ and $\mathbf{M}''$. By definition of reflexivity we then still have

$\mathbf{M}_0$	0
0	0

$\mathbf{M}_1$	0	1
0	0	1
1	1	1

$\mathbf{M}_2$	0	1
0	0	1
1	1	0

$\mathbf{M}_3$	0	1	2
0	0	1	2
1	1	2	2
2	2	2	2

$\mathbf{M}_4$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$\mathbf{M}_5$	0	1	2
0	0	1	2
1	1	0	2
2	2	2	2

$\mathbf{M}_6$	0	1	2
0	0	1	2
1	1	2	1
2	2	1	2

$\mathbf{M}_7$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Table 2.1: Cayley tables of the monoids $\mathbf{M}_0, \dots, \mathbf{M}_7$.

to check if $a \mapsto L_a$ from (2.2) is indeed one such monoid isomorphism. As $\mathbf{M}$ and $\mathbf{M}''$ are isomorphic, they have the same cardinality and it suffices to show that for all $a, a' \in M$ with $a \neq a'$ also $L_a \neq L_{a'}$. If $\mathbf{M}$ is indeed $\mathbf{T}$-reflexive, then, by Proposition 2.19 (ii), $\mathbf{M}$ is $\mathbf{T}$-dual to $\mathbf{M}'$ with respect to ψ from (2.19). By Proposition 2.19 (i) all dualities of finite commutative monoids arise in this way.

We start by listing all commutative monoids with at most four elements. For those with precisely four elements, we have used [Forsythe, 1955] as our source. For a monoid of cardinality n, we have enumerated its elements $0, \dots, n-1$ where 0 always denotes the neutral element. To enumerate the other elements, we have applied a specific set of rules outlined in [Latz and Swart, 2023b, Section 5.1].[4]

We have named the monoids $\mathbf{M}_0, \dots, \mathbf{M}_{26}$, where $\mathbf{M}_0$ is the one monoid with 1 element, $\mathbf{M}_1$ and $\mathbf{M}_2$ are the commutative monoids with 2 elements, $\mathbf{M}_3, \dots, \mathbf{M}_7$ are the ones with 3 elements and $\mathbf{M}_8, \dots, \mathbf{M}_{26}$ are the ones with 4 elements. Also for the ordering of the monoids within such a group we have used specific rules, again outlined in [Latz and Swart, 2023b, Section 5.1]. We list the Cayley tables (or operation tables) of $\mathbf{M}_0, \dots, \mathbf{M}_7$ in Table 2.1. The Cayley tables of $\mathbf{M}_8, \dots, \mathbf{M}_{26}$ are given in Table 2.4 (found at the end of Chapter 2).

The above explained strategy how to compute all dualities of commutative monoids with cardinality at most four was implemented in Mathematica [Wolfram Research Inc., 2024]. The corresponding programs can be accessed as attachments to the online version of this thesis, which is available in the online repository `https://dspace.cuni.cz/`.[5]

We add some details regarding the implementation. The list of Cayley tables of the monoids $\mathbf{M}_1, \dots, \mathbf{M}_{26}$ was given as an input. If one were to extend the code to also handle monoids of cardinality five one would probably have to compute the (Cayley tables of the) 78 commutative monoids of cardinality five as we are not aware of a corresponding source. For $\mathbf{M} = (M, +), \mathbf{T} = (T, \otimes) \in \{\mathbf{M}_1, \dots, \mathbf{M}_{26}\}$ we compute $\mathcal{H}(\mathbf{M}, \mathbf{T})$ by brute force, by checking for every function from M

[4]Since this set of rules also uses notions from the theory of semirings, which we have moved in this thesis to Chapter 3, we do not repeat the rules in this thesis.

[5]To locate this thesis, simply type its title or the author's name in the search bar.

to T whether it is a monoid homomorphism. In all cases where $\mathcal{H}(\mathbf{M},\mathbf{T})$ has at most four elements, we calculate the Cayley table of $(\mathcal{H}(\mathbf{M},\mathbf{T}),\otimes)$ and find the commutative monoid from our list that $(\mathcal{H}(\mathbf{M},\mathbf{T}),\otimes)$ is isomorphic to. The result of this is a table of size 26×26 that lists for each pair $(\mathbf{M},\mathbf{T})$ the monoid $\mathbf{N}$ such that $(\mathcal{H}(\mathbf{M},\mathbf{T}),\otimes)\cong\mathbf{N}$, if $\mathbf{N}\in\{\mathbf{M}_0,\ldots,\mathbf{M}_{26}\}$. Using this table, we find all triples $(\mathbf{M},\mathbf{N},\mathbf{T})$ of monoids of cardinality between two and four such that $\mathbf{N}\cong(\mathcal{H}(\mathbf{M},\mathbf{T}),\otimes)$ and $\mathbf{M}\cong(\mathcal{H}(\mathbf{N},\mathbf{T}),\otimes)$.

For each such triple $(\mathbf{M},\mathbf{N},\mathbf{T})=((M,+),(N,\oplus),(T,\otimes))$ there exists a monoid isomorphism $N\ni b\mapsto f_b\in\mathcal{H}(\mathbf{M},\mathbf{T})$ and we define $\psi: M\times N\to T$ as

$$\psi(a,b):=f_b(a)\qquad(a\in M,\ b\in N). \tag{2.20}$$

Recall from the outlined strategy above that we are left to check whether for $a,a'\in M$ with $a\neq a'$ also $L_a\neq L_{a'}$. Equivalently, we can check whether the functions $\psi(a,\,\cdot\,)$ with a ranging through M are all different from each other.[6]

In this way we identified 110 triples $(\mathbf{M},\mathbf{N},\mathbf{T})=((M,+),(N,\oplus),(T,\otimes))$ of monoids of cardinality between two and four such that $\mathbf{N}\cong(\mathcal{H}(\mathbf{M},\mathbf{T}),\otimes)$ and $\mathbf{M}\cong(\mathcal{H}(\mathbf{N},\mathbf{T}),\otimes)$. In all 110 cases the check for reflexivity came back positive, implying that $\mathbf{M}$ is indeed $\mathbf{T}$-dual to $\mathbf{N}$ with respect to ψ from (2.20). A lot of these 110 cases are trivially related to each other. We will use the following reductions to restrict the number of duality functions and then list only those that are "essentially" different.

- In many of the 110 examples we have found, it turns out that $\mathbf{T}$ contains a smaller submonoid $\tilde{\mathbf{T}}$ so that the duality function ψ takes values in $\tilde{\mathbf{T}}$. For this reason, we will only list examples that are "minimal" in the sense that the function values $\{\psi(a,b): a\in M,\ b\in N\}$ generate the monoid $\mathbf{T}$.

- If $\mathbf{M}$ is $\mathbf{T}$-dual to $\mathbf{N}$ with respect to ψ and $N\ni b\mapsto b'\in N$ is an isomorphism, then $\mathbf{M}$ is $\mathbf{T}$-dual to $\mathbf{N}$ with respect to ψ' defined as $\psi'(a,b):=\psi(a,b')$ $(a\in M,\ b\in N)$. If several duality functions are related in this way, then we will list only one of them.

- If $\mathbf{M}$ is $\mathbf{T}$-dual to $\mathbf{N}$ with respect to ψ, then $\mathbf{N}$ is $\mathbf{T}$-dual to $\mathbf{M}$ with respect to $\psi^\dagger$ defined as $\psi^\dagger(b,a):=\psi(a,b)$ $(b\in N,\ a\in M)$. If two duality functions are related in this way, then we will list only one of them.

After these reductions, we end up with 22 dualities that are "essentially" different. In all examples that are minimal in the sense defined above, we observe that the cardinalities of M and N are the same and that the cardinality of T is not larger than the cardinalities of M and N. Table 2.2 lists all dualities of commutative monoids of cardinality two or three. Each time the duality is stated first. Below, the corresponding duality function is given in table form, where the upper left corner indicates the monoids from which the outer elements come. For a $\mathbf{T}$-duality with $\mathbf{T}=(T,\otimes)$, the entries in the inner part of the tables are to be interpreted as elements of T. All dualities of commutative monoids of cardinality four are listed in Table 2.5 (found at the end of Chapter 2).

[6] Note that the functions $\psi(\,\cdot\,,b)$ with b ranging through N are trivially all different from each other, since $N\ni b\mapsto f_b\in\mathcal{H}(\mathbf{M},\mathbf{T})$ is a monoid isomorphism.

$\mathbf{M}_1$ is $\mathbf{M}_1$-dual to $\mathbf{M}_1$ w.r.t. ψ_1:

$\mathbf{M}_1 \backslash \mathbf{M}_1$	0	1
0	0	0
1	0	1

$\mathbf{M}_2$ is $\mathbf{M}_2$-dual to $\mathbf{M}_2$ w.r.t. ψ_2:

$\mathbf{M}_2 \backslash \mathbf{M}_2$	0	1
0	0	0
1	0	1

$\mathbf{M}_3$ is $\mathbf{M}_3$-dual to $\mathbf{M}_3$ w.r.t. ψ_3:

$\mathbf{M}_3 \backslash \mathbf{M}_3$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	2

$\mathbf{M}_4$ is $\mathbf{M}_1$-dual to $\mathbf{M}_4$ w.r.t. ψ_4:

$\mathbf{M}_4 \backslash \mathbf{M}_4$	0	1	2
0	0	0	0
1	0	0	1
2	0	1	1

$\mathbf{M}_5$ is $\mathbf{M}_5$-dual to $\mathbf{M}_6$ w.r.t. ψ_5:

$\mathbf{M}_5 \backslash \mathbf{M}_6$	0	1	2
0	0	0	0
1	0	1	0
2	0	2	2

$\mathbf{M}_6$ is $\mathbf{M}_6$-dual to $\mathbf{M}_6$ w.r.t. ψ_6:

$\mathbf{M}_6 \backslash \mathbf{M}_6$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	2

$\mathbf{M}_7$ is $\mathbf{M}_7$-dual to $\mathbf{M}_7$ w.r.t. ψ_7:

$\mathbf{M}_7 \backslash \mathbf{M}_7$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Table 2.2: Dualities of monoids of cardinality 2 and 3. Green tables indicate cases where $\boldsymbol{\psi}_k$, defined as in (2.9) but with ψ replaced by $\boldsymbol{\psi}_k$, is weakly informative, while red tables indicate cases where $\boldsymbol{\psi}_k$ is not weakly informative. Compare Chapter 2.6.

If ψ_k denotes one of the 22 identified duality functions we denote by $\boldsymbol{\psi}_k$ the duality function defined as in (2.9) but with ψ replaced by ψ_k. Recall that if $\mathbf{S}$ is $\mathbf{T}$-dual to $\mathbf{R}$ with respect to ψ_k, then, by Proposition 2.8, $\mathbf{S}^\Lambda$ is dual to $\mathbf{R}^\Lambda_{\text{fin}}$ with respect to $\boldsymbol{\psi}_k$.

We note some observations. The monoids corresponding to the four lattices with 2–4 elements are $\mathbf{M}_1 = (\{0,1\}, \vee)$, $\mathbf{M}_4 = (\{0,1,2\}, \vee)$, $\mathbf{M}_{11} \cong \mathbf{M}_1 \times \mathbf{M}_1$, and $\mathbf{M}_{15} = (\{0,1,2,3\}, \vee)$, where $\vee$ each time is the usual maximum. Their corresponding duality functions $\psi_1, \psi_4, \psi_{11}$ and ψ_{15} are hence of the form described in Lemma 2.14. Note that these are also the only $\mathbf{M}_1$-dualities we found.

The operators of the monoids $\mathbf{M}_2, \mathbf{M}_7$ and $\mathbf{M}_{26}$ are addition modulo 2,3 and 4, respectively. The duality functions ψ_2, ψ_7 and ψ_{26} correspond to multiplication modulo n, as we saw in Lemma 2.17.

Since $\mathbf{M}_k$ is $\mathbf{M}_k$-dual to $\mathbf{M}_k$ $(k = 1,2)$, Proposition 2.6 tells us that $\mathbf{M}_{11} \cong \mathbf{M}_1 \times \mathbf{M}_1$ is $\mathbf{M}_1$-dual to $\mathbf{M}_{11} \cong \mathbf{M}_1 \times \mathbf{M}_1$ and that $\mathbf{M}_{25} \cong \mathbf{M}_2 \times \mathbf{M}_2$ is $\mathbf{M}_2$-dual to $\mathbf{M}_{25} \cong \mathbf{M}_2 \times \mathbf{M}_2$. The corresponding duality functions are ψ_{11} and ψ_{25}. Since $\mathbf{M}_1$ and $\mathbf{M}_2$ are naturally submonoids of $\mathbf{M}_{23} \cong \mathbf{M}_1 \times \mathbf{M}_2$, using the fact that $\mathbf{M}_k$ is $\mathbf{M}_k$-dual to $\mathbf{M}_k$ $(k = 1,2)$, one can check that $\mathbf{M}_k$ is $\mathbf{M}_{23}$-dual to $\mathbf{M}_k$ $(k = 1,2)$ and hence, by Proposition 2.6, $\mathbf{M}_{23} \cong \mathbf{M}_1 \times \mathbf{M}_2$ is $\mathbf{M}_{23}$-dual to $\mathbf{M}_{23} \cong \mathbf{M}_1 \times \mathbf{M}_2$. It is easy to check that $\mathbf{M}_1$ and $\mathbf{M}_2$ are also both submonoids of $\mathbf{M}_5$, and by the same argument $\mathbf{M}_{23}$ is also $\mathbf{M}_5$-dual to $\mathbf{M}_{23}$. The duality functions in these last two cases are ψ_{23} and ψ_{235}.

2.6 Representations of monoids

Having identified several new dualities, we now aim to study their usefulness in the sense of Chapter 1.5, i.e., we check whether the corresponding duality functions are (weakly) informative. Recall that informativeness is only defined for duality functions that take values in a vector space. As already stated in Chapter 1.5, there exists a concrete strategy how to prove informativeness for a duality function that takes values in $T \subset \mathbb{C}$.

Proposition 2.20 (Informativeness of monoid dualities)**.** *Let* $\mathbf{S} = (S, +)$*,* $\mathbf{R} = (R, \oplus)$ *and* $\mathbf{T} = (T, \otimes)$ *be finite commutative monoids and assume that* $\mathbf{S}$ *is* $\mathbf{T}$*-dual to* $\mathbf{R}$*. Then* $\boldsymbol{\psi}$ *from (2.9) is informative if* $\mathbf{T}$ *is a submonoid of* $(\mathbb{C}, \cdot\,)$*, where* $\cdot$ *denotes the usual multiplication.*

Proof. By definition, we have to prove that the family

$$\mathcal{H} := (\boldsymbol{\psi}(\,\cdot\,, y))_{y \in R^\Lambda_{\text{fin}}}$$

is distribution determining. Both properties from Lemma 1.12 follow directly from the duality of S^Λ and R^Λ_{fin}. Indeed, property (i) of the definition of duality implies that

$$\boldsymbol{\psi}(x, y_1) \cdot \boldsymbol{\psi}(x, y_2) = \boldsymbol{\psi}(x, y_1 \oplus y_2) \qquad (x \in S^\Lambda,\ y_1, y_2 \in R^\Lambda_{\text{fin}}),$$

and the fact that $\mathcal{H}$ separates points follows from property (ii) of the definition of duality. Hence, the claim follows from Lemma 1.12. □

It is easy to see that all finite submonoids of $(\mathbb{C}, \cdot\,)$ (apart from $(\{0\}, \cdot\,)$) consist of the multiplicative group of n-th roots of unity for some $n \in \mathbb{N}$, either with or without $0 \in \mathbb{C}$ added. Those with cardinality up to four are isomorphic to $\mathbf{M}_0, \mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_5, \mathbf{M}_7, \mathbf{M}_{18}$ and $\mathbf{M}_{26}$ from Chapter 2.5. The $\mathbf{T}$-dualities with $\mathbf{T} \in \{\mathbf{M}_0, \mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_5, \mathbf{M}_7, \mathbf{M}_{18}\}$ identified in Table 2.2 and Table 2.5 are those with duality function ψ_k for

$$k \in \{1, 2, 4, 5, 7, 11, 15, 16, 17, 18, 21, 235, 25, 26\}.$$

If follows that for the k's above there exist monoid isomorphisms γ_k into the corresponding submonoid of $(\mathbb{C}, \cdot\,)$ such that $\gamma_k \circ \boldsymbol{\psi}_k$ is informative. This makes $\boldsymbol{\psi}_k$ weakly informative. Note that if $\boldsymbol{\psi}_k : M^\Lambda \times N^\Lambda_{\text{fin}} \to T$, then

$$\gamma_k \circ \boldsymbol{\psi}_k(x, y) = \prod_{i \in \Lambda} \gamma_k\Big(\psi_k(x(i), y(i))\Big) \qquad (x \in M^\Lambda,\ y \in N^\Lambda_{\text{fin}}).$$

We consider the examples of Chapter 2.4 in more detail. The dualities from Lemma 2.14 are all $\mathbf{M}_1$-dualities and $\gamma_1 : \{0, 1\} \to \{0, 1\}$ can be written as $\gamma_1(z) = 1 - z$, where $-$ denotes the usual subtraction in $\mathbb{R}$. Hence, $\gamma_1 \circ \boldsymbol{\psi}_{\text{add}} = 1 - \boldsymbol{\psi}_{\text{add}}$ is informative for $\boldsymbol{\psi}_{\text{add}}$ from (2.17). If follows that also $\boldsymbol{\psi}_{\text{add}}$ itself is informative.

The dualities from Lemma 2.17 are $(\{0, \ldots, n-1\}, \oplus)$-dualities ($n \in \mathbb{N}$), were $\oplus$ denotes addition modulo n. Let $n \in \mathbb{N}$ and fix a primitive n-th root of unity $z \in \mathbb{C}$, i.e., $z^n = 1$ but $z^m \neq 1$ for $m \in \{1, \ldots, n-1\}$. Then $\gamma : \{0, \ldots, n-1\} \to \{1, z, \ldots, z^{n-1}\}$ defined as $\gamma(k) := z^k$ ($k \in \{0, \ldots, n-1\}$) is a monoid isomorphism from $(\{0,\ldots,n-1\},\oplus)$ to $(\{1,z,\ldots,z^{n-1}\},\,\cdot\,)$. Hence, $\gamma\circ\boldsymbol{\psi}_{\rm canc}$ is informative for $\boldsymbol{\psi}_{\rm canc}$ from (2.18). It is natural to ask whether $\boldsymbol{\psi}_{\rm canc}$ itself is also informative, where one would compute the expectation in $[0,n-1]\subset\mathbb{R}$. For $n=2$ this is clearly the case but for $n\geq 3$ it is not clear how to prove or disprove this. For applications to interacting particle systems, however, this question does not really matter, as one can always work with the informative duality function $\gamma\circ\boldsymbol{\psi}_{\rm canc}$.

To further investigate $\mathbf{T}$-dualities in the case that the monoid $\mathbf{T}$ is not isomorphic to a submonoid of $(\mathbb{C},\,\cdot\,)$, we provide some additional notions. Let $\mathbb{A}=(A,+,\,\cdot\,)$ be a unital commutative algebra with (multiplicative) unit I.[7] A *multiplicative representation* of a commutative monoid $\mathbf{M}=(M,+)$ with neutral element 0 is a map $\gamma:M\to A$ so that $\gamma(a+a')=\gamma(a)\cdot\gamma(a')$ and $\gamma(0)=I$. Then $(\gamma(M),\,\cdot\,)$ is a submonoid of $(A,\,\cdot\,)$ and $\gamma:M\to\gamma(M)$ is a (surjective) monoid homomorphism. We say that γ is *faithful* if this map (with codomain $\gamma(M)$) is injective (and thus a monoid isomorphism).

Let $\mathbf{M}=(M,+)$, $\mathbf{N}=(N,\oplus)$ and $\mathbf{T}=(T,\otimes)$ be (topological) monoids. Assume that T is finite and that $\mathbf{M}$ is $\mathbf{T}$-dual to $\mathbf{N}$ with respect to $\psi:M\times N\to T$. Let $\gamma:T\to A$ be a faithful multiplicative representation of $\mathbf{T}$. It follows from the definition of duality of (topological) monoids that $\mathbf{M}$ is also $(\gamma(T),\,\cdot\,)$-dual to $\mathbf{N}$ with duality function $\gamma\circ\psi$. If ψ is weakly informative and if the elements of $\gamma(T)$ are affinely independent, then Proposition 1.10 and the faithfulness of γ imply that $\gamma\circ\psi$ is informative.

We say that γ is a *good multiplicative representation* of ψ if γ is a faithful multiplicative representation of $\mathbf{T}$ and $\gamma\circ\psi$ is informative. Hence, all the γ's we have already seen in this subchapters are in fact good multiplicative representations of the corresponding duality function in the unital commutative algebra $(\mathbb{C},+,\,\cdot\,)$. The next result shows that for each weakly informative duality function arising from a duality of (topological) commutative monoids, we can find a good multiplicative representation, at least if the duality function maps to a finite monoid.

Proposition 2.21 (Existence of good representations)**.** *Let* $\mathbf{M}=(M,+)$*,* $\mathbf{N}=(N,\oplus)$ *and* $\mathbf{T}=(T,\otimes)$ *be (topological) commutative monoids, assume that* T *is finite and that* $\mathbf{M}$ *is* $\mathbf{T}$*-dual to* $\mathbf{N}$ *with respect to* $\psi:M\times N\to T$*. Then there exist a finite-dimensional real unital commutative algebra* $\mathbb{A}=(A,+,\,\cdot\,)$ *and a faithful representation* $\gamma:T\to A$ *such that* $\gamma\circ\psi$ *is informative if* ψ *is weakly informative.*

Proof. Let $\mathbb{R}^T$ be the collection of all functions mapping from T to $\mathbb{R}$. The space $(\mathbb{R}^T,+)$, where $+$ denotes the usual (pointwise) sum of real-valued functions, is a finite-dimensional vector space on which we can define the product $*$ as

$$(g*h)(a):=\sum_{b,c\in T}g(b)h(c)\mathbb{1}_{\{a\}}(b\otimes c)\qquad(g,h\in\mathbb{R}^T,\ a\in T),$$

where the sum is the usual sum in $\mathbb{R}$. One readily checks that this makes $(\mathbb{R}^T,+,*)$ a finite-dimensional real unital algebra with unit $\mathbb{1}_{\{0\}}$. Defining $\gamma:T\to\mathbb{R}^T$ as $\gamma(a)=\mathbb{1}_{\{a\}}$ $(a\in T)$ then gives a faithful multiplicative representation of $\mathbf{T}$, and the elements of $\gamma(T)$ are clearly affinely independent. The claim then follows from Proposition 1.10 and the faithfulness of γ as stated above. $\square$

[7] Recall that an algebra over a field is a vector space equipped with a bilinear product.

Proposition 2.21 effectively says that weak informativeness is sufficient for applications to interacting particle systems: For a weakly informative duality function we can always use the faithful representation given in the proof of Proposition 2.21 in order to work only with expectations.

Proposition 2.21 moreover yields a procedure to systematically check the duality functions from Chapter 2.5 that do not map into a (monoid isomorphic to a) submonoid of $(\mathbb{C}, \cdot)$ for weak informativeness. Let $\mathbf{M} = (M, +)$, $\mathbf{N} = (N, \oplus)$ and $\mathbf{T} = (T, \otimes)$ be finite commutative monoids and assume that $\mathbf{M}$ is $\mathbf{T}$-dual to $\mathbf{N}$ with respect to $\psi : M \times N \to T$. One can write ψ in matrix form and then replace all entries by unit column vectors as indicated in the proof above. Let the resulting zero-one matrix be denoted by B. If the equation $Bx = 0$ has a non-trivial solution whose entries sum to zero, then one can conclude that ψ is not weakly informative and the same must hold for $\boldsymbol{\psi} : M^\Lambda \times N^\Lambda_{\text{fin}} \to T$. On the other hand, if $Bx = 0$ only has the trivial solution, one can continue with $\psi^2 : M^2 \times N^2 \to T$ defined as

$$\psi^2((a_1, a_2), (b_1, b_2)) := \psi(a_1, b_1) \otimes \psi(a_2, b_2) \qquad (a_1, a_2 \in M,\ b_1, b_2 \in N)$$

and repeat the above procedure. Again, if there is a non-trivial solution of the corresponding linear equation, one can conclude that $\boldsymbol{\psi}$ is not weakly informative. In the opposite case one can continue with ψ^3, defined analogously to ψ^2. Of course, one can never conclude weak informativeness from this iterative procedure.

We illustrate the procedure on ψ_3 from Table 2.2. As ψ_3 written in matrix form contains a row with three different entries, the above indicated test matrix B has to contain the rows $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, so $Bx = 0$ cannot have a non-trivial solution. Hence, we continue with ψ_3^2. Ordering the elements of $\{0, 1, 2\}^2$ as $(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)$, the matrix form of ψ_3^2 and a test matrix B are given as

$$\psi_3^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 2 & 2 & 0 & 2 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 1 & 2 & 2 & 2 & 2 & 2 \\ 0 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here, B was constructed by adding for each row ψ_3^2 first a zero-one row indicating the position of the 0's, and then a zero-one row indicating the position of the 1's. Afterwards, repeating rows were deleted. As B has only 8 rows, the equation $Bx = 0$ has to have a non-trivial solution. In fact, the solution set of $Bx = 0$ (over $\mathbb{R}^9$) is given as

$$L = \Big\{(0, 0, 0, 0, a, -a, 0, -a, a)^\intercal : a \in \mathbb{R}\Big\}.$$

It follows that $\boldsymbol{\psi}_3 : \{0, 1, 2\}^\Lambda \times \{0, 1, 2\}^\Lambda_{\text{fin}} \to \{0, 1, 2\}$ is not weakly informative. Indeed, pick some $i, j \in \Lambda$ with $i \neq j$ and let $X \sim 1/2\delta_{x_1} + 1/2\delta_{x_2}$ and $X' \sim 1/2\delta_{x_3} + 1/2\delta_{x_4}$ be random variables with values in $\{0, 1, 2\}^\Lambda$, where $\delta_x \in$

$\mathcal{M}_1(\{0,1,2\}^\Lambda)$ is the Dirac measure on $x \in \{0,1,2\}^\Lambda$ and $x_1, x_2, x_3, x_4 \in \{0,1,2\}^\Lambda$ are given as

$$x_1(k) := \begin{cases} 1 & \text{if } k = i, \\ 1 & \text{if } k = j, \\ 0 & \text{else}, \end{cases} \qquad x_2(k) := \begin{cases} 2 & \text{if } k = i, \\ 2 & \text{if } k = j, \\ 0 & \text{else}, \end{cases}$$

$$x_3(k) := \begin{cases} 1 & \text{if } k = i, \\ 2 & \text{if } k = j, \\ 0 & \text{else}, \end{cases} \qquad x_4(k) := \begin{cases} 2 & \text{if } k = i, \\ 1 & \text{if } k = j, \\ 0 & \text{else}, \end{cases} \qquad (k \in \Lambda).$$

Then, the distributions of $\boldsymbol{\psi}_3(X, y)$ and $\boldsymbol{\psi}_3(X', y)$ are identical for all y. Indeed,

$$\boldsymbol{\psi}_3(X, y) \stackrel{d}{=} \boldsymbol{\psi}_3(X', y) \stackrel{d}{=} \begin{cases} 0 & \text{if } y(i,j) = (0,0), \\ Z & \text{if } y(i,j) \in \{(0,1),(1,0)\}, \\ 2 & \text{else}, \end{cases} \qquad (y \in \{0,1,2\}^\Lambda_{\text{fin}}),$$

where $Z \sim 1/2\delta_1 + 1/2\delta_2$ is a random variable with values in $\{1, 2\}$.

In a very similar way one finds that also $\boldsymbol{\psi}_k$ for

$$k \in \{6, 9, 10, 13, 22, 24\}$$

is not weakly informative, where every time the second step in the iterative procedure yields the counterexample. We do not have an explanation why this is the case. Corresponding examples of two random variables X and X' are collected in Appendix A.1.

We are left with $\boldsymbol{\psi}_{23}$ as the only duality function from Chapter 2.5 for which we have not yet decided whether it is weakly informative. However, the existence of the duality function $\boldsymbol{\psi}_{235}$ helps. By Table 2.5 one has that

$$\psi_{23}(a, b) = \psi_{23}(a', b') \qquad \text{implies} \qquad \psi_{235}(a, b) = \psi(a', b')$$

for $a, a', b, b' \in \{0, 1, 2\}$. It follows that for two random variables X, X' with values in $\{0,1,2\}^\Lambda$ and for $y \in \{0,1,2\}^\Lambda_{\text{fin}}$,

$$\boldsymbol{\psi}_{23}(X, y) \stackrel{d}{=} \boldsymbol{\psi}_{23}(X', y) \quad \text{implies} \quad \boldsymbol{\psi}_{235}(X, y) \stackrel{d}{=} \boldsymbol{\psi}_{235}(X', y),$$

and, due to the weak informativeness of $\boldsymbol{\psi}_{235}$, the duality function $\boldsymbol{\psi}_{23}$ is also weakly informative. Hence, we have for all duality functions from Chapter 2.5 checked whether they are weakly informative. The results are summarized in Table 2.3 and encoded with colors in Table 2.2 and in Table 2.5. We see that the majority of the identified duality functions are weakly informative. However, the only weakly informative duality function not mapping into a (monoid isomorphic to a) submonoid of $(\mathbb{C}, \cdot)$ is $\boldsymbol{\psi}_{23}$, and its weak informativeness follows from a duality function mapping into a (monoid isomorphic to a) submonoid of $(\mathbb{C}, \cdot)$. Thus, judging from the examples we have identified, (weak) informativeness seems to be closely related to mapping into a submonoid of $(\mathbb{C}, \cdot)$.

$\boldsymbol{\psi}_k$	weakly informative	not weakly informative
k	1,2,4,5,7,11,15,16,17,18,21,23,235,25,26	3,6,9,10,13,22,24

Table 2.3: Weak informativeness of the duality functions from Chapter 2.5.

2.7 Applying monoid duality

To illustrate the application of monoid duality in the study of interacting particle systems, we consider an example. In the first two subchapters below we repeat some (mostly) known statements for contact processes from the point of view of the present thesis. In Chapter 2.7.3 we introduce a new type of contact process, whose invariant laws we study by means of monoid duality in Chapter 2.7.4 and Chapter 2.7.5.

2.7.1 Contact processes

Assume that $S = \{0,1\}$ and $\Lambda = \mathbb{Z}^d$ for some $d \in \mathbb{N}$. Let $\vee$ and $\oplus$ denote the binary operators of $\mathbf{M}_1$ and $\mathbf{M}_2$, respectively. In words, this says that for $a, a' \in \{0,1\}$ the quantity $a \vee a'$ is the maximum of a and a', and $a \oplus a'$ is the sum of a and a' modulo 2. For all $i, j \in \mathbb{Z}^d$, we define "infection maps" $\texttt{inf}^*_{ij} : \{0,1\}^{\mathbb{Z}^d} \to \{0,1\}^{\mathbb{Z}^d}$ $(* \in \{\vee, \oplus\})$ and a "death map" $\texttt{dth}_i : \{0,1\}^{\mathbb{Z}^d} \to \{0,1\}^{\mathbb{Z}^d}$ as

$$\texttt{inf}^*_{ij}(x)(k) := \begin{cases} x(i) * x(j) & \text{if } k = j, \\ x(k) & \text{else,} \end{cases}, \quad \texttt{dth}_i(x)(k) := \begin{cases} 0 & \text{if } k = i, \\ x(k) & \text{else,} \end{cases} \tag{2.21}$$
$$(x \in \{0,1\}^{\mathbb{Z}^d},\ k \in \mathbb{Z}^d).$$

We say that $i, j \in \mathbb{Z}^d$ are nearest neighbors and write $i \sim j$ if $\|i - j\|_1 = 1$. As in Chapter 1.1 we define generators via

$$G_* f(x) := \lambda \sum_{i,j \in \mathbb{Z}^d : i \sim j} \Big\{ f(\texttt{inf}^*_{ij}(x)) - f(x) \Big\} + \delta \sum_{i \in \mathbb{Z}^d} \Big\{ f(\texttt{dth}_i(x)) - f(x) \Big\} \tag{2.22}$$

for $* \in \{\vee, \oplus\}$, where $\lambda, \delta \geq 0$ are model parameters. Note that, independently of the choice of $*$,

$$\sup_{i \in \Lambda} \sum_{m \in \mathcal{G}} r_m \Big(\mathbb{1}_{\mathcal{D}(m)}(i) + |\mathcal{R}^{\downarrow}_i(m)| \Big) = \delta + 6d\lambda < \infty,$$

i.e., (1.7) is satisfied. Hence, due to Theorem 1.3, there exist interacting particle systems with generators $G_\vee$ and $G_\oplus$. The process $C = (C_t)_{t \geq 0}$ with generator $G_\vee$ is the well-known *contact process* on $\mathbb{Z}^d$ with infection rate λ and death rate δ (introduced by Harris [1974]). We denote this process shortly as $\mathrm{CP}(\lambda, \delta)$. The process $D = (D_t)_{t \geq 0}$ with generator $G_\oplus$ was introduced as the *annihilating branching process* in [Bramson et al., 1991]. We refer to it as the *cancellative contact process* ($\mathrm{cCP}(\lambda, \delta)$) to stress the similarity of the two processes, which differ only in the type of operator used in the definition of the infection maps $\texttt{inf}^*_{ij}$ $(* \in \{\vee, \oplus\})$.

In words, we can describe the dynamics of the (cancellative) contact process as follows:

- At each site $i \in \mathbb{Z}^d$ sit two "exponential clocks", one with rate $2d\lambda$ for *reproduction* and one with rate δ for *death*.
- If the clock for reproduction at site $i \in \mathbb{Z}^d$ rings, the corresponding individual *reproduces* by choosing a neighboring site j uniformly at random and adding its local state to the local state at j, where addition has to be interpreted in the sense of the operator $* \in \{\vee, \oplus\}$.
- If the "death clock" at site i rings, individual i *dies*, which means that its local state is replaced by 0, regardless of its previous value.

As clearly

$$\mathtt{inf}^*_{ij}(\underline{0}) = \mathtt{dth}_i(\underline{0}) = \underline{0} \qquad (i, j \in \mathbb{Z}^d,\ * \in \{\vee, \oplus\}),$$

we may define $\mathrm{supp}(x)$ $(x \in \{0,1\}^{\mathbb{Z}^d})$, δ_i $(i \in \mathbb{Z}^d)$, and $\{0,1\}^{\mathbb{Z}^d}_{\mathrm{fin}}$ as in Chapter 1.2, now based on $0 \in \{0, 1\}$.

It follows from Lemma 2.10 that, for all $i, j \in \mathbb{Z}^d$, $\mathtt{inf}^{\vee}_{ij}$ is additive, that $\mathtt{inf}^{\oplus}_{ij}$ is cancellative, and that $\mathtt{dth}_i$ is both additive and cancellative. This makes the CP an additive interacting particle system and the cCP a cancellative interacting system in the sense of the definitions from Chapter 2.4.

Let $\boldsymbol{\psi}_{\mathrm{add}} : \{0,1\}^{\mathbb{Z}^d} \times \{0,1\}^{\mathbb{Z}^d}_{\mathrm{fin}} \to \{0,1\}$ be the additive duality function from (2.17) and let $\boldsymbol{\psi}_{\mathrm{canc}} : \{0,1\}^{\mathbb{Z}^d} \times \{0,1\}^{\mathbb{Z}^d}_{\mathrm{fin}} \to \{0,1\}$ be the cancellative duality function from (2.18). Then it follows from Proposition 2.11 that $\mathtt{dth}_i$ is "self-dual" with respect to both $\boldsymbol{\psi}_{\mathrm{add}}$ and $\boldsymbol{\psi}_{\mathrm{canc}}$ in the sense that

$$\boldsymbol{\psi}_{\mathrm{add}}(\mathtt{dth}_i(x), y) = \boldsymbol{\psi}_{\mathrm{add}}(x, \mathtt{dth}_i(y)) \quad \text{and} \quad \boldsymbol{\psi}_{\mathrm{canc}}(\mathtt{dth}_i(x), y) = \boldsymbol{\psi}_{\mathrm{canc}}(x, \mathtt{dth}_i(y))$$

for all $i \in \mathbb{Z}^d$, $x \in \{0,1\}^{\mathbb{Z}^d}$, $y \in \{0,1\}^{\mathbb{Z}^d}_{\mathrm{fin}}$.[8] The same result implies that $\mathtt{inf}^{\vee}_{ij}$ is dual to (the restriction to $\{0,1\}^{\mathbb{Z}^d}_{\mathrm{fin}}$ of) $\mathtt{inf}^{\vee}_{ji}$ with respect to $\boldsymbol{\psi}_{\mathrm{add}}$ and that $\mathtt{inf}^{\oplus}_{ij}$ is dual to (the restriction to $\{0,1\}^{\mathbb{Z}^d}_{\mathrm{fin}}$ of) $\mathtt{inf}^{\oplus}_{ji}$ with respect to $\boldsymbol{\psi}_{\mathrm{canc}}$ $(i, j \in \{0,1\}^{\mathbb{Z}^d})$. One concludes from Theorem 2.15 that the contact process C is (pathwise) "self-dual" with respect to $\boldsymbol{\psi}_{\mathrm{add}}$ and from Theorem 2.18 that the cancellative contact process D is (pathwise) "self-dual" with respect to $\boldsymbol{\psi}_{\mathrm{canc}}$ in the sense that their dual processes are càglàd versions of the original processes restricted to $\{0,1\}^{\mathbb{Z}^d}_{\mathrm{fin}}$.

2.7.2 Long-time behavior of contact process

To speak about the long-time behavior of the CP and the cCP, we define shift operators $\theta_i : \{0,1\}^{\mathbb{Z}^d} \to \{0,1\}^{\mathbb{Z}^d}$ by

$$(\theta_i x)(j) := x(j - i) \qquad (i, j \in \mathbb{Z}^d,\ x \in \{0,1\}^{\mathbb{Z}^d}). \tag{2.23}$$

We say that a probability measure μ on $\{0,1\}^{\mathbb{Z}^d}$ is *homogeneous* if $\mu = \mu \circ \theta_i^{-1}$ $(i \in \mathbb{Z}^d)$.[9] We denote the CP again by $C = (C_t)_{t \geq 0}$ and the cCP by $D = (D_t)_{t \geq 0}$.

