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Besov-Orlicz Path Regularity of Non-Gaussian Processes

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Abstract

In the article, Besov-Orlicz regularity of sample paths of stochastic processes that are represented by multiple integrals of order $n \in \mathbb{N}$ is treated. We assume that the considered processes belong to the Hölder space

$$C^\alpha([0,T]; L^2(\Omega)) \quad \text{with} \quad \alpha \in (0,1),$$

and we give sufficient conditions for them to have paths in the exponential Besov-Orlicz space

$$B^\alpha_{\Phi_{2/n},\infty}(0,T) \qquad \text{with} \qquad \Phi_{2/n}(x) = \mathrm{e}^{x^{2/n}} - 1.$$

These results provide an extension of what is known for scalar Gaussian stochastic processes to stochastic processes in an arbitrary finite Wiener chaos. As an application, the Besov-Orlicz path regularity of fractionally filtered Hermite processes is studied. But while the main focus is on the non-Gaussian case, some new path properties are obtained even for fractional Brownian motions.

Keywords Besov-Orlicz space · Hermite process · multiple Wiener-Itô integral · path regularity

Mathematics Subject Classification (2010) Primary 60G17; Secondary 60G22 · 60G18 · 60H07

1 Introduction

It is well-known that the paths of the Wiener process belong to the Besov-Orlicz space $B^{1/2}_{\Phi_2,\infty}(0,T)$ where $\Phi_2(x) = \mathrm{e}^{x^2}-1$. The original proof of this result in [7] relies on intricate equivalences for Besov norms but a different proof is also available in [11]. From this result, one immediately obtains, for example, that Brownian paths belong to both the Besov space

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$B^{1/2}_{p,\infty}(0,T)$ for all $p \in [1,\infty)$ and the modulus Hölder space $C^{|r \log r|^{1/2}}([0,T])$ although historically, these two results came first; see [6] and [15].

There is a number of generalizations of this result in various directions. In [22], it is shown that any continuous local martingale with Lipschitz continuous quadratic variation as well as solutions to stochastic differential equations with locally bounded non-linearities have paths in the space $B^{1/2}_{\Phi_2,\infty}(0,T)$. In [21], it is shown that such regularity is also retained by stochastic convolutions (with values in 2-smooth Banach spaces) and the result is also shown for strong solutions to stochastic p-Laplace systems in [30]. In [27], it is shown that the fractional Brownian motion of Hurst parameter $\alpha \in (0,1)$ has paths in the Besov-Orlicz space $B^{\alpha}_{\Phi_2,\infty}(0,T)$. There are also other Besov regularity results, e.g., for bifractional Brownian motions [5] or Lévy processes [2, 8, 9].

The purpose of this article is to extend the results of [27] to non-Gaussian stochastic processes. In particular, we consider stochastic processes that fit the following scheme: Let H be a real separable Hilbert space and $W = \{W(h)\}_{h\in H}$ an H-isonormal Gaussian process. For $n \in \mathbb{N}_0$, denote the multiple divergence operator of order n by W_n and the completion of the n^{th} tensor power of H by $H^{\otimes n}$. Let $G = \{G(t)\}_{t\in[0,T]}$, $T > 0$, be the process represented by $G(t) = W_n(A_t)$, $t \in [0,T]$, with $A \in C^{\alpha}([0,T]; H^{\otimes n})$ for some $\alpha \in (0,1)$. For example, one can consider the family of fractional Brownian motions, Hermite (and, in particular, Rosenblatt) processes, or the so-called fractionally filtered generalized Hermite processes; see [3].

1.1 Main results

A main result of the present article is Theorem 3.3 which provides sufficient conditions under which process G has paths in the Besov-Orlicz space

$$B^{\alpha}_{\Phi_{2/n},\infty}(0,T) \quad \text{where} \quad \Phi_{2/n}(x) = \mathrm{e}^{x^{2/n}} - 1.$$

These conditions are formulated in terms of the kernel A. In the scalar case, Theorem 3.3 extends [27, Theorem 5.1], where Gaussian processes are considered, to a non-Gaussian setting. We thus make precise how the order of the Wiener chaos in which process G lives influences the regularity of its paths. In particular, it is clear that the higher the order of the Wiener chaos, the worse regularity of paths we get. Moreover, in Theorem 3.9, the result is refined and the precise pathwise behavior of the integral of the increments is obtained. Our results cover Gaussian processes (e.g. standard and fractional Brownian motions) but also non-Gaussian processes (e.g. higher-order Hermite processes). For example, we show that the Rosenblatt process with Hurst parameter $\alpha \in (1/2,1)$ has paths in the space $B^{\alpha}_{\Phi_1,\infty}(0,T)$.

1.2 Proof Method and Invalidity of the Strong Gebelein's Inequality in Higher-Order Wiener Chaoses

In order to establish such Besov-Orlicz regularity results, one would hope to proceed as in [27] (or [5]). The proofs there rely on Gebelein's inequality [10] (see also [4]):

Theorem 1.1 (Gebelein's inequality) *Let $(\xi,\eta)^{\top}$ be a centered Gaussian vector in $\mathbb{R}^2$ with $\mathbb{E}\xi^2 = \mathbb{E}\eta^2 = 1$. Then the inequality*

$$|\mathbb{E} f(\xi)g(\eta)| \le |\rho_{\xi,\eta}|\, \|f(\xi)\|_{L^2(\Omega)} \|g(\eta)\|_{L^2(\Omega)}, \tag{1}$$

where $\rho_{\xi,\eta}$ is the correlation coefficient between ξ and η, holds for all functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $\mathbb{E}f(\xi)^2 < \infty$, $\mathbb{E}g(\eta)^2 < \infty$, and $\mathbb{E}f(\xi) = \mathbb{E}g(\eta) = 0$.

However, while the original Gebelein's inequality can be used in a Gaussian setting, in our case, the process $G(t) = W_n(A_t)$, $t \in [0, T]$, is not Gaussian for $n \geq 2$. It is therefore natural to ask whether some analogue of Gebelein's inequality, that could be used for investigating Besov-Orlicz regularity of the paths of G, holds even in higher-order Wiener chaoses. In that respect, there is the following generalization of Gebelein's inequality (see [19, Theorem 2.3] and [17, Lemma 2.6]).

Theorem 1.2 (Generalized Gebelein's inequality) *Let $W = \{W(k),\ k \in \mathfrak{K}\}$ be an isonormal Gaussian process over some real separable Hilbert space $\mathfrak{K}$, and let $\mathfrak{K}_1$, $\mathfrak{K}_2$ be two Hilbert subspaces of $\mathfrak{K}$. Define W_1 and W_2 as the restrictions of W to $\mathfrak{K}_1$ and $\mathfrak{K}_2$, respectively. Then the inequality*

$$|\mathbb{E}\,[F_1F_2]| \leq \theta \|F_1\|_{L^p(\Omega)} \|F_2\|_{L^q(\Omega)}$$

holds with

$$\theta = \sup\{|\langle k_1, k_2\rangle_{\mathfrak{K}}| : k_1 \in \mathfrak{K}_1,\ k_2 \in \mathfrak{K}_2,\ \|k_1\|_{\mathfrak{K}} = \|k_2\|_{\mathfrak{K}} = 1\}$$

for every centered random variables F_1 and F_2 that are measurable with respect to the σ-algebra generated by W_1 and W_2, respectively, and for every Hölder conjugate exponents p, q. Moreover,

$$|\mathbb{E}\,[F_1F_2]| \leq \theta^d \|F_1\|_{L^2(\Omega)} \|F_2\|_{L^2(\Omega)}$$

holds for $d \in \mathbb{N}$ such that the projections of F_1 to the first $d-1$ chaoses are equal to zero and the projection to the d^{th} chaos is non-trivial.

On the other hand, there is also the following negative result.

Counterexample 1.3 *Let W be an $\mathbb{R}^2$-isonormal Gaussian process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ where the σ-algebra $\mathcal{F}$ is generated by W. Let $\{e_1, e_2\}$ be an orthonormal basis of $\mathbb{R}^2$ and set $X_i = W(e_i)$, $i = 1, 2$. Then both X_1 and X_2 are standard Gaussian random variables. Define $\xi = \frac{1}{\sqrt{2}}(X_1^2 - 1)$ and $\eta = X_1X_2$. Then both ξ and η belong to the second Wiener chaos of W (write*

$$\frac{1}{2}(X_1^2 - 1) + \frac{1}{2}(X_2^2 - 1) - \left[\left(\frac{X_1 - X_2}{\sqrt{2}}\right)^2 - 1\right] = X_1X_2$$

for η), $\mathbb{E}\xi = \mathbb{E}\eta = 0$, and $\mathbb{E}\xi^2 = \mathbb{E}\eta^2 = 1$. Then we have that

$$|\mathbb{E}(\xi^2 - \mathbb{E}\xi^2)(\eta^2 - \mathbb{E}\eta^2)| = 4$$

but $\mathbb{E}\xi\eta = 0$, hence $\rho_{\xi,\eta} = 0$, and an inequality of the type (1) does not hold.

Altogether, despite that Gebelein's inequality holds even in higher-order Wiener chaoses, it turns out that it loses the necessary power already in the second Wiener chaos and therefore, in order to prove Theorem 3.3, we need to proceed differently. Instead, we use orthogonality of Wiener chaoses and tensor calculus. In particular, we work directly with the Besov-Orlicz norm; we initially use the generalized product formula for multiple integrals (given in Theorem 5.9) to obtain the Wiener chaos expansion of

$$Y_{\ell,\delta}^{\ell} = \|G(\cdot + \delta) - G(\cdot)\|_{L^{\ell}(0,T-\delta)}^{\ell}$$

for an even integer ℓ in terms of tensor cancellations of the kernel A_t, i.e.

$$Y_{\ell,\delta}^{\ell} = \underbrace{\mathbb{E}Y_{\ell,\delta}^{\ell}}_{\in\mathscr{H}_0} + \underbrace{\xi_2(\delta)}_{\in\mathscr{H}_2} + \underbrace{\xi_4(\delta)}_{\in\mathscr{H}_4} + \ldots + \underbrace{\xi_{\ell n}(\delta)}_{\in\mathscr{H}_{\ell n}}$$

where $\mathscr{H}_k$ denotes the k^{th} Wiener chaos of the isonormal process W. Subsequently, by using several results regarding the cancellation operator (and, in particular, the key Lemma 5.6 that is used instead of Gebelein's inequality), we show[1] that

$$|\xi_2(\delta)| + |\xi_4(\delta)| + \ldots + |\xi_{\ell n}(\delta)| \ll \mathbb{E}Y_{\ell,\delta}^{\ell} \quad \text{as} \quad \delta \to 0+$$

so that

$$Y_{\ell,\delta}^{\ell} \sim \mathbb{E}Y_{\ell,\delta}^{\ell} \quad \text{as} \quad \delta \to 0+.$$

From this, we obtain that

$$Y_{\ell,\delta} \lesssim \delta^{\alpha}\ell^{\frac{n}{2}}$$

and the result follows.

1.3 Organisation of the Article

In Section 2, the definitions of Besov, Orlicz, and Besov-Orlicz spaces are recalled. In Section 3, the main results of the article are collected. In particular, we state the main Theorem 3.3 where, along some moment estimates, we give sufficient conditions for the considered process to have paths in the Besov-Orlicz space $B^{\alpha}_{\Phi_{2/n},\infty}(0,T)$ and sufficient conditions for the process not to have paths in any of the Besov spaces $B^{\alpha}_{p,q}(0,T)$ for $p \in [1,\infty]$ and $q \in [1,\infty)$. In addition, we also assess modular Hölder continuity of the process in Remark 3.4. Subsequently, we discuss the assumptions of Theorem 3.3 in Remark 3.5 and Remark 3.6 and compare the theorem to the known results in Remark 3.7. In there, we also note that some alternative assumptions, that correspond to those in [27, Theorem 5.1], can be considered. The section is concluded by Theorem 3.9 which refines Theorem 3.3 and in which pathwise asymptotics of the integral increments of the considered process is treated. In Section 4, the conditions are verified for fractionally filtered Hermite processes with a product kernel. In Section 5, we review elements of tensor calculus, provide several motivating examples and the necessary technical tools for the proofs of Theorem 3.3 and Theorem 3.9 (Lemma 5.3, Lemma 5.4, and Lemma 5.6 regarding the properties of the cancellation operator and Theorem 5.9 that contains the generalized product formula for multiple integrals). The proofs of Theorem 3.3, Theorem 3.3 with alternative assumptions, and Theorem 3.9 are given at the end of the article in Sections 6.1, 6.2, and 6.3, respectively.

2 Preliminaries: Besov, Orlicz, and Besov-Orlicz Spaces

We begin by recalling some facts about Besov, Orlicz, and Besov-Orlicz spaces. For a thorough exposition on Besov spaces, we refer the reader to, e.g., [26]. Orlicz spaces are covered in, e.g., [24, 31] and Besov-Orlitz spaces in, e.g., [21, 23]. Let $I \subseteq [0,\infty)$ be a bounded interval and for $h \in \mathbb{R}$, we denote $I(h) = \{s \in I : s + h \in I\}$. For $s \in (0,1)$

[1]More precisely, we show that the variances $\mathbb{E}[\xi_k(\delta)]^2$, $k = 2, 4, \ldots, \ell n$, are negligible so that the mean-square distance $\mathbb{E}(Y_{\ell,\delta}^{\ell} - \mathbb{E}Y_{\ell,\delta}^{\ell})^2$ is small by orthogonality of Wiener chaoses.

and $p, q \in [1, \infty)$ (with the usual modifications for $p = \infty$ or $q = \infty$), the Besov space $B^s_{p,q}(I)$ is defined as the linear space

$$B^s_{p,q}(I) = \{f \in L^p(I) : (f)_{B^s_{p,q}(I)} < \infty\}$$

where

$$(f)_{B^s_{p,q}(I)} = \left(\int_0^\infty [t^{-s} \sup_{|h|\le t} \|f(\cdot + h) - f(\cdot)\|_{L^p(I(h))}]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}$$

and the space $B^s_{p,q}(I)$ is a Banach space when endowed with the norm

$$\|f\|_{B^s_{p,q}(I)} = \|f\|_{L^p(I)} + (f)_{B^s_{p,q}(I)}.$$

Besov spaces can be generalized as follows. A function $\mathcal{N} : [0, \infty) \to \mathbb{R}$ is called a *Young function* if it is non-negative, non-decreasing, continuous, convex, and satisfies $\mathcal{N}(0) = 0$ and $\mathcal{N}(\infty-) = \infty$. For a Young function $\mathcal{N}$ and a measure space $(D, \mathcal{D}, \mu)$, where μ is a σ-finite measure, the *Orlicz space (with the Young function $\mathcal{N}$)* $L^{\mathcal{N}}(D)$ is defined as the linear space

$$L^{\mathcal{N}}(D) = \{f \in L^0(D) : \|f\|_{L^{\mathcal{N}}(D)} < \infty\}$$

where

$$\|f\|_{L^{\mathcal{N}}(D)} = \inf\left\{\lambda \ge 0 : \int_D \mathcal{N}\left(\frac{|f(x)|}{\lambda}\right) \mu(\mathrm{d}x) \le 1\right\}$$

is the so-called *Luxemburg norm*. Endowed with this norm, $L^{\mathcal{N}}(D)$ is a Banach space. (Note that with this notation, $L^{x^p}(D)$, $p \in [1, \infty)$, is the usual Lebesgue space $L^p(D)$.) Finally, for $s \in (0, 1)$, a Young function $\mathcal{N}$, and $q \in [1, \infty)$ (with the usual modification for $q = \infty$), the *Besov-Orlicz* space $B^s_{\mathcal{N},q}(I)$ is defined as the linear space

$$B^s_{\mathcal{N},q}(I) = \{f \in L^{\mathcal{N}}(I) : (f)_{B^s_{\mathcal{N},q}(I)} < \infty\}$$

where

$$(f)_{B^s_{\mathcal{N},q}(I)} = \left(\int_0^\infty [t^{-s} \sup_{|h|\le t} \|f(\cdot + h) - f(\cdot)\|_{L^{\mathcal{N}}(I(h))}]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}.$$

The space $B^s_{\mathcal{N},q}(I)$ is a Banach space when endowed with the norm

$$\|f\|_{B^s_{\mathcal{N},q}(I)} = \|f\|_{L^{\mathcal{N}}(I)} + (f)_{B^s_{\mathcal{N},q}(I)}.$$

(Note also, that with this notation, $B^s_{x^p,q}(I)$, $p \in [1, \infty)$, is the usual Besov space $B^s_{p,q}(I)$.)

Remark 2.1 There are equivalent (semi)norms that may be more convenient in certain problems. In particular, for $s \in (0, 1)$ and $p, q \in [1, \infty)$ (with the usual modifications for $p = \infty$ or $q = \infty$), the seminorm $(f)_{B^s_{p,q}(I)}$ is equivalent to

$$[f]_{B^s_{p,q}(I)} = \left(\sum_{j\ge 0} 2^{jsq} \|f(\cdot + 2^{-j}) - f(\cdot)\|^q_{L^p(I(2^{-j}))}\right)^{\frac{1}{q}}$$

by dyadic approximation; see, e.g., [14, Corollary 3.b.9]. Moreover, in the present article, particular attention will be given to the *exponential* Orlicz and Besov-Orlicz spaces $L^{\Phi_\beta}(D)$ and $B^s_{\Phi_\beta,\infty}(I)$ for $s \in (0, 1)$ where Φ_β, $\beta > 0$, is a Young function that satisfies $\Phi_\beta(x) =$

$e^{x^\beta} - 1$ for $x \in [\tau_\beta, \infty)$ with some $\tau_\beta \geq 0$. In this case, the (semi)norms $\|f\|_{L^{\Phi_\beta}(D)}$ and $(f)_{B^s_{\Phi_\beta,\infty}(I)}$ are equivalent to

$$|||f|||_{L^{\Phi_\beta}(D)} = \sup_{p\geq 1} p^{-1/\beta}\|f\|_{L^p(D)}$$

and

$$[f]_{B^s_{\Phi_\beta,\infty}(I)} = \sup_{j\geq 1} 2^{js}\left|\left|\left| f(\cdot + 2^{-j}) - f(\cdot)\right|\right|\right|_{L^{\Phi_\beta}(I(2^{-j}))},$$

respectively, by, e.g., [21, Proposition 2.3]. It follows that the norm $\|f\|_{B^s_{\Phi_\beta,\infty}(I)}$ is equivalent to

$$|||f|||_{B^s_{\Phi_\beta,\infty}(I)} = |||f|||_{L^{\Phi_\beta}(I)} + [f]_{B^s_{\Phi_\beta,\infty}(I)}.$$