[8]More precisely, $\mathtt{dth}_i : \{0,1\}^{\mathbb{Z}^d} \to \{0,1\}^{\mathbb{Z}^d}$ $(i \in \mathbb{Z}^d)$ is dual to its restriction to $\{0,1\}^{\mathbb{Z}^d}_{\mathrm{fin}}$ with respect to both $\boldsymbol{\psi}_{\mathrm{add}}$ and $\boldsymbol{\psi}_{\mathrm{canc}}$.

[9]The property to be homogeneous was call "shift-invariant" in [Latz and Swart, 2023a]. As we investigate (time-) invariant distribution, we decided to avoid this double use of the word "invariant" in this thesis.

As $\underline{0}$ is the only trap for both the CP and the cCP, a probability measure $\mu \in \mathcal{M}_1(\{0,1\}^{\mathbb{Z}^d})$ is C-non-trivial in the sense of the definition in Chapter 1.2 if and only if it is D-non-trivial if and only if $\mu(\{\underline{0}\}) = 0$. Moreover, the notion of survival from Chapter 1.2 applies to both the CP and the cCP.

It is well-known [Swart, 2022, Theorem 6.35] that the CP(λ, δ) with $\lambda + \delta > 0$ started in a C-non-trivial homogeneous distribution converges weakly to a (time-) invariant distribution $\bar{\nu}$ called the *upper invariant law* of the contact process (compare Chapter 4.5). Similarly, it is known [Bramson et al., 1991, Theorem 1.2 & Theorem 1.3] that the cCP(λ, δ) with $\lambda + \delta > 0$ started in a D-non-trivial homogeneous distribution converges weakly to an invariant distribution $\dot{\nu}$, which we call, in accordance with [Sturm and Swart, 2008a], the *odd upper invariant law* of the cancellative contact process.

Letting $\delta_{\underline{0}}$ denote the Dirac measure concentrated on the "all 0" configuration $\underline{0}$, $\bar{\nu}$ and $\dot{\nu}$ may or may not differ from $\delta_{\underline{0}}$ depending on the choice of the model parameters λ and δ. For a CP(λ, δ) $(\lambda + \delta > 0)$ there exists a critical value $\lambda_{\mathrm{CP}} = \lambda_{\mathrm{CP}}(d) \in (0, \infty)$ (dependent on the dimension d) such that $\bar{\nu} \neq \delta_{\underline{0}}$ if and only if $\lambda/\delta > \lambda_{\mathrm{CP}}$ [Liggett, 1985, Chapter IV.1], [Bezuidenhout and Grimmett, 1990]. Here and in the following we set $x/0 = \infty$ for $x \in (0, \infty)$. For the cCP we can define $\lambda^{\pm}_{\mathrm{cCP}} = \lambda^{\pm}_{\mathrm{cCP}}(d)$ as

$$\lambda^{-}_{\mathrm{cCP}} := \inf\{\lambda \geq 0 : \text{the odd upper inv. law of the cCP}(\lambda, 1) \text{ does not equal } \delta_{\underline{0}}\},$$
$$\lambda^{+}_{\mathrm{cCP}} := \sup\{\lambda \geq 0 : \text{the odd upper inv. law of the cCP}(\lambda, 1) \text{ equals } \delta_{\underline{0}}\}.$$

It is known that $\lambda^{+}_{\mathrm{cCP}} < \infty$ ([Bramson et al., 1991, Theorem 1.1] & Proposition 2.23 below). By coupling the CP and the cCP in such a way that infections and deaths only occur in both processes simultaneously (see below) one shows that $\lambda_{\mathrm{CP}} \leq \lambda^{-}_{\mathrm{cCP}}$. Thus, it is established that

$$0 < \lambda_{\mathrm{CP}} \leq \lambda^{-}_{\mathrm{cCP}} \leq \lambda^{+}_{\mathrm{cCP}} < \infty.$$

Simulations suggest that $\lambda^{-}_{\mathrm{cCP}} = \lambda^{+}_{\mathrm{cCP}}$ and $\lambda_{\mathrm{CP}} < \lambda^{-}_{\mathrm{cCP}}$ in all dimensions. The first assertion is a long-standing open problem that due to the non-monotone nature of the cancellative contact process seems very difficult. Using the bound $\lambda_{\mathrm{CP}}(1) \leq 1.942$, proved by Liggett [1995], and the following result, we can conclude the latter assertion at least in dimension one.

Proposition 2.22 (Lower bound for $\lambda^{-}_{\mathrm{cCP}}(1)$)**.** *One has* $\lambda^{-}_{\mathrm{cCP}}(1) \geq 2$.

To prove Proposition 2.22, we use the following characterization for the survival of the cCP. Such characterization are known to hold for several processes. In particular, the result below is stated as [Sturm and Swart, 2008a, Lemma 1] for an important class of cancellative processes. However, the cCP does not fit into this class, so we provide a short proof below. We will study similar characterization of survival also in Chapter 4.5.

Proposition 2.23 (Survival of the cCP)**.** *One has* $\dot{\nu} \neq \delta_{\underline{0}}$ *if and only if the* cCP *survives.*

Proof. Let $\gamma : \{0,1\} \to \{-1,1\}$ be the faithful representation of $\mathbf{M}_2$ as a submonoid of $(\mathbb{C}, \cdot\,)$ from Chapter 2.6, i.e., $\gamma(0) = 1$ and $\gamma(1) = -1$. We prove this statement using $\gamma\circ\boldsymbol{\psi}_{\mathrm{canc}}$, a good multiplicative representation of the cancellative duality function $\boldsymbol{\psi}_{\mathrm{canc}}$ defined in (2.18). Let $D=(D_t)_{t\geq 0}$ be a cCP(λ,δ) ($\lambda,\delta\geq 0$, $\lambda+\delta>0$) and let $y\in\{0,1\}^{\mathbb{Z}^d}_{\mathrm{fin}}$. Moreover, let $\nu_{1/2}$ denote the product measure with density 1/2, i.e., $\nu_{1/2}(\{x:x(i)=1\})=1/2$ independently for all $i\in\mathbb{Z}^d$, let ν_t denote the law of a D started in $\nu_{1/2}$ and let μ_t denote the law of D started deterministically in y. The (pathwise) "self-duality" of the cCP and the fact that if $X\sim\mathrm{Bin}(n,1/2)$ is binomially distributed, then

$$\mathbb{P}[X \text{ is odd}]=\frac{1}{2}$$

independently of $n\in\mathbb{N}$, imply that

$$\begin{aligned}\dot{\nu}\Big(\Big\{x\in\{0,1\}^{\mathbb{Z}^d}:|x\wedge y| \text{ is odd}\Big\}\Big)&=\int\boldsymbol{\psi}_{\mathrm{canc}}(x,y)\,\mathrm{d}\dot{\nu}(x)\\&=\lim_{t\to\infty}\int\boldsymbol{\psi}_{\mathrm{canc}}(x,y)\,\mathrm{d}\nu_t(x)\\&=\lim_{t\to\infty}\int\int\boldsymbol{\psi}_{\mathrm{canc}}(x,y')\,\mathrm{d}\mu_t(y')\,\mathrm{d}\nu_{1/2}(x)\qquad(2.24)\\&=\lim_{t\to\infty}\frac{1}{2}\mathbb{P}^y[D_t\neq\underline{0}]\\&=\frac{1}{2}\mathbb{P}^y[D_t\neq\underline{0}\ \forall t\geq 0],\end{aligned}$$

where $\wedge$ denotes the pointwise minimum on $\{0,1\}^{\mathbb{Z}^d}$ (compare [Bramson et al., 1991, Equation (1.4)]). Hence,

$$\begin{aligned}\int\gamma\circ\boldsymbol{\psi}_{\mathrm{canc}}(x,y)\,\mathrm{d}\dot{\nu}(x)&=1-2\dot{\nu}\Big(\Big\{x\in\{0,1\}^{\mathbb{Z}^d}:|x\wedge y| \text{ is odd}\Big\}\Big)\\&=\mathbb{P}^y[\exists t\geq 0:D_t=\underline{0}]\qquad(y\in\{0,1\}^{\mathbb{Z}^d}_{\mathrm{fin}}).\end{aligned}$$

If $\dot{\nu}=\delta_{\underline{0}}$, then $\mathbb{P}^y[\exists t\geq 0:D_t=\underline{0}]=1$ for all $y\in\{0,1\}^{\mathbb{Z}^d}_{\mathrm{fin}}$, thus D does not survive. On the other hand, if D does not survive and $\dot{Y}$ is a random variable with law $\dot{\nu}$, then (2.24) with $y=\delta_i$, defined via (1.16), implies that $\mathbb{P}[\dot{Y}(i)=0]=1$ ($i\in\mathbb{Z}^d$). Hence $\dot{\nu}=\delta_{\underline{0}}$ as measures on $\{0,1\}^{\mathbb{Z}^d}$ are characterized by their finite-dimensional marginals. □

By Proposition 2.23, to prove Proposition 2.22, it suffices to show that the cCP(λ,δ) does not survive when $\lambda\leq 2\delta$. Let now $d=1$. Following [Sudbury, 1998] (compare the definition of L in [Sudbury, 1998, Section 2]), the idea for the proof of Proposition 2.22 is to construct a supermartingale applying Dynkin's formula to the function $g:\{0,1\}^{\mathbb{Z}}_{\mathrm{fin}}\setminus\{\underline{0}\}\to\mathbb{N}_0$ defined as

$$g(x):=\max\{i\in\mathbb{Z}:x(i)=1\}-\min\{i\in\mathbb{Z}:x(i)=1\}\qquad(x\in\{0,1\}^{\mathbb{Z}}_{\mathrm{fin}}).\tag{2.25}$$

To be able to apply Dynkin's formula, one can "reduce" the cCP to a finite state space similarly as in [Sturm and Swart, 2008b, Proof of Lemma 3]. A full proof including the technical details is given below.

Proof of Proposition 2.22. Let $d=1$ and assume that $D=(D_t)_{t\geq 0}$ is a cCP(λ,δ) with $\lambda\leq 2\delta$. Using the g from (2.25) we define $f:\{0,1\}^{\mathbb{Z}}_{\mathrm{fin}}\to\mathbb{N}_0$ as

$$f(x)=\begin{cases}g(x)+4&\text{if } x\neq\underline{0},\\0&\text{else,}\end{cases}\qquad(x\in\{0,1\}^{\mathbb{Z}}_{\mathrm{fin}}).$$

One then has that $G_\oplus f(x) \leq 0$ for all $x \in \{0,1\}^{\mathbb{Z}}_{\mathrm{fin}}$, where $G_\oplus$ denotes the generator of the cCP from (2.22). To see this, we first look at $x_{101}, x_{11} \in \{0,1\}^{\mathbb{Z}}_{\mathrm{fin}}$ defined as

$$x_{101}(k) = \begin{cases} 1 & \text{if } k \in \{0,2\}, \\ 0 & \text{else,} \end{cases} \qquad x_{11}(k) = \begin{cases} 1 & \text{if } k \in \{0,1\}, \\ 0 & \text{else,} \end{cases} \qquad (k \in \mathbb{Z}).$$

In the configuration x_{101} the one at the origin reproduces with rate λ to the left, increasing the function f by one and it dies with rate δ, decreasing f by two. A reproduction to the right has no effect on f. By symmetry, an analogous statement holds for the one at $2 \in \mathbb{Z}$ so that $G_\oplus f(x_{101}) = 2\lambda - 4\delta$. For x_{11} on the other hand, both a death of the one at the origin and a reproduction of the one at the origin to the right reduce f by one, while a reproduction to the left again increases f by one. Hence $G_\oplus f(x_{11}) = -2\delta$. Let now $x \in \{0,1\}^{\mathbb{Z}}_{\mathrm{fin}}$ be an arbitrary configuration with at least two ones. As f is shift-invariant, i.e., $f = f \circ \theta_i^{-1}$ for all $i \in \mathbb{Z}$, one has that $G_\oplus f(x) \leq G_\oplus f(x_{101})$ if x has the form $010\ldots010$, $G_\oplus f(x) = G_\oplus f(x_{11})$ if x has the form $011\ldots110$ and $G_\oplus f(x) \leq (G_\oplus f(x_{11}) + G_\oplus f(x_{101}))/2$ if x has the form $010\ldots110$ or $011\ldots010$. Note that we had to use inequalities above as a death event of a one at the edge of a configuration reduces f by the number of zeros "to the inside" of this one, hence by at least two if there is a zero directly to the inside of the one. Finally, we consider the special case $x = \delta_0$ (defined by (1.16)) in which with rate 2λ the lone individual reproduces (either to the left or to the right) and with rate δ it dies. Hence $G_\oplus f(\delta_0) = G_\oplus f(x_{101}) = 2\lambda - 4\delta$, which was the reason to add the 4 in the definition of f. This completes the argument that $\lambda \leq 2\delta$ implies that $G_\oplus f(x) \leq 0$ for all $x \in \{0,1\}^{\mathbb{Z}}_{\mathrm{fin}}$.

The rest of the proof is a standard argument from the theory of continuous-time Markov chains, but, for the sake of completeness, we state it in detail. Let $N \in \mathbb{N}$ be arbitrary and set $\tau_N := \inf\{t \geq 0 : f(D_t) \geq N + 4\}$. We claim that $M^N = (M^N_t)_{t \geq 0}$ defined as

$$M^N_t := f(D_{t \wedge \tau_N}) - \int_0^{t \wedge \tau_N} G_\oplus f(D_s)\, \mathrm{d}s \qquad (t \geq 0)$$

is a martingale. Let

$$\{0,1\}^{\mathbb{Z}}_N := \Big\{x \in \{0,1\}^{\mathbb{Z}}_{\mathrm{fin}} : x(i) = 0 \text{ if } i \notin \{0,\ldots,N-1\}\Big\} \cup \{x_N\},$$

where

$$x_N(i) := \begin{cases} 1 & \text{if } i \in \{0,N\}, \\ 0 & \text{else,} \end{cases} \qquad (i \in \mathbb{Z}).$$

By shifting every $x \in \{0,1\}^{\mathbb{Z}}_{\mathrm{fin}}$ so that its leftmost 1 lies at the origin, we can construct a continuous-time Markov chain $Y = (Y_t)_{t \geq 0}$ on the finite state space $\{0,1\}^{\mathbb{Z}}_N$ so that

$$M^N_t = f(Y_t) - \int_0^t G_\oplus f(Y_s)\, \mathrm{d}s \qquad (t \geq 0).$$

As a continuous-time Markov chain on a finite state space, the process Y is a Feller process, and Dynkin's formula implies that M^N is indeed a martingale.

As $G_\oplus f(x) \leq 0$ for all $x \in \{0,1\}^{\mathbb{Z}}_{\text{fin}}$, we conclude that $M^s = (f(D_{t\wedge\tau_N}))_{t\geq 0}$ is a uniformly integrable supermartingale, and the martingale convergence theorem implies that M^s converges almost surely and in L_1 to a random variable M_∞. The random variable M_∞ is supported on $\{0, N+4\}$, as $M_\infty \in \{1,\dots,N+3\}$ would imply that there exists a $t_0 \geq 0$ such that $M^s_t = M^s_{t_0} \in \{1,\dots,N+3\}$ for all $t \geq t_0$, which has probability zero. Hence,

$$4 = \mathbb{E}^{\delta_0}[f(D_0)] \geq \mathbb{E}[M_\infty] = (N+4)(1-\mathbb{P}[M_\infty = 0]),$$

and we conclude that

$$\mathbb{P}^{\delta_0}[\exists t \geq 0 : D_t = \underline{0}] \geq \mathbb{P}^{\delta_0}[\exists t \leq \tau_N : D_t = \underline{0}] = \mathbb{P}[M_\infty = 0] \geq \frac{N}{N+4}.$$

As N was arbitrary, it follows that $\mathbb{P}^{\delta_0}[\exists t \geq 0 : D_t = \underline{0}] = 1$ and Proposition 2.23 implies that $\dot{\nu} = \delta_{\underline{0}}$. This establishes that $\lambda_{\text{cCP}} \geq 2$. □

As the methods in the proof of Proposition 2.22 are essentially one-dimensional in nature, it is not clear how to generalize the result to higher dimensions.

2.7.3 The double contact processes

Let, for the rest of this chapter, $S := \{0,1\} \times \{0,1\}$ and let $\veebar$ denote the product operator of $\vee$ and $\oplus$, i.e., its Cayley table is given as

$\veebar$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,0)$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,1)$	$(0,1)$	$(0,0)$	$(1,1)$	$(1,0)$
$(1,0)$	$(1,0)$	$(1,1)$	$(1,0)$	$(1,1)$
$(1,1)$	$(1,1)$	$(1,0)$	$(1,1)$	$(1,0)$

.

Thus, $(S, \veebar) \cong \mathbf{M}_{23}$ from Table 2.4.

We will be interested in a joint process, consisting of a CP and a cCP, that are coupled in such a way that some of the infections and deaths happen for both processes at the same times. Our motivation to study this coupled process comes primarily from the theoretical side of view and further couplings of "classic" interacting particle systems can be studied in a similar way. However, in order to prevent the reader from getting lost in abstract statements, we stick to this one process.

Informally, the coupled process of interest will behave in the following way. With rates $\lambda, \delta \geq 0$ infections and deaths, respectively, happen simultaneously for the CP and the cCP. With rates $\lambda_\vee, \delta_\vee \geq 0$ they only happen for the CP and with rates $\lambda_\oplus, \delta_\oplus \geq 0$ only for the cCP.

It will be helpful to write the generator of the coupled process in a form similar to (2.22). We define infection maps $\texttt{INF}_{ij}, \texttt{inf}^1{}_{ij}, \texttt{inf}^2{}_{ij} : S^{\mathbb{Z}^d} \to S^{\mathbb{Z}^d}$ and death maps $\texttt{DTH}_i, \texttt{dth}^1{}_i, \texttt{dth}^2{}_i : S^{\mathbb{Z}^d} \to S^{\mathbb{Z}^d}$ as

$$\begin{aligned} \texttt{INF}_{ij}(x) &:= (\texttt{inf}^{\vee}_{ij}(x_1), \texttt{inf}^{\oplus}_{ij}(x_2)), & \texttt{DTH}_i(x) &:= (\texttt{dth}_i(x_1), \texttt{dth}_i(x_2)), \\ \texttt{inf}^1{}_{ij}(x) &:= (\texttt{inf}^{\vee}_{ij}(x_1), x_2), & \texttt{dth}^1{}_i(x) &:= (\texttt{dth}_i(x_1), x_2), \\ \texttt{inf}^2{}_{ij}(x) &:= (x_1, \texttt{inf}^{\oplus}_{ij}(x_2)), & \texttt{dth}^2{}_i(x) &:= (x_1, \texttt{dth}_i(x_2)), \end{aligned} \tag{2.26}$$

for $x=(x_1,x_2)\in S^{\mathbb{Z}^d}$, where the maps on the right hand sides are the maps from (2.21). By Lemma 2.10, all these maps are monoid homomorphisms from $(S^{\mathbb{Z}^d},\veebar)$ to itself. In particular, they map $\underline{(0,0)}$ to itself, so we may define $S^{\mathbb{Z}^d}_{\rm fin}$ in parallel to (1.15). We define the generator $G_{\veebar}$ as

$$\begin{aligned}G_{\veebar}f(x):=\lambda\sum_{i,j\in\mathbb{Z}^d:i\sim j}&\Big\{f(\texttt{INF}_{ij}(x))-f(x)\Big\}+\delta\sum_{i\in\mathbb{Z}^d}\Big\{f(\texttt{DTH}_i(x))-f(x)\Big\}\\ +\lambda_\vee\sum_{i,j\in\mathbb{Z}^d:i\sim j}&\Big\{f(\texttt{inf}^{\texttt{1}}{}_{ij}(x))-f(x)\Big\}+\delta_\vee\sum_{i\in\mathbb{Z}^d}\Big\{f(\texttt{dth}^{\texttt{1}}{}_i(x))-f(x)\Big\}\\ +\lambda_\oplus\sum_{i,j\in\mathbb{Z}^d:i\sim j}&\Big\{f(\texttt{inf}^{\texttt{2}}{}_{ij}(x))-f(x)\Big\}+\delta_\oplus\sum_{i\in\mathbb{Z}^d}\Big\{f(\texttt{dth}^{\texttt{2}}{}_i(x))-f(x)\Big\},\end{aligned}\tag{2.27}$$

where $\lambda,\delta,\lambda_\vee,\delta_\vee,\lambda_\oplus,\delta_\oplus\geq 0$ are model parameters. It is easy to see that (1.7) is again satisfied. Hence, by Theorem 1.3, there exists an interacting particle system $X=(X^1,X^2)=(X^1_t,X^2_t)_{t\geq 0}$ with generator $G_{\veebar}$. We call X the *double contact process* and denote it shortly as $2\mathrm{CP}(\lambda,\delta,\lambda_\vee,\delta_\vee,\lambda_\oplus,\delta_\oplus)$. If X is a $2\mathrm{CP}(\lambda,\delta,\lambda_\vee,\delta_\vee,\lambda_\oplus,\delta_\oplus)$, then X^1 is a $\mathrm{CP}(\lambda+\lambda_\vee,\delta+\delta_\vee)$ and X^2 is a $\mathrm{cCP}(\lambda+\lambda_\oplus,\delta+\delta_\oplus)$.

In particular, if $\lambda=\delta=0$, then X^1 and X^2 are independent processes. On the other extreme, if $\delta_\vee=\lambda_\vee=\delta_\oplus=\lambda_\oplus=0$, then X^1 and X^2 are fully coordinated in the sense that their infections and deaths happen at the same times. An interesting consequence of this choice of parameters is that the CP stochastically dominates the cCP. The following lemma says that this holds a bit more generally.

Lemma 2.24 (Special parameters)**.** *Assume that $X=(X^1,X^2)=(X^1_t,X^2_t)_{t\geq 0}$ is a $2\mathrm{CP}(\lambda,\delta,\lambda_\vee,\delta_\vee,\lambda_\oplus,\delta_\oplus)$ with $\delta_\vee=\lambda_\oplus=0$. Then*

$$X^1_0(k)\geq X^2_0(k)\quad(k\in\mathbb{Z}^d)\quad\textit{implies}\quad X^1_t(k)\geq X^2_t(k)\quad(k\in\mathbb{Z}^d,\ t\geq 0).\tag{2.28}$$

Proof. This follows directly from the definition of the maps in (2.26). □

We are going to use the fact that $\mathbf{M}_{23}$ is $\mathbf{M}_5$-dual to itself with respect to ψ_{235} from Table 2.5. Let $\gamma:\{0,1,2\}\to\{-1,0,1\}$ denote the faithful representation of $\mathbf{M}_5$ as a submonoid of $(\mathbb{C},\cdot\,)$ from Chapter 2.6, i.e., $\gamma(0)=1$, $\gamma(1)=-1$, $\gamma(2)=0$. It follows that $(S,\veebar)$ is $(\{-1,0,1\},\cdot\,)$-dual to itself with respect to $\gamma\circ\psi_{235}:S\times S\to\{-1,0,1\}$, given in matrix form as

$$\gamma\circ\psi_{235}:=\begin{pmatrix}1&1&1&1\\1&-1&1&-1\\1&1&0&0\\1&-1&0&0\end{pmatrix}\tag{2.29}$$

when the elements of S are ordered as $(0,0),(0,1),(1,0),(1,1)$. Let $\Psi:S^{\mathbb{Z}^d}\times S^{\mathbb{Z}^d}_{\rm fin}\to\{-1,0,1\}$ denote the "global duality function" based on $\gamma\circ\psi_{235}$, i.e.,

$$\Psi(x,y):=\prod_{k\in\mathbb{Z}^d}\gamma\circ\psi_{235}(x(k),y(k))\qquad(x\in S^{\mathbb{Z}^d},\ y\in S^{\mathbb{Z}^d}_{\rm fin}).\tag{2.30}$$

Using the duality of the maps identified at the end of Chapter 2.7.1 one concludes from Theorem 2.9 that the 2CP is (pathwise) "self-dual" with respect to Ψ in the same sense as the CP and the cCP are (pathwise) self-dual (compare the end of Chapter 2.7.1). Recall that Ψ is, due to Proposition 2.20, informative.

2.7.4 Invariant laws of the double contact process

We are interested in the long-time behavior of the 2CP started in a homogeneous distribution. Let the 2CP again be denoted by $X = (X_t)_{t\geq 0} = (X_t^1, X_t^2)_{t\geq 0}$. With a slight abuse of notation we define shift operators $\theta_i : S^{\mathbb{Z}^d} \to S^{\mathbb{Z}^d}$ by applying the operators from (2.23) in both coordinates of $S = \{0,1\} \times \{0,1\}$. As for distributions on $\{0,1\}^{\mathbb{Z}^d}$, we say that a probability measure μ on $S^{\mathbb{Z}^d}$ is homogeneous if $\mu = \mu \circ \theta_i^{-1}$ $(i \in \mathbb{Z}^d)$. Since $\underline{(0,0)}$ is the only trap for the 2CP, a probability measure $\mu \in \mathcal{M}_1(S^{\mathbb{Z}^d})$ is X-non-trivial in the sense of the definition in Chapter 1.2 if and only if

$$\mu\Big(\{\underline{(0,0)}\}\Big) = 0.$$

We set

$$\begin{aligned}
\mathcal{S}_{(0,*)} &:= \Big\{x = (x_1, x_2) \in S^{\mathbb{Z}^d} : x_1 = \underline{0}\Big\}, \\
\mathcal{S}_{(*,0)} &:= \Big\{x = (x_1, x_2) \in S^{\mathbb{Z}^d} : x_2 = \underline{0}\Big\}, \\
\mathcal{S}_{\text{mix}} &:= S^{\mathbb{Z}^d} \setminus \big(\mathcal{S}_{(0,*)} \cup \mathcal{S}_{(*,0)}\big).
\end{aligned}$$

The known results for the CP and the cCP imply that the 2CP X started in an X-non-trivial homogeneous distribution on $\mathcal{S}_{(*,0)}$ converges weakly to $\bar{\nu} \otimes \delta_{\underline{0}}$. Analogously, the 2CP started in an X-non-trivial homogeneous distribution on $\mathcal{S}_{(0,*)}$ converges weakly to $\delta_{\underline{0}} \otimes \dot{\nu}$. If X is started in an X-non-trivial homogeneous distribution on $\mathcal{S}_{\text{mix}}$, then the laws of X_t^1 and X_t^2 individually converge weakly to $\bar{\nu}$ and $\dot{\nu}$, respectively. However, as a measure on a product space is in general not determined by its marginals, the long-time behavior of the joint law of $X_t = (X_t^1, X_t^2)$ is less straightforward. We will prove the following result.

Theorem 2.25 (Joint invariant law). *Let $X = (X^1, X^2) = (X_t^1, X_t^2)_{t\geq 0}$ be a* 2CP *with parameters $\lambda, \delta, \lambda_\vee, \delta_\vee, \lambda_\oplus, \delta_\oplus \geq 0$ so that $\lambda + \lambda_\vee + \delta + \delta_\vee > 0$ and $\lambda + \lambda_\oplus + \delta + \delta_\oplus > 0$. Then X has an invariant law ν that is uniquely characterized by the relation*

$$\int \Psi(x, y)\, \mathrm{d}\nu(x) = \mathbb{P}^y\Big[\exists t \geq 0 : X_t = \underline{(0,0)}\Big] \qquad (y \in S_{\text{fin}}^{\mathbb{Z}^d}). \tag{2.31}$$

If X is started in a homogeneous initial law that is concentrated on $\mathcal{S}_{\text{mix}}$, then

$$\mathbb{P}[X_t \in \cdot\,] \underset{t\to\infty}{\Longrightarrow} \nu, \tag{2.32}$$

i.e., the law of X converges weakly to ν.

Note that (2.32) implies that ν is (as $\bar{\nu}$ and $\dot{\nu}$) homogeneous. In the special case that $\delta_\vee = \lambda_\oplus = 0$, corresponding to the monotone coupling of CP and cCP, one has that

$$\nu\Big(\Big\{x \in S^{\mathbb{Z}^d} : \exists k \in \mathbb{Z}^d : x(k) = (0,1)\Big\}\Big) = 0,$$

as we can chose a homogeneous initial law that is concentrated on $\mathcal{S}_{\text{mix}}$ with the above property. This property is then preserved by the dynamics. One example of such an initial law would be the Dirac measure concentrated on $\underline{(1,1)}$. Thus, as long as the initial distribution of this special 2CP is homogeneous and concentrated on $\mathcal{S}_{\text{mix}}$, the law of this 2CP converges weakly to a monotonically coupled law, no matter how high the density of $(0,1)$'s was in the initial distribution.

Taking into account our earlier remarks about initial laws on $\mathcal{S}_{(0,*)}$ and $\mathcal{S}_{(*,0)}$, one can conclude (compare [Swart, 2022, Corollary 6.39]) that all homogeneous invariant laws of the 2CP are convex combinations of $\delta_{\underline{0}} \otimes \delta_{\underline{0}}$, $\bar{\nu} \otimes \delta_{\underline{0}}$, $\delta_{\underline{0}} \otimes \dot{\nu}$ and ν.

2.7.5 The main convergence result

We devote this subchapter to the proof of Theorem 2.25. We are going to use several auxiliary lemmas. The first one is [Swart, 2022, Lemma 6.37]. As already indicated in Chapter 2.7.1, we set $\text{supp}(z) := \{i \in \mathbb{Z}^d : z(i) = 1\}$ for $z \in \{0,1\}^{\mathbb{Z}^d}$. Throughout this subsection we will use $|z|$ as a shorthand for $|\text{supp}(z)|$. Moreover, as in the proof of Proposition 2.23, the symbol $\wedge$ below denotes the pointwise minimum, i.e., $(z_1 \wedge z_2)(i) = \min\{z_1(i), z_2(i)\}$ for $i \in \mathbb{Z}^d,\ z_1, z_2 \in \{0,1\}^{\mathbb{Z}^d}$.

Lemma 2.26 (Non-zero intersection: CP)**.** *Let $C = (C_t)_{t\geq 0}$ be a $\text{CP}(\lambda, \delta)$ ($\lambda > 0$, $\delta \geq 0$) with C-non-trivial homogeneous initial distribution. Given $\varepsilon > 0$, for each time $s > 0$ there exists an $N_{\text{CP}} \in \mathbb{N}$ such that for any $z \in \{0,1\}^{\mathbb{Z}^d}$ with $|z| \geq N_{\text{CP}}$ one has*

$$\mathbb{P}[C_s \wedge z = \underline{0}] \leq \varepsilon.$$

Additionally, we are going to use the following application of [Sturm and Swart, 2008a, Corollary 9].

Lemma 2.27 (Parity indeterminacy)**.** *Let $D = (D_t)_{t\geq 0}$ be a $\text{cCP}(\lambda, \delta)$ ($\lambda > 0$, $\delta \geq 0$) with D-non-trivial homogeneous initial distribution. Given $\varepsilon > 0$, for each time $s > 0$ there exists an $N_{\text{cCP}} \in \mathbb{N}$ such that for any $z \in \{0,1\}^{\mathbb{Z}^d}_{\text{fin}}$ with $|z| \geq N_{\text{cCP}}$ one has*

$$\left|\mathbb{P}[|D_s \wedge z| \textit{ is odd}] - \frac{1}{2}\right| \leq \varepsilon.$$

As [Sturm and Swart, 2008a, Corollary 9] is not stated in the most accessible form, we elaborate. Let $\mathbf{M}_2 = (\{0,1\}, \oplus)$ be the monoid from Table 2.1, where $\oplus$ denotes addition modulo 2. Lemma 2.10 implies that each local $m \in \mathcal{H}(\mathbf{M}_2^{\mathbb{Z}^d}, \mathbf{M}_2^{\mathbb{Z}^d})$ can be identified with an infinite matrix $(M_{ij})_{i,j\in\mathbb{Z}^d}$ with values in $\mathcal{H}(\mathbf{M}_2, \mathbf{M}_2) = \{o, \text{id}\}$, where $o : \{0,1\} \to \{0,1\}$ denotes the function constantly 0 and $\text{id} : \{0,1\} \to \{0,1\}$ denotes the identity. Then, by (2.12),

$$m(z)(i) = \bigoplus_{j\in\mathbb{Z}^d} M_{ij}(z(j)) \qquad (i \in \mathbb{Z}^d,\ z \in \{0,1\}^{\mathbb{Z}^d}).$$

As $\mathcal{H}(\mathbf{M}_2, \mathbf{M}_2)$ is of this simple form, we can equivalently define an infinite matrix $(A_{ij})_{i,j\in\mathbb{Z}^d}$ with values in $\{0,1\}$ such that

$$m(z)(i) = \bigoplus_{j\in\mathbb{Z}^d} (A_{ij} \cdot z(j)) \qquad (i \in \mathbb{Z}^d,\ z \in \{0,1\}^{\mathbb{Z}^d}), \tag{2.33}$$

where $\cdot$ denotes the usual product in $\mathbb{R}$. Corresponding to the usual matrix-vector multiplication, we denote the right hand side of (2.33) shortly as $Az(i)$ ($i \in \mathbb{Z}^d$). Hence, with this notation, $m(z) = Az$ ($z \in \{0,1\}^{\mathbb{Z}^d}$).

As $\mathbf{M}_2$ is $\mathbf{M}_2$-dual to itself, $\hat{m}$, the dual map of a local map $m \in \mathcal{H}(\mathbf{M}_2^{\mathbb{Z}^d}, \mathbf{M}_2^{\mathbb{Z}^d})$ with respect to $\boldsymbol{\psi}_{\text{canc}} : \{0,1\}^{\mathbb{Z}^d} \times \{0,1\}^{\mathbb{Z}^d}_{\text{fin}} \to \{0,1\}$ from (2.18) is, due to (2.13) and the self-duality of o and id, given as

$$\hat{m}(y)(i) = \bigoplus_{j \in \mathbb{Z}^d} M_{ji}(y(j)) \qquad (i \in \mathbb{Z}^d,\ y \in \{0,1\}^{\mathbb{Z}^d}_{\text{fin}}).$$

Hence, $\hat{m}(y) = A^\dagger y$ ($y \in \{0,1\}^{\mathbb{Z}^d}$), where $A^\dagger$ denotes the adjoint of A, i.e., $A^\dagger_{ij} := A_{ji}$ ($i,j \in \mathbb{Z}^d$).

Let $\mathcal{A}$ be the collection of all (infinite) matrices of the form $A = (A_{ij})_{i,j \in \mathbb{Z}^d}$ with $A_{ij} \in \{0,1\}$ for all $i,j \in \mathbb{Z}^d$ and $\sum_{i,j} A_{ij} < \infty$. In Sturm and Swart [2008a] the authors are interested in an interacting particle system $Z = (Z_t)_{t \geq 0}$ with state space $\{0,1\}^{\mathbb{Z}^d}$, that jumps from its current state $z \in \{0,1\}^{\mathbb{Z}^d}$ as

$$z \mapsto z \oplus Az \quad \text{with rate} \quad a(A), \tag{2.34}$$

where $(a(A))_{A \in \mathcal{A}}$ are non-negative rates and the operator $\oplus$ has to be interpreted in a pointwise sense, as well as an interacting particle system $Y = (Y_t)_{t \geq 0}$ that jumps as

$$y \mapsto y \oplus A^\dagger y \quad \text{with rate} \quad a(A).$$

In order for these interacting particle systems to be well-defined, Sturm and Swart [2008a] assume (compare [Sturm and Swart, 2008a, Condition (3.1)]) that

$$\sup_{i \in \mathbb{Z}^d} \sum_{A \in \mathcal{A}} a(A) \left|\{j : A_{ij} = 1\}\right| < \infty \quad \text{and} \quad \sup_{i \in \mathbb{Z}^d} \sum_{A \in \mathcal{A}} a(A) \left|\{j : A^\dagger_{ij} = 1\}\right| < \infty. \tag{2.35}$$

Evidently, there is a one-to-one correspondence between a map acting as in (2.34) and a map defined as in (2.33). Thus, the interacting particle systems studied by Sturm and Swart [2008a] are cancellative interacting particle systems in the sense of Chapter 2.4.2. Moreover, the summability conditions in (2.35) imply the summability condition (1.7) for both Z and Y. Indeed, denoting the map acting as in (2.34) by $m_A : \{0,1\}^{\mathbb{Z}^d} \to \{0,1\}^{\mathbb{Z}^d}$ one has

$$\mathcal{D}(m_A) = \left\{ i \in \mathbb{Z}^d : \exists j \in \mathbb{Z}^d : A_{ij} \neq 0 \right\}$$

and

$$\mathcal{R}(m_A[i]) = \left\{ j \in \mathbb{Z}^d : A_{ij} \neq 0 \right\} \qquad (i \in \mathbb{Z}^d).$$

We will restate [Sturm and Swart, 2008a, Corollary 9]. By definition, we say that the rates $(a(A))_{A \in \mathcal{A}}$ are *translation-invariant* if

$$a(\theta_i A) = a(A) \qquad (i \in \mathbb{Z}^d,\ A \in \mathcal{A}), \tag{2.36}$$

where the "translated" matrix $\theta_i A \in \mathcal{A}$ is defined as $(\theta_i A)_{jk} := A_{j-i,k-i}$ ($j,k \in \mathbb{Z}^d$). Moreover, we say that a *state* $z \in \{0,1\}^{\mathbb{Z}^d}$ is *Z-non-trivial* if

$$\mathbb{P}^z[(Z_t(i))_{i \in \Delta} = (x(i))_{i \in \Delta}] > 0 \quad \forall t > 0,\ \text{finite } \Delta \subset \mathbb{Z}^d,\ \text{and } (x(i))_{i \in \Delta} \in \{0,1\}^\Delta, \tag{2.37}$$

i.e., if Z started in z reaches at any positive time any configuration on any finite subset of $\mathbb{Z}^d$ with positive probability. For any finite subset $\mathcal{B} \subset \mathcal{A}$ such that $a(B) > 0$ for all $B \in \mathcal{B}$, we define, for $z \in \{0,1\}^{\mathbb{Z}^d}$,

$$\|z\|_{\mathcal{B}} := \left|\left\{i \in \mathbb{Z}^d : \exists y \in \{0,1\}^{\mathbb{Z}^d} \text{ and } B \in \mathcal{B} \text{ s.t. } \boldsymbol{\psi}_{\mathrm{canc}}(z, (\theta_i B)y) = 1\right\}\right|.$$

With these definitions, [Sturm and Swart, 2008a, Corollary 9] can be restated as follows.