Remark 2.2 There are, of course, some relations between Besov and Besov-Orlicz spaces. Let us briefly comment on the particular case of exponential Besov-Orlicz spaces as they are central to the present article. If $s \in (0,1)$, $\beta \in (0,\infty)$, and $p \in [1,\infty)$, then it is immediate from the definitions of these spaces that there is the embedding

$$B^s_{\Phi_\beta,q}(I) \subseteq B^s_{p,q}(I), \qquad q \in [1,\infty],$$

and, in particular,

$$B^s_{\Phi_\beta,\infty}(I) \subseteq B^s_{p,\infty}(I).$$

On the other hand, it follows from [27, Corollary 5.3] or Corollary 4.2 below that if $s \in (0,1)$, $\beta \in (0,2]$, and $p \in [1,\infty)$, then

$$B^s_{\Phi_\beta,\infty}(I) \nsubseteq B^s_{p,q}(I), \qquad q \in [1,\infty).$$

3 Main Results: Path Regularity

The main results of the article are collected in this section. Let H be a real separable Hilbert space, $W = \{W(h)\}_{h\in H}$ be an H-isonormal Gaussian process, i.e. a centered Gaussian process with the covariance

$$\mathbb{E}[W(h_1)W(h_2)] = \langle h_1, h_2\rangle_H, \qquad h_1, h_2 \in H,$$

defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and assume that the σ-algebra $\mathcal{F}$ is generated by process W. For $k \in \mathbb{N}_0$, we denote by

$$W_k : H^{\otimes k} \to L^2(\Omega)$$

the multiple divergence operator of order k defined via the duality

$$\mathbb{E}\langle D^k X, A\rangle_{H^{\otimes k}} = \mathbb{E}[XW_k(A)], \qquad A \in H^{\otimes k},$$

for all $X \in \mathbb{D}^{k,2}$ where D^k denotes the k^{th} Malliavin derivative and $\mathbb{D}^{k,2}$ its domain and where $H^{\otimes k}$ denotes the completion of the algebraic tensor power of the Hilbert space H. (If U is another real separable Hilbert space, the tensor product $U \otimes H$ is defined, throughout this paper, as the completion of the algebraic tensor product of U and H with respect to the inner product $\langle u_1 \otimes h_1, u_2 \otimes h_2\rangle_{U\otimes H} = \langle u_1, u_2\rangle_U \langle h_1, h_2\rangle_H$, $u_1, u_2 \in U$, $h_1, h_2 \in H$.) We refer to, e.g., [18] or [20] for details on Malliavin calculus and to [29] for details on tensor products of Hilbert spaces.

Let us fix $T > 0$, $n \in \mathbb{N}$, and $\alpha \in (0, 1)$ for the rest of this section. Let $G = \{G(t)\}_{t\in[0,T]}$ be a jointly measurable process in the n^{th} Wiener chaos of W represented by

$$G(t) = W_n(A_t), \qquad t \in [0, T], \tag{2}$$

where $A_t \in H^{\otimes n}$ is (the unique) symmetric tensor and $t \mapsto A_t$ is a Bochner measurable function from $[0, T]$ to $H^{\otimes n}$. We define

$$A_{x,s} = A_{x+s} - A_x$$

for $x, s \geq 0$, $x + s \leq T$, and

$$C_{s,t}(x, y) = s^{-2\alpha} t^{-2\alpha} \|A_{x,s} \otimes_1 A_{y,t}\|^2_{H^{\otimes(2n-2)}} + s^{-\alpha} t^{-\alpha} \sum_{j=2}^{n} \|A_{x,s} \otimes_j A_{y,t}\|_{H^{\otimes(2n-2j)}} \tag{3}$$

for $x, y \geq 0$, $s, t > 0$ that satisfy $x + s \leq T$, $y + t \leq T$. Here, $\otimes_j$, $j = 0, \ldots, n$, is the cancellation operator defined by

$$\begin{aligned}
(h_1 \otimes \cdots \otimes h_n) \otimes_0 (k_1 \otimes \cdots \otimes k_n) &= h_1 \otimes \cdots \otimes h_n \otimes k_1 \otimes \cdots \otimes k_n,\\
(h_1 \otimes \cdots \otimes h_n) \otimes_j (k_1 \otimes \cdots \otimes k_n) &= \langle h_1, k_1\rangle_H \ldots \langle h_j, k_j\rangle_H \, h_{j+1} \otimes \cdots \otimes h_n \otimes k_{j+1} \otimes \cdots \otimes k_n,\\
(h_1 \otimes \cdots \otimes h_n) \otimes_n (k_1 \otimes \cdots \otimes k_n) &= \langle h_1, k_1\rangle_H \ldots \langle h_n, k_n\rangle_H
\end{aligned}$$

for $h_i, k_i \in H$, $i = 1, \ldots, n$. We also define

$$F(s, t) = \int_0^{T-t} \int_0^{T-s} C_{s,t}(x, y) \mathrm{d}x \mathrm{d}y$$

for $s, t \in (0, T)$.

In the rest of the article, various combinations of assumptions (G1)–(G4) below are considered and these assumptions are formulated now.

Assumption 3.1 There exist $\kappa \in [1, \infty)$, $\kappa' \in (0, \infty)$, $\kappa'' \in [1, \infty)$, and $\varepsilon \in (0, 1)$ such that

(G1) $\|A_{x,s}\|_{H^{\otimes n}} \leq \kappa s^{\alpha}$ for every $x \geq 0$, $s > 0$, $x + s \leq T$,
(G2) $\liminf_{s\to 0} \inf_{x\in[0,T)} s^{-\alpha} \|A_{x,s}\|_{H^{\otimes n}} \geq \kappa'$,
(G3) $\sum_{j=j_0}^{\infty} F(2^{-j}, 2^{-j}) < \infty$ where $j_0 = \min\{j : 2^{-j} < T\}$,
(G4) $F(s, t) \leq \kappa'' s^{\varepsilon} t^{\varepsilon}$ for every $s, t \in (0, T)$.

Remark 3.2 Note that if process G satisfies condition (G1), then it has paths in the Besov space

$$B^r_{p,q}(0, T)$$

for every $p, q \in [1, \infty]$ and $r \in (0, \alpha)$. Indeed, by [18, Theorem 2.7.2] and Kolmogorov's continuity criterion, G has paths in the Hölder space $C^r([0, T])$ for every $r \in (0, \alpha)$ and this is equivalent to the claim by the embedding of Besov spaces from [26, Theorem 3.3.1 and Proposition 3.2.4].

Consider now Young functions Φ_β for $\beta > 0$ that satisfy $\Phi_\beta(x) = e^{x^\beta} - 1$ for $x \in [\tau_\beta, \infty)$ with some $\tau_\beta \geq 0$. The main result of the paper follows. Its proof is postponed to Section 6.1.

Theorem 3.3 *If process G satisfies (G1) and (G3), then*

$$\|G\|_{B^{\alpha}_{\Phi_{2/n},\infty}(0,T)} \in L^{\Phi_{2/n}}(\Omega)$$

and, in particular, it has paths in the Besov-Orlicz space $B^{\alpha}_{\Phi_{2/n},\infty}(0,T)$ almost surely. If, additionally, process G satisfies (G2), then its paths do not belong to the Besov space $B^{\alpha}_{p,q}(0,T)$ for any $p \in [1,\infty]$ and $q \in [1,\infty)$ almost surely.

Remark 3.4 By the results contained in [21, Section 2.4.1], exponential Besov-Orlicz spaces can be embedded into certain modular Hölder spaces. In particular, there is the embedding

$$B^{\alpha}_{\Phi_{2/n},\infty}(0,T) \subseteq C^{r^{\alpha}|\log r|^{\frac{n}{2}}}([0,T])$$

where $C^{r^{\alpha}|\log r|^{\frac{n}{2}}}([0,T])$ denotes the space of functions $g : [0,T] \to \mathbb{R}$ such that

$$|g(s) - g(t)| \le c\,|s-t|^{\alpha}|\log|s-t||^{\frac{n}{2}}, \qquad s,t \in [0,T],\ 0 < |s-t| < 1/2,$$

holds with some $c \in (0,\infty)$. Therefore, the paths of the process G from Theorem 3.3 lie in the space $C^{r^{\alpha}|\log r|^{\frac{n}{2}}}([0,T])$ almost surely. This can be also inferred directly from [28, Corollary 5.5].

Remark 3.5 Note that every α-self-similar process G of the form (2) that has stationary increments satisfies conditions (G1) and (G2). Indeed, in this case, we have

$$\|A_{x,s}\|^2_{H^{\otimes n}} = \frac{1}{n!}\,\mathbb{E}[G(x+s) - G(x)]^2 = \frac{1}{n!}\,\mathbb{E}[G(s) - G(0)]^2 = \frac{1}{n!}\,\mathbb{E}[G(1)]^2 s^{2\alpha}$$

for every $x \ge 0$, $s > 0$, $x + s \le T$ where stationarity of increments of G, the fact that $G(0) = 0$ holds almost surely, and self-similarity of G were used successively.

Remark 3.6 Note also that it follows from the proof of Theorem 3.3 that we are permitted to take a square of the first term in the definition of function $C_{s,t}(x,y)$ in formula (3). This allows to obtain better results for Gaussian processes. Of course, as there is the estimate

$$s^{-2\alpha}t^{-2\alpha}\|A_{x,s} \otimes_1 A_{y,t}\|^2_{H^{\otimes(2n-2)}} \le \kappa^2 s^{-\alpha}t^{-\alpha}\|A_{x,s} \otimes_1 A_{y,t}\|_{H^{\otimes(2n-2)}}$$

for $x, y \ge 0$ and $s,t > 0$ such that $x + s \le T$, $y + t \le T$ by (G1) and Lemma 5.4 below, we have

$$F(s,t) = \int_0^{T-t}\int_0^{T-s} C_{s,t}(x,y)\mathrm{d}x\mathrm{d}y \le \kappa^2 \int_0^{T-t}\int_0^{T-s} C'_{s,t}(x,y)\mathrm{d}x\mathrm{d}y = F'(s,t)$$

for $s,t \in (0,T)$ where

$$C'_{s,t}(x,y) = s^{-\alpha}t^{-\alpha}\sum_{j=1}^{n}\|A_{x,s} \otimes_j A_{y,t}\|_{H^{\otimes(2n-2j)}}$$

so that it is sufficient that condition (G3) is verified with F' in place of F.