Proposition 2.28 (General parity indeterminacy)**.** *Let $Z = (Z_t)_{t\geq 0}$ satisfying (2.34) be given via non-negative translation invariant rates $(a(A))_{A\in\mathcal{A}}$ that satisfy (2.35). Assume that Z is started in a homogeneous initial law that is concentrated on Z-non-trivial configurations. Then, for any finite subset $\mathcal{B} \subset \mathcal{A}$ such that $a(B) > 0$ for all $B \in \mathcal{B}$, any $\varepsilon > 0$ and $t > 0$, there exists an $N \in \mathbb{N}$ such that*

$$\left|\mathbb{P}[|Z_t \wedge y| \textit{ is odd}] - \frac{1}{2}\right| \leq \varepsilon \tag{2.38}$$

for all $y \in \{0,1\}^{\mathbb{Z}^d}_{\mathrm{fin}}$ with $\|y\|_{\mathcal{B}} \geq N$.

Proof. This is a simple reformulation of [Sturm and Swart, 2008a, Corollary 9]. There it is proved that if $(y_n)_{n\in\mathbb{N}} \subset \{0,1\}^{\mathbb{Z}^d}_{\mathrm{fin}}$ satisfy $\|y_n\|_{\mathcal{B}} \to \infty$, then

$$\mathbb{P}[|Z_t \wedge y_n| \text{ is odd}] \underset{n\to\infty}{\longrightarrow} \frac{1}{2}. \tag{2.39}$$

To see that this implies the claim of Proposition 2.28, note that if the claim would be false, then there exists an $\varepsilon > 0$ such that for all $n \in \mathbb{N}$ one can find $y_n \in \{0,1\}^{\mathbb{Z}^d}_{\mathrm{fin}}$ with $\|y_n\|_{\mathcal{B}} \geq n$ such that the left-hand side of (2.38) is greater than ε, contradicting (2.39). □

We show that applying Proposition 2.28 to the cCP yields Lemma 2.27.

Proof of Lemma 2.27. We first show that the jump rates of the cancellative contact process $D = (D_t)_{t\geq 0}$ can be cast in the form (2.34). Let $e_1, \ldots, e_d \in \mathbb{Z}^d$ denote the unit vectors and let $0 \in \mathbb{Z}^d$ denote the origin. For $1 \leq k \leq d$, we define $A^{\pm}_k \in \mathcal{A}$ by $(A^{\pm}_k)_{ij} := 1$ if $(i,j) = (\pm e_k, 0)$ and $(A^{\pm}_k)_{ij} := 0$ otherwise. Also, we define $A^0 \in \mathcal{A}$ by $(A^0)_{ij} := 1$ if $(i,j) = (0,0)$ and $(A^0)_{ij} := 0$ otherwise. Finally, we define rates $(a(A))_{A\in\mathcal{A}}$ by

$$a(\theta_i A^{\pm}_k) := \lambda \quad \text{and} \quad a(\theta_i A^0) := \delta \qquad (i \in \mathbb{Z}^d,\ 1 \leq k \leq d),$$

and $a(A) := 0$ in all other cases. Clearly, these rates are translation-invariant in the sense of (2.36) and satisfy the summability condition (2.35). Also, a jump of the form $x \mapsto x \oplus (\theta_{-i} A^{\pm}_k)x$ corresponds to a jump of the form $x \mapsto \mathtt{inf}^{\oplus}_{i,i\pm e_k}(x)$ in the notation of Section 2.7.1 and a jump of the form $x \mapsto x \oplus (\theta_{-i} A^0)x$ corresponds to a jump of the form $x \mapsto \mathtt{dth}_i(x)$, so the process defined by these rates is a cCP(λ, δ). The claim of Lemma 2.27 will now follow from Proposition 2.28 provided we show that: (i) each configuration $z \in \{0,1\}^{\mathbb{Z}^d} \setminus \{\underline{0}\}$ is D-non-trivial and: (ii) we can choose $\mathcal{B}$ such that $\|z\|_{\mathcal{B}} = |z|$ $(z \in \{0,1\}^{\mathbb{Z}^d})$.

We start by proving (ii). We set $\mathcal{B} = \{A_1^+\}$, where A_1^+ as defined above is one of the matrices corresponding to an infection next to the origin. Then $a(A_1^+) = \lambda > 0$. Moreover,

$$\boldsymbol{\psi}_{\text{canc}}(z, (\theta_{-i}A_1^+)y) = z(i + e_1) \cdot y(i) \qquad (z, y \in \{0,1\}^{\mathbb{Z}^d}).$$

Hence,

$$z(i) = 1$$

if and only if

$$i - e_1 \in \Big\{k \in \mathbb{Z}^d : \exists y \in \{0,1\}^{\mathbb{Z}^d} \text{ and } B \in \mathcal{B} \text{ s.t. } \boldsymbol{\psi}_{\text{canc}}(z, (\theta_i B)y) = 1\Big\},$$

which shows that $\|z\|_{\mathcal{B}} = |z|$ $(z \in \{0,1\}^{\mathbb{Z}^d})$.

It remains to prove (i). Fix $z \in \{0,1\}^{\mathbb{Z}^d} \setminus \{\underline{0}\}$, a finite set $\Delta \subset \mathbb{Z}^d$, and $(x(i))_{i\in\Delta} \in \{0,1\}^{\Delta}$. Using the fact that $z \neq \underline{0}$ and $\lambda > 0$, in a finite number of infection steps, we can infect each site in $\Delta \cup \{i \in \mathbb{Z}^d : \exists j \in \Delta : j \sim i\}$. Starting with the sites in Δ with the highest graph distance to $\mathbb{Z}^d \setminus \Delta$, we then can remove the infection from all sites $i \in \Delta$ with $x(i) = 0$ only using further infections, proving that (2.37) holds. □

The true strength of Proposition 2.28 lies in the fact that it can be applied even in situations where the definitions of Z-non-triviality and the norm $\|\cdot\|_{\mathcal{B}}$ are more complicated. In particular, [Sturm and Swart, 2008a, Theorem 3] is based on an application of Proposition 2.28 in a situation where the Z-non-trivial configurations are all $z \neq \underline{0}, \underline{1}$, and

$$\|z\|_{\mathcal{B}} = \Big|\Big\{(i,j) : |i-j| = 1,\ y(i) \neq y(j)\Big\}\Big|.$$

Instead of proving Lemma 2.27, we could have also followed the strategy of the proof of [Bramson et al., 1991, Theorem 1.2]. There the authors use the graphical representation of the cCP explicitly to work around proving Lemma 2.27.

The third and final lemma extends [Swart, 2022, Lemma 6.36] and [Bramson et al., 1991, Lemma 2.1].

Lemma 2.29 (Extinction or unbounded growth)**.** *Let $Z = (Z_t)_{t\geq 0}$ be either a* CP(λ, δ) *or a* cCP(λ, δ) *$(\lambda, \delta \geq 0,\ \lambda + \delta > 0)$. For each $z \in \{0,1\}^{\mathbb{Z}^d}_{\text{fin}}$ and $N \in \mathbb{N}$ one has*

$$\lim_{t\to\infty} \mathbb{P}^z[0 < |Z_t| < N] = 0. \tag{2.40}$$

Proof. If $z = \underline{0}$ the statement is trivial, so let $z \in \{0,1\}^{\mathbb{Z}^d}_{\text{fin}} \setminus \{\underline{0}\}$. In the case $\lambda, \delta > 0$ [Swart, 2022, Lemma 6.36] and [Bramson et al., 1991, Lemma 2.1] imply

$$\mathbb{P}^z\Big[\exists t \geq 0 : Z_t = \underline{0} \quad \text{or} \quad |Z_t| \to \infty \text{ as } t \to \infty\Big] = 1 \tag{2.41}$$

for the CP and the cCP, respectively, and (2.41) clearly implies (2.40). In fact, the two proofs are just reformulations of each other, both based on Lévy's 0-1 law.

In the case $\lambda = 0$, $\delta > 0$ there is no difference between a CP and a cCP and

$$\mathbb{P}^z[\exists t \geq 0 : Z_t = \underline{0}] = \lim_{t\to\infty} \mathbb{P}^z[Z_t = \underline{0}] = \lim_{t\to\infty} \left(1 - e^{-\delta t}\right)^{|z|} = 1$$

since $\underline{0}$ is absorbing. This implies (2.41) and hence also (2.40).

In the case $\lambda > 0$, $\delta = 0$, and if Z is a CP, the function $t \mapsto |Z_t|$ is non-decreasing, hence it converges in $\mathbb{N} \cup \{\infty\}$. Let $N \in \mathbb{N}$. One has

$$\mathbb{P}^z[\lim_{t\to\infty} |Z_t| \leq N] = 1 - \mathbb{P}^z[\exists t \geq 0 : |Z_t| > N] = 1 - \lim_{t\to\infty} \mathbb{P}^z[|Z_t| > N] = 0 \tag{2.42}$$

as choosing a suitable sequence of neighbors and neighbors of neighbors of the infected individuals in z yields that

$$\mathbb{P}^z[|Z_t| > N] \geq \left(1 - \mathbb{1}_{\{1,\ldots,N\}}(|z|) e^{-\frac{\lambda t}{N+1-|z|}}\right)^{N+1-|z|}$$

for $t > 0$. Here, in the case that $|z| \leq N$, we have divided time into $N + 1 - |z|$ subintervals and used the fact that $1 - e^{-\lambda t}$ is the probability to infect a previously chosen neighbor of an infected individual during a time interval of length t. Finally, (2.42) implies that

$$\begin{aligned}\mathbb{P}^z[|Z_t| \to \infty \text{ as } t \to \infty] &= 1 - \mathbb{P}^z[\exists N \in \mathbb{N} : \lim_{t\to\infty} |Z_t| = N] \\ &\geq 1 - \sum_{N\in\mathbb{N}} \mathbb{P}^z[\lim_{t\to\infty} |Z_t| \leq N] = 1,\end{aligned}$$

again implying (2.41) and hence also (2.40).

To treat the cCP in the case $\lambda > 0$, $\delta = 0$, we use [Bramson et al., 1991, Theorem 1.3]. It says that a cCP$(1, 0)$, started in *any* initial state other than $\underline{0}$, converges weakly to $\nu_{1/2}$, the product law assigning probability 1/2 to both 0 and 1 at every node. By changing the time scale, the same holds for a cCP$(\lambda, 0)$ with an arbitrary $\lambda > 0$. Let $N \in \mathbb{N}$ and $\varepsilon > 0$. Choose now an $M = M(N, \varepsilon) > N$ so that $p_N := \mathbb{P}[X \leq N] < \varepsilon$ if $X \sim \text{Bin}(M, 1/2)$ is a binomially distributed random variable. Additionally, choose an arbitrary $x \in \{0,1\}^{\mathbb{Z}^d}_{\text{fin}}$ with $|x| = M$. Then, by the weak convergence,

$$\limsup_{t\to\infty} \mathbb{P}^z[|Z_t| \leq N] \leq \lim_{t\to\infty} \mathbb{P}^z[|Z_t \wedge x| \leq N] = p_N < \varepsilon,$$

implying $\lim_{t\to\infty} \mathbb{P}^z[|Z_t| \leq N] = 0$ (i.e., convergence in probability to ∞). Thus, (2.40) holds. □

Using the three lemmas stated in this subchapter, we are able to prove Theorem 2.25. Recall that $S = \{0,1\} \times \{0,1\}$ and that Ψ is defined by (2.30).

Proof of Theorem 2.25. Let $Y = (Y^1, Y^2) = (Y^1_t, Y^2_t)_{t\geq 0}$ be a 2CP with the same parameters as the 2CP $X = (X^1, X^2) = (X^1_t, X^2_t)_{t\geq 0}$ in the formulation of the theorem, but started in the deterministic state $y = (y_1, y_2) \in S^{\mathbb{Z}^d}_{\text{fin}}$. Fix $t > 0$. Following [Jansen and Kurt, 2014, Proposition 4.1], we can construct a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ on which there exist *independent* processes $\widetilde{X} = (\widetilde{X}_t)_{t\geq 0}$ and

$\widetilde{Y} = (\widetilde{Y}_t)_{t\geq 0}$ whose finite-dimensional distributions coincide with those of X and Y, respectively, and

$$\mathsf{E}\Big[\Psi(\widetilde{X}_s, \widetilde{Y}_{t+1-s})\Big] = \mathsf{E}\Big[\Psi(\widetilde{X}_u, \widetilde{Y}_{t+1-u})\Big] \tag{2.43}$$

holds for all $s, u \in [0, t+1]$, where E denotes taking expectation with respect to P. For example, one could construct $\widetilde{X}$ as in Chapter 1.1 but then construct $\widetilde{Y}$ using an *independent* Poisson point set $\hat{\omega}$. Then one could take $(\Omega, \mathcal{F}, \mathsf{P})$ as the product of the two resulting probability spaces. The duality between $\widetilde{X}$ and $\widetilde{Y}$ is then, of course, not pathwise.

Below, we drop the tildes from the notation. Recall that, in contrast to P, the symbol $\mathbb{P}^y$ denotes the law of Y started in $y \in S_{\text{fin}}^{\mathbb{Z}^d}$. Due to the informativeness of Ψ it follows from Lemma 1.11 that

$$\mathcal{H} := \Big\{\Psi(\,\cdot\,, y) : y \in S_{\text{fin}}^{\mathbb{Z}^d}\Big\}$$

is also convergence determining, i.e., showing

$$\lim_{s\to\infty} \mathsf{E}[\Psi(X_s, y)] = \mathbb{P}^y\Big[\exists s \geq 0 : Y_s = \underline{(0,0)}\Big] \tag{2.44}$$

for all $y \in S_{\text{fin}}^{\mathbb{Z}^d}$ implies (2.31) and (2.32). If $y = \underline{(0,0)}$, (2.44) follows trivially from the definition of Ψ, so assume $y \neq \underline{(0,0)}$. We set

$$\lambda_1 := \lambda + \lambda^{\vee}, \quad \delta_1 := \delta + \delta^{\vee}, \quad \lambda_2 := \lambda + \lambda^{\oplus}, \quad \delta_2 := \delta + \delta^{\oplus},$$

so that X^1 and Y^1 are both a $\mathrm{CP}(\lambda_1, \delta_1)$, and X^2 and Y^2 are both a $\mathrm{cCP}(\lambda_2, \delta_2)$. Assume, for now, that $\lambda_1, \lambda_2 > 0$, so that all three auxiliary lemmas above are applicable. Let $\varepsilon > 0$ be arbitrary. Choose N_{CP} and N_{cCP} as in Lemma 2.26 and Lemma 2.27 in dependence of the chosen ε, $s = 1$, and the model parameters. We have, using the duality equation (2.43) and the law of total expectation, that

$$\begin{aligned}
&\mathsf{E}[\Psi(X_{t+1}, y)] \\
&= \mathsf{E}[\Psi(X_1, Y_t)] \\
&= \mathsf{E}\Big[\Psi(X_1, Y_t) \mid Y_t^1 = Y_t^2 = \underline{0}\Big]\mathbb{P}^y\Big[Y_t^1 = Y_t^2 = \underline{0}\Big] \\
&\quad + \mathsf{E}\Big[\Psi(X_1, Y_t) \mid Y_t^1 = \underline{0},\ 0 < |Y_t^2| < N_{\mathrm{cCP}}\Big] \underbrace{\mathbb{P}^y\Big[Y_t^1 = \underline{0},\ 0 < |Y_t^2| < N_{\mathrm{cCP}}\Big]}_{=:p_1(y,t)} \\
&\quad + \underbrace{\mathsf{E}\Big[\Psi(X_1, Y_t) \mid Y_t^1 = \underline{0},\ |Y_t^2| \geq N_{\mathrm{cCP}}\Big]}_{=:E_1(y,t)} \mathbb{P}^y\Big[Y_t^1 = \underline{0},\ |Y_t^2| \geq N_{\mathrm{cCP}}\Big] \\
&\quad + \mathsf{E}\Big[\Psi(X_1, Y_t) \mid 0 < |Y_t^1| < N_{\mathrm{CP}}\Big] \underbrace{\mathbb{P}^y\Big[0 < |Y_t^1| < N_{\mathrm{CP}}\Big]}_{=:p_2(y,t)} \\
&\quad + \underbrace{\mathsf{E}\Big[\Psi(X_1, Y_t) \mid |Y_t^1| \geq N_{\mathrm{CP}}\Big]}_{=:E_2(y,t)} \mathbb{P}^y\Big[|Y_t^1| \geq N_{\mathrm{CP}}\Big].
\end{aligned} \tag{2.45}$$

Depending on the choice of the model parameters and y, the deterministic initial state of Y, it might happen that some of the events on which we condition above have probability zero. The cases that either $y_1 = \underline{0}$ or $y_2 = \underline{0}$, or the monotonely coupled case $\delta_\vee = \lambda_\oplus = 0$ when y satisfies $y(i) \neq (0,1)$ for all $i \in \mathbb{Z}^d$ are such examples. In these cases we define the corresponding conditioned expectation (arbitrarily) to equal 1. As these conditioned expectations are then multiplied by 0, the lines in (2.45) where they occur drop out. For the remaining ones we can argue as below.

From the definition of Ψ it is clear that $\mathsf{E}[\Psi(X_1, Y_t) \mid Y_t^1 = Y_t^2 = \underline{0}] = 1$ and

$$\mathbb{P}^y\Big[Y_t^1 = Y_t^2 = \underline{0}\Big] \nearrow \mathbb{P}^y\Big[\exists t \geq 0 : Y_t = \underline{(0,0)}\Big]$$

as $t \to \infty$. Moreover, Lemma 2.29 implies that

$$\lim_{t\to\infty} p_1(y,t) = \lim_{t\to\infty} p_2(y,t) = 0.$$

As in the proof of [Swart, 2022, Theorem 6.35] we use Lemma 2.26 to compute that

$$\begin{aligned} |E_2(y,t)| &= \Big|\mathsf{P}\Big[\Psi(X_1,Y_t) = 1 \mid |Y_t^1| \geq N_{\mathrm{CP}}\Big] - \mathsf{P}\Big[\Psi(X_1,Y_t) = -1 \mid |Y_t^1| \geq N_{\mathrm{CP}}\Big]\Big| \\ &\leq \mathsf{P}\Big[\Psi(X_1,Y_t) \neq 0 \mid |Y_t^1| \geq N_{\mathrm{CP}}\Big] \\ &= \mathsf{P}\Big[X_1^1 \wedge Y_t^1 = \underline{0} \mid |Y_t^1| \geq N_{\mathrm{CP}}\Big] \leq \varepsilon \end{aligned} \tag{2.46}$$

by the choice of N_{CP}. For $E_1(y,t)$ one has that

$$\begin{aligned} E_1(y,t) &= 1 - 2\mathsf{P}\Big[\Psi(X_1,Y_t) = -1 \mid Y_t^1 = \underline{0},\ |Y_t^2| \geq N_{\mathrm{cCP}}\Big] \\ &= 1 - 2\mathsf{P}\Big[|X_1^2 \wedge Y_t^2| \text{ is odd } \Big|\ Y_t^1 = \underline{0},\ |Y_t^2| \geq N_{\mathrm{cCP}}\Big] \end{aligned}$$

and, due to the independence of X and Y, we can apply Lemma 2.27 and conclude that

$$|E_1(y,t)| \leq 2\varepsilon.$$

Plugging then back into (2.45) and computing the limit inferior and the limit superior, one concludes (2.44) as ε was arbitrary.

To finish the proof, we consider the case that $\lambda_1 = 0$ and/or $\lambda_2 = 0$. By assumption, λ_i ($i \in \{1,2\}$) can only equal zero if $\delta_i > 0$. The idea is to still use (2.45), where we used $\lambda_1 > 0$ for the treatment of $E_2(y,t)$ and $\lambda_2 > 0$ for the treatment of $E_1(y,t)$. However, if $\lambda_1 = 0$, then Y^1 is a $\mathrm{CP}(0,\delta_1)$ with $\delta_1 > 0$, so the number of infected individuals can only decrease. Choosing $N_{\mathrm{CP}} := |y_1| + 1$ makes the line in (2.45) in which $E_2(y,t)$ appears vanish. Analogously, choosing $N_{\mathrm{cCP}} := |y_2| + 1$ makes the line in which $E_1(y,t)$ appears vanish if $\lambda_2 = 0$. For the rest of the terms one then can argue as above.

We conclude that in all cases (2.44) holds, thus also (2.31) and (2.32) as explained above. Lastly, it is well-known (compare [Swart, 2022, Lemma 4.40]) that (2.32) implies that ν is indeed invariant and the proof is complete. □

$\mathbf{M}_8$	0	1	2	3
0	0	1	2	3
1	1	3	3	3
2	2	3	3	3
3	3	3	3	3

$\mathbf{M}_9$	0	1	2	3
0	0	1	2	3
1	1	2	3	3
2	2	3	3	3
3	3	3	3	3

$\mathbf{M}_{10}$	0	1	2	3
0	0	1	2	3
1	1	3	3	3
2	2	3	2	3
3	3	3	3	3

$\mathbf{M}_{11}$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
3	3	3	3	3

$\mathbf{M}_{12}$	0	1	2	3
0	0	1	2	3
1	1	0	2	3
2	2	2	3	3
3	3	3	3	3

$\mathbf{M}_{13}$	0	1	2	3
0	0	1	2	3
1	1	3	1	3
2	2	1	2	3
3	3	3	3	3

$\mathbf{M}_{14}$	0	1	2	3
0	0	1	2	3
1	1	2	2	3
2	2	2	2	3
3	3	3	3	3

$\mathbf{M}_{15}$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	3

$\mathbf{M}_{16}$	0	1	2	3
0	0	1	2	3
1	1	0	2	3
2	2	2	2	3
3	3	3	3	3

$\mathbf{M}_{17}$	0	1	2	3
0	0	1	2	3
1	1	2	1	3
2	2	1	2	3
3	3	3	3	3

$\mathbf{M}_{18}$	0	1	2	3
0	0	1	2	3
1	1	2	0	3
2	2	0	1	3
3	3	3	3	3

$\mathbf{M}_{19}$	0	1	2	3
0	0	1	2	3
1	1	2	2	3
2	2	2	2	3
3	3	3	3	2

$\mathbf{M}_{20}$	0	1	2	3
0	0	1	2	3
1	1	3	1	1
2	2	1	2	3
3	3	1	3	3

$\mathbf{M}_{21}$	0	1	2	3
0	0	1	2	3
1	1	3	1	1
2	2	1	0	3
3	3	1	3	3

$\mathbf{M}_{22}$	0	1	2	3
0	0	1	2	3
1	1	3	3	2
2	2	3	3	2
3	3	2	2	3

$\mathbf{M}_{23}$	0	1	2	3
0	0	1	2	3
1	1	3	3	1
2	2	3	0	1
3	3	1	1	3

$\mathbf{M}_{24}$	0	1	2	3
0	0	1	2	3
1	1	2	3	1
2	2	3	1	2
3	3	1	2	3

$\mathbf{M}_{25}$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

$\mathbf{M}_{26}$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Table 2.4: Cayley tables of the monoids $\mathbf{M}_8, \ldots, \mathbf{M}_{26}$.

$\mathbf{M}_9$ is $\mathbf{M}_9$-dual to $\mathbf{M}_9$ w.r.t. ψ_9:

$\mathbf{M}_9$ \ $\mathbf{M}_9$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	3
3	0	3	3	3

$\mathbf{M}_{10}$ is $\mathbf{M}_3$-dual to $\mathbf{M}_{10}$ w.r.t. ψ_{10}:

$\mathbf{M}_{10}$ \ $\mathbf{M}_{10}$	0	1	2	3
0	0	0	0	0
1	0	1	2	2
2	0	2	0	2
3	0	2	2	2

$\mathbf{M}_{11}$ is $\mathbf{M}_1$-dual to $\mathbf{M}_{11}$ w.r.t. ψ_{11}:

$\mathbf{M}_{11}$ \ $\mathbf{M}_{11}$	0	1	2	3
0	0	0	0	0
1	0	0	1	1
2	0	1	0	1
3	0	1	1	1

$\mathbf{M}_{13}$ is $\mathbf{M}_3$-dual to $\mathbf{M}_{14}$ w.r.t. ψ_{13}:

$\mathbf{M}_{13}$ \ $\mathbf{M}_{14}$	0	1	2	3
0	0	0	0	0
1	0	1	2	2
2	0	0	0	2
3	0	2	2	2

$\mathbf{M}_{15}$ is $\mathbf{M}_1$-dual to $\mathbf{M}_{15}$ w.r.t. ψ_{15}:

$\mathbf{M}_{15}$ \ $\mathbf{M}_{15}$	0	1	2	3
0	0	0	0	0
1	0	0	0	1
2	0	0	1	1
3	0	1	1	1

$\mathbf{M}_{16}$ is $\mathbf{M}_5$-dual to $\mathbf{M}_{20}$ w.r.t. ψ_{16}:

$\mathbf{M}_{16}$ \ $\mathbf{M}_{20}$	0	1	2	3
0	0	0	0	0
1	0	1	0	0
2	0	2	0	2
3	0	2	2	2

$\mathbf{M}_{17}$ is $\mathbf{M}_5$-dual to $\mathbf{M}_{17}$ w.r.t. ψ_{17}:

$\mathbf{M}_{17}$ \ $\mathbf{M}_{17}$	0	1	2	3
0	0	0	0	0
1	0	1	0	2
2	0	0	0	2
3	0	2	2	2

$\mathbf{M}_{18}$ is $\mathbf{M}_{18}$-dual to $\mathbf{M}_{24}$ w.r.t. ψ_{18}:

$\mathbf{M}_{18}$ \ $\mathbf{M}_{24}$	0	1	2	3
0	0	0	0	0
1	0	1	2	0
2	0	2	1	0
3	0	3	3	3

$\mathbf{M}_{21}$ is $\mathbf{M}_5$-dual to $\mathbf{M}_{21}$ w.r.t. ψ_{21}:

$\mathbf{M}_{21}$ \ $\mathbf{M}_{21}$	0	1	2	3
0	0	0	0	0
1	0	2	1	2
2	0	1	0	0
3	0	2	0	2

$\mathbf{M}_{22}$ is $\mathbf{M}_{22}$-dual to $\mathbf{M}_{22}$ w.r.t. ψ_{22}:

$\mathbf{M}_{22}$ \ $\mathbf{M}_{22}$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	2	3
3	0	3	3	3

$\mathbf{M}_{23}$ is $\mathbf{M}_{23}$-dual to $\mathbf{M}_{23}$ w.r.t. ψ_{23}:

$\mathbf{M}_{23}$ \ $\mathbf{M}_{23}$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	2	0
3	0	3	0	3

$\mathbf{M}_{23}$ is $\mathbf{M}_5$-dual to $\mathbf{M}_{23}$ w.r.t. ψ_{235}:

$\mathbf{M}_{23}$ \ $\mathbf{M}_{23}$	0	1	2	3
0	0	0	0	0
1	0	2	1	2
2	0	1	1	0
3	0	2	0	2

$\mathbf{M}_{24}$ is $\mathbf{M}_{24}$-dual to $\mathbf{M}_{24}$ w.r.t. ψ_{24}:

$\mathbf{M}_{24}$ \ $\mathbf{M}_{24}$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	1	3
3	0	3	3	3

$\mathbf{M}_{25}$ is $\mathbf{M}_2$-dual to $\mathbf{M}_{25}$ w.r.t. ψ_{25}:

$\mathbf{M}_{25}$ \ $\mathbf{M}_{25}$	0	1	2	3
0	0	0	0	0
1	0	0	1	1
2	0	1	0	1
3	0	1	1	0

$\mathbf{M}_{26}$ is $\mathbf{M}_{26}$-dual to $\mathbf{M}_{26}$ w.r.t. ψ_{26}:

$\mathbf{M}_{26}$ \ $\mathbf{M}_{26}$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Table 2.5: Dualities of monoids of cardinality 4. Green tables indicate cases where $\boldsymbol{\psi}_k$, defined as in (2.9) but with ψ replaced by $\boldsymbol{\psi}_k$, is weakly informative, while red tables indicate cases where $\boldsymbol{\psi}_k$ is not weakly informative. Compare Chapter 2.6.

3. Module duality

In Chapter 2 we have equipped the local state space S with a binary operator, making it a monoid, and we were able to define a useful notion of duality based on this structure. As many important mathematical structures have *two* binary operators, it is natural to ask if we can also develop a useful duality theory if S is equipped with two binary operators. Recall that, by definition, a *semiring* is a triple $(R, +, \cdot\,)$ such that:

(i) $(R, +)$ is a commutative monoid with neutral element 0,

(ii) $(R, \cdot\,)$ is a monoid with neutral element 1,

(iii) $a \cdot 0 = 0 = 0 \cdot a$ for all $a \in R$,

(iv) $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$ for all $a, b, c \in R$.

Property (iii) makes 0 an *absorbing element* of $(R, \cdot\,)$. Property (iv) is called *distributivity*. The semiring $(R, +, \cdot\,)$ is called *commutative* if the monoid $(R, \cdot\,)$ is commutative. We will sometimes call $(R, +)$ the *additive monoid* of $(R, +, \cdot\,)$ and $(R, \cdot\,)$ the *multiplicative monoid* of $(R, +, \cdot\,)$.

Assume that the local state space S is equipped with binary operators $+$ and $\cdot$, making $(S, +, \cdot\,)$ a semiring. Then we can equip the monoid $\mathbf{S}^\Lambda = (S^\Lambda, +)$, defined via (2.1), with an additional structure by defining multiplication by scalars from the left and right as

$$(a \cdot x)(i) := a \cdot x(i) \quad \text{and} \quad (x \cdot a)(i) := x(i) \cdot a \qquad (a \in S,\ x \in S^\Lambda,\ i \in \Lambda). \tag{3.1}$$

One can check that with this definition $\mathbf{S}^\Lambda$ becomes both a left S-module and a right S-module. Recall that a *left R-module* over the semiring $(R, +, \cdot\,)$ is a commutative monoid $\mathbf{M} = (M, +)$ equipped with a *scalar multiplication* $* : R \times M \to M$ such that

(i) $a * (x + x') = a * x + a * x' \qquad (a \in R,\ x, x' \in M)$,

(ii) $(a + a') * x = a * x + a' * x \qquad (a, a' \in R,\ x \in M)$,

(iii) $(a \cdot a') * x = a * (a' * x) \qquad (a, a' \in R,\ x \in M)$,

(iv) $1 * x = x \qquad (x \in M)$,

where 1 denotes the neutral element of $(R, \cdot\,)$. A *right R-module* over the semiring $(R, +, \cdot\,)$ has a scalar multiplication $* : M \times R \to M$ and is defined analogously. If $(R, +, \cdot\,)$ is commutative, left R-modules and right R-modules coincide and are simply called *R-modules*. In particular, if $(R, +, \cdot\,)$ is a field, then each R-module is a vector space.

We now follow the same ideas as in Chapter 2 in order to construct a pathwise duality based on $\boldsymbol{\psi}_{\text{basic}}$ from (1.36), now under the assumption that S^Λ is equipped with the structure of a module over a semiring. Thus, we construct a pathwise duality for an interacting particle system that has a generator G represented as in (1.8) with $\mathcal{G}$ consisting only of local module homomorphisms mapping from S^Λ to itself. In particular, each such $m : S^\Lambda \to S^\Lambda$ satisfies $m(\underline{0}) = \underline{0}$, where 0 is the neutral element of the additive monoid $(S, +)$ of the semiring $(S, +, \cdot)$. Hence, we may use the notions of $\mathrm{supp}(x)$, the support of $x \in S^\Lambda$, S^Λ_{fin}, and $\delta^a_i \in S^\Lambda_{\mathrm{fin}}$ ($i \in \Lambda$, $a \in S$) from Chapter 1.2, now based on 0, the neutral element of $(S, +)$.

In contrast to Chapter 2, we will have to assume that T, the finite set to which $\boldsymbol{\psi}_{\mathrm{basic}}$ maps, is equal to the local state space S. On the other hand, it will turn out that for *any* choice of the underlying semiring $(S, +, \cdot)$ we can identify the space of all continuous left S-module homomorphisms from S^Λ to itself with S^Λ_{fin}. Also the corresponding duality function will be completely determined by the underlying semiring on S.

As most results of this chapter have direct analogues from Chapter 2, their proofs in this section will be stated in a shortened version: If some arguments work the same way as in a proof from Chapter 2, the arguments are not repeated, but the corresponding proof is referenced.

3.1 Module homomorphisms

Assume that $(S, +, \cdot)$ is a semiring and let $\mathbf{S}^\Lambda = (S^\Lambda, +)$ denote the monoid defined via (2.1). We define

$$\mathcal{L}(\mathbf{S}^\Lambda, \mathbf{S}^\Lambda) := \Big\{ h \in \mathcal{H}(\mathbf{S}^\Lambda, \mathbf{S}^\Lambda) : h(a \cdot x) = a \cdot h(x) \ (a \in S,\ x \in S^\Lambda) \Big\},$$
$$\mathcal{R}(\mathbf{S}^\Lambda, \mathbf{S}^\Lambda) := \Big\{ h \in \mathcal{H}(\mathbf{S}^\Lambda, \mathbf{S}^\Lambda) : h(x \cdot a) = h(x) \cdot a \ (a \in S,\ x \in S^\Lambda) \Big\}.$$

In words, $\mathcal{L}(\mathbf{S}^\Lambda, \mathbf{S}^\Lambda)$ is the collection of all left S-module homomorphisms from S^Λ to itself, and likewise $\mathcal{R}(\mathbf{S}^\Lambda, \mathbf{S}^\Lambda)$ is the collection of all right S-module homomorphisms from S^Λ to itself. If $(S, +, \cdot)$ is a commutative semiring, then $\mathcal{L}(\mathbf{S}^\Lambda, \mathbf{S}^\Lambda) = \mathcal{R}(\mathbf{S}^\Lambda, \mathbf{S}^\Lambda)$. In particular, if $(S, +, \cdot)$ is a field, then $\mathcal{L}(\mathbf{S}^\Lambda, \mathbf{S}^\Lambda)$ is the space of linear functions from S^Λ to itself. The following lemma is similar to Lemma 2.10.

Lemma 3.1 (Local module homomorphisms)**.** *Let $(S, +, \cdot)$ be a finite semiring and let $\mathbf{S} = (S, +)$. Let $(M_{ij})_{i,j \in \Lambda}$ be an infinite matrix with values in $\mathcal{L}(\mathbf{S}, \mathbf{S})$ such that the set*

$$\Delta := \Big\{ (i, j) \in \Lambda^2 : i \neq j,\ M_{ij} \neq o \Big\} \cup \Big\{ (i, i) \in \Lambda^2 : M_{ii} \neq \mathrm{id} \Big\}$$

is finite, where $o \in \mathcal{L}(\mathbf{S}, \mathbf{S})$ denotes the function constantly equal to the neutral element of $\mathbf{S}$ and $\mathrm{id} \in \mathcal{L}(\mathbf{S}, \mathbf{S})$ denotes the identity. Then setting

$$m[i](x) := \sum_{j \in \Lambda} M_{ij}(x(j)) \qquad (i \in \Lambda,\ x \in S^\Lambda) \tag{3.2}$$

defines a local map $m \in \mathcal{L}(\mathbf{S}^\Lambda, \mathbf{S}^\Lambda)$. Conversely, each local map $m \in \mathcal{L}(\mathbf{S}^\Lambda, \mathbf{S}^\Lambda)$ is of this form.

Proof. Using the arguments of the proof of Lemma 2.10, it suffices to show that for any finite $\Delta \subset \Lambda$ and any function $m' : S^\Delta \to S$ one has that $m' \in \mathcal{L}(\mathbf{S}^\Delta, \mathbf{S})$ if and only if there exists a vector $(M_j)_{j \in \Delta}$ with $M_j \in \mathcal{L}(\mathbf{S}, \mathbf{S})$ such that

$$m'(x) = \sum_{j \in \Delta} M_j(x(j)) \qquad (x \in S^\Delta). \tag{3.3}$$

It is straightforward to check that $m' : S^\Delta \to S$ from (3.3) defines a map $m' \in \mathcal{L}(\mathbf{S}^\Delta, \mathbf{S})$. On the other hand, given $m' \in \mathcal{L}(\mathbf{S}^\Delta, \mathbf{S})$, we define $M_i : S \to S$ by $M_i(a) := m(a^i)$ ($a \in S$, $i \in \Delta$), where $a^i \in S^\Delta$ is defined as in (2.4). Then it is straightforward to check that $M_i \in \mathcal{L}(\mathbf{S}, \mathbf{S})$ and m' is of the form (3.3). □

Lemma 3.1, of course, also holds if we replace left S-modules by right ones.

3.2 Duality of (topological) modules

In parallel to Chapter 2.2, we can define a notion of duality of modules. Let $(R, +, \cdot\,)$ be a semiring. We say that the left R-module $\mathbf{M} = (M, +)$ is *dual* to the right R-module $\mathbf{N} = (N, \oplus)$ *with respect to* $\psi : M \times N \to R$ if ψ has the following properties:

(i) $\psi(a, b) = \psi(a', b)$ for all $b \in N$ implies $a = a'$ ($a, a' \in M$),

(ii) $\mathcal{L}(\mathbf{M}, (R, +)) = \Big\{\psi(\,\cdot\,, b) : b \in N\Big\}$,

(iii) $\psi(a, b) = \psi(a, b')$ for all $a \in M$ implies $b = b'$ ($b, b' \in N$),

(iv) $\mathcal{R}(\mathbf{N}, (R, +)) = \Big\{\psi(a, \,\cdot\,) : a \in M\Big\}$.

Thus, duality of modules is defined as duality of monoids, with the sole distinction being that monoid homomorphisms are replaced by R-module homomorphisms, a special type of monoid homomorphisms. This definition has the following consequence.

Lemma 3.2 (Semiring duality)**.** *Let $(R, +, \cdot\,)$ be a semiring. Then $(R, +)$, seen as a left R-module over itself, is dual to $(R, +)$, seen as a right R-module over itself, with respect to $\psi : R \times R \to R$ defined as*

$$\psi(a, b) := a \cdot b \qquad (a, b \in R). \tag{3.4}$$

Proof. Due to symmetry, it suffices to show properties (i) and (ii) of the definition of duality of modules. If $\psi(a, b) = \psi(a', b)$ for all $b \in R$, then, in particular,

$$a = \psi(a, 1) = \psi(a', 1) = a' \qquad (a, a' \in R),$$

where 1 denotes, as usual, the neutral element of $(R, \cdot\,)$. This shows property (i).

If $h \in \mathcal{L}((R, +), (R, +))$, then

$$h(a) = h(a \cdot 1) = a \cdot h(1) = \psi(a, h(1)) \qquad (a \in R).$$

On the other hand,

$$\begin{aligned} \psi(a + a', b) &= (a + a') \cdot b = a \cdot b + a' \cdot b = \psi(a, b) \cdot \psi(a', b), \\ \psi(a \cdot a', b) &= a \cdot a' \cdot b = a \cdot \psi(a', b) \qquad\qquad (a, a', b \in R), \end{aligned}$$

completing the proof of property (ii). □

Lemma 3.2 implies the following analogue of Proposition 2.5.

Lemma 3.3 (Maps with a dual IV). *Let $(R, +, \cdot)$ be a semiring. Then a map $m : R \to R$ has a dual map $\hat{m} : R \to R$ with respect to ψ from (3.4) (in the sense that $\psi(m(a), b) = \psi(a, \hat{m}(b))$ for all $a, b \in R$) if and only if $m \in \mathcal{L}((R, +), (R, +))$. The dual map $\hat{m}$, if it exists, is unique and satisfies $\hat{m} \in \mathcal{R}((R, +), (R, +))$.*

Proof. This follows essentially in the same way as Proposition 2.5. Note, however, that we cannot use the argument with $\psi^\dagger$ due to the difference between the left and right R-module. However, one can easily show that $\hat{m} \in \mathcal{R}((R, +), (R, +))$ if it exists with the same arguments used for m. □

In parallel to Chapter 2.3 we say that a left or right S-module $\mathbf{M}$ over a semiring $(R, +, \cdot)$ is a *topological (left or right) R-module* if it is a topological monoid. Let $(R, +, \cdot)$ be a semiring. We say that the topological left R-module $\mathbf{M} = (M, \boldsymbol{+})$ is *dual* to the topological right R-module $\mathbf{N} = (N, \oplus)$ *with respect to $\boldsymbol{\psi} : M \times N \to R$* if $\boldsymbol{\psi}$ has the following properties:

(i) $\boldsymbol{\psi}(x, y) = \boldsymbol{\psi}(x', y)$ for all $y \in N$ implies $x = x'$ $(x, x' \in M)$,

(ii) $\mathcal{L}(\mathbf{M}, (R, +)) \cap \mathcal{C}(\mathbf{M}, (R, +)) = \big\{\boldsymbol{\psi}(\,\cdot\,, y) : y \in N\big\}$,

(iii) $\boldsymbol{\psi}(x, y) = \boldsymbol{\psi}(x, y')$ for all $x \in M$ implies $y = y'$ $(y, y' \in N)$,

(iv) $\mathcal{R}(\mathbf{N}, (R, +)) \cap \mathcal{C}(\mathbf{N}, (R, +)) = \big\{\boldsymbol{\psi}(x, \,\cdot\,) : x \in M\big\}$.

Assume that the local state space S is equipped with binary operators $+$ and $\cdot$, making $(S, +, \cdot)$ a semiring. Let $\mathbf{S}^\Lambda = (S^\Lambda, +)$ denote the commutative topological monoid defined via (2.1), and let $\mathbf{S}^\Lambda_{\text{fin}} = (S^\Lambda_{\text{fin}}, +)$ be the monoid from Chapter 2.3. Equipping $\mathbf{S}^\Lambda_{\text{fin}}$ with the same scalar multiplication as $\mathbf{S}^\Lambda$, i.e., with the one defined in (3.1), clearly also $\mathbf{S}^\Lambda_{\text{fin}}$ becomes both a left S-module and a right S-module. According to our conventions we equip it with the discrete topology, making it a topological (left and right) S-module. Similarly as in (2.9) we define a function $\boldsymbol{\psi} : S^\Lambda \times S^\Lambda_{\text{fin}} \to S$ by

$$\boldsymbol{\psi}(x, y) := \sum_{i \in \Lambda} x(i) \cdot y(i) \qquad (x \in S^\Lambda,\ y \in S^\Lambda_{\text{fin}}). \tag{3.5}$$

Note that, in difference to the duality function in (2.9), the function $\boldsymbol{\psi}$ above is entirely characterized by the underlying structure of a semiring. In parallel to Proposition 2.8 we have the following result.

Proposition 3.4 (Duality of product modules). *Let $(S, +, \cdot)$ be a finite semiring. Then the topological left S-module $\mathbf{S}^\Lambda$ is dual to the topological right S-module $\mathbf{S}^\Lambda_{\text{fin}}$ with respect to $\boldsymbol{\psi}$ from (3.5).*

Proof. One has that $\boldsymbol{\psi}(x, y) = \boldsymbol{\psi}(x', y)$ for all $y \in S^\Lambda_{\text{fin}}$ implies

$$x(i) = \boldsymbol{\psi}(x, \delta^1_i) = \boldsymbol{\psi}(x', \delta^1_i) = x'(i) \qquad (i \in \Lambda),$$

proving property (i) of the definition of duality of topological modules. Property (iii) follows in the same way.

Using the distributivity of the product and the commutativity of the sum, we see that

$$\begin{aligned}\boldsymbol{\psi}(x+x',y) &= \sum_{i\in\mathrm{supp}(y)} (x(i)+x'(i))\cdot y(i) = \sum_{i\in\mathrm{supp}(y)} (x(i)\cdot y(i) + x'(i)\cdot y(i)) \\ &= \sum_{i\in\mathrm{supp}(y)} x(i)\cdot y(i) + \sum_{i\in\mathrm{supp}(y)} x'(i)\cdot y(i) = \boldsymbol{\psi}(x,y)+\boldsymbol{\psi}(x',y)\end{aligned}$$

for all $x, x' \in S^\Lambda$ and $y \in S^\Lambda_{\mathrm{fin}}$. Using the associativity and distributivity of the product, we moreover obtain that

$$\begin{aligned}\boldsymbol{\psi}(a\cdot x,y) &= \sum_{i\in\mathrm{supp}(y)} (a\cdot x(i))\cdot y(i) = \sum_{i\in\mathrm{supp}(y)} a\cdot(x(i)\cdot y(i)) \\ &= a\cdot \sum_{i\in\mathrm{supp}(y)} x(i)\cdot y(i) = a\cdot\boldsymbol{\psi}(x,y) \qquad (a\in S,\ x\in S^\Lambda,\ y\in S^\Lambda_{\mathrm{fin}}).\end{aligned}$$

We conclude that $\{\boldsymbol{\psi}(\,\cdot\,,y) : y\in S^\Lambda_{\mathrm{fin}}\} \subset \mathcal{L}(\mathbf{S}^\Lambda,(S,+))$. The fact that $\{\boldsymbol{\psi}(\,\cdot\,,y) : y \in S^\Lambda_{\mathrm{fin}}\} \subset \mathcal{C}(\mathbf{S}^\Lambda,(S,+))$ follows as in the proof of Proposition 2.8.

Conversely, assume that $h \in \mathcal{L}(\mathbf{S}^\Lambda,(S,+)) \cap \mathcal{C}(\mathbf{S}^\Lambda,(S,+))$. As in the proof of Proposition 2.8, there then exists a finite set $\Delta\subset\Lambda$ such that

$$h(x) = h(x\lfloor_\Delta) = h\left(\sum\nolimits_{i\in\Delta} x(i)\cdot\delta^1_i\right) = \sum_{i\in\Delta} x(i)\cdot h(\delta^1_i) \qquad (x\in S^\Lambda),$$

where $x\lfloor_\Delta \in S^\Lambda_{\mathrm{fin}}$ is defined as in (2.10). Hence, defining $y\in S^\Lambda_{\mathrm{fin}}$ as

$$y(i) = \begin{cases} h(\delta^1_i) & \text{if } i\in\Delta, \\ 0 & \text{else,}\end{cases} \qquad (i\in\Lambda)$$

yields $h=\boldsymbol{\psi}(\,\cdot\,,y)$, and property (ii) of the definition of duality of topological modules follows. Property (iv) follows similarly (without relying on the continuity) and the proof is complete. □

Thus, by Proposition 3.4, for *any* semiring $(S,+,\,\cdot\,)$ we may identify the space $\mathcal{L}(\mathbf{S}^\Lambda,(S,+))\cap\mathcal{C}(\mathbf{S}^\Lambda,(S,+))$ with S^Λ_{fin}. Under this identification, restricting $\boldsymbol{\psi}_{\mathrm{basic}} : S^\Lambda\times\mathcal{C}(S^\Lambda,S)\to S$ from (1.36) in the second entry to $\mathcal{L}(\mathbf{S}^\Lambda,(S,+))\cap \mathcal{C}(\mathbf{S}^\Lambda,(S,+))$ yields $\boldsymbol{\psi}$ from (3.5).

Combining the proofs of Proposition 2.5, Proposition 2.7 and Proposition 2.13, one concludes the following.

Proposition 3.5 (Maps with a dual V)**.** *Let $(S,+,\,\cdot\,)$ be a finite semiring. Then a map $m : S^\Lambda\to S^\Lambda$ has a dual map $\hat m : S^\Lambda_{\mathrm{fin}}\to S^\Lambda_{\mathrm{fin}}$ with respect to the duality function $\boldsymbol{\psi}$ defined in (3.5) if and only if $m\in\mathcal{L}(\mathbf{S}^\Lambda,\mathbf{S}^\Lambda)\cap\mathcal{C}(\mathbf{S}^\Lambda,\mathbf{S}^\Lambda)$. The dual map $\hat m$, if it exists, is unique and satisfies $\hat m\in\mathcal{R}(\mathbf{S}^\Lambda_{\mathrm{fin}},\mathbf{S}^\Lambda_{\mathrm{fin}})$.*

Proof. First one shows, as in the proof of Proposition 2.7, that a map $m : S^\Lambda\to S^\Lambda$ has a dual map $\hat m : S^\Lambda_{\mathrm{fin}}\to S^\Lambda_{\mathrm{fin}}$ with respect to the function $\boldsymbol{\psi}$ if and only if m preserves $\mathcal{L}(\mathbf{S}^\Lambda,(S,+))\cap\mathcal{C}(\mathbf{S}^\Lambda,(S,+))$. As in the proof of Proposition 2.5, one then shows that any map $m : S^\Lambda\to S^\Lambda$ that preserves $\mathcal{L}(\mathbf{S}^\Lambda,(S,+))$ has to satisfy $m\in\mathcal{L}(\mathbf{S}^\Lambda,\mathbf{S}^\Lambda)$. The proof of Proposition 2.13 shows that no discontinuous map can preserve $\mathcal{L}(\mathbf{S}^\Lambda,(S,+))\cap\mathcal{C}(\mathbf{S}^\Lambda,(S,+))$. The part regarding $\hat m$ also follows from the arguments of the proof of Proposition 2.5. □

One concludes the following analogue of Theorem 2.9.

Theorem 3.6 (Pathwise module duality)**.** *Let there exist two associative binary operators* $+$ *and* $\cdot$ *on the local state space* S *such that* $(S, +, \cdot\,)$ *is a semiring. Let* G *and* $\widehat{G}$ *be the generators from (1.8) and (1.34) defined via* $\mathcal{G} \subset \mathcal{L}(\mathbf{S}^\Lambda, \mathbf{S}^\Lambda)$*, a countable collection of local module homomorphisms. Assuming, as usual, that* G *satisfies (1.7), there exists a continuous-time Markov chain* $Y = (Y_t)_{t \geq 0}$ *with generator* $\widehat{G}$*, state space* S^Λ_{fin} *and càglàd sample paths such that* $X = (X_t)_{t\geq 0}$*, the interacting particle system defined in Chapter 1.1, is pathwise dual to* Y *with respect to* $\boldsymbol{\psi}$*, the function defined in (3.5).*

Proof. The theorem follows with exactly the same arguments as Theorem 2.9. □

As in Chapter 2, we need to know how to effectively compute dual maps with respect to $\boldsymbol{\psi}$ in order to apply Theorem 3.6 in practice. Due to Lemma 3.1, the following analogue of Proposition 2.11 follows readily.

Proposition 3.7 (Dual local homomorphisms II)**.** *Let* $(S, +, \cdot\,)$ *be a finite semiring. For each local* $m \in \mathcal{L}(\mathbf{S}^\Lambda, \mathbf{S}^\Lambda)$ *there exists a local map* $\hat{m} \in \mathcal{R}(\mathbf{S}^\Lambda, \mathbf{S}^\Lambda)$ *so that the restriction of* $\hat{m}$ *to* S^Λ_{fin} *is the unique dual map of* m *with respect to* $\boldsymbol{\psi}$ *from (3.5). If* $(M_{ij})_{i,j\in\Lambda}$ *denotes the matrix from Lemma 3.1 such that (3.2) holds, then* $\hat{m}$ *is given via*

$$\hat{m}[i](y) = \sum_{j\in\Lambda} \widehat{M}_{ji}(y(j)) \qquad (i \in \Lambda,\ y \in S^\Lambda),$$

where, for $i, j \in \Lambda$*,* $\widehat{M}_{ij} \in \mathcal{R}((S, +), (S, +))$ *is the (unique) dual map of* $M_{ij} \in \mathcal{L}((S, +), (S, +))$ *with respect to* ψ *from (3.4).*

Proof. The proposition follows with exactly the same arguments as Proposition 2.11. □

3.3 Previously known special cases

Some dualities of modules can be identified with dualities of their additive monoids. Comparing the definition of duality of modules from Chapter 3.2 with the one of duality of monoids from Chapter 2.2, Lemma 3.2 has the following direct consequence.

Lemma 3.8 (Semirings and monoids)**.** *Let* $(R, +, \cdot\,)$ *be a semiring. Then the monoid* $(R, +)$ *is* $(R, +)$*-dual to itself with respect to* ψ *from (3.4) if and only if*

$$\mathcal{H}((R, +), (R, +)) = \mathcal{L}((R, +), (R, +)). \tag{3.6}$$

Let $(R, +, \cdot\,)$ be a semiring with $1 \in R$ being the neutral element of its multiplicative monoid $(R, \cdot\,)$. We say that 1 *generates* $(R, +)$ if each $a \in R$ with $a \neq 0$ is of the form

$$a = \underbrace{1 + \cdots + 1}_{n \text{ times}}$$

for some $n \in \mathbb{N}$. It is easy to see that $(R, +, \cdot\,)$ must be commutative if 1 generates $(R, +)$.

Lemma 3.9 (Semirings generated by 1)**.** *Assume that* $(R, +, \cdot)$ *is a commutative semiring and that* 1*, the neutral element of* $(R, \cdot)$*, generates* $(R, +)$*. Then the monoid* $(R, +)$ *is* $(R, +)$*-dual to itself with respect to* ψ *from (3.4).*

Proof. Due to Lemma 3.8 it suffices to verify that one has $\mathcal{H}((R, +), (R, +)) = \mathcal{L}((R, +), (R, +))$, i.e., that each $h \in \mathcal{H}((R, +), (R, +))$ satisfies $h(a \cdot b) = a \cdot h(b)$ $(a, b \in R)$. For $a = 0$, the neutral element of $(R, +)$, this is clear since $h(0) = 0$. Otherwise, we can write $a = 1 + \cdots + 1$ and observe that

$$\begin{aligned} h(a \cdot b) &= h((1 + \cdots + 1) \cdot b) = h(b + \cdots + b) = h(b) + \cdots + h(b) \\ &= (1 + \cdots + 1) \cdot h(b) = a \cdot h(b) \qquad (a \in R \setminus \{0\},\ b \in R), \end{aligned}$$

completing the proof. □

Let $n \in \mathbb{N}$ and denote, as in Chapter 2.4.2, addition modulo n by $\oplus$ and multiplication modulo n by $\odot$. Then $(\{0, \ldots, n-1\}, \oplus, \odot)$ is a semiring and 1 generates $(\{0, \ldots, n-1\}, \oplus)$. It follows that the corresponding duality function $\boldsymbol{\psi}$ from (3.5) is $\boldsymbol{\psi}_{\text{canc}}$ from (2.18).

Next, we consider the lattice duality from Chapter 2.4.1. One says that a lattice $(L, \leq)$ with join $\vee$ and meet $\wedge$ is *distributive* if

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \qquad (a, b, c \in L).$$

Clearly, each distributive lattice $(L, \leq)$ is a semiring $(L, \vee, \wedge)$, where the least element $0 \in L$ is the neutral element of its additive monoid $(L, \vee)$ and the greatest element $\top \in L$ is the neutral element of its multiplicative monoid $(L, \wedge)$. The only lattice $(L, \leq)$ in which $\top$ generates $(L, \vee)$ is $(\{0, 1\}, \vee, \wedge)$, where $\vee$ and $\wedge$ denote the usual maximum and minimum, respectively. In this special case the duality function $\boldsymbol{\psi}$ from (3.5) is $\boldsymbol{\psi}_1$ from Chapter 2.5, a special case of $\boldsymbol{\psi}_{\text{add}}$ from (2.17). For all other distributive lattices $(L, \leq)$, however, $\top \in L$ does not generate $(L, \vee)$ and $\boldsymbol{\psi}$ from (3.5) does not have the form of (2.17).

It is also worth noting that if one drops our usual assumption that the local state space S has to be finite, one could set $S = \mathbb{R}$. Then the duality function $\boldsymbol{\psi}$ from (3.5) is the standard inner product on ℓ_2. Linear duality with this duality function has long been used in the study of linear interacting particle systems (see [Liggett, 1985, Chapter IX] for an overview). As the focus of the present thesis lies on interacting particle systems with finite local state spaces, the topic of linear duality will not be further explored in this work.

3.4 Computing module dualities

In parallel to Chapter 2.5, we list all possible dualities arising from Proposition 3.4 if S^Λ is equipped with the structure of a module and S has cardinality between two and four. This task boils down to finding all possible ways to define a product $\cdot$ on one commutative monoid $(M, +)$ of those identified in Chapter 2.5 so that $(M, +, \cdot)$ is a semiring. Unfortunately, we have not found a sequence of the

$\mathbf{N}_1$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	3	3	3
3	3	3	3	3

$\mathbf{N}_2$	0	1	2	3
0	0	1	2	3
1	1	1	1	3
2	2	2	2	3
3	3	3	3	3

Table 3.1: Cayley tables of the monoids $\mathbf{N}_1$ and $\mathbf{N}_2$.

number of semirings (up to (anti-) isomorphisms) with n elements in the OEIS [OEIS Foundation Inc., 2024].[1]

Recall that if $(M, +, \cdot)$ is a semiring, then the monoid $(M, \cdot)$ does not have to be commutative. On the other hand, the monoid $(M, \cdot)$ has to have an absorbing element, which is the neutral element 0 of $(M, +)$. It turns out that all monoids with two or three elements that contain an absorbing element are commutative, but there exist two monoids with four elements that contain an absorbing element and are not commutative. We have named these $\mathbf{N}_1$ and $\mathbf{N}_2$. Their Cayley tables are given in Table 3.1.

Again using Mathematica [Wolfram Research Inc., 2024], we have checked for all pairs of monoids $((M, +), (N, \cdot))$ so that $(M, +)$ is commutative, $(N, \cdot)$ contains an absorbing element and M and N have the same cardinality that lies between two and four, if it is possible to identify the elements of M and N in such a way that $(M, +, \cdot)$ is a semiring. Again, the corresponding code can be accessed in the attachments to the online version of this thesis at `https://dspace.cuni.cz/`.

In Table 3.2 we list all possible ways to define a multiplication $\cdot$ on the commutative monoids $\mathbf{M}_k = (M_k, +)$ with $k \in \{1, \dots, 7\}$ such that $(M_k, +, \cdot)$ is a semiring. Below each Cayley table, we have indicated to which monoid $(M_k, \cdot)$ is (anti-) isomorphic. Note that each Cayley table gives rise to a duality function of the form (3.5). The corresponding tables for the monoids $\mathbf{M}_8, \dots, \mathbf{M}_{26}$ are given in Table 3.3 (found at the end of Chapter 3). We have only listed semirings that are not (anti-) isomorphic to each other. In other words, on some of the monoids it may be possible to define a multiplication in a way that is not listed, but in such a case the resulting semiring is (anti-) isomorphic to a semiring that occurs in our list. In this way we have identified two semirings with 2 elements, six semirings with 3 elements and thirty eight semirings with 4 elements.

In Chapter 3.3 we saw that the method to identify dualities via dual modules partially overlaps with the method to identify dualities via dual monoids presented in Chapter 2. By Lemma 3.9, if $(R, +, \cdot)$ is a semiring in which 1, the neutral element of $(R, \cdot)$, generates $(R, +)$, then the monoid $(R, +)$ is $(R, +)$-dual to itself with respect to $\psi : R \times R \to R$ defined as $\psi(a, b) := a \cdot b$ $(a, b \in R)$. The monoid dualities with respect to the duality functions ψ_1, ψ_2, ψ_3, ψ_6, ψ_7, ψ_9, ψ_{22}, ψ_{24} and ψ_{26} from Table 2.2 and Table 2.5 are of this special form and hence occur also in our tables of multiplications in semirings.

[1] A *semiring isomorphism* between $(R, +, \cdot)$ and $(S, \oplus, \otimes)$ is a bijection $f : R \to S$ such that $f(0) = 0$, $f(1) = 1$, $f(a + b) = f(a) \oplus f(b)$ and $f(a \cdot b) = f(a) \otimes f(b)$ $(a, b \in R)$. A *semiring antiisomorphism* is a bijection $f : R \to S$ with the same first three properties, but requiring $f(a \cdot b) = f(b) \otimes f(a)$ $(a, b \in R)$.

$(M_1, \cdot)$	0	1
0	0	0
1	0	1

mult. $\cong \mathbf{M}_1$

$(M_2, \cdot)$	0	1
0	0	0
1	0	1

mult. $\cong \mathbf{M}_1$

$(M_3, \cdot)$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	2

mult. $\cong \mathbf{M}_4$

$(M_4, \cdot)$	0	1	2
0	0	0	0
1	0	0	1
2	0	1	2

mult. $\cong \mathbf{M}_3$

$(M_4, \cdot)$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	2

mult. $\cong \mathbf{M}_4$

$(M_4, \cdot)$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

mult. $\cong \mathbf{M}_4$

$(M_6, \cdot)$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	2

mult. $\cong \mathbf{M}_4$

$(M_7, \cdot)$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

mult. $\cong \mathbf{M}_5$

Table 3.2: Cayley tables of all products on $(M_k, +)$ such that $(M_k, +, \cdot\,)$ becomes a semiring ($k \in \{1, \ldots, 7\}$). Green tables indicate cases where $\boldsymbol{\psi}$, defined as in (3.5), is weakly informative, while red tables indicate cases where $\boldsymbol{\psi}$ is not weakly informative. Orange tables indicate cases where our methods were not able to decide whether $\boldsymbol{\psi}$ is weakly informative.

Interestingly, we have found one more duality between commutative monoids that also occurs in our tables of multiplications in semirings. This is ψ_{23}, which also occurs in Table 3.3 as the multiplication on $\mathbf{M}_{23} = (M_{23}, +)$ that is isomorphic to $\mathbf{M}_{11}$. In this example, the neutral element of $(M_{23}, \cdot\,)$ does not generate $(M_{23}, +) \cong \mathbf{M}_1 \times \mathbf{M}_2$. Nevertheless, one can check that $\mathcal{L}(\mathbf{M}_{23}, \mathbf{M}_{23}) = \mathcal{R}(\mathbf{M}_{23}, \mathbf{M}_{23}) = \mathcal{H}(\mathbf{M}_{23}, \mathbf{M}_{23})$ if the multiplication is given as in the corresponding Cayley table.

We point out some additional observations. Recall that the four lattices with 2–4 elements correspond to $\mathbf{M}_1$, $\mathbf{M}_4$, $\mathbf{M}_{11}$ and $\mathbf{M}_{15}$ from Table 2.1 and Table 2.4. All of these lattices are distributive and hence are semirings of the form $(L, \vee, \wedge)$ as outlined in Chapter 3.3. One should moreover note that for more than half of the identified semirings either the additive monoid or the multiplicative monoid is isomorphic to one of the four monoids $\mathbf{M}_1$, $\mathbf{M}_4$, $\mathbf{M}_{11}$ and $\mathbf{M}_{15}$.

Defining a *ring* $(R, +, \cdot\,)$ to be a semiring in which each $a \in R$ has an *additive inverse*, i.e., an element $-a \in R$ such that $a + (-a) = 0$, the neutral element of $(R, +)$, there exist one ring with 2 elements, one ring with 3 elements and four rings with 4 elements (sequence A037291 in the OEIS [OEIS Foundation Inc., 2024]). Since, for each $n \in \mathbb{N}$, the semiring $(\{0, \ldots, n-1\}, \oplus, \odot)$, discussed in Chapter 3.3, is a ring, three of the six rings were already looked at. The other three rings are those semirings from Table 3.3, whose additive monoid is isomorphic to $\mathbf{M}_{25}$. In particular, the ring whose multiplicative monoid is isomorphic to $\mathbf{M}_{18}$ is the finite field $(\mathbb{F}_4, +, \cdot\,)$ with 4 elements. As 1 does not generate $\mathbf{M}_{25}$, none of these dualities satisfies Lemma 3.9 and one can check that in all three cases (3.6) is not satisfied. Indeed, $\mathcal{H}(\mathbf{M}_{25}, \mathbf{M}_{25})$ contains sixteen elements while, for all three choices of the product, $\mathcal{L}(\mathbf{M}_{25}, \mathbf{M}_{25}) = \mathcal{R}(\mathbf{M}_{25}, \mathbf{M}_{25})$ contains, due to Lemma 3.2, only four elements.

3.5 Representation of semirings

As in Chapter 2.6, we investigate in which cases the duality function $\boldsymbol{\psi}$ from (3.5) is informative. Let $(S, +, \otimes)$ be a finite semiring. Since we can write $\boldsymbol{\psi}(x, y) = \sum_{i\in\Lambda} \psi(x(i), y(i))$ ($x \in S^{\Lambda}$, $y \in S^{\Lambda}_{\text{fin}}$) for ψ from (3.4), $\boldsymbol{\psi}$ from (3.5) has the same structure as $\boldsymbol{\psi}$ from (2.9) and the additive monoid $(S, +)$ is the one that determines the informativeness. In particular, one has the following analogue of Proposition 2.20.

Proposition 3.10 (Informativeness of module dualities)**.** *Let $(S, +, \otimes)$ be a finite semiring. Then $\boldsymbol{\psi}$ from (3.5) is informative if $(S, +)$ is a submonoid of $(\mathbb{C}, \cdot)$, where $\cdot$ denotes the usual multiplication.*

Proof. Using Proposition 3.4 this follows in the same way as Proposition 2.20. □

Also Proposition 2.21 has a direct analogue for dualities between modules if one replaces the commutative monoid $\mathbf{T}$ with $(S, +)$. In particular, we can apply the same iterative procedure to check if a duality function arising from a duality between modules is weakly informative. Unfortunately, in contrast to the results in Chapter 2.6, there exist cases in which we cannot rule out weak informativeness in the second step of the iteration while Proposition 3.10 is not applicable either. For example, consider the semiring from Table 3.2, whose additive monoid is $\mathbf{M}_4$ and whose multiplicative monoid is isomorphic to $\mathbf{M}_3$. Ordering the elements of $\{0, 1, 2\}^2$ again as $(0, 0)$, $(0, 1)$, $(0, 2)$, $(1, 0)$, $(1, 1)$, $(1, 2)$, $(2, 0)$, $(2, 1)$, $(2, 2)$, the matrix form of ψ^2, computed as in Chapter 2.6 based on ψ from (3.4) for the chosen product, and a test matrix B are given as

$$\psi^2 = \begin{pmatrix} 0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&1&0&0&1 \\ 0&1&2&0&1&2&0&1&2 \\ 0&0&0&0&0&0&1&1&1 \\ 0&0&1&0&0&1&1&1&1 \\ 0&1&2&0&1&2&1&1&2 \\ 0&0&0&1&1&1&2&2&2 \\ 0&0&1&1&1&1&2&2&2 \\ 0&1&2&1&1&2&2&2&2 \end{pmatrix}, \quad B = \begin{pmatrix} 1&1&1&1&1&1&1&1&1 \\ 1&1&0&1&1&0&1&1&0 \\ 1&0&0&1&0&0&1&0&0 \\ 1&0&0&1&0&0&0&0&0 \\ 1&1&1&0&0&0&0&0&0 \\ 0&0&0&1&1&1&0&0&0 \\ 1&1&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0 \\ 0&1&0&1&1&0&0&0&0 \end{pmatrix}.$$

Clearly, the equation $Bx = 0$ only has the trivial solution and we cannot rule out the (weak) informativeness of the corresponding duality function $\boldsymbol{\psi}$.

We have computed ψ^2 for all found semirings and colored Table 3.2 and Table 3.3 accordingly. Examples of two random variables that rule out the weak informativeness of the semirings colored in red are collected in Appendix A.2. Note that Proposition 3.10 is only applicable to duality functions that also appeared in Chapter 2.5. Moreover, the considerable number of duality functions for which our methods are unable to determine whether they are weakly informative highlights the necessity for the development of further methods.

$(M_8,\cdot)$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	3
3	0	3	3	3

mult. $\cong \mathbf{M}_{14}$

$(M_8,\cdot)$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	2	3
3	0	3	3	3

mult. $\cong \mathbf{M}_{15}$

$(M_8,\cdot)$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	1	3
3	0	3	3	3

mult. $\cong \mathbf{M}_{16}$

$(M_9,\cdot)$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	3
3	0	3	3	3

mult. $\cong \mathbf{M}_{14}$

$(M_{10},\cdot)$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	3

mult. $\cong \mathbf{M}_{13}$

$(M_{10},\cdot)$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	2	2
3	0	3	2	3

mult. $\cong \mathbf{M}_{15}$

$(M_{11},\cdot)$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

mult. $\cong \mathbf{M}_{11}$

$(M_{11},\cdot)$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	3

mult. $\cong \mathbf{M}_{13}$

$(M_{11},\cdot)$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	3
3	0	3	3	3

mult. $\cong \mathbf{M}_{14}$

$(M_{11},\cdot)$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	2	2
3	0	3	2	3

mult. $\cong \mathbf{M}_{15}$

$(M_{11},\cdot)$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	1	3
3	0	3	3	3

mult. $\cong \mathbf{M}_{16}$

$(M_{13},\cdot)$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	3

mult. $\cong \mathbf{M}_{13}$

$(M_{13},\cdot)$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	2	2
3	0	3	2	3

mult. $\cong \mathbf{M}_{15}$

$(M_{14},\cdot)$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	2	3
3	0	3	3	3

mult. $\cong \mathbf{M}_{15}$

$(M_{15},\cdot)$	0	1	2	3
0	0	0	0	0
1	0	0	0	1
2	0	0	0	2
3	0	1	2	3

mult. $\cong \mathbf{M}_{8}$

$(M_{15},\cdot)$	0	1	2	3
0	0	0	0	0
1	0	0	0	1
2	0	0	1	2
3	0	1	2	3

mult. $\cong \mathbf{M}_{9}$

$(M_{15},\cdot)$	0	1	2	3
0	0	0	0	0
1	0	0	0	1
2	0	0	2	2
3	0	1	2	3

mult. $\cong \mathbf{M}_{10}$

$(M_{15},\cdot)$	0	1	2	3
0	0	0	0	0
1	0	0	1	1
2	0	1	2	3
3	0	1	3	3

mult. $\cong \mathbf{M}_{13}$

$(M_{15},\cdot)$	0	1	2	3
0	0	0	0	0
1	0	0	1	1
2	0	1	2	2
3	0	1	2	3

mult. $\cong \mathbf{M}_{13}$

$(M_{15},\cdot)$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	3
3	0	3	3	3

mult. $\cong \mathbf{M}_{14}$

$(M_{15},\cdot)$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	1	2
3	0	1	2	3

mult. $\cong \mathbf{M}_{14}$

$(M_{15},\cdot)$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	2	3
3	0	3	3	3

mult. $\cong \mathbf{M}_{15}$

$(M_{15},\cdot)$	0	1	2	3
0	0	0	0	0
1	0	1	1	3
2	0	1	2	3
3	0	3	3	3

mult. $\cong \mathbf{M}_{15}$

$(M_{15},\cdot)$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	3
3	0	1	3	3

mult. $\cong \mathbf{M}_{15}$

$(M_{15},\cdot)$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	2
3	0	1	2	3

mult. $\cong \mathbf{M}_{15}$

$(M_{15},\cdot)$	0	1	2	3
0	0	0	0	0
1	0	0	0	1
2	0	1	2	2
3	0	1	2	3

mult. $\cong \mathbf{N}_{1}$

$(M_{15},\cdot)$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	3
3	0	3	3	3

mult. $\cong \mathbf{N}_{2}$

$(M_{17},\cdot)$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	2	3
3	0	3	3	3

mult. $\cong \mathbf{M}_{15}$

$(M_{20},\cdot)$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	3

mult. $\cong \mathbf{M}_{13}$

$(M_{20},\cdot)$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	2	2
3	0	3	2	3

mult. $\cong \mathbf{M}_{15}$

$(M_{21},\cdot)$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	0
3	0	3	0	3

mult. $\cong \mathbf{M}_{10}$

$(M_{22},\cdot)$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	2	3
3	0	3	3	3

mult. $\cong \mathbf{M}_{15}$

$(M_{23},\cdot)$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	2	0
3	0	3	0	3

mult. $\cong \mathbf{M}_{11}$

$(M_{24},\cdot)$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	1	3
3	0	3	3	3

mult. $\cong \mathbf{M}_{16}$

$(M_{25},\cdot)$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	2	0
3	0	3	0	3

mult. $\cong \mathbf{M}_{11}$

$(M_{25},\cdot)$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	1	3
3	0	3	3	0

mult. $\cong \mathbf{M}_{12}$

$(M_{25},\cdot)$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	1
3	0	3	1	2

mult. $\cong \mathbf{M}_{18}$

$(M_{26},\cdot)$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

mult. $\cong \mathbf{M}_{12}$

Table 3.3: Cayley tables of all products on $(M_k,+)$ such that $(M_k,+,\cdot)$ becomes a semiring ($k \in \{8,\dots,26\}$). Green tables indicate cases where $\boldsymbol{\psi}$, defined as in (3.5), is weakly informative, while red tables indicate cases where $\boldsymbol{\psi}$ is not weakly informative. Orange tables indicate cases where our methods were not able to decide whether $\boldsymbol{\psi}$ is weakly informative. Compare Chapter 3.5.

4. Monotone duality

In this chapter we assume that the local state space S is equipped with a partial order $\leq$ and that is has a least element 0, i.e., $0 \leq a$ for all $a \in S$. We equip S^Λ with the product order and we write $x < y$ if $x \leq y$ and $x \neq y$ $(x, y \in S^\Lambda)$. For any set $A \subset S^\Delta$ $(\Delta \subset \Lambda)$, we call

$$A^\uparrow := \Big\{x \in S^\Delta : \exists x' \in A \text{ s.t. } x' \leq x\Big\} \tag{4.1}$$

the *upset* of A, and we say that A is *increasing* if $A^\uparrow = A$. In analogy with (4.1), the *downset* of a set $A \subset S^\Delta$ $(\Delta \subset \Lambda)$ is defined as

$$A^\downarrow = \Big\{x \in S^\Delta : \exists x' \in A \text{ s.t. } x \leq x'\Big\},$$

and we say that A is *decreasing* if $A^\downarrow = A$.

In this setup we will construct a dual process for interacting particle systems that have a random mapping representation, where each $m \in \mathcal{G}$ is *monotone* in the sense that

$$x \leq x' \quad \text{implies} \quad m(x) \leq m(x') \qquad (x, x' \in S^\Lambda). \tag{4.2}$$

The staring point of the construction is $\boldsymbol{\psi}_{\text{basic}} : S^\Lambda \times \mathcal{P}(S^\Lambda) \to \{0, 1\}$ from Chapter 1.4 in the form of (1.38). Recall that every map $m : S^\Lambda \to S^\Lambda$ has a unique dual map with respect to $\boldsymbol{\psi}_{\text{basic}}$ that is given by its preimage map m^{-1}. If m is monotone and $A \subset S^\Lambda$ is increasing, then it turns out (compare the proof of [Sturm and Swart, 2018, Lemma 5]) that $m^{-1}(A)$ is increasing as well. Moreover, if m is continuous, then $m^{-1}(A)$ is, of course, open if A is open. This gives rise to the idea to restrict $\boldsymbol{\psi}_{\text{basic}}$ in the second argument to all open and increasing subsets of S^Λ. Constructing the interacting particle system $X = (X_t)_{t \geq 0}$ then via its stochastic flow, as done in (1.13), the idea is to define a backward stochastic flow consisting of the preimage maps of the maps that make up the original stochastic flow.

The dual process based on the idea outlined above would have the collection of all open increasing subsets of S^Λ as its state space. Since this would be a rather abstract state space, we describe in Chapter 4.1 a more accessible characterization of this space.

We will also assume that the interacting particle system X has $\underline{0}$, the configuration constantly equal to the least element $0 \in S$, as a trap. This will prevent the about to be constructed dual process from jumping into a trap that we are not interested in. Due to the existence of the trap $\underline{0}$, we are again in the setup of Chapter 1.2 and can repeat the definitions of $\text{supp}(x)$, S^Λ_{fin} and δ^a_i $(i \in \Lambda, a \in S)$, now based on the least element 0. Moreover, we are going to use $x\lfloor_\Gamma \in S^\Lambda$ $(\Gamma \subset \Lambda)$, defined as in (2.10) but based on the least element 0.