Remark 3.7 Note moreover that there is a discrete version of condition (G3) in Theorem 3.3. In particular, Theorem 3.3 remains valid if condition (G3) is replaced by

(G3′) There exists a function $K : [0,T]^2 \to [0,\infty)$ such that

$$C_{s,s}(x,y) \le K(s,|x-y|)$$

holds for $x, y \geq 0, s > 0$ that satisfy $x+s \leq T$, $y+s \leq T$, $(x, x+s)\cap(y, y+s) = \emptyset$, and

$$\sum_{j=1}^{\infty} \delta_j^2 \sum_{\substack{m,m'=1 \\ m\neq m'}}^{J_{\delta_j}} K(\delta_j, |m-m'|\delta_j) < \infty$$

holds with

$$\delta_j = T2^{-j} \quad \text{and} \quad J_{\delta_j} = 2^j - 1.$$

The modified proof can be found in Section 6.2. In the Gaussian case, the above condition (G3$'$) is implied by the conditions in [27, Theorem 5.1] and, consequently, Theorem 3.3 provides an extension of that result in the scalar case to the general setting.

Clearly, to verify condition (G3) it suffices to find $\varepsilon \in (0, 1)$ and $\kappa'' > 0$ such that $F(s, s) \leq \kappa'' s^{2\varepsilon}$ holds for every $s \in (0, T)$. If this condition is slightly strengthened, we obtain the exact pathwise asymptotics of the integral increments of G. The proof is postponed to Section 6.3.

Notation 3.8 We write

$$f(x) \in \Theta(g(x)) \quad \text{as} \quad x \to a+ \qquad \text{if} \qquad 0 < \liminf_{x\to a+} \frac{f(x)}{g(x)} \leq \limsup_{x\to a+} \frac{f(x)}{g(x)} < \infty.$$

Theorem 3.9 *If process G satisfies (G1), (G2), and (G4), then, in addition to the assertions of Theorem 3.3 being true, we have that*

$$\|G(\cdot + s) - G(\cdot)\|_{L^p(0,T-s)} \in \Theta(s^\alpha) \quad as \quad s \to 0+ \quad a.s.$$

holds for every $p \in [1, \infty)$ and

$$\|G(\cdot + s) - G(\cdot)\|_{L^{\Phi_{2/n}}(0,T-s)} \in \Theta(s^\alpha) \quad as \quad s \to 0+ \quad a.s.$$

Remark 3.10 Although condition (G4) is by no means sharp, it suffices for the demonstration of the method in Theorem 3.9 and it covers all our examples. Therefore, only this criterion is given here for simplicity of the exposition.

Notation 3.11 Throughout the article, we write $A \lesssim B$ (and $A \gtrsim B$) whenever there exists a finite positive constant C such that $A \leq CB$ (and $A \geq CB$) whose precise value is not important. This constant can change from line to line. We also write $A \lesssim_K B$ (and $A \gtrsim_K B$) to indicate the dependence of constant C on a different quantity K. If both relations $A \lesssim B$ and $A \gtrsim B$ hold, then we simply write $A \eqsim B$.

4 Example: Fractionally Filtered Generalized Hermite Processes

In this section, the results are applied to a specific class of stochastic processes that contains some well-studied examples such as fractional Brownian motions, Rosenblatt, or, more generally, Hermite processes. Let $n \in \mathbb{N}$ and let β_1 and β_2 be real numbers such that

$$0 < \beta_1 + \frac{n}{2}(\beta_2 - 1) + 1 < 1 \qquad \text{and} \qquad 1 - \frac{1}{n} < \beta_2 < 1. \tag{4}$$

Let us set $H = L^2(\mathbb{R})$,

$$\alpha = \beta_1 + \frac{n}{2}(\beta_2 - 1) + 1,$$

and

$$A_t(x_1, \dots, x_n) = \int_{\mathbb{R}} k_t^{\beta_1}(u) \prod_{i=1}^{n} \phi^{\beta_2}(u - x_i) \mathrm{d}u, \qquad x_1, \dots, x_n \in \mathbb{R}, \quad t \in [0, T], \quad (5)$$

for the function $k_t^{\beta_1} : \mathbb{R} \to \mathbb{R}$ defined by

$$k_t^{\beta_1}(u) = \begin{cases} \mathbf{1}_{(0,t]}(u), & \beta_1 = 0, \\ \frac{1}{\beta_1}[(t-u)_+^{\beta_1} - (-u)_+^{\beta_1}], & \beta_1 \neq 0, \end{cases} \quad (6)$$

where $(u)_+^{\beta_1} = u^{\beta_1}$ if $u > 0$ and $(u)_+^{\beta_1} = 0$ otherwise, and for some measurable function $\phi^{\beta_2} : \mathbb{R} \to \mathbb{R}$ that satisfies the estimate

$$\int_{\mathbb{R}} |\phi^{\beta_2}(x)\phi^{\beta_2}(x+u)| \mathrm{d}x \lesssim u^{\beta_2 - 1}, \qquad u > 0. \quad (7)$$

In this situation, the following result is obtained.

Corollary 4.1 *Process G defined by* (2) *with kernel A defined by* (5) *for which assumptions* (4), (6)*, and* (7) *hold satisfies conditions (G1) and (G4), and, consequently, it holds that*

$$\|G\|_{B^{\alpha}_{\Phi_{2/n},\infty}(0,T)} \in L^{\Phi_{2/n}}(\Omega).$$

Proof We only treat the case of $\beta_1 \neq 0$; the case $\beta_1 = 0$ follows by similar arguments. Denote

$$K(u) = \int_{\mathbb{R}} \phi^{\beta_2}(x)\phi^{\beta_2}(x+u)\mathrm{d}x.$$

Then K is a symmetric locally bounded function on $\mathbb{R} \setminus \{0\}$ and with the notation introduced in Section 3, we have that

$$\|A_{x,s}\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^2} k_s^{\beta_1}(u) k_s^{\beta_1}(v) K(|u-v|)^n \mathrm{d}u\mathrm{d}v$$

holds for every $x \geq 0$ and $s > 0$ such that $x + s \leq T$. As there is the inequality

$$\int_{\mathbb{R}^2} |k_s^{\beta_1}(u)||k_s^{\beta_1}(v)||K(|u-v|)|^n \mathrm{d}u\mathrm{d}v \lesssim s^{2\alpha} \quad (8)$$

by assumptions (4), (6), and (7), it follows that process G is well-defined and satisfies condition (G1). In order to verify condition (G4) notice first that

$$|k_s^{\beta_1}(-v)| \eqsim \begin{cases} (s+v)^{\beta_1}, & v \in (-s, 0], \\ \begin{cases} s^{\beta_1}, & \beta_1 > 0, \\ v^{\beta_1}, & \beta_1 < 0, \end{cases} & v \in (0, s], \\ s v^{\beta_1 - 1}, & v \in [s, \infty), \end{cases}$$

where the approximation constants do not depend on either s or v. Now, denote

$$|x|^{\gamma_1,\gamma_2} = \mathbf{1}_{[|x|<1]}|x|^{\gamma_1} + \mathbf{1}_{[|x|\geq 1]}|x|^{\gamma_2}, \qquad x \in \mathbb{R} \setminus \{0\},$$

for $\gamma_1, \gamma_2 \in \mathbb{R}$. It follows from the above approximation that

$$\int_{\mathbb{R}} |k_s^{\beta_1}(-v)||u-v|^{0,n(\beta_2-1)} \mathrm{d}v \eqsim s^{1+\min\{\beta_1,0\}}|u|^{0,n(\beta_2-1)+\max\{\beta_1,0\}}$$

for $u \in \mathbb{R} \setminus \{0\}$, from which we obtain the estimate

$$Q(s,t) = \int_{\mathbb{R}^2} |k_t^{\beta_1}(u)||k_s^{\beta_1}(v)||u-v|^{0,n(\beta_2-1)}\mathrm{d}v\mathrm{d}u \lesssim s^{1+\min\{\beta_1,0\}}t^{1+\min\{\beta_1,0\}}. \tag{9}$$

for $s,t \in (0,T)$. We are in position to verify condition (G4) now. By Remark 3.6, it suffices to find $\varepsilon \in (0,1)$ such that

$$\int_0^{T-t}\int_0^{T-s}\sum_{j=1}^{n} \|A_{x,s}\otimes_j A_{y,t}\|_{L^2(\mathbb{R}^{2(n-j)})}\mathrm{d}x\mathrm{d}y \lesssim s^{\alpha+\varepsilon}t^{\alpha+\varepsilon}$$

for $s,t \in (0,T)$. We have that

$$\|A_{x,s}\otimes_j A_{y,t}\|^2_{L^2(\mathbb{R}^{2n-2j})} = \int_{\mathbb{R}^4} k_s^{\beta_1}(r_1)k_t^{\beta_1}(r_2)k_s^{\beta_1}(r_3)k_t^{\beta_1}(r_4)$$
$$\cdot K(|r_1-r_3|)^{n-j}K(|r_2-r_4|)^{n-j}K(|r_1-r_2+x-y|)^jK(|r_3-r_4+x-y|)^j\mathrm{d}r$$

and, consequently, that

$$\begin{aligned}
&\|A_{x,s}\otimes_j A_{y,t}\|^2_{L^2(\mathbb{R}^{2n-2j})}\\
&\le \int_{\mathbb{R}^4}\left(|k_s^{\beta_1}(r_1)k_t^{\beta_1}(r_2)k_s^{\beta_1}(r_3)k_t^{\beta_1}(r_4)||K(|r_1-r_3|)|^n|K(|r_2-r_4|)|^n\right)^{1-\frac{j}{n}}\\
&\quad\cdot\left(|k_s^{\beta_1}(r_1)k_t^{\beta_1}(r_2)k_s^{\beta_1}(r_3)k_t^{\beta_1}(r_4)||K(|r_1-r_2+x-y|)|^n|K(|r_3-r_4+x-y|)|^n\right)^{\frac{j}{n}}\mathrm{d}r\\
&\le \left(\int_{\mathbb{R}^4}|k_s^{\beta_1}(r_1)k_t^{\beta_1}(r_2)k_s^{\beta_1}(r_3)k_t^{\beta_1}(r_4)||K(|r_1-r_3|)|^n|K(|r_2-r_4|)|^n\mathrm{d}r\right)^{1-\frac{j}{n}}\\
&\quad\cdot\left(\int_{\mathbb{R}^4}|k_s^{\beta_1}(r_1)k_t^{\beta_1}(r_2)k_s^{\beta_1}(r_3)k_t^{\beta_1}(r_4)||K(|r_1-r_2+x-y|)|^n|K(|r_3-r_4+x-y|)|^n\mathrm{d}r\right)^{\frac{j}{n}}\\
&= \left(\int_{\mathbb{R}^2}|k_s^{\beta_1}(u)||k_s^{\beta_1}(v)||K(|u-v|)|^n\mathrm{d}u\mathrm{d}v\right)^{1-\frac{j}{n}}\left(\int_{\mathbb{R}^2}|k_t^{\beta_1}(u)||k_t^{\beta_1}(v)||K(|u-v|)|^n\mathrm{d}u\mathrm{d}v\right)^{1-\frac{j}{n}}\\
&\quad\cdot\left(\int_{\mathbb{R}^2}|k_s^{\beta_1}(u)||k_t^{\beta_1}(v)||K(|u-v+x-y|)|^n\mathrm{d}u\mathrm{d}v\right)^{\frac{2j}{n}}
\end{aligned}$$

hold for every $x, y \ge 0$ and $s, t > 0$ such that $x+s \le T$, $y+t \le T$, and $j \in \{1,\dots,n\}$ by using Hölder's inequality to obtain the second estimate. Hence, the successive use of the above estimate, inequality (8), Jensen's inequality, assumption (7), and estimate (9) yields

$$\begin{aligned}
&\int_0^{T-t}\int_0^{T-s}\|A_{x,s}\otimes_j A_{y,t}\|_{L^2(\mathbb{R}^{2n-2j})}\mathrm{d}x\mathrm{d}y\\
&\qquad\lesssim s^{\alpha\left(1-\frac{j}{n}\right)}t^{\alpha\left(1-\frac{j}{n}\right)}\left(\int_0^T\int_0^T\int_{\mathbb{R}^2}|k_s^{\beta_1}(u)||k_t^{\beta_1}(v)||K(|u-v+x-y|)|^n\mathrm{d}u\mathrm{d}v\mathrm{d}x\mathrm{d}y\right)^{\frac{j}{n}}\\
&\qquad\lesssim s^{\alpha\left(1-\frac{j}{n}\right)}t^{\alpha\left(1-\frac{j}{n}\right)}[Q(s,t)]^{\frac{j}{n}}\\
&\qquad\lesssim s^{\alpha+[1+\min\{\beta_1,0\}-\alpha]\frac{j}{n}}t^{\alpha+[1+\min\{\beta_1,0\}-\alpha]\frac{j}{n}}
\end{aligned}$$

for $s,t \in (0,T)$ and $j \in \{1,\dots,n\}$. As $1+\min\{\beta_1,0\}-\alpha > 0$ by (4), condition (G4) is verified and the claim of the corollary is obtained by appealing to Theorem 3.3. □

The following family of stochastic processes, called the fractionally filtered Hermite process, provides prototypical examples to which our results apply. In particular, let $z_n^{\beta_1,\beta_2} = \{z_n^{\beta_1,\beta_2}(t)\}_{t\in[0,1]}$ be the stochastic process defined by

$$z_n^{\beta_1,\beta_2}(t) = c_{n,\beta_1,\beta_2} \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}} k_t^{\beta_1}(u) \prod_{i=1}^n (u - x_i)_+^{\frac{\beta_2}{2}-1} \mathrm{d}u \right] \mathrm{d}W_x, \qquad t \in [0, 1],$$

where $\int_{\mathbb{R}^n}(\cdots)\mathrm{d}W_x$ is the multiple Wiener-Itô integral with respect to the Wiener process $\{W_t\}_{t\in\mathbb{R}}$ and c_{n,β_1,β_2} is a suitable normalizing constant. We remark that

- $z_1^{\beta_1,\beta_2}$ is a fractional Brownian motion with Hurst parameter $\alpha = \beta_1 + \frac{\beta_2}{2} + \frac{1}{2} \in (0, 1)$,
- z_2^{0,β_2} is a Rosenblatt process with Hurst parameter $\alpha = \beta_2 \in (\frac{1}{2}, 1)$,
- and z_n^{0,β_2} is a Hermite process of order n and Hurst parameter $\alpha = \frac{n}{2}(\beta_2 - 1) + 1 \in (\frac{1}{2}, 1)$

upon suitable choices of the constant c_{n,β_1,β_2}; see, e.g., [3] or [16].

Corollary 4.2 *The fractionally filtered Hermite process $z_n^{\beta_1,\beta_2}$ satisfies*

$$\|z_n^{\beta_1,\beta_2}\|_{B^\alpha_{\Phi_{2/n},\infty}(0,1)} \in L^{\Phi_{2/n}}(\Omega),$$

so that it has sample paths in the Besov-Orlicz space $B^\alpha_{\Phi_{2/n},\infty}(0, 1)$ almost surely, but it does not have sample paths in the Besov space $B^\alpha_{p,q}(0, 1)$ with $p \in [1, \infty]$ and $q \in [1, \infty)$ almost surely. Moreover, we have that

$$\|z_n^{\beta_1,\beta_2}(\cdot + s) - z_n^{\beta_1,\beta_2}(s)\|_{L^p(0,1-s)} \in \Theta(s^\alpha) \quad \textit{as} \quad s \to 0+$$

holds for every $p \in [1, \infty)$ almost surely and

$$\|z_n^{\beta_1,\beta_2}(\cdot + s) - z_n^{\beta_1,\beta_2}(s)\|_{L^{\Phi_{2/n}}(0,1-s)} \in \Theta(s^\alpha) \quad \textit{as} \quad s \to 0+$$

holds almost surely.

Proof It follows from [3, Theorem 3.27] that $z_n^{\beta_1,\beta_2}$ is a well-defined α-self-similar process with stationary increments. Thus, conditions (G1) and (G2) are verified by appealing to Remark 3.5. Condition (G4) holds by Corollary 4.1 upon verifying (7) with

$$\phi^{\beta_2}(x) = c_{n,\beta_1,\beta_2}^{\frac{1}{n}}(x)_+^{\frac{\beta_2}{2}-1}, \qquad x \in \mathbb{R}.$$

Subsequently, the claim follows by Theorem 3.9. □

Remark 4.3 The fact that sample paths of the fractional Brownian motion $z_1^{\beta_1,\beta_2}$ belong to the Besov-Orlicz space $B^{\beta_1+\beta_2/2+1/2}_{\Phi_2,\infty}(0, 1)$ and do not belong to the Besov space $B^{\beta_1+\beta_2/2+1/2}_{p,q}(0, 1)$ for any $p \in [1, \infty]$ and $q \in [1, \infty)$ almost surely has already been established in [27, Corollary 5.3]. Moreover, it is shown in [27, Corollary 5.8] that

$$\mathbb{E}\|z_1^{\beta_1,\beta_2}\|_{B^{\beta_1+\beta_2/2+1/2}_{\Phi_2,\infty}(0,1)} \eqsim \mathbb{E}|z_1^{\beta_1,\beta_2}(1)|.$$

The above Corollary 4.2 complements these results by providing an upper bound on the asymptotic behavior of the moments of the Besov-Orlicz norm and the almost sure asymptotic behavior of the integral modulus of continuity of the fractional Brownian motion. As far as Rosenblatt processes, other higher-order Hermite processes, and, more generally, sub-n^{th}-Gaussian chaos processes are concerned, there also exist results on their finer path properties. In particular, an almost sure estimate of the supremal modulus of continuity of the sample paths of sub-n^{th}-Gaussian chaos fields is established in [28, Corollary 5.5] and almost sure upper and lower bounds on the asymptotic behavior of local oscillations of Hermite processes are provided in [1, formulae (1.7) and (1.8)]. The lower bounds then allow to obtain almost sure non-differentiability of the paths of Hermite processes; see [1, Theorem 1.1] and [13, Corollary 1.7]. In contrast, it is shown in Corollary 4.2 that the sample paths of the Hermite process z_n^{0,β_2} belong to the exponential Besov-Orlicz space $B^{n(\beta_2-1)/2+1}_{\Phi_{2/n},\infty}(0,1)$ almost surely. The supremal modulus of continuity of these processes, known already from [28, Corollary 5.5], can be recovered as a consequence of this fact - we obtain that z_n^{0,β_2} has paths in the modular Hölder space $C^{|r|^{n(\beta_2-1)/2+1}|\log r|^{n/2}}([0,1])$ almost surely by Remark 3.4 - but additionally, the result is improved in the sense that Corollary 4.2 provides a smaller space to which Hermite paths belong. Moreover, we obtain an upper bound on the asymptotic behavior of the moments of the Besov-Orlicz norm and the asymptotic behavior of the integral modulus of continuity of the Hermite process.