4.1 The monotone dual space

Recall that a *minimal element* of a set $A \subset S^\Lambda$ is a configuration $x \in A$ such that there does not exist an $x' \in A$ with $x' < x$. We let

$$A^\circ := \Big\{x : x \text{ is a minimal element of } A\Big\} \tag{4.3}$$

denote the set of minimal elements of A. Recall that S^Λ is equipped with the product topology. We set

$$\mathcal{I}(S^\Lambda) := \Big\{A \subset S^\Lambda : A \text{ is open and increasing}\Big\},$$
$$\mathcal{H}(S^\Lambda) := \Big\{Y \subset S^\Lambda_{\text{fin}} : Y^\circ = Y\Big\}.$$

The following proposition describes a bijection between $\mathcal{I}(S^\Lambda)$ and $\mathcal{H}(S^\Lambda)$.

Proposition 4.1 (Encoding open increasing sets)**.** *The map $Y \mapsto Y^\uparrow$ is a bijection from $\mathcal{H}(S^\Lambda)$ to $\mathcal{I}(S^\Lambda)$ and the map $A \mapsto A^\circ$ is its inverse.*

Proof. First note that $Y^\uparrow$ is indeed open for $Y \in \mathcal{H}(S^\Lambda)$. Indeed, if $Y = \emptyset$, then also $Y^\uparrow = \emptyset$. If $Y = \{y\}$ for some $y \in S^\Lambda_{\text{fin}}$, then

$$Y^\uparrow = \{y\}^\uparrow = \Big\{x \in S^\Lambda : y(i) \leq x(i) \text{ for } i \in \operatorname{supp}(y)\Big\}.$$

By the definitions of the product topology and of the discrete topology on S, all finite-dimensional cylinder sets are open and hence, since $\operatorname{supp}(y)$ is finite, $Y^\uparrow$ is open in the product topology.[1] If Y consists of more than one element, then we can write

$$Y^\uparrow = \bigcup_{y \in Y} \{y\}^\uparrow,$$

and $Y^\uparrow$ is open as a union of open sets.

It then suffices to show that $(Y^\uparrow)^\circ = Y$ for $Y \in \mathcal{H}(S^\Lambda)$, $A^\circ \subset S^\Lambda_{\text{fin}}$ for $A \in \mathcal{I}(S^\Lambda)$ and $(A^\circ)^\uparrow = A$ for $A \in \mathcal{I}(S^\Lambda)$. The first assertion is easy to verify. We will show the third assertion and the arguments along the way will imply the second one as well.

Let $A \in \mathcal{I}(S^\Lambda)$. Then $A^\circ \subset A$ implies that $(A^\circ)^\uparrow \subset A^\uparrow = A$. If $A = \emptyset$, then $\emptyset \subset (\emptyset^\circ)^\uparrow$ trivially and there is nothing left to show. Hence, assume that $A \neq \emptyset$ and let $x \in A$. Let $(\Delta_n)_{n\in\mathbb{N}}$ be a sequence of finite subsets of Λ with the property that $\Delta_n \nearrow \Lambda$. Then $x\lfloor_{\Delta_n} \to x$ in the product topology. As A is open, there exists an $N \in \mathbb{N}$ so that $x\lfloor_{\Delta_n} \in A$ for all $n \geq N$. As $x\lfloor_{\Delta_N} \in S^\Lambda_{\text{fin}}$, we can find an $x' \in A^\circ$ such that $x' \leq x\lfloor_{\Delta_N} \leq x$, thus $x \in (A^\circ)^\uparrow$. In particular, this shows that there cannot exist an $x \in A^\circ \cap (S^\Lambda_{\text{fin}})^{\text{c}}$. □

We will use the space

$$\mathcal{H}_-(S^\Lambda) := \mathcal{H}(S^\Lambda) \setminus \{\{\underline{0}\}\}$$

as the state space of the dual process. The main advantage of choosing $\mathcal{H}_-(S^\Lambda)$ over $\mathcal{H}(S^\Lambda)$ lies in the more interesting greatest element of $\mathcal{H}_-(S^\Lambda)$, based on which we will construct an upper invariant law of the dual process in Chapter 4.5. For the aim to use $\mathcal{H}_-(S^\Lambda)$ as the state space of the dual process, we need to equip $\mathcal{H}_-(S^\Lambda)$ with a topology. Note that $\mathcal{H}_-(S^\Lambda)$ is uncountable, so this task is not as straightforward as in the previous chapters. We first equip $\mathcal{I}(S^\Lambda)$ with a topology and then use the bijection from Proposition 4.1 to transfer it to $\mathcal{H}(S^\Lambda)$. We will use the topology described in the following proposition. Note that $\{\underline{0}\}^\uparrow = S^\Lambda$ so that the space $\mathcal{I}_-(S^\Lambda)$ defined below corresponds to $\mathcal{H}_-(S^\Lambda)$ via the bijection of Proposition 4.1.

[1]In fact, $Y^\uparrow = \{y\}^\uparrow$ is clopen (i.e., both closed and open).

Proposition 4.2 (Dual topology)**.** *There exists a unique metrizable topology on $\mathcal{I}(S^\Lambda)$ such that a sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{I}(S^\Lambda)$ converges to a limit $A \in \mathcal{I}(S^\Lambda)$ if and only if*

$$\mathbb{1}_{A_n}(x) \to \mathbb{1}_A(x) \quad \textit{for all} \quad x \in S^\Lambda_{\text{fin}}. \tag{4.4}$$

The space $\mathcal{I}(S^\Lambda)$ is compact in this topology, and so is $\mathcal{I}_-(S^\Lambda) := \mathcal{I}(S^\Lambda) \setminus \{S^\Lambda\}$.

We construct the topology on $\mathcal{I}(S^\Lambda)$ that satisfies Proposition 4.2 explicitly using several lemmas. The staring point is the metric d from (1.2). Recall that the metric d generates the product topology on S^Λ. Note that $d(x,y) \leq 1/2$ $(x, y \in S^\Lambda)$ with equality if and only if $x(i) \neq y(i)$ for all $i \in \Lambda$. Moreover, $d(x,y) < 1/3^n$ $(x, y \in S^\Lambda)$ implies that $d(x,y) \leq 1/(2 \cdot 3^n) = \sum_{k=n+1}^\infty 1/3^k$, so that the open ball

$$\mathcal{B}_{1/3^n}(x) := \Big\{ y \in S^\Lambda : d(x,y) < \frac{1}{3^n} \Big\} \qquad (x \in S^\Lambda)$$

is actually clopen, i.e., both closed and open.

Let $\mathcal{K}(S^\Lambda)$ denote the space of all compact subsets of S^Λ. On $\mathcal{K}_+(S^\Lambda) := \mathcal{K}(S^\Lambda) \setminus \{\emptyset\}$ one defines the *Hausdorff metric* d_{H} as

$$d_{\text{H}}(A,B) := \max \Big\{ \sup_{x\in A} d(x,B), \sup_{y \in B} d(y,A) \Big\} \qquad (A, B \in \mathcal{K}_+(S^\Lambda)),$$

where d is the metric from (1.2) and (as usual)

$$d(x,B) := \inf_{y\in B} d(x,y) \qquad (x \in S^\Lambda,\ B \subset S^\Lambda).$$

The corresponding topology on $(\mathcal{K}_+(S^\Lambda), d_{\text{H}})$ is called the *Hausdorff topology* and it is well-known (see, e.g., [Schertzer et al., 2014, Lemma B.1]) that it only depends on the topology on S^Λ, i.e., the product topology, and not on the exact definition of the underlying metric. However, using the metric d_{H} based on the concrete metric d from (1.2) will be useful in the following. We extend the metric d_{H} to $\mathcal{K}(S^\Lambda)$ by setting $d_{\text{H}}(\emptyset, A) := 1$ for all $A \in \mathcal{K}_+(S^\Lambda)$ so that $\emptyset$ is an isolated point. By [Schertzer et al., 2014, Lemma B.3] the space $\mathcal{K}(S^\Lambda)$ is then compact since S^Λ is compact.

We want to identify $\mathcal{I}(S^\Lambda)$ with a subspace of $\mathcal{K}(S^\Lambda)$. The assertion $A_n^\updownarrow \to A^\updownarrow$ in the following lemma is to be understood to mean that both $A_n^\uparrow \to A^\uparrow$ and $A_n^\downarrow \to A^\downarrow$.

Lemma 4.3 (Convergence of up- and downset)**.** *Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{K}(S^\Lambda)$ and assume that $A_n \to A \in \mathcal{K}(S^\Lambda)$. Then also $A_n^\updownarrow \to A^\updownarrow$ in $\mathcal{K}(S^\Lambda)$.*

For the proof of Lemma 4.3 we will need a classical result. Let E be a compact metrizable space that is equipped with a partial order $\leq$ that is *compatible with the topology* in the sense that the set

$$\Big\{ (x,y) \in E^2 : x \leq y \Big\} \quad \text{is a closed subset of } E^2, \tag{4.5}$$

where E^2 is equipped with the product topology.[2] The following result is a special case of [Nachbin, 1965, Proposition 4].

[2]This notion is also used in [Liggett, 1985]. In the more classical references [Nachbin, 1965] and [Kamae and Krengel, 1978] an order that satisfies (4.5) is called *closed*.

Lemma 4.4 (Closedness of upset and downset)**.** *Let E be a compact Hausdorff space that is equipped with a partial order $\leq$ that is compatible with the topology. Assume that $A \subset E$ is closed. Then $A^{\uparrow}$ and $A^{\downarrow}$ are also closed.*

Proof of Lemma 4.3. It is easy to check that the product order on S^{Λ} is compatible with the (product) topology. Moreover, every $A \in \mathcal{K}(S^{\Lambda})$ is closed (in S^{Λ}), so Lemma 4.4 implies that the sets $A^{\updownarrow}$ are closed as well. Hence indeed $A^{\updownarrow} \in \mathcal{K}(S^{\Lambda})$, as closed subsets of a compact topological space are compact.

Now assume that $A = \emptyset$. As $\emptyset$ is isolated in $\mathcal{K}(S^{\Lambda})$, $A_n \to A$ implies that there exists an $N \in \mathbb{N}$ so that $A_n = \emptyset$ for all $n \geq N$. As $\emptyset^{\uparrow} = \emptyset^{\downarrow} = \emptyset$ it follows that also $A_n^{\updownarrow} \to A^{\updownarrow}$.

Let now $A, B \in \mathcal{K}_+(S^{\Lambda})$ and $n \in \mathbb{N}$. We show that $d_{\mathrm{H}}(A, B) < 1/3^n$ implies that $d_{\mathrm{H}}(A^{\uparrow}, B^{\uparrow}) < 1/3^n$. Let $x \in B^{\uparrow}$ and $b \in B$ so that $b \leq x$. Then, as $d_{\mathrm{H}}(A, B) < 1/3^n$, there exists an $a \in A$ such that $d(a, b) < 1/3^n$, which implies that $a_{\gamma^{-1}(\{1,\ldots,n\})} = b_{\gamma^{-1}(\{1,\ldots,n\})}$. Let now $a^x \in S^{\Lambda}$ be defined as

$$a^x(i) := \begin{cases} x(i) & \text{if } i \in \gamma^{-1}(\{1, \ldots, n\}), \\ a(i) & \text{else,} \end{cases} \qquad (i \in \Lambda).$$

Then $b \leq x$ implies that $a \leq a^x$ and the construction of a^x implies that $d(x, a^x) \leq 1/(2 \cdot 3^n)$ and hence $d(x, A^{\uparrow}) \leq 1/(2 \cdot 3^n)$. As $x \in B^{\uparrow}$ was arbitrary we conclude that $\sup_{x \in B^{\uparrow}} d(x, A^{\uparrow}) \leq 1/(2 \cdot 3^n)$. Interchanging the roles of B and A yields that $d_{\mathrm{H}}(A^{\uparrow}, B^{\uparrow}) \leq 1/(2 \cdot 3^n) < 1/3^n$. The argument for $\uparrow$ replaced by $\downarrow$ works analogously. □

Let $A \in \mathcal{I}(S^{\Lambda})$. Then A^{c}, being a closed subset of a compact topological space, is compact. We have the following.

Lemma 4.5 (Closedness within compact sets)**.** *The set $\{A^{\mathrm{c}} : A \in \mathcal{I}(S^{\Lambda})\}$ is closed in $\mathcal{K}(S^{\Lambda})$.*

Proof. Let $(B_n)_{n \in \mathbb{N}} \subset \{A^{\mathrm{c}} : A \in \mathcal{I}(S^{\Lambda})\}$ and assume that $B_n \to B \in \mathcal{K}(S^{\Lambda})$. As each B_n ($n \in \mathbb{N}$) is decreasing, Lemma 4.3 shows that also $B_n \to B^{\downarrow}$. The Hausdorff property of $\mathcal{K}(S^{\Lambda})$ then implies that $B = B^{\downarrow}$ and the proof is complete. □

Now we are ready to prove Proposition 4.2.

Proof of Proposition 4.2. Using Lemma 4.5 we can equip $\mathcal{I}(S^{\Lambda})$ with the metric

$$d_{\mathcal{I}}(A, B) := d_{\mathrm{H}}(A^{\mathrm{c}}, B^{\mathrm{c}}) \qquad (A, B \in \mathcal{I}(S^{\Lambda})), \tag{4.6}$$

making $(\mathcal{I}(S^{\Lambda}), d_{\mathcal{I}})$ and $(\mathcal{I}_-(S^{\Lambda}), d_{\mathcal{I}})$ compact metric spaces, isometric to some closed subspaces of the metric space $(\mathcal{K}(S^{\Lambda}), d_{\mathrm{H}})$.

Next, we prove the convergence criterion. To start with, we consider the case $A = S^{\Lambda}$. As S^{Λ} is isolated in $\mathcal{I}(S^{\Lambda})$, $A_n \to S^{\Lambda}$ implies that there exists an $N \in \mathbb{N}$ such that $A_n = S^{\Lambda}$ for all $n \geq N$ so that (4.4) is trivial. On the other hand, assuming (4.4) and taking $x = \underline{0}$ implies that there has to exist an $N \in \mathbb{N}$ such that $\underline{0} \in A_n$ for all $n \geq N$. But $\underline{0} \in A_n$ implies $\underline{0} \in A_n^{\circ}$ and by minimality $A_n^{\circ} = \{\underline{0}\}$. Hence, by Proposition 4.1, $A_n = S^{\Lambda}$ for all $n \geq N$ so that $A_n \to S^{\Lambda}$.

Assume now that $A \in \mathcal{I}_{-}(S^\Lambda)$. If $x = \underline{0}$ violates (4.4), then, by the arguments above, $A_n = S^\Lambda$ for infinitely many $n \in \mathbb{N}$ and A_n cannot converge to A. Now assume that there exists an $x \in S^\Lambda_{\text{fin}} \setminus \{\underline{0}\}$ such that $\mathbb{1}_{A_n}(x)$ does not converge to $\mathbb{1}_A(x)$. This implies that for all $n \in \mathbb{N}$ there exists an $N \geq n$ such that $x \in A_N \,\Delta\, A = (A_N \setminus A) \cup (A \setminus A_N)$. Let now $a_* := \min\{a_i : i \in \text{supp}(x)\}$ and $m := \log_{1/3}(a_*)$. We claim that $x \in A_N \,\Delta\, A$ implies that $d_{\text{H}}(A^{\text{c}}_N, A^{\text{c}}) \geq a_*$ and hence also

$$\limsup_{n\to\infty} d_{\text{H}}(A^{\text{c}}_n, A^{\text{c}}) \geq a_*.$$

To check the claim, due to symmetry we may w.l.o.g. assume that $x \in A_N \cap A^{\text{c}}$. Would there now be a $y \in A^{\text{c}}_N$ with $d(x, y) < a_* = 1/3^m$, then, as $\text{supp}(x) \subset \gamma^{-1}(\{1, \dots, m\})$, we would have $y\lfloor_{\text{supp}(x)} = x$. But, as A^{c}_N is decreasing, $x = y\lfloor_{\text{supp}(x)} \leq y$ implies $x \in A^{\text{c}}_N$, a contradiction.

For the reverse direction let $\varepsilon > 0$. Choose an $m \in \mathbb{N}$ such that $1/(2 \cdot 3^m) < \varepsilon$. By assumption, for all $x \in S^\Lambda_{\text{fin}}$ there exists an $N(x) \in \mathbb{N}$ such that $\mathbb{1}_{A_n}(x) = \mathbb{1}_A(x)$ for all $n \geq N(x)$. Set now

$$N_0 := \max\Big\{N(x) : \text{supp}(x) \subset \gamma^{-1}(\{1, \dots, m\})\Big\}.$$

We claim that this implies that $d_{\text{H}}(A^{\text{c}}_n, A^{\text{c}}) < \varepsilon$ for all $n \geq N_0$ and hence

$$\limsup_{n\to\infty} d_{\text{H}}(A^{\text{c}}_n, A^{\text{c}}) = 0,$$

as ε was arbitrary. To check the claim, let $n \geq N_0$ and assume that there exists an $x \in S^\Lambda$ with arbitrary support satisfying $x \in A^{\text{c}}_n \cap A$. We then also have that $x\lfloor_{\gamma^{-1}(\{1,\dots,m\})} \in A^{\text{c}}_n$ which implies that $x\lfloor_{\gamma^{-1}(\{1,\dots,m\})} \in A^{\text{c}}$ as $n \geq N_0$. Due to the construction of d this implies that $d(x, A^{\text{c}}) \leq 1/(2 \cdot 3^{N_0})$. By symmetry, an arbitrary $x \in A_n \cap A^{\text{c}}$ has to satisfy $d(x, A^{\text{c}}_n) \leq 1/(2 \cdot 3^{N_0})$ and the claim follows.

The uniqueness of the metrizable topology that satisfies (4.4) follows from the fact that convergence of sequences characterizes a topology in metrizable spaces. □

We equip $\mathcal{H}(S^\Lambda)$ with a topology so that the bijection from Proposition 4.1 is a homeomorphism. Then both $\mathcal{H}(S^\Lambda)$ and $\mathcal{H}_{-}(S^\Lambda)$ are compact metrizable spaces and a sequence $(Y_n)_{n\in\mathbb{N}} \subset \mathcal{H}(S^\Lambda)$ converges to a limit $Y \in \mathcal{H}(S^\Lambda)$ if and only if $\mathbb{1}_{Y^\uparrow_n}(x) \to \mathbb{1}_{Y^\uparrow}(x)$ for all $x \in S^\Lambda_{\text{fin}}$.

As S^Λ is equipped with a partial order, it makes sense to also have a partial order on $\mathcal{H}_{-}(S^\Lambda)$. We equip the space $\mathcal{I}(S^\Lambda)$ with the partial order of set inclusion, which through the bijection of Proposition 4.1 defines a partial order $\leq$ on $\mathcal{H}(S^\Lambda)$ such that

$$Y \leq Z \quad \Leftrightarrow \quad Y^\uparrow \subset Z^\uparrow \qquad (Y, Z \in \mathcal{H}(S^\Lambda)). \tag{4.7}$$

Since $\{\underline{0}\}^\uparrow = S^\Lambda$, it is clear that $\{\underline{0}\}$ is the greatest element of $\mathcal{H}(S^\Lambda)$. It turns out that $\mathcal{H}_{-}(S^\Lambda)$ also has a greatest element, which is more interesting. We set

$$S_{\text{sec}} := (S \setminus \{0\})^\circ.$$

Elements of S_{sec} are "second from below" in the order on S. We define $Y_{\mathrm{sec}} \in \mathcal{H}(S^\Lambda)$ as

$$Y_{\mathrm{sec}} := \left\{\delta_i^a : i \in \Lambda,\ a \in S_{\mathrm{sec}}\right\}, \tag{4.8}$$

with δ_i^a being defined by (1.16). The following lemma describes some elementary properties of the partial order on $\mathcal{H}(S^\Lambda)$.

Lemma 4.6 (Order on the dual state space)**.** *The partial order $\leq$ defined in (4.7) is compatible with the topologies on $\mathcal{H}(S^\Lambda)$ and $\mathcal{H}_-(S^\Lambda)$. Moreover, Y_{sec} is the greatest element of $\mathcal{H}_-(S^\Lambda)$.*

Proof. Let $Y, Z \in \mathcal{H}(S^\Lambda)$. As

$$Y \leq Z \quad \Leftrightarrow \quad Y^\uparrow \subset Z^\uparrow \quad \Leftrightarrow \quad (Z^\uparrow)^{\mathrm{c}} \subset (Y^\uparrow)^{\mathrm{c}},$$

it suffices to show that $\mathcal{K}(S^\Lambda)$ satisfies (4.5) if it is equipped with the partial order $\subset$. Let $(A_n)_{n\in\mathbb{N}}, (B_n)_{n\in\mathbb{N}} \subset \mathcal{K}(S^\Lambda)$ be two sequences such that $A_n \subset B_n$ for all $n \in \mathbb{N}$ and assume that $A_n \to A \in \mathcal{K}(S^\Lambda)$ and $B_n \to B \in \mathcal{K}(S^\Lambda)$. If $A = \emptyset$, then trivially $A \subset B$. Hence, assume that $A \neq \emptyset$ and let $a \in A$. From the definition of the Hausdorff metric we can conclude that there exist $a_n \in A_n$ ($n \in \mathbb{N}$) so that $a_n \to a$ in S^Λ. But then $a_n \in B_n$ ($n \in \mathbb{N}$) and by [Schertzer et al., 2014, Lemma B.1] this implies that $a \in B$. From this one concludes that $A \subset B$. Hence, $\mathcal{K}(S^\Lambda)$ satisfies (4.5).

Finally, we consider Y_{sec}. Clearly, $\underline{0} \notin Y_{\mathrm{sec}}$, so $Y_{\mathrm{sec}} \in \mathcal{H}_-(S^\Lambda)$. Let $Y \in \mathcal{H}_-(S^\Lambda)$. Since $\underline{0} \notin Y$, for every $y \in Y$ there has to exist an $i \in \Lambda$ satisfying $y(i) \neq 0$. But then there also exists an $a \in S_{\mathrm{sec}}$ such that $a \leq y(i)$. Hence also $\delta_i^a \leq y$ and thus $y \in Y_{\mathrm{sec}}^\uparrow$. It follows that $Y^\uparrow \subset Y_{\mathrm{sec}}^\uparrow$, i.e., $Y \leq Y_{\mathrm{sec}}$. □

4.2 The monotone backward flow

After having equipped $\mathcal{H}_-(S^\Lambda)$ in the last subchapter with a topology and a partial order, we are now ready to construct the dual process. Recall that, in contrast to the last two chapters, Proposition 1.8 and Theorem 1.9 are not applicable since $\mathcal{H}_-(S^\Lambda)$ is uncountable. Therefore, we take a more direct approach than in the last two chapters. Instead of working with the local maps that make up the collection $\mathcal{G}$ and first prove duality between local maps, we work directly with the stochastic flow $(\mathbf{X}_{s,u})_{s\leq u}$. As indicated at the start of this chapter, the idea is to define a backward stochastic flow consisting of the preimage maps of the maps that make up $(\mathbf{X}_{s,u})_{s\leq u}$ while utilizing the homeomorphism from Proposition 4.1. The next lemma shows that this idea indeed leads to an almost surely well-defined backward stochastic flow in the sense of Chapter 1.3.

Lemma 4.7 (Dual flow)**.** *Assume the summability condition (1.7) and that every map $m \in \mathcal{G}$ is monotone in the sense of (4.2) and satisfies (1.22). Let $(\mathbf{X}_{s,u})_{s\leq u}$ be the stochastic flow from (1.12). Then, almost surely, setting*

$$\mathbf{Y}_{u,s}(Y) := \left\{y \in S^\Lambda : \mathbf{X}_{s,u}(y) \in Y^\uparrow\right\}^\circ \qquad (Y \in \mathcal{H}(S^\Lambda),\ u \geq s) \tag{4.9}$$

yields a well-defined map $\mathbf{Y}_{u,s} : \mathcal{H}(S^\Lambda) \to \mathcal{H}(S^\Lambda)$ for all $u \geq s$.

Proof. Let $u \geq s$, $Y \in \mathcal{H}(S^\Lambda)$, and let $A := \{y \in S^\Lambda : \mathbf{X}_{s,u}(y) \in Y^\uparrow\}$ be the preimage of $Y^\uparrow$ under the map $\mathbf{X}_{s,u}$. By Lemma 1.4, $\mathbf{X}_{s,u}$ is almost surely continuous. Similarly, it follows from Lemma 1.2 that $\mathbf{X}_{s,u}$ is almost surely monotone. The continuity of $\mathbf{X}_{s,u}$ implies that A is open and the monotonicity of $\mathbf{X}_{s,u}$ implies that A is increasing. By (4.9) and Proposition 4.1, it now follows that $\mathbf{Y}_{u,s}(Y)^\uparrow = A$. Hence $\mathbf{Y}_{u,s}(Y)^\uparrow \in \mathcal{I}(S^\Lambda)$ and $\mathbf{Y}_{u,s}(Y) \in \mathcal{H}(S^\Lambda)$ by Proposition 4.1. □

It follows readily that $(\mathbf{Y}_{u,s})_{u \geq s}$ is a backward stochastic flow in the sense of Chapter 1.3. Note that $\mathbf{Y}_{u,s}(Y)$ is the collection of minimal configurations y with the property that if we start the interacting particle system $X = (X_t)_{t \geq 0}$ from Theorem 1.3 at time s in the initial state y and evolve it under the graphical representation, then at time u the state of the interacting particle system lies in $Y^\uparrow$.

Due to the construction we instantly get a duality of the stochastic flows $(\mathbf{X}_{s,u})_{s \leq u}$ and $(\mathbf{Y}_{u,s})_{u \geq s}$ in the sense of (1.27). Let $\boldsymbol{\psi}_{\rm mon} : S^\Lambda \times \mathcal{H}(S^\Lambda) \to \{0,1\}$ be defined as

$$\boldsymbol{\psi}_{\rm mon}(x, Y) := \mathbb{1}_{Y^\uparrow}(x) \qquad (x \in S^\Lambda,\ Y \in \mathcal{H}(S^\Lambda)), \tag{4.10}$$

i.e., by first restricting $\boldsymbol{\psi}_{\rm basic}$ in the form of (1.38) in the second coordinate to $\mathcal{I}(S^\Lambda)$ and then applying the homeomorphism between $\mathcal{I}(S^\Lambda)$ and $\mathcal{H}(S^\Lambda)$ from Proposition 4.1. It follows from the definition of the backward stochastic flow $(\mathbf{Y}_{u,s})_{u \geq s}$ that

$$\boldsymbol{\psi}_{\rm mon}(\mathbf{X}_{s,u}(x), Y) = \boldsymbol{\psi}_{\rm mon}(x, \mathbf{Y}_{u,s}(Y)) \tag{4.11}$$

holds almost surely for all $s \leq u$, $x \in S^\Lambda$ and $Y \in \mathcal{H}(S^\Lambda)$ simultaneously.

It turns out that we can define the backward stochastic flow $(\mathbf{Y}_{u,s})_{u \geq s}$ also if we assume the summability condition (1.18) instead of (1.7). As a consequence, (4.11) then holds almost surely for all $s \leq u$, $x \in S^\Lambda_{\rm fin}$ (!) and $Y \in \mathcal{H}(S^\Lambda)$ simultaneously.

Lemma 4.8 (Alternative definition)**.** *Assume the summability condition (1.18) and that every map $m \in \mathcal{G}$ is monotone in the sense of (4.2) and satisfies (1.22). Let $(\mathbf{X}_{s,u})_{s \leq u}$ be the stochastic flow from (1.20). Then, almost surely, setting*

$$\mathbf{Y}_{u,s}(Y) := \left\{y \in S^\Lambda_{\rm fin} : \mathbf{X}_{s,u}(y) \in Y^\uparrow\right\}^\circ \qquad (Y \in \mathcal{H}(S^\Lambda),\ u \geq s) \tag{4.12}$$

yields a well-defined map $\mathbf{Y}_{u,s} : \mathcal{H}(S^\Lambda) \to \mathcal{H}(S^\Lambda)$ for all $u \geq s$. If also the summability condition (1.7) holds, then the maps from (4.9) and (4.12) are (a.s.) identical.

Proof. It follows from the proofs of Proposition 1.5 and Theorem 1.6 that under (1.18), almost surely, for all $s \leq u$ the random map $\mathbf{X}_{s,u} : S^\Lambda_{\rm fin} \to S^\Lambda_{\rm fin}$ is a concatenation of finitely many continuous and monotone maps. Hence, $\mathbf{X}_{s,u}$ itself is continuous and monotone. The rest of the argument for the first assertion follows in the same way as in the proof of Lemma 4.7.

If the summability conditions (1.7) and (1.18) both hold, then the definitions of $\mathbf{X}_{s,u}(y)$ for $y \in S^\Lambda_{\rm fin}$ via (1.12) and (1.20) yield, by definition, the same (random) element of $S^\Lambda_{\rm fin}$. Moreover, any element of the right-hand side of (4.9) has to be an element of $S^\Lambda_{\rm fin}$ as we have seen in the proof of Lemma 4.7. It follows that the right-hand sides of (4.9) and (4.12) (a.s.) coincide. □

4.2.1 The monotone dual process

With this we have shown two of the three bullet points from Chapter 1.3 that were needed in order to have a pathwise duality. The main result of this subchapter says that under the assumptions from Chapter 1.2 also the last remaining bullet point is satisfied. Namely, we show that, for fixed $u \in \mathbb{R}$ and a random variable Y_0, setting $\mathsf{Y}_s := \mathbf{Y}_{u,u-s}(\mathsf{Y}_0)$ as in (1.33) defines a Feller process $\mathsf{Y} = (\mathsf{Y}_s)_{s\geq 0}$ with state space $\mathcal{H}_-(S^\Lambda)$ and càglàd sample paths. In the upcoming theorem $\mathbb{P}$ again denotes the probability measure on the probability space where the graphical representation ω from Chapter 1.1 lives.

Theorem 4.9 (Dual stochastic flow and Markov process)**.** *Assume the summability condition (1.18), and that every map $m \in \mathcal{G}$ is monotone and satisfies (1.22). Then, almost surely, (4.12) defines a map $\mathbf{Y}_{u,s} : \mathcal{H}_-(S^\Lambda) \to \mathcal{H}_-(S^\Lambda)$ for all $u \geq s$, and setting*

$$Q_t(Y, \cdot\,) := \mathbb{P}[\mathbf{Y}_{t,0}(Y) \in \cdot\,] \qquad (Y \in \mathcal{H}_-(S^\Lambda),\ t \geq 0)$$

defines a Feller semigroup on $\mathcal{H}_-(S^\Lambda)$. If $u \in \mathbb{R}$ and Y_0 is a random variable with values in $\mathcal{H}_-(S^\Lambda)$ that is independent of the graphical representation ω, then setting

$$\mathsf{Y}_t := \mathbf{Y}_{u,u-t}(\mathsf{Y}_0) \qquad (t \geq 0) \tag{4.13}$$

defines a Feller process $\mathsf{Y} = (\mathsf{Y}_t)_{t\geq 0}$ with càglàd sample paths whose transition probabilities are $(Q_t)_{t\geq 0}$.

Proof. Recall from Theorem 1.6 and its proof that the summability condition (1.18) implies that

$$\sum_{\substack{m\in\mathcal{G}:\\ m(x)\neq x}} r_m < \infty \quad \text{for all } x \in S^\Lambda_{\text{fin}} \tag{4.14}$$

holds and that one can (a.s.) define random maps $\mathbf{X}_{s,u} : S^\Lambda_{\text{fin}} \to S^\Lambda_{\text{fin}}$ for all $s \leq u$ via (1.20).

We begin the proof by showing that $\mathbf{Y}_{u,s}(Y) \in \mathcal{H}_-(S^\Lambda)$ for all $u \geq s$ and $Y \in \mathcal{H}_-(S^\Lambda)$. Let $Y \in \mathcal{H}_-(S^\Lambda)$. Using (4.11), Lemma 1.4 and the definition of $\boldsymbol{\psi}_{\text{mon}}$,

$$0 = \boldsymbol{\psi}_{\text{mon}}(\underline{0}, Y) = \boldsymbol{\psi}_{\text{mon}}(\mathbf{X}_{s,u}(\underline{0}), Y) = \boldsymbol{\psi}_{\text{mon}}(\underline{0}, \mathbf{Y}_{u,s}(Y)), \tag{4.15}$$

thus $\mathbf{Y}_{u,s}(Y) \neq \{\underline{0}\}$, i.e., $\mathbf{Y}_{u,s}(Y) \in \mathcal{H}_-(S^\Lambda)$.

Moreover, by construction, the stochastic flow $(\mathbf{Y}_{u,s})_{u\geq s}$ has independent increments meaning that $\mathbf{Y}_{t_1,t_0}, \mathbf{Y}_{t_2,t_1}, \ldots, \mathbf{Y}_{t_n,t_{n-1}}$ are independent for all $t_0 < t_1 < \ldots < t_n$ $(n \in \mathbb{N})$ and $\mathbf{Y}_{u,s}$ and $\mathbf{Y}_{u+t,s+t}$ $(u \geq s,\ t \in \mathbb{R})$ are identically distributed. These two facts imply that $\mathsf{Y} = (\mathsf{Y}_t)_{t\geq 0}$ defined by (4.13) is a Markov process with Markov semigroup $(Q_t)_{t\geq 0}$ (compare [Swart, 2022, Proofs of Theorem 4.20 and Proposition 2.7]).

Hence, to conclude that Y is a Feller process it suffices to show that

$$(Y, t) \mapsto Q_t(Y, \cdot\,) \text{ is a continuous map from } \mathcal{H}_-(S^\Lambda) \times [0, \infty) \text{ to } \mathcal{M}_1(\mathcal{H}_-(S^\Lambda)).$$

Let $((Y_n, t_n))_{n\in\mathbb{N}} \subset \mathcal{H}_-(S^\Lambda)\times[0,\infty)$ such that $(Y_n, t_n) \to (Y, t) \in \mathcal{H}_-(S^\Lambda)\times[0,\infty)$ as $n \to \infty$ (where $\mathcal{H}_-(S^\Lambda)\times[0,\infty)$ is equipped with the product topology). Since almost sure convergence implies weak convergence in law, it suffices to show that

$$\mathbf{Y}_{t_n,0}(Y_n) \underset{n\to\infty}{\longrightarrow} \mathbf{Y}_{t,0}(Y) \quad \text{a.s.}$$

By Proposition 4.2 we have to show that

$$\mathbb{1}_{(\mathbf{Y}_{t_n,0}(Y_n))^\uparrow}(x) \underset{n\to\infty}{\longrightarrow} \mathbb{1}_{(\mathbf{Y}_{t,0}(Y))^\uparrow}(x) \quad \text{a.s.}$$

for each $x \in S^\Lambda_{\text{fin}}$. By (4.11) this is equivalent to

$$\mathbb{1}_{Y_n^\uparrow}(\mathbf{X}_{0,t_n}(x)) \underset{n\to\infty}{\longrightarrow} \mathbb{1}_{Y^\uparrow}(\mathbf{X}_{0,t}(x)) \quad \text{a.s.} \tag{4.16}$$

for each $x \in S^\Lambda_{\text{fin}}$. Let, as in (1.21),

$$I(x) := \Big\{u \in \mathbb{R} : \exists (m,u) \in \omega \text{ with } m(x) \neq x\Big\}.$$

Then, due to (4.14), $I(x)$ is a Poison point set on $\mathbb{R}$ with finite intensity. Let now

$$t_- := \sup\{u \in I(x) : u \le t\},$$
$$t_+ := \inf\{u \in I(x) : u \ge t\}.$$

Since t is a deterministic time, $t_- < t < t_+$ a.s. Since $\mathbf{X}_{0,t_n}(x) = \mathbf{X}_{0,t}(x)$ for all n large enough so that $t_- < t_n < t_+$, (4.16) follows from Proposition 4.2.

Finally, we show that Y has (a.s.) càglàd sample paths. Fix $u \in \mathbb{R}$. We show that Y has (a.s.) càglàd sample paths by proving that $(-\infty, u] \ni t \mapsto \mathbf{Y}_{u,t}(Y) \in \mathcal{H}_-(S^\Lambda)$ has (a.s.) càdlàg sample paths for all $Y \in \mathcal{H}_-(S^\Lambda)$. As indicated above, (4.14) implies that (a.s.)

$$I_{s_1,s_2}(x) := I(x) \cap (s_1, s_2] \text{ is finite for all } s_1 \le s_2 \text{ and } x \in S^\Lambda_{\text{fin}}. \tag{4.17}$$

For any $Y \in \mathcal{H}_-(S^\Lambda)$ and $t < u$ choose an arbitrary sequence $(s_n)_{n\in\mathbb{N}} \subset (t, u]$ with $s_n \searrow t$. We show that $\mathbf{Y}_{u,s_n}(Y) \to \mathbf{Y}_{u,t}(Y)$ in $\mathcal{H}_-(S^\Lambda)$ as $n \to \infty$. Using Proposition 4.2 and the definition of $\boldsymbol{\psi}_{\text{mon}}$, this is equivalent to showing that

$$\boldsymbol{\psi}_{\text{mon}}(x, \mathbf{Y}_{u,s_n}(Y)) \underset{n\to\infty}{\longrightarrow} \boldsymbol{\psi}_{\text{mon}}(x, \mathbf{Y}_{u,t}(Y)) \quad \text{for all } x \in S^\Lambda_{\text{fin}}.$$

By (4.11) this again is equivalent to showing

$$\boldsymbol{\psi}_{\text{mon}}(\mathbf{X}_{s_n,u}(x), Y) \underset{n\to\infty}{\longrightarrow} \boldsymbol{\psi}_{\text{mon}}(\mathbf{X}_{t,u}(x), Y) \quad \text{for all } x \in S^\Lambda_{\text{fin}}. \tag{4.18}$$

By (4.17) for all $t \in \mathbb{R}$ and $x \in S^\Lambda_{\text{fin}}$ there (a.s.) exists an $\varepsilon > 0$ such that $I_{t,t+\varepsilon}(x) = \emptyset$. Hence, in this case $\mathbf{X}_{s_n,u}(x) = \mathbf{X}_{t,u}(x)$ for all $s_n \in (t, t+\varepsilon]$ and (4.18) follows. Thus, $t \mapsto \mathbf{Y}_{u,t}(Y)$ is (a.s.) right-continuous.