Remark 4.4 In [3], the following family of processes is considered. Define the kernel A by

$$A_t(x_1,\ldots,x_n) = \int_{x_1\vee\ldots\vee x_n}^{\infty} k_t^{\gamma_1}(u)\phi^{\gamma_2}(u-x_1,\ldots,u-x_n)\mathrm{d}u, \quad x_1,\ldots,x_n\in\mathbb{R}, \quad t\in[0,T], \tag{10}$$

where $k_t^{\gamma_1}$ is the function defined by formula (6) and where $\phi^{\gamma_2}:\mathbb{R}_+^n\to\mathbb{R}$ is a non-zero function for which there exists $\gamma_2\in(-\frac{n+1}{2},-\frac{n}{2})$ such that

$$\phi^{\gamma_2}(\lambda x) = \lambda^{\gamma_2}\phi^{\gamma_2}(x)$$

holds for every $x\in\mathbb{R}_+^n$ and every $\lambda>0$ and such that

$$\int_{\mathbb{R}_+^n} |\phi^{\gamma_2}(x)\phi^{\gamma_2}(1+x)|\mathrm{d}x < \infty.$$

The corresponding process G is a well-defined γ-self-similar process with stationary increments (here $\gamma=\gamma_1+\gamma_2+\frac{n}{2}+1$) provided that $-1<-\gamma_2-\frac{n}{2}-1<\gamma_1<-\gamma_2-\frac{n}{2}<\frac{1}{2}$; see [3, Theorem 3.27]. This family of stochastic processes generalizes the one treated above; however, without assuming the product structure in (10) as in (5), it remains unclear whether our results can be applied in this case.

5 Tensor Calculus: Cancellations of Tensors and Expansion Formula

In this section, we review elements of tensor calculus and an explicit formula for Wiener chaos expansion of products of random variables. We adopt the following

Convention 5.1 When working in the field of integers, $[a,b]$ shall denote the set $\{i\in\mathbb{Z}: a\le i \text{ and } i\le b\}$. When working in the field of reals, $[a,b]$ shall denote the set $\{t\in\mathbb{R}: a\le t \text{ and } t\le b\}$.

5.1 Cancellation of Tensors

5.1.1 Motivating Example

Consider a product of Hilbert spaces

$$H^{\otimes 5} \times H^{\otimes 5} \times H^{\otimes 3} \times H^{\otimes 1} \times H^{\otimes 4},$$

define sets of indices corresponding to the coordinates of this space

$$I_1 = \{1, 2, 3, 4, 5\}, \quad I_2 = \{6, 7, 8, 9, 10\}, \quad I_3 = \{11, 12, 13\}, \quad I_4 = \{14\}, \quad I_5 = \{15, 16, 17, 18\},$$

and consider a set V of unordered pairs of indices that obey the following rules:

(1) Every index is in one pair at most.
(2) No index is paired with an index from the same set nor with itself.

For example, one can consider

$$V = \{\{3, 6\}, \{5, 10\}, \{9, 11\}, \{13, 14\}\}.$$

The example can be graphically visualized as in Fig. 1.

For the element

$$(h_1 \otimes h_2 \otimes h_3 \otimes h_4 \otimes h_5) \times (h_6 \otimes h_7 \otimes h_8 \otimes h_9 \otimes h_{10}) \times (h_{11} \otimes h_{12} \otimes h_{13}) \times h_{14} \times (h_{15} \otimes h_{16} \otimes h_{17} \otimes h_{18}),$$

the (V-)cancellation is done as follows: The vectors with indices forming a pair in V are multiplied and the remaining vectors are shrunken, i.e.

$$\underbrace{h_1 \otimes h_2 \otimes h_3 \otimes h_4 \otimes h_5}_{I_1} \times \underbrace{h_6 \otimes h_7 \otimes h_8 \otimes h_9 \otimes h_{10}}_{I_2} \times \underbrace{h_{11} \otimes h_{12} \otimes h_{13}}_{I_3} \times \underbrace{h_{14}}_{I_4} \times \underbrace{h_{15} \otimes h_{16} \otimes h_{17} \otimes h_{18}}_{I_5}$$

which results in

$$\langle h_3, h_6 \rangle_H \langle h_5, h_{10} \rangle_H \langle h_9, h_{11} \rangle_H \langle h_{13}, h_{14} \rangle_H h_1 \otimes h_2 \otimes h_4 \otimes h_7 \otimes h_8 \otimes h_{12} \otimes h_{15} \otimes h_{16} \otimes h_{17} \otimes h_{18}.$$

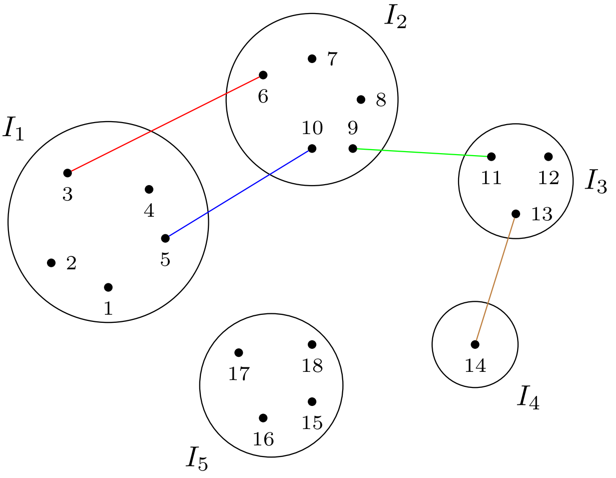

Fig. 1 Example of tensor cancellation

The cancellation extends to a 5-linear operator from $H^{\otimes 5} \times H^{\otimes 5} \times H^{\otimes 3} \times H^{\otimes 1} \times H^{\otimes 4}$ to $H^{\otimes 10}$; see also Corollary 5.5 below.

5.1.2 General Case

Consider an integer $\ell \geq 2$, positive integers $d_1, \dots, d_\ell$ and decompose $\{1, \dots, N\}$ to subsequent intervals $I_1, \dots, I_\ell$ of lengths $d_1, \dots, d_\ell$, respectively, where $N = d_1 + \dots + d_\ell$.

Definition 5.2 A set V of unordered pairs of numbers in $\{1, \dots, N\}$ is said to be *a set of admissible pairs* for intervals $I_1, \dots, I_\ell$ if it is either empty, or if it can be enumerated as

$$V = \{\{m_1, n_1\}, \dots, \{m_k, n_k\}\} \tag{11}$$

for some integer $k \geq 1$ where

(F1) $m_1, \dots, m_k, n_1, \dots, n_k$ are all distinct, i.e. $|\{m_1, \dots, m_k, n_1, \dots, n_k\}| = 2k$, and
(F2) m_j and n_j do not belong to the same interval $I_1, \dots, I_\ell$ for every $j \in \{1, \dots, k\}$.

For a set V of admissible pairs for intervals $I_1, \dots, I_\ell$ with $|V| = k$, we define $V_* = \{1, \dots, N\}$ if $V = \emptyset$ or, if $V \neq \emptyset$,

$$V_* = \{1, \dots, N\} \setminus \{m_1, \dots, m_k, n_1, \dots, n_k\}. \tag{12}$$

If $2k < N$, we enumerate V_* in an increasing order as

$$\{o_1, \dots, o_{N-2k}\}. \tag{13}$$

The cancellation operator is then defined for $h_1, \dots, h_N \in H$ as follows: If $V = \emptyset$, then

$$R^V(\otimes_{i_1 \in I_1} h_{i_1}, \dots, \otimes_{i_\ell \in I_\ell} h_{i_\ell}) = h_{o_1} \otimes \dots \otimes h_{o_N} = h_1 \otimes \dots \otimes h_N,$$

if $0 < 2k < N$, then

$$R^V(\otimes_{i_1 \in I_1} h_{i_1}, \dots, \otimes_{i_\ell \in I_\ell} h_{i_\ell}) = \prod_{j=1}^{k} \langle h_{m_j}, h_{n_j} \rangle_H h_{o_1} \otimes \dots \otimes h_{o_{N-2k}},$$

and if $2k = N$, then

$$R^V(\otimes_{i_1 \in I_1} h_{i_1}, \dots, \otimes_{i_\ell \in I_\ell} h_{i_\ell}) = \prod_{j=1}^{k} \langle h_{m_j}, h_{n_j} \rangle_H.$$

Due to symmetry, the definition of R^V is independent of the enumeration of the set V and R^V extends to a unique ℓ-linear operator

$$R^V : H^{\otimes d_1} \times \dots \times H^{\otimes d_\ell} \to H^{\otimes (N-2|V|)};$$

see also Corollary 5.5 below.

5.2 Permutations

It is often convenient to use a relation between cancellations and permutations. Under the assumptions made in Section 5.1.2, let us assume additionally that π_j is a permutation on $\{1, \dots, d_j\}$ for every $j \in \{1, \dots, \ell\}$ and define

$$\pi(i) = s_j + \pi_j(i - s_j), \qquad i \in I_j, \quad j \in \{1, \dots, \ell\},$$

where $I_j = s_j + \{1, \dots, d_j\}$ for $j \in \{1, \dots, \ell\}$. Then π is a permutation on $\{1, \dots, N\}$ such that $\pi|_{I_j}$ is a permutation on I_j for every $j \in \{1, \dots, \ell\}$. We define $V^\pi = \emptyset$ if $V = \emptyset$ and

$$V^\pi = \{\{\pi(m_1), \pi(n_1)\}, \dots, \{\pi(m_k), \pi(n_k)\}\}$$

otherwise, with the notation from (11). It is clear that V^π is admissible for the intervals $I_1, \dots, I_\ell$. If $2k < N$, let σ be the permutation on $\{1, \dots, N-2k\}$ such that $\pi \circ o \circ \sigma$ is increasing. Then

$$R^{V^\pi}(A_1, \dots, A_\ell) = P_\sigma R^V(P_{\pi_1} A_1, \dots, P_{\pi_\ell} A_\ell), \qquad A_1 \in H^{\otimes d_1}, \dots, A_\ell \in H^{\otimes d_\ell}, \tag{14}$$

where the permutation operator $P_\theta : H^{\otimes n} \to H^{\otimes n}$ is defined in a standard manner by

$$P_\theta(h_1 \otimes \cdots \otimes h_n) = h_{\theta_1} \otimes \cdots \otimes h_{\theta_n} \quad \text{if} \quad n \in \mathbb{N}$$

and by

$$P_\emptyset(t) = t \quad \text{if} \quad n = 0.$$

If $2k = N$, then (14) still holds with $\sigma = \emptyset$ and $P_\emptyset = 1$.

5.3 Composition of Cancellations

5.3.1 Motivating Example - Continued

Let us start again by a sequel of the previous example in Section 5.1.1 after cancellation. We renumerate the remaining indices in the ascending order and we add a new interval $I_6 = \{11, 12, 13\}$ with two more pairs $V' = \{\{2, 11\}, \{9, 13\}\}$ obeying rules (F1) and (F2). This situation is now depicted in Fig. 2.

In the original picture (with the original enumeration), the new indices are enumerated as $I_6 = \{19, 20, 21\}$ and the new pairs correspond to the set $V'' = \{\{2, 19\}, \{17, 21\}\}$ as shown in Fig. 3.

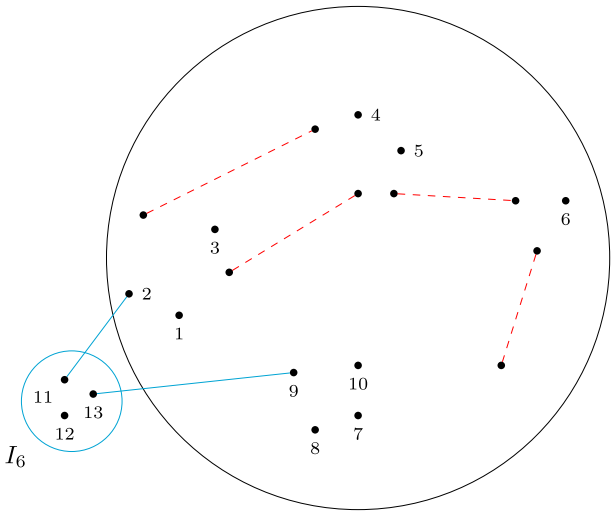

Fig. 2 Example of tensor cancellation with composition - new enumeration

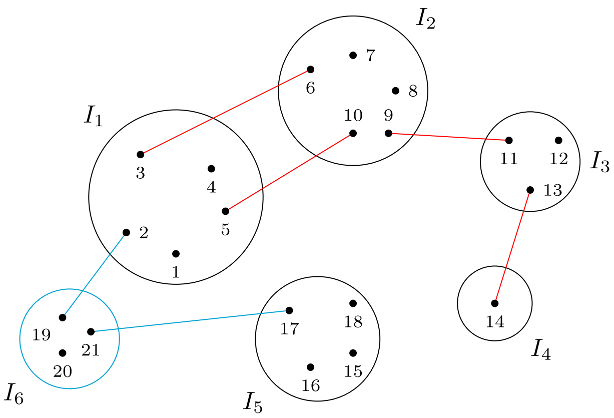

Fig. 3 Example of tensor cancellation with composition - original enumeration

The extended cancellation $R^{V\cup V''}: H^{\otimes 5}\times H^{\otimes 5}\times H^{\otimes 3}\times H^{\otimes 1}\times H^{\otimes 4}\times H^{\otimes 3}\to H^{\otimes 9}$ then satisfies

$$R^{V\cup V''}(A_1,A_2,A_3,A_4,A_5,A_6)=R^{V'}(R^V(A_1,A_2,A_3,A_4,A_5),A_6)$$

as

$$\underbrace{h_1\otimes h_2\otimes h_3\otimes h_4\otimes h_5}_{I_1}\times\underbrace{h_6\otimes h_7\otimes h_8\otimes h_9\otimes h_{10}}_{I_2}\times\underbrace{h_{11}\otimes h_{12}\otimes h_{13}}_{I_3}\times\underbrace{h_{14}}_{I_4}\times\underbrace{h_{15}\otimes h_{16}\otimes h_{17}\otimes h_{18}}_{I_5}\times\underbrace{h_{19}\otimes h_{20}\otimes h_{21}}_{I_6}$$

results in

$$\langle h_3,h_6\rangle_H\langle h_5,h_{10}\rangle_H\langle h_9,h_{11}\rangle_H\langle h_{13},h_{14}\rangle_H\langle h_2,h_{19}\rangle_H\langle h_{17},h_{21}\rangle_H$$
$$h_1\otimes h_4\otimes h_7\otimes h_8\otimes h_{12}\otimes h_{15}\otimes h_{16}\otimes h_{18}\otimes h_{20}.$$

5.3.2 General Case

Under the assumptions in Section 5.1.2, assume additionally that $2k<N$, consider two intervals

$$I_1'=\{1,\ldots,N-2k\},\quad I_2'=\{N-2k+1,\ldots,N-2k+d_{\ell+1}\},$$

and consider the interval $I_{\ell+1}=\{N+1,\ldots,N+d_{\ell+1}\}$ which completes $I_1,\ldots,I_\ell$ to a subsequent decomposition of $\{1,\ldots,N'\}$ where $N'=N+d_{\ell+1}=d_1+\cdots+d_{\ell+1}$. Consider a set of admissible pairs V' for intervals I_1' and I_2'. If $V'=\emptyset$, we set $V''=\emptyset$. If V' is non-empty, we enumerate it

$$V'=\{\{m_1',n_1'\},\ldots,\{m_{k'}',n_{k'}'\}\}$$

in such a way that $\{m_1',\ldots,m_{k'}'\}\subseteq I_1'$ and $\{n_1',\ldots,n_{k'}'\}\subseteq I_2'$, and we define

$$m_{k+1}=o_{m_1'},\ldots,m_{k+k'}=o_{m_{k'}'},\qquad n_{k+1}=n_1'+2k,\ldots,n_{k+k'}=n_{k'}'+2k$$

and

$$V''=\{\{m_{k+1},n_{k+1}\},\ldots,\{m_{k+k'},n_{k+k'}\}\}.$$

Lemma 5.3 *The set of pairs $V \cup V''$ is admissible for the intervals $I_1, \dots, I_{\ell+1}$ and*

$$R^{V\cup V''}(A_1, \dots, A_{\ell+1}) = R^{V'}(R^V(A_1, \dots, A_\ell), A_{\ell+1})$$

holds for every $A_1 \in H^{\otimes d_1}, \dots, A_{\ell+1} \in H^{\otimes d_{\ell+1}}$.

Proof If $V' = \emptyset$, then the assertion is trivial. If $V' \neq \emptyset$ but $V = \emptyset$, then $I_1' = \{1, \dots, N\}$, $I_2' = I_{\ell+1}$, $V' = V''$ so the assertion is obvious. Now assume that $V \neq \emptyset$ and $V' \neq \emptyset$. The admissibility of the pairs $V \cup V''$ is rather straightforward since it is a mere renumeration and, as for the identity, it suffices to show that it holds for elementary tensors, i.e. that

$$\begin{aligned} R^{V'}(h_{o_1} \otimes \cdots \otimes h_{o_{N-2k}}, h_{N+1} \otimes \cdots \otimes h_{N+d_{\ell+1}}) \\ = \langle h_{m_{k+1}}, h_{n_{k+1}}\rangle \dots \langle h_{m_{k+k'}}, h_{n_{k+k'}}\rangle h_{o_1''} \otimes \cdots \otimes h_{o''_{N'-2k-2k'}} \end{aligned} \tag{15}$$

where o'' is the increasing enumeration of $\{1, \dots, N'\} \setminus \{m_1, \dots, m_{k+k'}, n_1, \dots, n_{k+k'}\}$. If we define

$$h_1' = h_{o_1}, \dots, h'_{N-2k} = h_{o_{N-2k}}, h'_{N-2k+1} = h_{N+1}, \dots, h'_{N-2k+d_{\ell+1}} = h_{N+d_{\ell+1}},$$

then the left-hand-side of (15) is equal to

$$\langle h'_{m_1'}, h'_{n_1'}\rangle \dots \langle h'_{m_{k'}'}, h'_{n_{k'}'}\rangle h'_{o_1'} \otimes \cdots \otimes h'_{o'_{N'-2k-2k'}} \tag{16}$$

where o' is the increasing enumeration of $\{1, \dots, N' - 2k\} \setminus \{m_1', \dots, m_{k'}', n_1', \dots, n_{k'}'\}$. Since $m_j' \le N - 2k < n_j'$, we have $h'_{m_j'} = h_{o_{m_j'}} = h_{m_{k+j}}$ and $h'_{n_j'} = h_{n_j'+2k} = h_{n_{k+j}}$. Let us also define

$$L = (o_1, \dots, o_{N-2k}, N+1, \dots, N + d_{\ell+1}).$$

Then

$$L : \{1, \dots, N' - 2k\} \to \{1, \dots, N'\} \setminus \{m_1, \dots, m_k, n_1, \dots n_k\}$$

is an increasing bijection and it can be checked that

$$\{m_{k+1}, \dots, m_{k+k'}, n_{k+1}, \dots, n_{k+k'}\} \cap \operatorname{Rng}(L(o')) = \emptyset.$$

Thus it is seen that

$$L(o') : \{1, \dots, N' - 2k - 2k'\} \to \{1, \dots, N'\} \setminus \{m_1, \dots, m_{k+k'}, n_1, \dots n_{k+k'}\}$$

is an increasing bijection. But such bijection is exactly one, hence $o'' = L(o')$. Now, $h_i' = h_{L(i)}$ for $i \in \{1, \dots, N' - 2k\}$ so $h'_{o_j'} = h_{L(o_j')} = h_{o_j''}$ for every $j \in \{1, \dots, N' - 2k - 2k'\}$ which proves that (16) coincides with the right-hand side of (15). □

We will prove the following two results using the composition Lemma 5.3.