For any $Y \in \mathcal{H}_-(S^\Lambda)$ and $t \le u$ choose an arbitrary sequence $(s_n)_{n\in\mathbb{N}} \subset (-\infty, t)$ with $s_n \nearrow t$. We show that $\mathbf{Y}_{u,s_n}(Y)$ has a limit as $n \to \infty$. With the arguments from above we can equivalently show that $\boldsymbol{\psi}_{\text{mon}}(\mathbf{X}_{s_n,u}(x), Y)$ has a limit as $n \to \infty$. Again, due to (4.17), for all $t \in \mathbb{R}$ and $x \in S^\Lambda_{\text{fin}}$ there (a.s.) exists an $\varepsilon > 0$ such that $I_{t-\varepsilon,t}(x)$ is either the empty set or equal to $\{t\}$. But in both cases there exists an $n_0 \in \mathbb{N}$ such that $\mathbf{X}_{s_n,u}(x) = \mathbf{X}_{s_{n_0},u}(x)$ for all $n \ge n_0$. It follows that $\boldsymbol{\psi}_{\text{mon}}(\mathbf{X}_{s_n,u}(x), Y)$ has a limit as $n \to \infty$ and hence $t \mapsto \mathbf{Y}_{u,t}(Y)$ has (a.s.) left limits. □

Before we continue we add a remark regarding the proof of Theorem 4.9. Going back to the proof of Theorem 1.6, one may notice that the conditions of Theorem 4.9 are sufficient but not necessary. A particularly interesting question is in what way one can weaken (1.22), i.e., the condition that every local map $m \in \mathcal{G}$ maps $\underline{0}$ to itself. Without the condition (1.22) we lose the property that $\mathbf{Y}_{u,s}(Y) \in \mathcal{H}_-(S^\Lambda)$ for all $u \geq s$ and $Y \in \mathcal{H}_-(S^\Lambda)$. Hence, we have to choose the whole set $\mathcal{H}(S^\Lambda)$ as the state space of the dual process. If started in $\mathcal{H}_-(S^\Lambda)$, the dual process can jump into the trap $\{\underline{0}\}$. In order for it to still be a Feller process we have to make sure that the first entrance time of the trap $\{\underline{0}\}$ is almost surely positive. It is straightforward to modify the proof of Theorem 1.6 to show that having

$$\sup_{i\in\Lambda} \sum_{\substack{m\in\mathcal{G}:\\ m(\underline{0})=\underline{0}}} r_m\Big(\mathbb{1}_{\mathcal{D}(m)}(i) + |\mathcal{R}_i^{\uparrow}(m)|\Big) < \infty$$

instead of (1.18) and

$$\sum_{j\in\Lambda} \sum_{\substack{m\in\mathcal{G}:\\ m(\underline{0})\neq\underline{0}}} r_m \mathbb{1}_{\mathcal{D}(m)}(j) < \infty \tag{4.19}$$

instead of (1.22) suffices to conclude (4.14) and that one can (a.s.) define random maps $\mathbf{X}_{s,u} : S^\Lambda_{\text{fin}} \to S^\Lambda_{\text{fin}}$ for all $s \leq u$ via (1.20). Note, however, that (4.19) implies that

$$\sum_{\substack{m\in\mathcal{G}:\\ m(\underline{0})\neq\underline{0}}} r_m < \infty,$$

disallowing many natural and interesting cases. Indeed, it seems that the first entrance time of $\{\underline{0}\}$ is a.s. zero in many cases where (4.19) is violated, e.g., for stochastic Ising models or more generally for processes on infinite grids in which zero spins jump with a constant rate to a nonzero value, independent of everything else.

In parallel to Theorem 2.9 and Theorem 3.6, we formulate the following result summarizing the work we have done up to now in this chapter.

Theorem 4.10 (Pathwise monotone duality)**.** *Let there exist a partial order $\leq$ on the local state space S and assume that there exists a least element $0 \in S$ with respect to this partial order. Let G be the generator from (1.8) defined via $\mathcal{G}$, a countable collection of local monotone maps satisfying (1.22). Assuming the summability condition (1.18), there exists a Feller process $\mathsf{Y} = (\mathsf{Y}_t)_{t\geq 0}$ with state space $\mathcal{H}_-(S^\Lambda)$ and càglàd sample paths such that $X = (X_t)_{t\geq 0}$, the continuous-time Markov chain with state space S^Λ_{fin}, defined in Chapter 1.2, is pathwise dual to Y with respect to $\boldsymbol{\psi}_{\text{mon}}$, the duality function defined in (4.10). If also the summability condition (1.7) holds, then X can be defined as an interacting particle system with state space S^Λ, and pathwise duality of X and Y with respect to $\boldsymbol{\psi}_{\text{mon}}$ still holds.*

Proof. The claims follow from the previous results in this subchapter, the construction of X as a continuous-time Markov chain in Chapter 1.2, the steps outlined in Chapter 1.3, and Theorem 1.3. □

Thus, if only the summability condition (1.18) holds, then the roles of the two processes X and Y are exchanged compared to the setup at the end of Chapter 1.3: X becomes a continuous-time Markov chain on a countable state space and Y is a Feller process on an uncountable state space. In this situation, Y may be regarded as the process of primary interest, that is studied through the continuous-time Markov chain X, which is comparatively easier to comprehend.

On the other hand, if only the summability condition (1.7) holds, then one can construct a version of Y on a countable state space as we will see in Chapter 4.3. The (pathwise) duality of the interacting particle system X from Theorem 1.3 to this continuous-time Markov chain had already been established in the literature.

4.2.2 Monotone dual maps

In difference to Theorem 2.9 and Theorem 3.6, Theorem 4.10 does not provide a generator of the dual process Y. Formally, its generator should look similar to the one in (1.34), i.e., it should have the form

$$\sum_{m\in\mathcal{G}} r_m\Big\{f(\hat{m}(Y)) - f(Y)\Big\} \qquad (Y \in \mathcal{H}_-(S^\Lambda)), \tag{4.20}$$

if $\hat{m} : \mathcal{H}_-(S^\Lambda) \to \mathcal{H}_-(S^\Lambda)$ is the unique dual map of $m : S^\Lambda \to S^\Lambda$. The existence of unique dual maps is guaranteed by the following result.

Lemma 4.11 (Maps with a dual V)**.** *A map $m : S^\Lambda \to S^\Lambda$ has a dual map $\hat{m} : \mathcal{H}(S^\Lambda) \to \mathcal{H}(S^\Lambda)$ with respect to $\boldsymbol{\psi}_{\rm mon}$ if and only if it preserves $\{f : S^\Lambda \to \{0,1\} : f$ is monotone and lower semi-continuous$\}$. The dual map $\hat{m}$, if it exists, is unique. In particular, the unique dual map $\hat{m}$ of a continuous monotone map $m : S^\Lambda \to S^\Lambda$ is given by*

$$\hat{m}(Y) := m^{-1}(Y^\uparrow)^\circ \qquad (Y \in \mathcal{H}(S^\Lambda)). \tag{4.21}$$

If m satisfies (1.22), then moreover $\hat{m}(Y) \in \mathcal{H}_-(S^\Lambda)$ for all $Y \in \mathcal{H}_-(S^\Lambda)$.

To prove Lemma 4.11, we again apply Lemma 1.7. We first prove the following small fact that implies (1.30). It will also be used at other points in the remainder of this chapter.

Lemma 4.12 (Separation of points)**.** *The collection $\{\boldsymbol{\psi}_{\rm mon}(x,\,\cdot\,) : x \in S^\Lambda_{\rm fin}\}$ of functions from $\mathcal{H}(S^\Lambda)$ to $\{0,1\}$ separates points.*

Proof. Let $Y_1, Y_2 \in \mathcal{H}(S^\Lambda)$ with $Y_1 \neq Y_2$. Then, by Proposition 4.1, there exists an $x \in Y_1^\uparrow \triangle Y_2^\uparrow$. W.l.o.g. we assume $x \in Y_1^\uparrow \backslash Y_2^\uparrow$ and we choose $y \in Y_1$ with $y \leq x$. Then $y \notin Y_2^\uparrow$ as else $x \in Y_2^\uparrow$, and $y \in S^\Lambda_{\rm fin}$ as $Y_1 \subset S^\Lambda_{\rm fin}$. Hence,

$$\boldsymbol{\psi}_{\rm mon}(y, Y_1) = \mathbb{1}_{Y_1^\uparrow}(y) = 1 \neq 0 = \mathbb{1}_{Y_2^\uparrow}(y) = \boldsymbol{\psi}_{\rm mon}(y, Y_2),$$

implying the claim. □

We continue with the proof of Lemma 4.11.

Proof of Lemma 4.11. By Lemma 4.12, the first two assertion of Lemma 4.11 follow from Lemma 1.7 provided we show that

$$\mathcal{H}_{\psi_{\text{mon}}} = \Big\{ f : S^\Lambda \to \{0,1\} : f \text{ is monotone and lower semi-continuous} \Big\}. \quad (4.22)$$

Note that a function $f : S^\Lambda \to \{0,1\}$ is lower semi-continuous if and only if $f^{-1}(\{1\}) = \{x \in S^\Lambda : f(x) = 1\}$ is open. Thus,

$$\Big\{ f : S^\Lambda \to \{0,1\} : f \text{ is lower semi-continuous} \Big\} = \Big\{ \mathbb{1}_A : A \subset S^\Lambda \text{ is open} \Big\}.$$

Moreover, $\mathbb{1}_A$ is monotone if $A \subset S^\Lambda$ is increasing, while, for all monotone $f : S^\Lambda \to \{0,1\}$, $A_f := \{x \in S^\Lambda : f(x) = 1\}$ is increasing and $f = \mathbb{1}_{A_f}$. Together with Proposition 4.1 this implies (4.22).

Assume now that $m : S^\Lambda \to S^\Lambda$ is monotone and continuous. Then one has that

$$\psi_{\text{mon}}(m(x), Y) = \mathbb{1}_{Y^\uparrow}(m(x)) = \mathbb{1}_{m^{-1}(Y^\uparrow)}(x) \qquad (Y \in \mathcal{H}(S^\Lambda)).$$

Using the same argument as in the proof of Lemma 4.7, the set $m^{-1}(Y^\uparrow)$ is open and increasing. Hence, by Proposition 4.1, $m^{-1}(Y^\uparrow)^\circ \in \mathcal{H}(S^\Lambda)$ and

$$m^{-1}(Y^\uparrow) = \Big[m^{-1}(Y^\uparrow)^\circ \Big]^\uparrow \qquad (Y \in \mathcal{H}(S^\Lambda)).$$

Thus, the map $\hat{m}$, defined in (4.21), is dual to m. The last assertion of the lemma follows analogously to the argument in (4.15). □

Recall that (4.20) does not suffice to define a generator. One must also specify its *domain*, usually characterized by a *core*, that has the property that the closure of the generator restricted to the core is the generator itself (compare, e.g., [Liggett, 1985, Chapter I.2], [Swart, 2022, Chapter 4.2]). It is not apparent to us which collection of continuous function defined on $\mathcal{H}_-(S^\Lambda)$ may be used to define a core for the generator of Y.

4.2.3 The backward evolution equation

While we leave the problem of giving a generator characterization of the dual process Y to future work, we do give a characterization of the backward stochastic flow as a solution of an evolution equation. Recall that for countable dual spaces we had seen this characterization in Chapter 1.3.

Proposition 4.13 (Backward flow evolution). *Under the assumptions of Theorem 4.9, almost surely, for each $u \in \mathbb{R}$ and $Y \in \mathcal{H}_-(S^\Lambda)$, there exists a unique càdlàg function $(-\infty, u] \ni t \mapsto Y_t \in \mathcal{H}_-(S^\Lambda)$ that solves the evolution equation*

$$Y_u = Y \quad \textit{and} \quad Y_{t-} = \begin{cases} \hat{m}(Y_t) & \textit{if } (m,t) \in \omega, \\ Y_t & \textit{else,} \end{cases} \qquad (t \le u). \qquad (4.23)$$

This function is given by $Y_t = \mathbf{Y}_{u,t}(Y)$ $(t \le u)$, where $(\mathbf{Y}_{u,s})_{u \ge s}$ is the backward stochastic flow defined in (4.12).

Proof. Recall that it was shown in the proof of Theorem 4.9 that under its assumptions for fixed $u \in \mathbb{R}$ and $Y \in \mathcal{H}_-(S^\Lambda)$, $t \mapsto \mathbf{Y}_{u,t}(Y)$ is (a.s.) indeed a càdlàg function from $(-\infty, u]$ to $\mathcal{H}_-(S^\Lambda)$.

Fix $u \in \mathbb{R}$. We show that $(-\infty, u] \ni t \mapsto \mathbf{Y}_{u,t}(Y) \in \mathcal{H}_-(S^\Lambda)$ (a.s.) solves (4.23). The first part of (4.23) follows directly from the definition of $\mathbf{Y}_{s,s}$ and the fact that $\mathbf{X}_{s,s}(y) = y$ for all $y \in S^\Lambda_{\text{fin}}$ (compare the construction of $(\mathbf{X}_{s,u})_{s \le u}$ via (1.20)). To show also the second part, we introduce further notation. As $\mathcal{G} \times \mathcal{G}$ is countable and all maps are applied with finite rates, almost surely

$$\nexists t \in \mathbb{R} : |\{m \in \mathcal{G} : (m,t) \in \omega\}| \ge 2.$$

Hence, we can almost surely define random maps $\mathfrak{m}^\omega_t : S^\Lambda_{\text{fin}} \to S^\Lambda_{\text{fin}}$ and $\hat{\mathfrak{m}}^\omega_t : \mathcal{H}_-(S^\Lambda) \to \mathcal{H}_-(S^\Lambda)$ for all $t \in \mathbb{R}$ as

$$\mathfrak{m}^\omega_t := \begin{cases} m & \text{if } (m,t) \in \omega, \\ \text{id} & \text{else,} \end{cases} \qquad \text{and} \qquad \hat{\mathfrak{m}}^\omega_t := \begin{cases} \hat{m} & \text{if } (m,t) \in \omega, \\ \text{id} & \text{else,} \end{cases}$$

where id denotes in both cases the identity and $\hat{m}$ is the dual map of $m \in \mathcal{G}$ from Lemma 4.11. Using the newly introduced notation, it follows from the arguments from the proof of Theorem 4.9 that, for any $t \le u$ and $x \in S^\Lambda_{\text{fin}}$,

$$\boldsymbol{\psi}_{\text{mon}}(x, \mathbf{Y}_{u,s}(Y)) = \boldsymbol{\psi}_{\text{mon}}(\mathbf{X}_{s,u}(x), Y) \to \boldsymbol{\psi}_{\text{mon}}(\mathbf{X}_{t,u} \circ \mathfrak{m}^\omega_t(x), Y) \quad \text{as } s \nearrow t.$$

But, by (4.11) and the duality of the maps,

$$\boldsymbol{\psi}_{\text{mon}}(\mathbf{X}_{t,u} \circ \mathfrak{m}^\omega_t(x), Y) = \boldsymbol{\psi}_{\text{mon}}(\mathfrak{m}^\omega_t(x), \mathbf{Y}_{u,t}(Y)) = \boldsymbol{\psi}_{\text{mon}}(x, \hat{\mathfrak{m}}^\omega_t(\mathbf{Y}_{u,t}(Y)))$$

and we conclude from Proposition 4.2 and the definition of $\boldsymbol{\psi}_{\text{mon}}$ that $\mathbf{Y}_{u,s}(Y) \to \hat{\mathfrak{m}}^\omega_t(\mathbf{Y}_{u,t}(Y))$ in $\mathcal{H}_-(S^\Lambda)$ as $s \nearrow t$. As this is just another way of writing the second part of (4.23), we conclude that $(-\infty, u] \ni t \mapsto \mathbf{Y}_{u,t}(Y) \in \mathcal{H}_-(S^\Lambda)$ (a.s.) solves (4.23).

Finally, we prove the uniqueness of the solutions of (4.23). We will show that if $(Y_t)_{t \le u}$, for fixed $u \in \mathbb{R}$ and $Y \in \mathcal{H}_-(S^\Lambda)$, solves (4.23), then

$$\boldsymbol{\psi}_{\text{mon}}(x, Y_s) = \boldsymbol{\psi}_{\text{mon}}(X_u, Y) \qquad (s \le u,\ x \in S^\Lambda_{\text{fin}}), \tag{4.24}$$

where $(X_t)_{t \ge s}$ solves (1.14) started at time s in state x. The uniqueness of the solutions of (1.14) (compare the comment below Theorem 1.6) together with Lemma 4.12 then implies the uniqueness of the solutions of (4.23).

We use a strategy similar to the proof of [Swart, 2022, Theorem 6.20]. We equip $S^\Lambda_{\text{fin}} \times \mathcal{H}_-(S^\Lambda)$ with the product topology consisting of the discrete topology on S^Λ_{fin} and of the topology from Proposition 4.2 on $\mathcal{H}_-(S^\Lambda)$. Then $\boldsymbol{\psi}_{\text{mon}}$ is, by Proposition 4.2, a continuous function from $S^\Lambda_{\text{fin}} \times \mathcal{H}_-(S^\Lambda)$ to $\{0,1\}$. Fix $u \in \mathbb{R}$ and $Y \in \mathcal{H}_-(S^\Lambda)$ and assume that $(Y_t)_{t \le u}$ solves (4.23). Fix moreover $s \le u$ and $x \in S^\Lambda_{\text{fin}}$ and assume that $(X_t)_{t \ge s}$ solves (1.14) (restricted to S^Λ_{fin}). Then the fact that $(Y_t)_{t \le u}$ and $(X_t)_{t \ge s}$ are càdlàg implies that

$$[s,u] \ni t \mapsto \boldsymbol{\psi}_{\text{mon}}(X_t, Y_t) \in \{0,1\} \tag{4.25}$$

is càdlàg as well. If $(m,t) \in \omega$ for some $t \in (s,u]$, then the evolution equations and the duality of the maps imply that

$$\boldsymbol{\psi}_{\text{mon}}(X_t, Y_t) = \boldsymbol{\psi}_{\text{mon}}(m(X_{t-}), Y_t) = \boldsymbol{\psi}_{\text{mon}}(X_{t-}, \hat{m}(Y_t)) = \boldsymbol{\psi}_{\text{mon}}(X_{t-}, Y_{t-}). \tag{4.26}$$

If there exists no $m \in \mathcal{G}$ such that $(m,t) \in \omega$, then (4.26) holds trivially. The finiteness of $\{0,1\}$ now implies that the function in (4.25) is constant. Plugging in $t = u$ and $t = s$ this implies (4.24) and the proof is complete. $\square$

4.3 Previously known construction

In previous work [Gray, 1986, Sturm and Swart, 2018], the dual process Y from Theorem 4.9 had already been constructed, but only for finite initial states. This more limited construction does not allow one to discuss the upper invariant law of the dual process, as we will do in Chapter 4.5. Nevertheless, for the sake of completeness, we demonstrate in this subchapter how this construction may be done using the previously established notation and results. Let

$$\mathcal{H}_{\mathrm{fin}}(S^\Lambda) := \Big\{ Y \in \mathcal{H}(S^\Lambda) : |Y| < \infty \Big\} \tag{4.27}$$

denote the subset of $\mathcal{H}(S^\Lambda)$ consisting of the *finite* subsets $Y \subset S^\Lambda$ with $Y^\circ = Y$. Note that in this case we do not remove the element $\{\underline{0}\}$, and that $\mathcal{H}_{\mathrm{fin}}(S^\Lambda)$ is countable. As usual, we equip $\mathcal{H}_{\mathrm{fin}}(S^\Lambda)$ with the discrete topology.

Proposition 4.14 (Dual chain). *Assume the summability condition (1.7) and that every map $m \in \mathcal{G}$ is monotone. Then, almost surely, (4.9) defines a map $\mathbf{Y}_{u,s} : \mathcal{H}_{\mathrm{fin}}(S^\Lambda) \to \mathcal{H}_{\mathrm{fin}}(S^\Lambda)$ for all $u \geq s$. If $u \in \mathbb{R}$ and Y_0 is a random variable with values in $\mathcal{H}_{\mathrm{fin}}(S^\Lambda)$ that is independent of the graphical representation ω, then setting*

$$\mathsf{Y}_t := \mathbf{Y}_{u,u-t}(\mathsf{Y}_0) \qquad (t \geq 0) \tag{4.28}$$

defines a continuous-time Markov chain $\mathsf{Y} = (\mathsf{Y}_t)_{t \geq 0}$ with state space $\mathcal{H}_{\mathrm{fin}}(S^\Lambda)$ and càglàd sample paths.

Comparing with [Sturm and Swart, 2018], the reader might notice that the random maps $\{\mathbf{Y}_{u,s}\}_{u \geq s}$ there are not defined as in (4.9), but as limits of concatenations of finitely many dual maps of the maps appearing in the Poison point set ω (compare [Sturm and Swart, 2018, Equation (143) & Proposition 28]). However, based on the proof of Proposition 4.14 that is about to follow, it is not hard to see that we can prove a version of Proposition 4.13 with $\mathcal{H}_{-}(S^\Lambda)$ replaced by $\mathcal{H}_{\mathrm{fin}}(S^\Lambda)$. The uniqueness in the modified version then implies that both approaches (a.s.) yield the same random maps.

In order to prove Proposition 4.14, we first state a lemma with an additional property of the dual map $\hat{m}$ from (4.21). In fact, the result below was already stated as part of [Sturm and Swart, 2018, Lemma 29]. However, in [Sturm and Swart, 2018] monotone dual maps are defined via the dual of a partially ordered set (compare Chapter 2.4.1). For the convenience of the reader, we present below a reformulated proof adapted to the notation and definitions of the present thesis. For any $A \subset S^\Lambda$, we set $\mathrm{supp}(A) := \bigcup_{x \in A} \mathrm{supp}(x)$ and call $\mathrm{supp}(A)$ the support of A.

Lemma 4.15 (Support of the dual map). *For each continuous monotone map $m : S^\Lambda \to S^\Lambda$, the map $\hat{m}$ from (4.21) satisfies*

$$\mathrm{supp}(\hat{m}(Y)) \subset \bigcup_{i \in \mathrm{supp}(Y)} \mathcal{R}(m[i]) \qquad (Y \in \mathcal{H}(S^\Lambda)). \tag{4.29}$$

Proof. Let $Y \in \mathcal{H}(S^\Lambda)$. As

$$\operatorname{supp}(\hat{m}(Y))$$
$$= \operatorname{supp}\left(\left[\bigcup_{y\in y} m^{-1}(\{y\}^\uparrow)\right]^\circ\right) \subset \operatorname{supp}\left(\bigcup_{y\in y} m^{-1}(\{y\}^\uparrow)^\circ\right) = \bigcup_{y\in y} \operatorname{supp}(m^{-1}(\{y\}^\uparrow)^\circ)$$

and $\operatorname{supp}(Y) = \bigcup_{y\in Y} \operatorname{supp}(y)$, it suffices to show that

$$\operatorname{supp}\Big(m^{-1}(\{y\}^\uparrow)^\circ\Big) \subset \bigcup_{i\in\operatorname{supp}(y)} \mathcal{R}(m[i]) \qquad (y \in S^\Lambda_{\mathrm{fin}}). \tag{4.30}$$

Hence, let $y \in S^\Lambda_{\mathrm{fin}}$ and assume that $k \in \operatorname{supp}(m^{-1}(\{y\}^\uparrow)^\circ)$. By the definition of the support, there then exists an $x \in m^{-1}(\{y\}^\uparrow)^\circ$ with $x(k) \neq 0$. We define $x_{k\rightsquigarrow 0} \in S^\Lambda$ via

$$x_{k\rightsquigarrow 0}(m) := \begin{cases} 0 & \text{if } m = k, \\ x(m) & \text{else,} \end{cases} \qquad (m \in \Lambda). \tag{4.31}$$

Then $y \not\leq m(x_{k\rightsquigarrow 0})$ as otherwise the minimality of $x \in m^{-1}(\{y\}^\uparrow)$ would be violated. Hence, there exists an $i \in \Lambda$ such that $y(i) \leq m(x)(i)$ but $y(i) \not\leq m(x_{k\rightsquigarrow 0})(i)$. This shows that $m(x)(i) \neq m(x_{k\rightsquigarrow 0})(i)$ and hence $k \in \mathcal{R}(m[i])$, and also that $y(i) \neq 0$ so that $i \in \operatorname{supp}(y)$. This establishes (4.30) and hence also (4.29). □

With this, we are ready to prove Proposition 4.14.

Proof of Proposition 4.14. By Lemma 1.4, $\mathbf{X}_{s,u}$ is almost surely a continuous map for all $s \leq u$. Hence, by Lemma 4.11, $\mathbf{X}_{s,u}$ possesses (a.s.) a dual map that we denote by $\widehat{\mathbf{X}}_{s,u}$. Then Lemma 4.15 (using also Lemma 1.1) implies that $\widehat{\mathbf{X}}_{s,u}$ (a.s.) maps $\mathcal{H}_{\mathrm{fin}}(S^\Lambda)$ into itself. Finally, (4.11) and the uniqueness of the dual map in Lemma 4.11 imply that $\mathbf{Y}_{u,s} = \widehat{\mathbf{X}}_{s,u}$.

Fix $u \in \mathbb{R}$, let Y_0 be a random variable with values in $\mathcal{H}_{\mathrm{fin}}(S^\Lambda)$ that is independent of the graphical representation ω, and let $\mathsf{Y} = (\mathsf{Y}_t)_{t\geq 0}$ be defined by (4.28). The fact that Y is a Markov process follows from the fact that it is constructed from a stochastic flow with independent increments (compare the proof of Theorem 4.9). It remains to show that Y has (a.s.) càglàd sample paths. As $\mathcal{H}_{\mathrm{fin}}(S^\Lambda)$ is equipped with the discrete topology, this amounts to showing that $(-\infty, u] \ni t \mapsto \mathbf{Y}_{u,t}(Y) \in \mathcal{H}_{\mathrm{fin}}(S^\Lambda)$ is (a.s.) piecewise constant and right-continuous for all $Y \in \mathcal{H}_{\mathrm{fin}}(S^\Lambda)$.

Generalizing the notation $m[i]$ introduced in Chapter 1.1, for any finite set $\Delta \subset \Lambda$ and map $m : S^\Lambda \to S^\Lambda$, let $m[\Delta] : S^\Lambda \to S^\Delta$ denote the map defined by $m[\Delta](x)(i) := m(x)(i)$ $(i \in \Delta)$. Recall (4.9), the definition of the backward stochastic flow under (1.7). We observe that $\mathbf{X}_{s,u}(x) \in Y^\uparrow$ if and only if

$$\exists y \in Y \text{ s.t. } y(i) \leq \mathbf{X}_{s,u}(x)(i)\ \forall i \in \operatorname{supp}(Y). \tag{4.32}$$

It follows from Lemma 1.2 that for fixed $u \in \mathbb{R}$ and $i \in \Lambda$, the function $s \mapsto \omega_{s,u}(i)$ is piecewise constant and right-continuous and hence the same is true for $s \mapsto \mathbf{X}_{s,u}[i]$. As a consequence, for any finite $\Delta \subset \Lambda$, also the map $s \mapsto \mathbf{X}_{s,u}[\Delta]$ is piecewise constant and right-continuous. Applying this to $\Delta = \operatorname{supp}(Y)$, we see from (4.9) and (4.32) that $t \mapsto \mathbf{Y}_{u,t}(Y)$ is piecewise constant and right-continuous for all $Y \in \mathcal{H}_{\mathrm{fin}}(S^\Lambda)$. □

4.4 Informativeness of monotone duality

As in the previous two chapters, we want to investigate whether (or in which cases) the duality function $\boldsymbol{\psi}_{\mathrm{mon}}$ from (4.10) is informative. It turns that $\boldsymbol{\psi}_{\mathrm{mon}}$ is always informative if we take $\mathcal{H}_{\mathrm{fin}}(S^\Lambda)$ from (4.27) as the dual space.

Proposition 4.16 (Informativeness of $\boldsymbol{\psi}_{\mathrm{mon}}$)**.** *The collection*

$$\mathcal{F} := \left\{ \boldsymbol{\psi}_{\mathrm{mon}}(\,\cdot\,, Y) : Y \in \mathcal{H}_{\mathrm{fin}}(S^\Lambda) \setminus \{\{\underline{0}\}\} \right\} \subset \mathcal{C}(S^\Lambda, \{0,1\}) \tag{4.33}$$

is distribution determining.

For the proof of Proposition 4.16 we are going to use the following lemma.

Lemma 4.17 (Clopen increasing subsets)**.** *Let $Y \in \mathcal{H}(S^\Lambda)$. Then Y is finite if and only if $Y^\uparrow$ is closed.*

Proof. Fist assume that $Y \in \mathcal{H}(S^\Lambda)$ is finite, i.e., that there exist an $n \in \mathbb{N}_0$ and $y_1, \dots, y_n \in S^\Lambda_{\mathrm{fin}}$ such that $Y = \{y_1, \dots, y_n\}$. Then

$$Y^\uparrow = \{y_1\}^\uparrow \cup \dots \cup \{y_n\}^\uparrow$$

is closed, as it is the union of finitely many closed sets (compare Lemma 4.4).

Conversely, assume that $Y^\uparrow$ is closed, and hence clopen (i.e., also open), as $Y^\uparrow \in \mathcal{I}(S^\Lambda)$ by Proposition 4.1. This implies that $\mathbb{1}_{Y^\uparrow} : S^\Lambda \to \{0,1\}$ is a continuous function. By Lemma 1.1, this implies that $\mathcal{R}(\mathbb{1}_{Y^\uparrow})$ from (1.3) is finite. We claim that

$$\mathrm{supp}(Y) \subset \mathcal{R}(\mathbb{1}_{Y^\uparrow}) \tag{4.34}$$

implying the finiteness of Y. To see (4.34), let $i \in \mathrm{supp}(Y)$. Then there exists a $y \in Y$ with $i \in \mathrm{supp}(y)$. By the minimality of Y then $y_{i\rightsquigarrow 0} \notin Y^\uparrow$, where $y_{i\rightsquigarrow 0}$, defined in (4.31), denotes the configuration obtained from y by changing the i-th coordinate to 0. Hence $\mathbb{1}_{Y^\uparrow}(y) \neq \mathbb{1}_{Y^\uparrow}(y_{i\rightsquigarrow 0})$ and $i \in \mathcal{R}(\mathbb{1}_{Y^\uparrow})$, implying (4.34). This completes the proof. □

With this, we are ready to prove Proposition 4.16.

Proof of Proposition 4.16. As usual, we want to apply Lemma 1.12. We start by showing that $\mathcal{F}$ from (4.33) is closed under products. Note that $\mathcal{F} \subset \mathcal{C}(S^\Lambda, \{0,1\})$ follows from Lemma 4.17. Let $Y_1, Y_2 \in \mathcal{H}_{\mathrm{fin}}(S^\Lambda)$. Noting that $Y_1^\uparrow \cap Y_2^\uparrow \in \mathcal{I}(S^\Lambda)$ and using Proposition 4.1, one has that

$$\boldsymbol{\psi}_{\mathrm{mon}}(\,\cdot\,, Y_1)\boldsymbol{\psi}_{\mathrm{mon}}(\,\cdot\,, Y_2) = \mathbb{1}_{Y_1^\uparrow}(\,\cdot\,)\mathbb{1}_{Y_2^\uparrow}(\,\cdot\,) = \mathbb{1}_{Y_1^\uparrow \cap Y_2^\uparrow}(\,\cdot\,) = \boldsymbol{\psi}_{\mathrm{mon}}(\,\cdot\,, (Y_1^\uparrow \cap Y_2^\uparrow)^\circ).$$

By Lemma 4.17, $(Y_1^\uparrow)^{\mathrm{c}}$ and $(Y_1^\uparrow)^{\mathrm{c}}$ are open and

$$(Y_1^\uparrow \cap Y_2^\uparrow)^{\mathrm{c}} = (Y_1^\uparrow)^{\mathrm{c}} \cup (Y_1^\uparrow)^{\mathrm{c}}$$

is open as well. Hence, using Lemma 4.17 in the converse direction, $(Y_1^\uparrow \cap Y_2^\uparrow)^\circ$ is finite.[3] Moreover, if $Y_1 \neq \{\underline{0}\} \neq Y_2$, then clearly also $(Y_1^\uparrow \cap Y_2^\uparrow)^\circ \neq \{\underline{0}\}$. Hence, $\mathcal{F}$ is closed under products.

[3]If S is a lattice, then one can check that $(Y_1^\uparrow \cap Y_2^\uparrow)^\circ = \{y_1 \vee y_2 : y_1 \in Y_1,\ y_2 \in Y_2\}^\circ$, providing an alternative proof that $(Y_1^\uparrow \cap Y_2^\uparrow)^\circ$ is finite.

Next, we show that $\mathcal{F}$ also separates points. Let $x_1, x_2 \in S^\Lambda$ and assume that $x_1 \neq x_2$. Then there has to exist an $i \in \Lambda$ such that $x_1(i) \neq x_2(i)$. Then either $x_1(i) \not\leq x_2(i)$ or $x_2(i) \not\leq x_1(i)$, so interchanging the roles of x_1 and x_2 if necessary, we can w.l.o.g. assume that $x_1(i) \not\leq x_2(i)$. Now, using the notion of (1.16),

$$\boldsymbol{\psi}_{\mathrm{mon}}\Big(x_2, \big\{\delta_i^{x_1(i)}\big\}\Big) = 0 \neq 1 = \boldsymbol{\psi}_{\mathrm{mon}}\Big(x_1, \big\{\delta_i^{x_1(i)}\big\}\Big).$$

This shows that $\mathcal{F}$ separates points and hence $\mathcal{F}$ is distribution determining by Lemma 1.12. □

By Theorem 4.10, the construction of Theorem 4.9 allows one to view $X = (X_t)_{t\geq 0}$ as a continuous-time Markov chain on the countable state space S^Λ_{fin} and $\mathsf{Y} = (\mathsf{Y}_t)_{t\geq 0}$ as a Feller process on the uncountable state space $\mathcal{H}_-(S^\Lambda)$. Hence, in order to characterize the law of the Feller process Y, one would like $\boldsymbol{\psi}_{\mathrm{mon}}$ to also be "informative" if one exchanges the roles of primal and dual space. Unfortunately, the collection

$$\big\{\boldsymbol{\psi}_{\mathrm{mon}}(x, \,\cdot\,) : x \in S^\Lambda_{\mathrm{fin}}\big\}$$

is *not* distribution determining. Indeed, let $S = \{0, 1\}$ and $i, j \in \Lambda$ with $i \neq j$. Define random variables Y^1 and Y^2 on $\mathcal{H}(\{0,1\}^\Lambda)$ via

$$\mathbb{P}[\mathsf{Y}^1 = \{\delta_i\}] = \mathbb{P}[\mathsf{Y}^1 = \{\delta_j\}] = \frac{1}{2}$$

and

$$\mathbb{P}[\mathsf{Y}^2 = \{\delta_i\}] = \mathbb{P}[\mathsf{Y}^2 = \{\delta_j\}] = \mathbb{P}[\mathsf{Y}^2 = \{\delta_i, \delta_j\}] = \mathbb{P}[\mathsf{Y}^2 = \{\delta_i + \delta_j\}] = \frac{1}{4},$$

where $\delta_i, \delta_j \in \{0,1\}^\Lambda_{\mathrm{fin}}$ are defined by (1.16) and $+$ denotes the usual pointwise addition on $\{0,1\}^\Lambda$. Then,

$$\mathbb{P}\Big[x \in (\mathsf{Y}^1)^\uparrow\Big] = \mathbb{P}\Big[x \in (\mathsf{Y}^2)^\uparrow\Big] = \begin{cases} 1 & \text{if } i, j \in \operatorname{supp}(x), \\ 0 & \text{if } i, j \notin \operatorname{supp}(x), \\ \frac{1}{2} & \text{else,} \end{cases}$$

for all $x \in \{0,1\}^\Lambda_{\mathrm{fin}}$.

However, if one takes a larger collection than just $\{\boldsymbol{\psi}_{\mathrm{mon}}(x, \,\cdot\,) : x \in S^\Lambda_{\mathrm{fin}}\}$, the property to be distribution determining follows easily from the already established results.

Proposition 4.18 (Dual informativeness of $\boldsymbol{\psi}_{\mathrm{mon}}$)**.** *The collection*

$$\mathcal{F} := \Big\{\textstyle\prod_{k=1}^n \boldsymbol{\psi}_{\mathrm{mon}}(x_k, \,\cdot\,) : n \in \mathbb{N},\, x_1, \dots, x_n \in S^\Lambda_{\mathrm{fin}}\Big\} \subset \mathcal{C}(\mathcal{H}(S^\Lambda), \{0,1\}) \quad (4.35)$$

is distribution determining.

Proof. Note that the continuity of the functions in $\mathcal{F}$ follows directly from Proposition 4.2 and the definition of $\boldsymbol{\psi}_{\mathrm{mon}}$. The closedness of $\mathcal{F}$ under products follows from its definition. The fact that $\mathcal{F}$ separates points follows from Lemma 4.12. Now, Lemma 1.12 again implies that $\mathcal{F}$ is distribution determining. □

4.5 Upper invariant laws and survival

As already stated at the beginning of the Chapter 4.3, one of the main advantages of the extended construction of the dual process Y from Theorem 4.9 is the possibility to study its upper invariant law. We begin this subchapter by recapping general theory for monotone Feller processes on compact metrizable spaces.

Let E be a compact metrizable space that is equipped with a partial order $\leq$ that is compatible with the topology. Two probability measure $\mu, \nu \in \mathcal{M}_1(E)$ are said to be *stochastically ordered*, denoted $\mu \leq \nu$, if they satisfy the following equivalent conditions [Liggett, 1985, Theorem II.2.4].

(i) $\int f(x)\,\mathrm{d}\nu(x) \leq \int f(x)\,\mathrm{d}\mu(x)$ for all continuous monotone $f : E \to \mathbb{R}$.

(ii) It is possible to couple random variables X, X' with laws μ, ν such that $X \leq X'$ a.s.

It is known [Kamae and Krengel, 1978, Theorem 2] that the stochastic order is a partial order on $\mathcal{M}_1(E)$. In particular, $\mu \leq \nu \leq \mu$ implies $\mu = \nu$ $(\mu, \nu \in \mathcal{M}_1(E))$. A Feller process with state space E and Feller semigroup $(P_t)_{t\geq 0}$ is said to be *monotone* if

$$P_t(x,\,\cdot\,) \leq P_t(y,\,\cdot\,) \qquad (x, y \in E,\ x \leq y).$$

The following result is well-known. It is stated for $E = \{0,1\}^\Lambda$ in [Liggett, 1985, Theorem III.2.3] and [Swart, 2022, Theorem 5.4]. Generalizing the proof to all compact metrizable spaces equipped with a compatible topology is straightforward. The measure $\overline{\nu}$ below is called the *upper invariant law.*

Proposition 4.19 (Upper invariant law)**.** *Let E be a compact metrizable space equipped with a partial order that is compatible with the topology. Assume that E possesses a greatest element $\top \in E$, i.e., $x \leq \top$ for all $x \in E$. Let $(P_t)_{t\geq 0}$ be the semigroup of a monotone Feller process $F = (F_t)_{t\geq 0}$ with state space E. Then there exists an invariant law $\overline{\nu}$ of F that is uniquely characterized by the property that $\nu \leq \overline{\nu}$ for each invariant law ν of F. Moreover, one has*

$$P_t(\top,\,\cdot\,) \underset{t\to\infty}{\Longrightarrow} \overline{\nu},$$

where $\Rightarrow$ denotes weak convergence of probability measures on E.

Returning to the previous setup, let $X = (X_t)_{t\geq 0}$ be an interacting particle system with generator of the form (1.8) and with its local state space S being equipped with a partial order $\leq$ and a least element 0. As already stated multiple times, it follows from Lemma 1.2 that if all maps $m \in \mathcal{G}$ are monotone, then (assuming (1.7)) the maps $\{\mathbf{X}_{s,u}\}_{s\leq u}$ are (a.s.) monotone for all $s \leq u$. This, in turn, implies that the interacting particle system X is monotone.[4] Thus, if the local state space S has a greatest element $\top$, then such an interacting particle system has an upper invariant law that is the long-time limit law started from the constant configuration $\underline{\top}$.

[4]Remarkably, the converse statement does not hold. Having a generator of the form (1.8) with all maps being monotone is strictly stronger than being monotone, see [Fill and Machida, 2001].

Recall that we equipped $\mathcal{H}(S^\Lambda)$ with a partial order defined in (4.7). It follows immediately from the definitions that the maps $\{\mathbf{Y}_{u,s}\}_{u\geq s}$ are monotone with respect to this partial order for all $s \leq u$ as long as they are well-defined. Hence, the Feller process $\mathsf{Y} = (\mathsf{Y}_t)_{t\geq 0}$ with state space $\mathcal{H}_-(S^\Lambda)$ defined in Theorem 4.9 is monotone. Recall the definition of $Y_{\rm sec}$ from (4.8). The abstract Proposition 4.19 implies (together with Lemma 4.6) that Y has an upper invariant law $\overline{\mu}$ and that

$$\mathsf{Y}_0 = Y_{\rm sec} \quad \text{implies that} \quad \mathbb{P}[\mathsf{Y}_t \in \cdot\,] \underset{t\to\infty}{\Longrightarrow} \overline{\mu}.$$

As far as we know, this upper invariant law has never been studied before, except in the special case when the interacting particle system X is additive. Compare Chapter 4.6.

Recall the definitions of $S^\Lambda_{\rm fin}$ and $\mathcal{H}_{\rm fin}(S^\Lambda)$ from (1.15) and (4.27). In view of Theorem 1.6 and Proposition 4.14, under the assumptions of the latter part of Theorem 4.10, the Markov processes X and Y, started in initial states in $S^\Lambda_{\rm fin}$ and $\mathcal{H}_{\rm fin}(S^\Lambda)$, respectively, stay in these spaces for all times $t \geq 0$. We will relate the upper invariant laws of X and Y to the behavior of Y and X (in this order) started from finite initial states.

Following the convention in Chapter 1.3, we denote by $\mathbb{P}_Y$ the law of Y started in $Y \in \mathcal{H}_{\rm fin}(S^\Lambda)$, while $\mathbb{P}^x$ denotes the law of X started in $x \in S^\Lambda$. We denote expectation with respect to $\mathbb{P}_Y$ by $\mathbb{E}_Y$. Recall the definition of survival for the interacting particle system X from Chapter 1.2. Similarly, we say that its monotone dual process Y *survives* if there exists a $Y \in \mathcal{H}_{\rm fin}(S^\Lambda)$ such that

$$\mathbb{P}_Y[\exists t \geq 0 : \mathsf{Y}_t = \emptyset] < 1.$$

Otherwise we say that Y *dies out*. Note that in order to speak about the survival of Y, it suffices to construct Y in the sense of Proposition 4.14. The following proposition is a simple consequence of duality. Similar results have been exploited to great length for additive interacting particle systems.

Proposition 4.20 (Upper invariant law of the particle system). *Assume the summability condition (1.7), that every map $m \in \mathcal{G}$ is monotone and that S has a greatest element $\top$. Let $\overline{X}$ be a random variable whose law is $\overline{\nu}$, the upper invariant law of the interacting particle system X from Theorem 1.3. Then $\overline{X} = \underline{0}$ a.s. if the dual process Y from Proposition 4.14 dies out and $\overline{X} \neq \underline{0}$ a.s. if the dual process Y survives.*

To prove Proposition 4.20, we show that $\overline{\nu}$ and can be characterized by how it integrates the duality function $\boldsymbol{\psi}_{\rm mon}$ in the following sense.

Lemma 4.21 (Characterizing $\overline{\nu}$). *Assume the summability condition (1.7), that every map $m \in \mathcal{G}$ is monotone and that S has a greatest element $\top$. Then the upper invariant law $\overline{\nu}$ of the interacting particle system X from Theorem 1.3 is uniquely characterized by the relation*

$$\int \boldsymbol{\psi}_{\rm mon}(x, Y)\, \mathrm{d}\overline{\nu}(x) = \mathbb{P}_Y[\mathsf{Y}_t \neq \emptyset \ \forall t \geq 0] \qquad (Y \in \mathcal{H}_{\rm fin}(S^\Lambda)), \tag{4.36}$$

where $\mathsf{Y} = (\mathsf{Y}_t)_{t\geq 0}$ the dual process from Proposition 4.14.

Proof. Let $Y \in \mathcal{H}_{\rm fin}(S^\Lambda)$. Since $\emptyset$ is a trap for the dual process Y, we have that

$$\mathbb{P}_Y\big[\mathsf{Y}_t \neq \emptyset\big] \searrow \mathbb{P}_Y\big[\mathsf{Y}_s \neq \emptyset \ \forall s \geq 0\big] \quad \text{as} \quad t \to \infty. \tag{4.37}$$

The duality between X and Y implies for $t \geq 0$ that

$$\mathbb{E}^{\underline{\top}}[\boldsymbol{\psi}_{\rm mon}(X_t, Y)] = \mathbb{E}_Y\big[\boldsymbol{\psi}_{\rm mon}(\underline{\top}, \mathsf{Y}_t)\big] = \mathbb{P}_Y[\mathsf{Y}_t \neq \emptyset].$$

Together with (4.37) this implies (4.36). The fact that (4.36) uniquely characterizes $\overline{\nu}$ follows from the fact that $\mathcal{F}$ from (4.33) is distribution determining. □

The proof of Proposition 4.20 now follows readily.

Proof of Proposition 4.20. Let $\delta_{\underline{0}}$ denote the Dirac measure on $\underline{0} \in S^\Lambda$. As

$$\int \boldsymbol{\psi}_{\rm mon}(x, Y)\, \mathrm{d}\delta_{\underline{0}}(x) = 0 \qquad (Y \in \mathcal{H}_{\rm fin}(S^\Lambda) \setminus \{\{\underline{0}\}\}),$$

the fact that $\mathcal{F}$ from (4.33) is distribution determining implies that $\overline{\nu} = \delta_{\underline{0}}$ if and only if $\int \boldsymbol{\psi}_{\rm mon}(x, Y)\, \mathrm{d}\overline{\nu}(x) = 0$ for all $Y \in \mathcal{H}_{\rm fin}(S^\Lambda) \setminus \{\{\underline{0}\}\}$. By (4.36), the latter statement is equivalent to survival of the dual process. Using the fact that $\overline{\nu}$ is an extremal invariant measure [Liggett, 1985, Theorem III.2.3], it is easy to see (compare [Swart, 2022, Lemma 5.10]) that if $\overline{\nu} \neq \delta_{\underline{0}}$, then $\overline{\nu}$ and $\delta_{\underline{0}}$ are mutually singular. Together, these observations imply the statements of Proposition 4.20. □

Thanks to the fact that we have constructed the dual process in Theorem 4.9 also for infinite initial states and have shown that it has an upper invariant law, we can now formulate an analogue result to Proposition 4.20 with the roles of X and Y reversed. Note that we only need X to be started in finite initial states, so we may drop the usual summability condition (1.7).

Theorem 4.22 (Upper invariant law of the dual process)**.** *Assume the summability condition (1.18), and that every map $m \in \mathcal{G}$ is monotone and satisfies (1.22). Let $\overline{\mathsf{Y}}$ be a random variable whose law is $\overline{\mu}$, the upper invariant law of the dual process Y from Theorem 4.9. Then $\overline{\mathsf{Y}} = \emptyset$ a.s. if X from below Theorem 1.6 dies out and $\overline{\mathsf{Y}} \neq \emptyset$ a.s. if X survives.*

In parallel to the proof above, we again first show that $\overline{\mu}$ can be characterized by how it integrates $\boldsymbol{\psi}_{\rm mon}$. As we will work with the collection $\mathcal{F}$ from (4.35), we will have to consider X started in multiple initial states at once. Due to this fact we will exclusively work on the probability space, where the graphical representation ω is defined. The probability measure of this probability space is again denoted by $\mathbb{P}$.

Lemma 4.23 (Characterizing $\overline{\mu}$)**.** *Assume the summability condition (1.18), and that every map $m \in \mathcal{G}$ is monotone and satisfies (1.22). Then the upper invariant law $\overline{\mu}$ of the dual process Y from Theorem 4.9 is uniquely characterized by the relation*

$$\int \prod_{k=1}^{n} \boldsymbol{\psi}_{\rm mon}(x_k, Y)\, \mathrm{d}\overline{\mu}(Y) = \mathbb{P}[\mathbf{X}_{0,t}(x_k) \neq \underline{0} \ \forall t \geq 0,\ k = 1, \ldots, n] \tag{4.38}$$
$$(n \in \mathbb{N},\ x_1, \ldots, x_n \in S^\Lambda_{\rm fin}),$$

where $(\mathbf{X}_{s,u})_{s \leq u}$ is the stochastic flow from (1.20).

Proof. Since, by Proposition 1.5 and Theorem 1.6, each random map $\mathbf{X}_{s,u}$ $(s \leq u)$ defined by (1.20) is (a.s.) a concatenation of finitely many maps, it (a.s.) maps $\underline{0}$ to itself. Hence, have that

$$\mathbb{P}[\mathbf{X}_{0,t}(x_k) \neq \underline{0}\ \forall k = 1, \ldots, n] \searrow \mathbb{P}[\mathbf{X}_{0,s}(x_k) \neq \underline{0}\ \forall s \geq 0,\ k = 1, \ldots, n] \text{ as } t \to \infty. \tag{4.39}$$

The duality of the stochastic flows, i.e., (4.11), implies for $t \geq 0$ that

$$\begin{aligned}\mathbb{E}\left[\prod_{k=1}^{n} \boldsymbol{\psi}_{\text{mon}}(x_k, \mathbf{Y}_{t,0}(Y_{\text{sec}}))\right] &= \mathbb{E}\left[\prod_{k=1}^{n} \boldsymbol{\psi}_{\text{mon}}(\mathbf{X}_{0,t}(x_k), Y_{\text{sec}})\right] \\ &= \mathbb{P}[\mathbf{X}_{0,t}(x_k) \neq \underline{0}\ \forall k = 1, \ldots, n],\end{aligned}$$

where $(\mathbf{Y}_{u,s})_{u \geq s}$ is the backward stochastic flow from (4.12). Together with (4.39) this implies (4.38). Analogously to the proof of Lemma 4.21, the fact that (4.38) uniquely characterizes $\overline{\mu}$ follows from the fact that $\mathcal{F}$ from (4.35) is distribution determining. □

Again, the proof of Theorem 4.22 follows readily.

Proof of Theorem 4.22. Let $\delta_\emptyset$ denote the Dirac measure on $\emptyset \in \mathcal{H}_-(S^\Lambda)$. As

$$\int \prod_{k=1}^{n} \boldsymbol{\psi}_{\text{mon}}(x_k, Y)\, \mathrm{d}\delta_\emptyset(Y) = 0 \qquad (x_1, \ldots, x_n \in S^\Lambda_{\text{fin}}),$$

the fact that $\mathcal{F}$ from (4.35) is distribution determining implies that $\overline{\mu} = \delta_\emptyset$ if and only if $\int \prod_{k=1}^{n} \boldsymbol{\psi}_{\text{mon}}(x_k, Y)\, \mathrm{d}\overline{\mu}(Y) = 0$ for all $x_1, \ldots, x_n \in S^\Lambda_{\text{fin}}$. By (4.38), the latter statement is equivalent to

$$\mathbb{P}\Big[\mathbf{X}_{0,t}(x_k) \neq \underline{0}\ \forall t \geq 0,\ k = 1, \ldots, n\Big] = 0 \quad (n \in \mathbb{N},\ x_1, \ldots, x_n \in S^\Lambda_{\text{fin}}),$$

which in turn is equivalent to

$$\mathbb{P}[\mathbf{X}_{0,t}(x) \neq \underline{0}\ \forall t \geq 0] = 0 \quad (x \in S^\Lambda_{\text{fin}}).$$

The rest of the proof is now the same as the proof of Proposition 4.20, where one can argue as in [Swart, 2022, Lemma 5.8] to see that $\overline{\mu}$ is an extremal invariant law and then as in [Swart, 2022, Lemma 5.10] to see that $\overline{\mu} \neq \delta_\emptyset$ implies that $\overline{\mu}$ and $\delta_\emptyset$ are mutually singular. □

4.6 Additive duality revisited

Recall that a lattice, defined in Chapter 2.4.1, is a special type of partially ordered set, based on which there already existed a known duality theory for interacting particle systems (compare Theorem 2.15). It turns out that, if the local state space is a lattice, then the dual process $\mathsf{Y} = (\mathsf{Y}_t)_{t \geq 0}$ from Theorem 4.9 can be identified with an interacting particle system and the pathwise duality from Theorem 4.10 reduces to the one from Theorem 2.15 (and Proposition 2.16). This fact was already observed in [Sturm and Swart, 2018]. However, to provide a complete picture of monotone pathwise duality, we show this fact in this subchapter by means developed in the previous ones.

4.6.1 General lattices

Recall from Chapter 2.4.1 that each finite lattice has a least and a greatest element. Moreover, recall the definition of an additive map. Clearly, each additive map between lattices is also monotone. Assume that S is equipped with a partial order $\leq$ so that $(S, \leq)$ is a lattice. In the following result $(\mathbf{X}_{s,u})_{s\leq u}$ denotes the stochastic flow from (1.20) and $(\widehat{S}, \leq)$ denotes the dual of the partially ordered set $(S, \leq)$ (compare Chapter 2.4.1).

Proposition 4.24 (Additive systems pathwise duality)**.** *Let there exist a partial order $\leq$ on the local state space S so that $(S, \leq)$ is a lattice. Assume the summability condition (1.18), and that every map $m \in \mathcal{G}$ is additive. Then there (a.s.) exists a backward stochastic flow $(\mathbf{Z}_{u,s})_{u\geq s}$, consisting of random maps from $\widehat{S}^\Lambda$ to itself, satisfying the relation*

$$\boldsymbol{\psi}_{\mathrm{add}}(\mathbf{X}_{s,u}(x), \hat{y}) = \boldsymbol{\psi}_{\mathrm{add}}(x, \mathbf{Z}_{u,s}(\hat{y})) \qquad (s \leq u,\ x \in S^\Lambda_{\mathrm{fin}},\ \hat{y} \in \widehat{S}^\Lambda). \tag{4.40}$$

Proof. Recall that, by Lemma 2.12, the summability condition (1.18) implies (2.14), i.e., the usual summability condition (1.7) for the dual process Y from Theorem 2.15. Exchanging the roles of S and $\widehat{S}$, the claim follows from (the results leading up to) Theorem 2.15, the construction in Chapter 1.3 and the fact that $x \not\leq y$ if and only if $\hat{y} \not\leq \hat{x}$ $(x, y \in S^\Lambda)$. □

We will show that the backward stochastic flow $(\mathbf{Z}_{u,s})_{u\geq s}$ can, in fact, be identified with the backward stochastic flow $(\mathbf{Y}_{u,s})_{u\geq s}$ from (4.12). A non-empty, decreasing subset $I \subset S^\Lambda$ is called an *ideal* if it is closed under taking the join, i.e., if $x \vee y \in I$ for all $x, y \in I$. A *principal ideal* is an ideal that has a greatest element. Let

$$\mathcal{H}_{\mathrm{pi}}(S^\Lambda) := \Big\{ Y \in \mathcal{H}(S^\Lambda) : (Y^\uparrow)^{\mathrm{c}} \text{ is a principal ideal} \Big\}$$

Note that $\mathcal{H}_{\mathrm{pi}}(S^\Lambda) \subset \mathcal{H}_-(S^\Lambda)$. The following proposition identifies the partially ordered set $\mathcal{H}_{\mathrm{pi}}(S^\Lambda)$ with the dual lattice $\widehat{S}^\Lambda$ and shows that in this identification the monotone duality function from (4.10) reduces to the additive duality function from (2.17). Note that the three occurrences of $\leq$ below denote three different partial orders: The first one is the underlying one on S, the second one is the product order based on the first one, and the third one is one defied via (4.7) (and Proposition 4.2).

Proposition 4.25 (Isomorphism to the dual lattice)**.** *If $(S, \leq)$ is a lattice, the partially ordered topological space $(\widehat{S}^\Lambda, \leq)$ is isomorphic to $(\mathcal{H}_{\mathrm{pi}}(S^\Lambda), \leq)$ via the monotone homeomorphism $\phi : \widehat{S}^\Lambda \to \mathcal{H}_{\mathrm{pi}}(S^\Lambda)$ defined as*

$$\phi(\hat{y}) := \Big[(\{y\}^\downarrow)^{\mathrm{c}} \Big]^\circ \qquad (\hat{y} \in \widehat{S}^\Lambda). \tag{4.41}$$

Moreover,

$$\boldsymbol{\psi}_{\mathrm{add}}(x, \hat{y}) = \boldsymbol{\psi}_{\mathrm{mon}}(x, \phi(\hat{y})) \qquad (x \in S^\Lambda,\ \hat{y} \in \widehat{S}^\Lambda). \tag{4.42}$$

Proof. Let $\mathcal{I}_{\mathrm{pi}}(S^\Lambda) := \{A \subset S^\Lambda : A^{\mathrm{c}} \text{ is a principal ideal}\}$ and note that $\mathcal{I}_{\mathrm{pi}}(S^\Lambda) \subset \mathcal{I}(S^\Lambda)$ as for all $y \in S^\Lambda$ the set $\{y\}^\downarrow$ is closed by Lemma 4.4 and decreasing by definition. It is obvious that the map $\hat{y} \mapsto (\{y\}^{\downarrow})^{\mathrm{c}}$ is a bijection from $\widehat{S}^{\Lambda}$ to $\mathcal{I}_{\mathrm{pi}}(S^{\Lambda})$ and it follows from Proposition 4.1 that also $\phi : \widehat{S}^{\Lambda} \to \mathcal{H}_{\mathrm{pi}}(S^{\Lambda})$ is a bijection.

To see that ϕ is monotone, let $\hat{y}, \hat{y}' \in \widehat{S}^{\Lambda}$ with $\hat{y} \leq \hat{y}'$. Then

$$y' \leq y \text{ in } S^{\Lambda} \quad \Rightarrow \quad \{y'\}^{\downarrow} \subset \{y\}^{\downarrow} \quad \Rightarrow \quad (\{y\}^{\downarrow})^{\mathrm{c}} \subset (\{y'\}^{\downarrow})^{\mathrm{c}}.$$

But the last assertion above says by definition that $\phi(\hat{y}) \leq \phi(\hat{y}')$ in $\mathcal{H}(S^{\Lambda})$. Hence, ϕ is monotone.

To prove the continuity of ϕ and ϕ^{-1}, we argue as follows. Let $y, y' \in S^{\Lambda}$. Then for all $x \in \{y\}^{\downarrow}$ there exists an $x' \in \{y'\}^{\downarrow}$ satisfying $x(i) = x'(i)$ for all $i \in \Lambda$ with $y(i) = y'(i)$, which implies $d(x, x') \leq d(y, y')$. Likewise, for all $x' \in \{y'\}^{\downarrow}$ there exists an $x \in \{y\}^{\downarrow}$ such that $d(x, x') \leq d(y, y')$. Hence

$$d_{\mathrm{H}}(\{y\}^{\downarrow}, \{y'\}^{\downarrow}) \leq d(y, y') \qquad (y, y' \in S^{\Lambda}),$$

implying the (Lipschitz) continuity of $y \mapsto \{y\}^{\downarrow}$ and consequently also of ϕ. Here we use that the maps

$$S^{\Lambda} \ni y \mapsto \hat{y} \in \widehat{S}^{\Lambda} \quad \text{and} \quad \mathcal{H}_{-}(S^{\Lambda}) \ni Y \mapsto (Y^{\uparrow})^{\mathrm{c}} \in \mathcal{K}_{+}(S^{\Lambda})$$

are, due to the definitions of the corresponding metrics, isometries. On the other hand, $d(y, y') \geq 1/3^{k}$ implies that there exists an $i \in \gamma^{-1}(\{1, \ldots, k\})$ such that $y(i) \neq y'(i)$. Hence $\{y\}^{\downarrow}_{\gamma^{-1}(\{1,\ldots,k\})} \neq \{y'\}^{\downarrow}_{\gamma^{-1}(\{1,\ldots,k\})}$, where, for $A \subset S^{\Lambda}$ and $\Delta \subset \Lambda$, $A_{\Delta} := \{a_{\Delta} : a \in A\}$, where a_{Δ} is the restriction of a to Δ, defined in Chapter 1.1. It follows that there exists either an $x \in \{y\}^{\downarrow}$ with $d(x, \{y'\}^{\downarrow}) \geq 1/3^{k}$ or an $x' \in \{y'\}^{\downarrow}$ with $d(x', \{y\}^{\downarrow}) \geq 1/3^{k}$. Hence, $d_{\mathrm{H}}(\{y\}^{\downarrow}, \{y'\}^{\downarrow}) \geq 1/3^{k}$. From this one concludes the continuity of ϕ^{-1}.

Finally, (4.42) follows directly from the definitions of $\boldsymbol{\psi}_{\mathrm{add}}$ and ϕ as

$$\boldsymbol{\psi}_{\mathrm{add}}(x, \hat{y}) = \mathbb{1}_{(\{y\}^{\downarrow})^{\mathrm{c}}}(x) = \mathbb{1}_{\phi(\hat{y})^{\uparrow}}(x) = \boldsymbol{\psi}_{\mathrm{mon}}(x, \phi(\hat{y})) \qquad (x \in S^{\Lambda},\ \hat{y} \in \widehat{S}^{\Lambda}),$$

where we used Proposition 4.1 in the second equality. □

The subspace $\mathcal{H}_{\mathrm{pi}}(S^{\Lambda})$ and the function ϕ from (4.41) are rather abstract. If S is a distributive lattice, we can give an alternative description of $\mathcal{H}_{\mathrm{pi}}(S^{\Lambda})$. However, in order not to distract from the main goal of this subchapter, we postpone these reformulations for distributive lattices in general and totally ordered lattices in particular until Chapter 4.6.2. The final result of this subchapter says that for additive interacting particle systems, the backward stochastic flow $(\mathbf{Y}_{u,s})_{u \geq s}$ defined in (4.12) preserves the space $\mathcal{H}_{\mathrm{pi}}(S^{\Lambda})$.

Proposition 4.26 (Preserved subspace). *Let there exist a partial order $\leq$ on the local state space S so that $(S, \leq)$ is a lattice. Assume the summability condition (1.18), and that every map $m \in \mathcal{G}$ is additive. Then, almost surely,*

$$\mathbf{Y}_{u,s}(Y) \in \mathcal{H}_{\mathrm{pi}}(S^{\Lambda}) \qquad (u \geq s,\ Y \in \mathcal{H}_{\mathrm{pi}}(S^{\Lambda})),$$

where $(\mathbf{Y}_{u,s})_{u \geq u}$ denotes the backward stochastic flow from (4.12).

Proof. Let $Y \in \mathcal{H}_{\mathrm{pi}}(S^\Lambda)$ and $u \leq s$. Due to (1.18), $\mathbf{Y}_{u,s}$ from (4.12) and $\mathbf{Z}_{u,s}$ from Proposition 4.24 are almost surely well-defined. One now computes that

$$\begin{aligned}\boldsymbol{\psi}_{\mathrm{mon}}(x, \mathbf{Y}_{u,s}(Y)) &= \boldsymbol{\psi}_{\mathrm{mon}}(\mathbf{X}_{s,u}(x), Y) = \boldsymbol{\psi}_{\mathrm{add}}(\mathbf{X}_{s,u}(x), \phi^{-1}(Y)) \\ &= \boldsymbol{\psi}_{\mathrm{add}}\Big(x, \mathbf{Z}_{u,s}(\phi^{-1}(Y))\Big) = \boldsymbol{\psi}_{\mathrm{mon}}\Big(x, \phi\Big(\mathbf{Z}_{u,s}(\phi^{-1}(Y))\Big)\Big)\end{aligned} \tag{4.43}$$

for all $x \in S^\Lambda_{\mathrm{fin}}$. Here we used (4.11) in the first equality, (4.42) in the second and fourth equality, and (4.40) in the third equality. By Lemma 4.12, (4.43) implies that

$$\mathbf{Y}_{u,s}(Y) = \phi\Big(\mathbf{Z}_{u,s}(\phi^{-1}(Y))\Big)$$

and, using that ϕ is a homeomorphism from $\widehat{S}^\Lambda$ to $\mathcal{H}_{\mathrm{pi}}(S^\Lambda)$, we conclude that $\mathbf{Y}_{u,s}(Y) \in \mathcal{H}_{\mathrm{pi}}(S^\Lambda)$. □

By grace of Proposition 4.25 and Proposition 4.26, if S is lattice we can identify the restriction of $(\mathbf{Y}_{u,s})_{u\geq s}$ to $\mathcal{H}_{\mathrm{pi}}(S^\Lambda)$ with the backward stochastic flow $(\mathbf{Z}_{u,s})_{u\geq s}$ from (4.40). It follows that we can identify the Feller process Y from Theorem 4.9 with an interacting particle system on $\widehat{S}^\Lambda$ with generator $\widehat{G}$ from (1.34).

4.6.2 Distributive lattices

An unpleasant feature of Proposition 4.25 is that the definitions of the space $\mathcal{H}_{\mathrm{pi}}(S^\Lambda)$ and the bijection ϕ are rather abstract. In this last subchapter we show that in the special case that S is a distributive lattice one can give a much more concrete description of these objects.

Recall from Chapter 3.3 that a lattice $(L, \leq)$ is distributive if

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \qquad (a, b, c \in L).$$

For example, partially ordered sets of the form $L = \{0, \dots, N\}^n$ (equipped with the product order) are distributive lattices ($N, n \in \mathbb{N}$). We call an element $a \in L$ *(join-)irreducible* if

$$a = b \vee c \quad \text{implies} \quad b = a \text{ or } c = a$$

for $b, c \in L$.[5]

Assume that the local state space S is equipped with a partial order $\leq$ so that $(S, \leq)$ is a distributive lattice. We define $S_{\mathrm{ir}} := \{a \in S : a \text{ is irreducible}\}$ and $S^\Lambda_{\mathrm{ir}} := \{x \in S^\Lambda : x \text{ is irreducible}\}$. It is easy to see that

$$S^\Lambda_{\mathrm{ir}} = \Big\{\delta^a_i : i \in \Lambda,\ a \in S_{\mathrm{ir}}\Big\}. \tag{4.44}$$

We define $\mathcal{H}_1(S^\Lambda) := \{Y \subset S^\Lambda_{\mathrm{ir}} : Y^\circ = Y\}$. The following result is the promised less abstract characterization of $\mathcal{H}_{\mathrm{pi}}(S^\Lambda)$ in case that $(S, \leq)$ is a distributive lattice.

[5] Often one excludes the least element 0 from the set of (join-)irreducible elements. However, for our purposes it is convenient to include it.

Proposition 4.27 (Ideals on distributive lattices)**.** *Assume that* $(S, \le)$ *is a distributive lattice and let* $Y \in \mathcal{H}(S^\Lambda)$*. Then* $(Y^\uparrow)^c \subset S^\Lambda$ *is a principal ideal if and only if* $Y \in \mathcal{H}_1(S^\Lambda)$*, i.e.,* $\mathcal{H}_{\mathrm{pi}}(S^\Lambda) = \mathcal{H}_1(S^\Lambda)$*.*

For the proof of Proposition 4.27 we need the following lemma. Recall that an ideal of a lattice $(L, \le)$ is a non-empty, decreasing subset $I \subset L$ that is closed under taking the join.

Lemma 4.28 (Closed ideals)**.** *Assume that* $(S, \le)$ *is a lattice. An ideal of the lattice* $(S^\Lambda, \le)$ *is closed in* S^Λ *if and only if it is principal.*

Proof. Let $I \subset S^\Lambda$ be an ideal. If I is a principal ideal, i.e., $I = \{y\}^\downarrow$ for some $y \in S^\Lambda$, then I is closed by Lemma 4.4.

Conversely, assume that I is closed. Recall that a *net* in I is an indexed collection of elements $(y_\alpha)_{\alpha \in \Gamma}$ of I whose index set Γ is equipped with a partial order $\le$ such that for each $\alpha, \beta \in \Gamma$, there exists a $\gamma \in \Gamma$ such that $\alpha, \beta \le \gamma$. In particular, if we let $y_x := x$ denote the identity map, then $(y_x)_{x \in I}$ is a net in I. Since I is a closed subset of the compact space S^Λ, it is compact, which implies that each net in I has a convergent subnet. Let $(y_x)_{x \in I'}$ be a convergent subnet of the net we have just described and let y be its limit. The definition of a subnet means that for each $x \in I$ there exists an $x' \in I'$ such that all $x'' \in I'$ with $x'' \ge x'$ satisfy $x'' \ge x$. Using this and the fact that the set $\{z \in I : z \ge x\}$ is closed, we see that $y \ge x$ for all $x \in I$. It follows that $I = \{y\}^\downarrow$, i.e., I is principal. □

Recall that, by Proposition 4.1, for any $Y \in \mathcal{H}(S^\Lambda)$ the subset $(Y^\uparrow)^c \subset S^\Lambda$ is closed in S^Λ. Hence, by Lemma 4.28, the proof of Proposition 4.27 reduces to showing that $(Y^\uparrow)^c \subset S^\Lambda$ is an ideal if and only if $Y \in \mathcal{H}_1(S^\Lambda)$.

Proof of Proposition 4.27. Let $Y \notin \mathcal{H}_1(S^\Lambda)$. Then, by (4.44), there exists a $y \in Y$ with either two non-zero coordinates, i.e., there exist $i, j \in \Lambda$ with $i \ne j$ and $y(i) \ne 0 \ne y(j)$ or with $y(l) \in S \setminus S_{\mathrm{ir}}$ for some $l \in \Lambda$. In both cases the minimality of Y implies that $(Y^\uparrow)^c$ cannot be an ideal. More precisely, in the first case we have $y = y_{i \rightsquigarrow 0} \vee y_{j \rightsquigarrow 0} \in Y^\uparrow$, where, for $k \in \Lambda$, $y_{k \rightsquigarrow 0}$ denotes the configuration obtained from y by changing the k-th coordinate to 0 defined in (4.31), while the minimality of Y implies that $y_{i \rightsquigarrow 0}, y_{j \rightsquigarrow 0} \notin Y^\uparrow$. In the second case we can write $y(l) = b \vee c$ with $b \ne y(l) \ne c$, change the value of y at l once to b and once to c and run a similar argument as in the first case.

Let now $Y \in \mathcal{H}_1(S^\Lambda)$. Then $(Y^\uparrow)^c$ is non-empty and decreasing. Hence, if $(Y^\uparrow)^c$ were not an ideal we could find $x_1, x_2 \in (Y^\uparrow)^c$ such that $x_1 \vee x_2 \in Y^\uparrow$, i.e., there would exist a $y \in Y$ with the property that $y \le x_1 \vee x_2$. As the distributivity of S implies the distributivity of S^Λ, we could conclude that

$$y = y \wedge (x_1 \vee x_2) = (y \wedge x_1) \vee (y \wedge x_2).$$

As y is irreducible it would follow that either $y \wedge x_1 = y$ or $y \wedge x_2 = y$. But the former implies that that $y \le x_1$ while the later implies $y \le x_2$, contradicting that $x_1, x_2 \in (Y^\uparrow)^c$. □

To close the subchapter, we compute the bijection ϕ from (4.41) explicitly for the important example $S = \{0, \dots, N\}$ $(N \in \mathbb{N})$ equipped with the natural total order $0 < 1 < \cdots < N$. We set $\widehat{S} = \{0, \ldots, N\}$ and $\hat{y} := \underline{N} - y$ for $y \in S^\Lambda$ (defined pointwise). One has that

$$(\{y\}^{\downarrow})^{\mathrm{c}} = \Big\{x \in S^\Lambda : \exists i \in \Lambda \text{ s.t. } y(i) < x(i)\Big\} \qquad (y \in S^\Lambda),$$

and hence

$$\phi(\hat{y}) = \Big\{\delta_i^{y(i)+1} : i \in \Lambda \text{ s.t. } y(i) \neq N\Big\} = \Big\{\delta_i^{N+1-\hat{y}(i)} : i \in \mathrm{supp}(\hat{y})\Big\} \qquad (\hat{y} \in \widehat{S}^\Lambda).$$

Compare also [Foxall, 2016, Example 1].

Conclusion

This thesis studies three types of pathwise dualities of interacting particle systems based on monoids, modules over a semiring, and partially ordered sets, respectively. In all three cases, first the local state space S is equipped with an additional structure (the one of a monoid, a semiring, and a partially ordered set, respectively), that then induces an additional structure on the global state space S^Λ. Any interacting particle system whose generator has a random mapping representation as in (1.8), where $\mathcal{G}$, the countable collection of local maps, consists only of maps that preserve the structure on S^Λ, is shown to have a pathwise dual process (see Theorem 2.9, Theorem 3.6, and Theorem 4.10).

The construction in the case that S^Λ is equipped with the structure of a monoid (Chapter 2) and in the case that S^Λ is equipped with the structure of a module over a semiring (Chapter 3) are very similar. In both cases, the dual process has the form of an interacting particle system restricted to S^Λ_{fin}, a countable space whose exact definition depends on the choice of the (additive) monoid with which the local state space S is equipped. The system on this reduced state space becomes a continuous-time Markov chain. This construction contains the well-known examples of additive duality and cancellative duality as special cases (see Chapter 2.4), and also yields new dualities.

The construction in the case that S^Λ is equipped with the structure of a partially ordered set (Chapter 4) differs considerable from the other two. In this case the dual process is not of the form described above. Its state space $\mathcal{H}(S^\Lambda)$ consists of all subsets of S^Λ_{fin} that are equal to their set of minimal elements. The definition of S^Λ_{fin} in this case depends on the least element of the partial order that the local state space S is equipped with. It is important to note that a continuous-time Markov chain $\mathsf{Y} = (\mathsf{Y}_t)_{t\geq 0}$ on $\mathcal{H}_{\text{fin}}(S^\Lambda)$, a countable subspace of $\mathcal{H}(S^\Lambda)$, that is dual to a monotone interacting particle system, had already been constructed by Gray [1986] and Sturm and Swart [2018]. The main contribution of Chapter 4 is to provide an extended construction of Y on $\mathcal{H}_-(S^\Lambda)$, an uncountable subspace of $\mathcal{H}(S^\Lambda)$, where Y becomes a Feller process. This allows one to change the roles of primary and dual process with respect to the guiding principle of duality formulated in the introduction: One can study Y started in an infinite initial configuration through the original monotone interacting particle system restricted to S^Λ_{fin}, where it again becomes a continuous-time Markov chain. The extended construction of the process Y allows one to study its upper invariant law and connect its non-triviality to the survival of the original interacting particle system (see Chapter 4.5).

As already mentioned, the introduction of the general duality theory for monoids in Chapter 2 and modules over a semiring in Chapter 3 made it possible to compute new dualities of interacting particle systems with respect to new duality functions (see Chapter 2.5 and Chapter 3.4). To study the "usefulness" of these duality functions the notion of (weak) informativeness was introduced in Chapter 1.5. Informativeness was studied for the duality functions arising from monoid duality in Chapter 2.6 and for those arising from module duality in Chapter 3.5. All duality functions arising from monoid duality have been classified as either weakly informative or not weakly informative, with the majority being weakly informative. For the duality function arising from module duality the situation turned out to be more complicated. For some duality functions the methods of this thesis were not able to decide whether they are weakly informative or not. The duality function $\boldsymbol{\psi}_{\mathrm{mon}}$ from (4.10) is always informative, and a similar property also holds if one exchanges the roles of both processes (see Chapter 4.4).

Chapter 2.7 presents a typical application of duality: One of the newly established monoid dualities is used to compute all homogeneous (i.e., shift-invariant) invariant measures of the double contact process, a variant of the well-known contact process. Moreover, (weak) convergence to the greatest of these invariant measures (with respect to the stochastic order) is established if the double contact process is started in a homogeneous initial law that has the property that neither of its two marginals is the Dirac measure on the all-zero configuration (see Theorem 2.25).

A. Appendix

In this appendix we collect examples that rule out the weak informativeness of those duality functions from Chapter 2 and Chapter 3 that were highlighted in the corresponding tables with a red color. The examples were computed using the method presented in Chapter 2.6. Let, throughout the appendix, $i, j \in \Lambda$ with $i \neq j$.

A.1 Duality functions from Chapter 2

A.1.1 Duality functions from Table 2.2

$\boldsymbol{\psi}_3$: The duality function $\boldsymbol{\psi}_3$ was already considered in detail in Chapter 2.6.

$\boldsymbol{\psi}_6$: Let $X \sim 1/2\delta_{x_1} + 1/2\delta_{x_2}$ and $X' \sim 1/2\delta_{x_3} + 1/2\delta_{x_4}$ be random variables with values in $\{0,1,2\}^\Lambda$, where $x_1, x_2, x_3, x_4 \in \{0,1,2\}^\Lambda$ are given as

$$x_1(k) := \begin{cases} 1 & \text{if } k = i, \\ 0 & \text{if } k = j, \\ 0 & \text{else,} \end{cases} \qquad x_2(k) := \begin{cases} 2 & \text{if } k = i, \\ 2 & \text{if } k = j, \\ 0 & \text{else,} \end{cases}$$

$$x_3(k) := \begin{cases} 1 & \text{if } k = i, \\ 2 & \text{if } k = j, \\ 0 & \text{else,} \end{cases} \qquad x_4(k) := \begin{cases} 2 & \text{if } k = i, \\ 0 & \text{if } k = j, \\ 0 & \text{else,} \end{cases}$$

for $k \in \Lambda$. Then

$$\boldsymbol{\psi}_6(X, y) \stackrel{d}{=} \boldsymbol{\psi}_6(X', y) \stackrel{d}{=} \begin{cases} 0 & \text{if } y(i,j) = (0,0), \\ Z_1 & \text{if } y(i,j) \in \{(0,1),(0,2)\}, \\ Z_2 & \text{if } y(i,j) \in \{(1,0),(1,1),(1,2)\}, \\ 2 & \text{else,} \end{cases}$$

for $y \in \{0,1,2\}^\Lambda_{\text{fin}}$, where $Z_1 \sim 1/2\delta_0 + 1/2\delta_2$ and $Z_2 \sim 1/2\delta_1 + 1/2\delta_2$ are random variables with values in $\{0,1,2\}$.

A.1.2 Duality functions from Table 2.5

$\boldsymbol{\psi}_9$: Let $X \sim 1/2\delta_{x_1} + 1/2\delta_{x_2}$ and $X' \sim 1/2\delta_{x_3} + 1/2\delta_{x_4}$ be random variables with values in $\{0,1,2,3\}^\Lambda$, where $x_1, x_2, x_3, x_4 \in \{0,1,2,3\}^\Lambda$ are given as

$$x_1(k) := \begin{cases} 1 & \text{if } k = i, \\ 2 & \text{if } k = j, \\ 0 & \text{else,} \end{cases} \qquad x_2(k) := \begin{cases} 2 & \text{if } k = i, \\ 3 & \text{if } k = j, \\ 0 & \text{else,} \end{cases}$$

$$x_3(k) := \begin{cases} 1 & \text{if } k = i, \\ 3 & \text{if } k = j, \\ 0 & \text{else,} \end{cases} \qquad x_4(k) := \begin{cases} 2 & \text{if } k = i, \\ 2 & \text{if } k = j, \\ 0 & \text{else,} \end{cases}$$

for $k \in \Lambda$. Then

$$\boldsymbol{\psi}_9(X,y) \overset{d}{=} \boldsymbol{\psi}_9(X',y) \overset{d}{=} \begin{cases} 0 & \text{if } y(i,j) = (0,0), \\ Z_1 & \text{if } y(i,j) = (1,0), \\ Z_2 & \text{if } y(i,j) \in \{(0,1),(2,0)\}, \\ 3 & \text{else}, \end{cases}$$

for $y \in \{0,1,2,3\}^{\Lambda}_{\text{fin}}$, where $Z_1 \sim 1/2\delta_1 + 1/2\delta_2$ and $Z_2 \sim 1/2\delta_2 + 1/2\delta_3$ are random variables with values in $\{1,2,3\}$.

$\boldsymbol{\psi}_{10}$: Let $X \sim 1/2\delta_{x_1} + 1/2\delta_{x_2}$ and $X' \sim 1/2\delta_{x_3} + 1/2\delta_{x_4}$ be random variables with values in $\{0,1,2,3\}^{\Lambda}$, where $x_1, x_2, x_3, x_4 \in \{0,1,2,3\}^{\Lambda}$ are given as

$$x_1(k) := \begin{cases} 1 & \text{if } k = i, \\ 1 & \text{if } k = j, \\ 0 & \text{else}, \end{cases} \qquad x_2(k) := \begin{cases} 3 & \text{if } k = i, \\ 3 & \text{if } k = j, \\ 0 & \text{else}, \end{cases}$$

$$x_3(k) := \begin{cases} 1 & \text{if } k = i, \\ 3 & \text{if } k = j, \\ 0 & \text{else}, \end{cases} \qquad x_4(k) := \begin{cases} 3 & \text{if } k = i, \\ 1 & \text{if } k = j, \\ 0 & \text{else}, \end{cases}$$

for $k \in \Lambda$. Then

$$\boldsymbol{\psi}_{10}(X,y) \overset{d}{=} \boldsymbol{\psi}_{10}(X',y) \overset{d}{=} \begin{cases} 0 & \text{if } y(i,j) = (0,0), \\ Z & \text{if } y(i,j) \in \{(1,0),(0,1)\}, \\ 2 & \text{else}, \end{cases}$$

for $y \in \{0,1,2,3\}^{\Lambda}_{\text{fin}}$, where $Z \sim 1/2\delta_1 + 1/2\delta_2$ is a random variable with values in $\{1,2\}$.

$\boldsymbol{\psi}_{13}$: Let X and X' be random variables defined as in the previous example. Then

$$\boldsymbol{\psi}_{13}(X,y) \overset{d}{=} \boldsymbol{\psi}_{10}(X,y) \overset{d}{=} \boldsymbol{\psi}_{10}(X',y) \overset{d}{=} \boldsymbol{\psi}_{13}(X',y).$$

$\boldsymbol{\psi}_{22}$: Let $X \sim 1/2\delta_{x_1} + 1/2\delta_{x_2}$ and $X' \sim 1/2\delta_{x_3} + 1/2\delta_{x_4}$ be random variables with values in $\{0,1,2,3\}^{\Lambda}$, where $x_1, x_2, x_3, x_4 \in \{0,1,2,3\}^{\Lambda}$ are given as

$$x_1(k) := \begin{cases} 1 & \text{if } k = i, \\ 1 & \text{if } k = j, \\ 0 & \text{else}, \end{cases} \qquad x_2(k) := \begin{cases} 2 & \text{if } k = i, \\ 2 & \text{if } k = j, \\ 0 & \text{else}, \end{cases}$$

$$x_3(k) := \begin{cases} 1 & \text{if } k = i, \\ 2 & \text{if } k = j, \\ 0 & \text{else}, \end{cases} \qquad x_4(k) := \begin{cases} 2 & \text{if } k = i, \\ 1 & \text{if } k = j, \\ 0 & \text{else}, \end{cases}$$

for $k \in \Lambda$. Then

$$\boldsymbol{\psi}_{22}(X,y) \overset{d}{=} \boldsymbol{\psi}_{22}(X',y) \overset{d}{=} \begin{cases} 0 & \text{if } y(i,j) = (0,0), \\ Z & \text{if } y(i,j) \in \{(1,0),(0,1)\}, \\ 2 & \text{if } y(i,j) \in \{(0,2),(1,3),(2,3), \\ & \qquad (2,0),(3,1),(3,2)\}, \\ 3 & \text{else}, \end{cases}$$

for $y \in \{0,1,2,3\}^\Lambda_{\text{fin}}$, where $Z \sim 1/2\delta_1 + 1/2\delta_2$ is a random variable with values in $\{1,2\}$.

$\boldsymbol{\psi}_{24}$: Let $X \sim 1/2\delta_{x_1} + 1/2\delta_{x_2}$ and $X' \sim 1/2\delta_{x_3} + 1/2\delta_{x_4}$ be random variables with values in $\{0,1,2,3\}^\Lambda$, where $x_1, x_2, x_3, x_4 \in \{0,1,2,3\}^\Lambda$ are given as

$$x_1(k) := \begin{cases} 1 & \text{if } k = i, \\ 0 & \text{if } k = j, \\ 0 & \text{else,} \end{cases} \qquad x_2(k) := \begin{cases} 2 & \text{if } k = i, \\ 3 & \text{if } k = j, \\ 0 & \text{else,} \end{cases}$$

$$x_3(k) := \begin{cases} 1 & \text{if } k = i, \\ 3 & \text{if } k = j, \\ 0 & \text{else,} \end{cases} \qquad x_4(k) := \begin{cases} 2 & \text{if } k = i, \\ 0 & \text{if } k = j, \\ 0 & \text{else,} \end{cases}$$

for $k \in \Lambda$. Then

$$\boldsymbol{\psi}_{24}(X, y) \stackrel{d}{=} \boldsymbol{\psi}_{24}(X', y) \stackrel{d}{=} \begin{cases} 0 & \text{if } y(i,j) = (0,0), \\ 3 & \text{if } y(i,j) \in \{(3,0),(3,1),(3,2),(3,3)\}, \\ Z_1 & \text{if } y(i,j) \in \{(0,1),(0,2),(0,3)\}, \\ Z_2 & \text{else,} \end{cases}$$

for $y \in \{0,1,2,3\}^\Lambda_{\text{fin}}$, where $Z_1 \sim 1/2\delta_0 + 1/2\delta_3$ and $Z_2 \sim 1/2\delta_1 + 1/2\delta_2$ are random variables with values in $\{0,1,2,3\}$.

A.2 Duality functions from Chapter 3

In Chapter 3 the duality function of interest is $\boldsymbol{\psi}$ from (3.5). Its exact definition depends on the underlying semiring $(S, +, \cdot,)$. In each of the following examples a different semiring is assumed, indicated by its position in Table 3.2 and Table 3.3. We refer to the entries of Table 3.2 and Table 3.3 in the following from: Entry **x-y** refers to the **y**-th entry in row **x** of either Table 3.2 or Table 3.3. We additionally indicate to which monoids from Chapter 2 the additive and the multiplicative monoid of the underlying semiring are (anti-) isomorphic to.

A.2.1 Duality functions from Table 3.2

2-1: add. $\cong \mathbf{M}_3$, mult. $\cong \mathbf{M}_4$. In this case $\boldsymbol{\psi}$ from (3.5) is the duality function $\boldsymbol{\psi}_3$ considered in Chapter 2.6.

3-1: add. $\cong \mathbf{M}_4$, mult. $\cong \mathbf{M}_4$. Let $X \sim 1/2\delta_{x_1} + 1/2\delta_{x_2}$ and $X' \sim 1/2\delta_{x_3} + 1/2\delta_{x_4}$ be random variables with values in $\{0,1,2\}^\Lambda$, where $x_1, x_2, x_3, x_4 \in \{0,1,2\}^\Lambda$ are given as

$$x_1(k) := \begin{cases} 1 & \text{if } k = i, \\ 0 & \text{if } k = j, \\ 0 & \text{else,} \end{cases} \qquad x_2(k) := \begin{cases} 2 & \text{if } k = i, \\ 1 & \text{if } k = j, \\ 0 & \text{else,} \end{cases}$$

$$x_3(k) := \begin{cases} 1 & \text{if } k = i, \\ 1 & \text{if } k = j, \\ 0 & \text{else,} \end{cases} \qquad x_4(k) := \begin{cases} 2 & \text{if } k = i, \\ 0 & \text{if } k = j, \\ 0 & \text{else,} \end{cases}$$

for $k \in \Lambda$. Then

$$\boldsymbol{\psi}(X,y) \overset{d}{=} \boldsymbol{\psi}(X',y) \overset{d}{=} \begin{cases} 0 & \text{if } y(i,j) = (0,0), \\ 1 & \text{if } y(i,j) \in \{(1,0),(1,1),(1,2)\}, \\ Z_1 & \text{if } y(i,j) \in \{(0,1),(0,2)\}, \\ Z_2 & \text{else}, \end{cases}$$

for $y \in \{0,1,2\}^{\Lambda}_{\text{fin}}$, where $Z_1 \sim 1/2\delta_0 + 1/2\delta_1$ and $Z_2 \sim 1/2\delta_1 + 1/2\delta_2$ are random variables with values in $\{0,1,2\}$.

3-2: add. $\cong \mathbf{M}_6$, mult. $\cong \mathbf{M}_4$. In this case $\boldsymbol{\psi}$ from (3.5) is the duality function $\boldsymbol{\psi}_6$ considered in Appendix A.1.

A.2.2 Duality functions from Table 3.3

1-1: add. $\cong \mathbf{M}_8$, mult. $\cong \mathbf{M}_{14}$. Let $X \sim 1/2\delta_{x_1} + 1/2\delta_{x_2}$ and $X' \sim 1/2\delta_{x_3} + 1/2\delta_{x_4}$ be random variables with values in $\{0,1,2,3\}^{\Lambda}$, where $x_1, x_2, x_3, x_4 \in \{0,1,2,3\}^{\Lambda}$ are given as

$$x_1(k) := \begin{cases} 1 & \text{if } k = i, \\ 1 & \text{if } k = j, \\ 0 & \text{else}, \end{cases} \qquad x_2(k) := \begin{cases} 2 & \text{if } k = i, \\ 2 & \text{if } k = j, \\ 0 & \text{else}, \end{cases}$$

$$x_3(k) := \begin{cases} 1 & \text{if } k = i, \\ 2 & \text{if } k = j, \\ 0 & \text{else}, \end{cases} \qquad x_4(k) := \begin{cases} 2 & \text{if } k = i, \\ 1 & \text{if } k = j, \\ 0 & \text{else}, \end{cases}$$

for $k \in \Lambda$. Then

$$\boldsymbol{\psi}(X,y) \overset{d}{=} \boldsymbol{\psi}(X',y) \overset{d}{=} \begin{cases} 0 & \text{if } y(i,j) = (0,0), \\ Z_1 & \text{if } y(i,j) \in \{(0,1),(1,0)\}, \\ Z_2 & \text{if } y(i,j) \in \{(0,2),(2,0)\}, \\ 3 & \text{else}, \end{cases}$$

for $y \in \{0,1,2,3\}^{\Lambda}_{\text{fin}}$, where $Z_1 \sim 1/2\delta_1 + 1/2\delta_2$ and $Z_2 \sim 1/2\delta_2 + 1/2\delta_3$ are random variables with values in $\{1,2,3\}$.

1-2: add. $\cong \mathbf{M}_8$, mult. $\cong \mathbf{M}_{15}$. Let X and X' be random variables defined as in the previous example. Then

$$\boldsymbol{\psi}(X,y) \overset{d}{=} \boldsymbol{\psi}(X',y) \overset{d}{=} \begin{cases} 0 & \text{if } y(i,j) = (0,0), \\ Z & \text{if } y(i,j) \in \{(0,1),(1,0)\}, \\ 2 & \text{if } y(i,j) \in \{(0,2),(2,0)\}, \\ 3 & \text{else}, \end{cases}$$

for $y \in \{0,1,2,3\}^{\Lambda}_{\text{fin}}$, where $Z \sim 1/2\delta_1 + 1/2\delta_2$ is a random variable with values in $\{1,2\}$.

1-3: add. $\cong \mathbf{M}_8$, mult. $\cong \mathbf{M}_{16}$. Let X and X' be random variables defined as in the previous two examples. Then

$$\boldsymbol{\psi}(X,y) \stackrel{d}{=} \boldsymbol{\psi}(X',y) \stackrel{d}{=} \begin{cases} 0 & \text{if } y(i,j) = (0,0), \\ Z & \text{if } y(i,j) \in \{(0,1),(0,2),(1,0),(2,0)\}, \\ 3 & \text{else,} \end{cases}$$

for $y \in \{0,1,2,3\}^{\Lambda}_{\text{fin}}$, where $Z \sim 1/2\delta_1 + 1/2\delta_2$ is a random variable with values in $\{1,2\}$.

1-4: add. $\cong \mathbf{M}_9$, mult. $\cong \mathbf{M}_{14}$. In this case $\boldsymbol{\psi}$ from (3.5) is the duality function $\boldsymbol{\psi}_9$ considered in Appendix A.1.

1-5: add. $\cong \mathbf{M}_{10}$, mult. $\cong \mathbf{M}_{13}$. Let $X \sim 1/2\delta_{x_1} + 1/2\delta_{x_2}$ and $X' \sim 1/2\delta_{x_3} + 1/2\delta_{x_4}$ be random variables with values in $\{0,1,2,3\}^{\Lambda}$, where $x_1, x_2, x_3, x_4 \in \{0,1,2,3\}^{\Lambda}$ are given as

$$x_1(k) := \begin{cases} 1 & \text{if } k = i, \\ 1 & \text{if } k = j, \\ 0 & \text{else,} \end{cases} \qquad x_2(k) := \begin{cases} 3 & \text{if } k = i, \\ 3 & \text{if } k = j, \\ 0 & \text{else,} \end{cases}$$

$$x_3(k) := \begin{cases} 1 & \text{if } k = i, \\ 3 & \text{if } k = j, \\ 0 & \text{else,} \end{cases} \qquad x_4(k) := \begin{cases} 3 & \text{if } k = i, \\ 1 & \text{if } k = j, \\ 0 & \text{else,} \end{cases}$$

for $k \in \Lambda$. Then

$$\boldsymbol{\psi}(X,y) \stackrel{d}{=} \boldsymbol{\psi}(X',y) \stackrel{d}{=} \begin{cases} 0 & \text{if } y(i,j) = (0,0), \\ Z & \text{if } y(i,j) \in \{(0,1),(1,0)\}, \\ 2 & \text{if } y(i,j) \in \{(0,2),(2,0),(2,2)\}, \\ 3 & \text{else,} \end{cases}$$

for $y \in \{0,1,2,3\}^{\Lambda}_{\text{fin}}$, where $Z \sim 1/2\delta_1 + 1/2\delta_3$ is a random variable with values in $\{1,3\}$.

2-1: add. $\cong \mathbf{M}_{10}$, mult. $\cong \mathbf{M}_{15}$. Let X and X' be random variables defined as in the previous example. Then $\boldsymbol{\psi}(X,y)$ and $\boldsymbol{\psi}(X',y)$ are distributed for all $y \in \{0,1,2,3\}^{\Lambda}_{\text{fin}}$ as in the previous example.

2-5: add. $\cong \mathbf{M}_{11}$, mult. $\cong \mathbf{M}_{15}$. Let $X \sim 1/2\delta_{x_1} + 1/2\delta_{x_2}$ and $X' \sim 1/2\delta_{x_3} + 1/2\delta_{x_4}$ be random variables with values in $\{0,1,2,3\}^{\Lambda}$, where $x_1, x_2, x_3, x_4 \in \{0,1,2,3\}^{\Lambda}$ are given as

$$x_1(k) := \begin{cases} 2 & \text{if } k = i, \\ 1 & \text{if } k = j, \\ 0 & \text{else,} \end{cases} \qquad x_2(k) := \begin{cases} 3 & \text{if } k = i, \\ 3 & \text{if } k = j, \\ 0 & \text{else,} \end{cases}$$

$$x_3(k) := \begin{cases} 2 & \text{if } k = i, \\ 3 & \text{if } k = j, \\ 0 & \text{else,} \end{cases} \qquad x_4(k) := \begin{cases} 3 & \text{if } k = i, \\ 1 & \text{if } k = j, \\ 0 & \text{else,} \end{cases}$$

for $k \in \Lambda$. Then

$$\psi(X,y) \overset{d}{=} \psi(X',y) \overset{d}{=} \begin{cases} 0 & \text{if } y(i,j) = (0,0), \\ Z_1 & \text{if } y(i,j) = (0,1), \\ Z_2 & \text{if } y(i,j) \in \{(0,2),(1,0),(1,2)\}, \\ 2 & \text{if } y(i,j) \in \{(2,0),(2,2)\}, \\ 3 & \text{else,} \end{cases}$$

for $y \in \{0,1,2,3\}^\Lambda_{\text{fin}}$, where $Z_1 \sim 1/2\delta_1 + 1/2\delta_3$ and $Z_2 \sim 1/2\delta_2 + 1/2\delta_3$ are random variables with values in $\{1,2,3\}$.

3-2: add. $\cong \mathbf{M}_{13}$, mult. $\cong \mathbf{M}_{13}$. Let X and X' be random variables defined as in the example for entry **1-5**. Then

$$\psi(X,y) \overset{d}{=} \psi(X',y) \overset{d}{=} \begin{cases} 0 & \text{if } y(i,j) = (0,0), \\ Z & \text{if } y(i,j) \in \{(0,1),(1,0),(1,2),(2,1)\}, \\ 2 & \text{if } y(i,j) \in \{(0,2),(2,0),(2,2)\}, \\ 3 & \text{else,} \end{cases}$$

for $y \in \{0,1,2,3\}^\Lambda_{\text{fin}}$, where $Z \sim 1/2\delta_1 + 1/2\delta_3$ is a random variable with values in $\{1,3\}$.

3-3: add. $\cong \mathbf{M}_{13}$, mult. $\cong \mathbf{M}_{15}$. Let X and X' be random variables defined as in the example for entry **1-5**. Then $\psi(X,y)$ and $\psi(X',y)$ are distributed for all $y \in \{0,1,2,3\}^\Lambda_{\text{fin}}$ as in the previous example.

3-4: add. $\cong \mathbf{M}_{14}$, mult. $\cong \mathbf{M}_{15}$. Let X and X' be random variables defined as in the example for entry **1-1**. Then

$$\psi(X,y) \overset{d}{=} \psi(X',y) \overset{d}{=} \begin{cases} 0 & \text{if } y(i,j) = (0,0), \\ Z & \text{if } y(i,j) \in \{(0,1),(1,0)\}, \\ 2 & \text{if } y(i,j) \in \{(0,2),(1,1),(1,2), \\ & \qquad\qquad (2,0),(2,1),(2,2)\}, \\ 3 & \text{else,} \end{cases}$$

for $y \in \{0,1,2,3\}^\Lambda_{\text{fin}}$, where $Z \sim 1/2\delta_1 + 1/2\delta_2$ is a random variable with values in $\{1,2\}$.

3-5: add. $\cong \mathbf{M}_{15}$, mult. $\cong \mathbf{M}_8$. Let $X \sim 1/2\delta_{x_1} + 1/2\delta_{x_2}$ and $X' \sim 1/2\delta_{x_3} + 1/2\delta_{x_4}$ be random variables with values in $\{0,1,2,3\}^\Lambda$, where $x_1, x_2, x_3, x_4 \in \{0,1,2,3\}^\Lambda$ are given as

$$x_1(k) := \begin{cases} 0 & \text{if } k = i, \\ 1 & \text{if } k = j, \\ 0 & \text{else,} \end{cases} \qquad x_2(k) := \begin{cases} 1 & \text{if } k = i, \\ 2 & \text{if } k = j, \\ 0 & \text{else,} \end{cases}$$

$$x_3(k) := \begin{cases} 0 & \text{if } k = i, \\ 2 & \text{if } k = j, \\ 0 & \text{else,} \end{cases} \qquad x_4(k) := \begin{cases} 1 & \text{if } k = i, \\ 1 & \text{if } k = j, \\ 0 & \text{else,} \end{cases}$$

for $k \in \Lambda$. Then

$$\boldsymbol{\psi}(X,y) \stackrel{d}{=} \boldsymbol{\psi}(X',y) \stackrel{d}{=} \begin{cases} Z_1 & \text{if } y(i,j) \in \{(3,0),(3,1),(3,2)\}, \\ Z_2 & \text{if } y(i,j) \in \{(0,3),(1,3),(2,3),(3,3)\}, \\ 0 & \text{else}, \end{cases}$$

for $y \in \{0,1,2,3\}^{\Lambda}_{\text{fin}}$, where $Z_1 \sim 1/2\delta_0 + 1/2\delta_1$ and $Z_2 \sim 1/2\delta_1 + 1/2\delta_2$ are random variables with values in $\{0,1,2\}$.

4-3: add. $\cong \mathbf{M}_{15}$, mult. $\cong \mathbf{M}_{13}$. Let $X \sim 1/2\delta_{x_1} + 1/2\delta_{x_2}$ and $X' \sim 1/2\delta_{x_3} + 1/2\delta_{x_4}$ be random variables with values in $\{0,1,2,3\}^{\Lambda}$, where $x_1, x_2, x_3, x_4 \in \{0,1,2,3\}^{\Lambda}$ are given as

$$x_1(k) := \begin{cases} 2 & \text{if } k = i, \\ 0 & \text{if } k = j, \\ 0 & \text{else}, \end{cases} \qquad x_2(k) := \begin{cases} 3 & \text{if } k = i, \\ 1 & \text{if } k = j, \\ 0 & \text{else}, \end{cases}$$

$$x_3(k) := \begin{cases} 2 & \text{if } k = i, \\ 1 & \text{if } k = j, \\ 0 & \text{else}, \end{cases} \qquad x_4(k) := \begin{cases} 3 & \text{if } k = i, \\ 0 & \text{if } k = j, \\ 0 & \text{else}, \end{cases}$$

for $k \in \Lambda$. Then

$$\boldsymbol{\psi}(X,y) \stackrel{d}{=} \boldsymbol{\psi}(X',y) \stackrel{d}{=} \begin{cases} 0 & \text{if } y(i,j) \in \{(0,0),(0,1)\}, \\ Z_1 & \text{if } y(i,j) \in \{(0,2),(0,3)\}, \\ 1 & \text{if } y(i,j) \in \{(1,0),(1,1),(1,2),(1,3)\}, \\ Z_2 & \text{if } y(i,j) \in \{(2,0),(2,1),(2,2),(2,3)\}, \\ 3 & \text{else}, \end{cases}$$

for $y \in \{0,1,2,3\}^{\Lambda}_{\text{fin}}$, where $Z_1 \sim 1/2\delta_0 + 1/2\delta_1$ and $Z_2 \sim 1/2\delta_2 + 1/2\delta_3$ are random variables with values in $\{0,1,2,3\}$.

4-4: add. $\cong \mathbf{M}_{15}$, mult. $\cong \mathbf{M}_{13}$. Let X and X' be random variables defined as in the previous example. Then

$$\boldsymbol{\psi}(X,y) \stackrel{d}{=} \boldsymbol{\psi}(X',y) \stackrel{d}{=} \begin{cases} 0 & \text{if } y(i,j) \in \{(0,0),(0,1)\}, \\ Z_1 & \text{if } y(i,j) \in \{(0,2),(0,3)\}, \\ 1 & \text{if } y(i,j) \in \{(1,0),(1,1),(1,2),(1,3)\}, \\ 2 & \text{if } y(i,j) \in \{(2,0),(2,1),(2,2),(2,3)\}, \\ Z_2 & \text{else}, \end{cases}$$

for $y \in \{0,1,2,3\}^{\Lambda}_{\text{fin}}$, where $Z_1 \sim 1/2\delta_0 + 1/2\delta_1$ and $Z_2 \sim 1/2\delta_2 + 1/2\delta_3$ are random variables with values in $\{0,1,2,3\}$.

5-1: add. $\cong \mathbf{M}_{15}$, mult. $\cong \mathbf{M}_{14}$. Let X and X' be random variables defined as in the previous two examples. Then

$$\boldsymbol{\psi}(X,y) \stackrel{d}{=} \boldsymbol{\psi}(X',y) \stackrel{d}{=} \begin{cases} 0 & \text{if } y(i,j)=(0,0), \\ Z_1 & \text{if } y(i,j)\in\{(0,1),(0,2),(0,3)\}, \\ 1 & \text{if } y(i,j)\in\{(1,0),(1,1),(1,2),(1,3)\}, \\ Z_2 & \text{if } y(i,j)\in\{(2,0),(2,1),(2,2),(2,3)\}, \\ Z_3 & \text{else,} \end{cases}$$

for $y \in \{0,1,2,3\}^{\Lambda}_{\text{fin}}$, where $Z_1 \sim 1/2\delta_0 + 1/2\delta_1$, $Z_2 \sim 1/2\delta_1 + 1/2\delta_2$ and $Z_2 \sim 1/2\delta_2 + 1/2\delta_3$ are random variables with values in $\{0,1,2,3\}$.

5-2: add. $\cong \mathbf{M}_{15}$, mult. $\cong \mathbf{M}_{15}$. Let $X \sim 1/2\delta_{x_1} + 1/2\delta_{x_2}$ and $X' \sim 1/2\delta_{x_3} + 1/2\delta_{x_4}$ be random variables with values in $\{0,1,2,3\}^{\Lambda}$, where $x_1,x_2,x_3,x_4 \in \{0,1,2,3\}^{\Lambda}$ are given as

$$x_1(k) := \begin{cases} 2 & \text{if } k=i, \\ 1 & \text{if } k=j, \\ 0 & \text{else,} \end{cases} \qquad x_2(k) := \begin{cases} 3 & \text{if } k=i, \\ 2 & \text{if } k=j, \\ 0 & \text{else,} \end{cases}$$

$$x_3(k) := \begin{cases} 2 & \text{if } k=i, \\ 2 & \text{if } k=j, \\ 0 & \text{else,} \end{cases} \qquad x_4(k) := \begin{cases} 3 & \text{if } k=i, \\ 1 & \text{if } k=j, \\ 0 & \text{else,} \end{cases}$$

for $k \in \Lambda$. Then

$$\boldsymbol{\psi}(X,y) \stackrel{d}{=} \boldsymbol{\psi}(X',y) \stackrel{d}{=} \begin{cases} 0 & \text{if } y(i,j)=(0,0), \\ Z_1 & \text{if } y(i,j)=(0,1), \\ 2 & \text{if } y(i,j)=(0,2), \\ Z_2 & \text{if } y(i,j)\in\{(1,0),(1,1),(1,2), \\ & \qquad\qquad (2,0),(2,1),(2,2)\}, \\ 3 & \text{else,} \end{cases}$$

for $y \in \{0,1,2,3\}^{\Lambda}_{\text{fin}}$, where $Z_1 \sim 1/2\delta_1 + 1/2\delta_2$ and $Z_2 \sim 1/2\delta_2 + 1/2\delta_3$ are random variables with values in $\{1,2,3\}$.

5-4: add. $\cong \mathbf{M}_{15}$, mult. $\cong \mathbf{M}_{15}$. Let X and X' be random variables defined as in the example for entry **4-3** (and entry **5-1**). Then $\boldsymbol{\psi}(X,y)$ and $\boldsymbol{\psi}(X',y)$ are distributed for all $y \in \{0,1,2,3\}^{\Lambda}_{\text{fin}}$ as in the example for entry **5-1**.

5-5: add. $\cong \mathbf{M}_{15}$, mult. $\cong \mathbf{M}_{15}$. Let X and X' be random variables defined as in the example for entry **4-3**. Then

$$\boldsymbol{\psi}(X,y) \stackrel{d}{=} \boldsymbol{\psi}(X',y) \stackrel{d}{=} \begin{cases} 0 & \text{if } y(i,j)=(0,0), \\ Z_1 & \text{if } y(i,j)\in\{(0,1),(0,2),(0,3)\}, \\ 1 & \text{if } y(i,j)\in\{(1,0),(1,1),(1,2),(1,3)\}, \\ 2 & \text{if } y(i,j)\in\{(2,0),(2,1),(2,2),(2,3)\}, \\ Z_2 & \text{else,} \end{cases}$$

for $y \in \{0,1,2,3\}^{\Lambda}_{\text{fin}}$, where $Z_1 \sim 1/2\delta_0 + 1/2\delta_1$ and $Z_2 \sim 1/2\delta_2 + 1/2\delta_3$ are random variables with values in $\{0,1,2,3\}$.

6-1: add. $\cong \mathbf{M}_{15}$, mult. $\cong \mathbf{N}_1$. Let X and X' be random variables defined as in the example for entry **4-3**. Then

$$\boldsymbol{\psi}(X,y) \overset{d}{=} \boldsymbol{\psi}(X',y) \overset{d}{=} \begin{cases} 0 & \text{if } y(i,j) \in \{(0,0),(0,1),(0,2)\}, \\ Z_1 & \text{if } y(i,j) = (0,3), \\ 1 & \text{if } y(i,j) \in \{(1,0),(1,1),(1,2),(1,3)\}, \\ 2 & \text{if } y(i,j) \in \{(2,0),(2,1),(2,2),(2,3)\}, \\ Z_2 & \text{else,} \end{cases}$$

for $y \in \{0,1,2,3\}^{\Lambda}_{\text{fin}}$, where $Z_1 \sim 1/2\delta_0 + 1/2\delta_1$ and $Z_2 \sim 1/2\delta_2 + 1/2\delta_3$ are random variables with values in $\{0,1,2,3\}$.

6-2: add. $\cong \mathbf{M}_{15}$, mult. $\cong \mathbf{N}_2$. Let X and X' be random variables defined as in the example for entry **4-3**. Then

$$\boldsymbol{\psi}(X,y) \overset{d}{=} \boldsymbol{\psi}(X',y) \overset{d}{=} \begin{cases} 0 & \text{if } y(i,j) = (0,0), \\ Z_1 & \text{if } y(i,j) \in \{(0,1),(0,2),(0,3)\}, \\ Z_2 & \text{if } y(i,j) \in \{(1,0),(1,1),(1,2),(1,3)\}, \\ Z_3 & \text{if } y(i,j) \in \{(2,0),(2,1),(2,2),(2,3)\}, \\ 3 & \text{else,} \end{cases}$$

for $y \in \{0,1,2,3\}^{\Lambda}_{\text{fin}}$, where $Z_1 \sim 1/2\delta_0 + 1/2\delta_1$, $Z_2 \sim 1/2\delta_1 + 1/2\delta_3$ and $Z_3 \sim 1/2\delta_2 + 1/2\delta_3$ are random variables with values in $\{0,1,2,3\}$.

6-4: add. $\cong \mathbf{M}_{20}$, mult. $\cong \mathbf{M}_{13}$. Let $X \sim 1/2\delta_{x_1} + 1/2\delta_{x_2}$ and $X' \sim 1/2\delta_{x_3} + 1/2\delta_{x_4}$ be random variables with values in $\{0,1,2,3\}^{\Lambda}$, where $x_1, x_2, x_3, x_4 \in \{0,1,2,3\}^{\Lambda}$ are given as

$$x_1(k) := \begin{cases} 0 & \text{if } k = i, \\ 1 & \text{if } k = j, \\ 0 & \text{else,} \end{cases} \qquad x_2(k) := \begin{cases} 2 & \text{if } k = i, \\ 3 & \text{if } k = j, \\ 0 & \text{else,} \end{cases}$$

$$x_3(k) := \begin{cases} 0 & \text{if } k = i, \\ 3 & \text{if } k = j, \\ 0 & \text{else,} \end{cases} \qquad x_4(k) := \begin{cases} 2 & \text{if } k = i, \\ 1 & \text{if } k = j, \\ 0 & \text{else,} \end{cases}$$

for $k \in \Lambda$. Then

$$\boldsymbol{\psi}(X,y) \overset{d}{=} \boldsymbol{\psi}(X',y) \overset{d}{=} \begin{cases} 0 & \text{if } y(i,j) \in \{(0,0),(2,0)\}, \\ Z_1 & \text{if } y(i,j) \in \{(1,0),(3,0)\}, \\ Z_2 & \text{if } y(i,j) \in \{(0,1),(1,1),(2,1),(3,1)\}, \\ 2 & \text{if } y(i,j) \in \{(0,2),(1,2),(2,2),(3,2)\}, \\ 3 & \text{else,} \end{cases}$$

for $y \in \{0,1,2,3\}^{\Lambda}_{\text{fin}}$, where $Z_1 \sim 1/2\delta_0 + 1/2\delta_2$ and $Z_2 \sim 1/2\delta_1 + 1/2\delta_3$ are random variables with values in $\{0,1,2,3\}$.

6-5: add. $\cong \mathbf{M}_{20}$, mult. $\cong \mathbf{M}_{15}$. Let X and X' be random variables defined as in the previous example. Then

$$\boldsymbol{\psi}(X,y) \overset{d}{=} \boldsymbol{\psi}(X',y) \overset{d}{=} \begin{cases} 0 & \text{if } y(i,j) = (0,0), \\ Z_1 & \text{if } y(i,j) \in \{(1,0),(2,0),(3,0)\}, \\ Z_2 & \text{if } y(i,j) \in \{(0,1),(1,1),(2,1),(3,1)\}, \\ 2 & \text{if } y(i,j) \in \{(0,2),(1,2),(2,2),(3,2)\}, \\ 3 & \text{else}, \end{cases}$$

for $y \in \{0,1,2,3\}^{\Lambda}_{\text{fin}}$, where $Z_1 \sim 1/2\delta_0 + 1/2\delta_2$ and $Z_2 \sim 1/2\delta_1 + 1/2\delta_3$ are random variables with values in $\{0,1,2,3\}$.

7-2: add. $\cong \mathbf{M}_{22}$, mult. $\cong \mathbf{M}_{15}$. In this case $\boldsymbol{\psi}$ from (3.5) is the duality function $\boldsymbol{\psi}_{22}$ considered in Appendix A.1.

7-4: add. $\cong \mathbf{M}_{24}$, mult. $\cong \mathbf{M}_{16}$. In this case $\boldsymbol{\psi}$ from (3.5) is the duality function $\boldsymbol{\psi}_{24}$ considered in Appendix A.1.

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