Lemma 5.4 *If $A_1 \in H^{\otimes d_1}, \dots, A_\ell \in H^{\otimes d_\ell}$, then*

$$\|R^V(A_1, \dots, A_\ell)\|_{H^{\otimes(N-2|V|)}} \le \|A_1\|_{H^{\otimes d_1}} \dots \|A_\ell\|_{H^{\otimes d_\ell}}.$$

Proof Let us proceed by induction on $\ell \ge 2$. Let U be an admissible set of pairs for intervals $I_1, \dots, I_{\ell+1}$ and define

$$V = \{\{m, n\} \in U : m \notin I_{\ell+1} \text{ and } n \notin I_{\ell+1}\}, \qquad V'' = U \setminus V,$$

with cardinalities $|V| = k$ and $|V''| = k'$, respectively. We distinguish two cases. First, if $V'' = \emptyset$, then define $V' = \emptyset$. If $V'' \neq \emptyset$, then necessarily $2k < N$ where $N = d_1 + \dots + d_\ell$ and $N' = N + d_{\ell+1}$, so we can enumerate

$$V'' = \{\{m_{k+1}, n_{k+1}\}, \dots, \{m_{k+k'}, n_{k+k'}\}\}$$

such that $m_{k+j} \le N < n_{k+j}$ for every $j \in [1, k']$. Since U is admissible, $m_{k+1}, \dots, m_{k+k'}$ belong to V_*, hence we can define $m'_j = o^{-1}_{m_{k+j}} \in [1, N-2k] = I'_1$ and $n'_j = n_{k+j} - 2k \in [N-2k+1, N-2k+d_{\ell+1}] = I'_2$ for $j \in [1, k']$, and $V' = \{\{m'_j, n'_j\}, j \in [1, k']\}$. After this construction, Lemma 5.3 is applicable. If $V'' = \emptyset$, then $U = V$ and

$$R^U(A_1, \dots, A_{\ell+1}) = R^V(A_1, \dots, A_\ell) \otimes A_{\ell+1}$$

holds if $2k < N$ or

$$R^U(A_1, \dots, A_{\ell+1}) = R^V(A_1, \dots, A_\ell) A_{\ell+1}$$

holds if $2k = N$. If $V'' \neq \emptyset$, then

$$R^U(A_1, \dots, A_{\ell+1}) = R^{V'}(R^V(A_1, \dots, A_\ell), A_{\ell+1}),$$

so the claim follows by the induction step. Let therefore $\ell = 2$. Since $R^{\emptyset}(A_1, A_2) = A_1 \otimes A_2$, it suffices to assume that $V \neq \emptyset$. Express

$$A_1 = \sum_{i_1,\dots,i_{d_1}} \alpha^1_{i_1,\dots,i_{d_1}} e_{i_1} \otimes \dots \otimes e_{i_{d_1}}, \qquad A_2 = \sum_{i_{d_1+1},\dots,i_{d_1+d_2}} \alpha^2_{i_{d_1+1},\dots,i_{d_1+d_2}} e_{i_{d_1+1}} \otimes \dots \otimes e_{i_{d_1+d_2}}$$

for some orthonormal system $\{e_\beta\}$ in H and let $V = \{\{j, d_1 + j\} : j \in \{1, \dots, k\}\}$ for some positive integer $k \le \min\{d_1, d_2\}$. Then, if $k < \min\{d_1, d_2\}$, we have that

$$R^V(A_1, A_2) = \sum_{i,j,l} \alpha^1_{i,j} \alpha^2_{i,l} e_{j_1} \otimes \dots \otimes e_{j_{d_1-k}} \otimes e_{l_1} \otimes \dots \otimes e_{l_{d_2-k}}$$

so the estimate

$$\|R^V(A_1, A_2)\|^2_{H^{\otimes(d_1+d_2-2k)}} = \sum_{j,l} \left(\sum_i \alpha^1_{i,j} \alpha^2_{i,l} \right)^2 \le \|A_1\|^2_{H^{\otimes d_1}} \|A_2\|^2_{H^{\otimes d_2}} \tag{17}$$

follows by the Cauchy-Schwarz inequality. If $k = \min\{d_1, d_2\}$, the same estimation is obtained analogously. In the general case, the result is obtaind from (14) and (17) by reordering the set V via a suitable permutation. □

Corollary 5.5 *The operator R^V extends to a continuous ℓ-linear operator*

$$R^V : H^{\otimes d_1} \times \dots \times H^{\otimes d_\ell} \to H^{\otimes(N-2|V|)}.$$

Now we are going to study the behaviour of the operator R further. Recall the numbers $s_1, \dots, s_\ell$ from Section 5.2 that were defined in such a way that

$$I_1 = s_1 + \{1, \dots, d_1\}, \dots, I_\ell = s_\ell + \{1, \dots, d_\ell\},$$

holds, consider the set V_* from (12), and define

$$V_*^{(1)} = (V_* \cap I_1) - s_1, \dots V_*^{(\ell)} = (V_* \cap I_\ell) - s_\ell.$$

In this way, $V_*^{(j)} \subseteq \{1, \dots, d_j\}$ for every $j \in \{1, \dots, \ell\}$ and these sets are actually the traces of V_* on I_j, but renumbered such that each interval begins with 1. Finally, define

$$V_j = \{\{i, d_j + i\} : i \in V_*^{(j)}\}, \qquad j \in \{1, \dots, \ell\}.$$

Lemma 5.6 *If* $A_1, B_1 \in H^{\otimes d_1}, \dots, A_\ell, B_\ell \in H^{\otimes d_\ell}$, *then*

$$|\langle R^V(A_1,\dots,A_\ell), R^V(B_1,\dots,B_\ell)\rangle_{H^{\otimes(N-2|V|)}}| \le \|R^{V_1}(A_1,B_1)\|_{H^{\otimes(N-2|V_1|)}} \dots \|R^{V_\ell}(A_\ell,B_\ell)\|_{H^{\otimes(N-2|V_\ell|)}}.$$

Proof If $V=\emptyset$, then

$$\begin{aligned}\langle R^V(A_1,\dots,A_\ell), R^V(B_1,\dots,B_\ell)\rangle_{H^{\otimes(N-2|V|)}} &= \langle A_1\otimes\cdots\otimes A_\ell, B_1\otimes\cdots\otimes B_\ell\rangle_{H^{\otimes(N-2|V|)}}\\ &= \langle A_1,B_1\rangle_{H^{\otimes d_1}}\dots\langle A_\ell,B_\ell\rangle_{H^{\otimes d_\ell}}\\ &= R^{V_1}(A_1,B_1)\dots R^{V_\ell}(A_\ell,B_\ell).\end{aligned}$$

Assume therefore that $V\neq\emptyset$. First assume additionally that, for every $j\in\{1,\dots,\ell\}$, the sets $V_*^{(j)}$ are either empty or $V_*^{(j)}=\{\lambda_j+1,\dots,d_j\}$ for some λ_j. We can cover both cases simultaneously by admitting that $\lambda_j\in\{0,\dots,d_j\}$. Now, consider an expansion

$$A_j=\sum_{i_1,\dots,i_{d_j}} a^j_{i_1,\dots,i_{d_j}} e_{i_1}\otimes\cdots\otimes e_{i_{d_j}}, \qquad B_j=\sum_{i_1,\dots,i_{d_j}} b^j_{i_1,\dots,i_{d_j}} e_{i_1}\otimes\cdots\otimes e_{i_{d_j}}$$

for some orthonormal system $\{e_\gamma\}$ in H. Then

$$\begin{aligned}R^V(A_1,\dots,A_\ell) &= \sum_i\left(\prod_{\alpha=1}^{\ell} a^\alpha_{i_{s_\alpha+1},\dots,i_{s_\alpha+d_\alpha}}\right)\left(\prod_{\beta=1}^{k}\delta_{i_{m_\beta},i_{n_\beta}}\right)\bigotimes_{\gamma=1}^{\ell}\left(e_{i_{s_\gamma+\lambda_\gamma+1}}\otimes\cdots\otimes e_{i_{s_\gamma+d_\gamma}}\right)\\ R^V(B_1,\dots,B_\ell) &= \sum_i\left(\prod_{\alpha=1}^{\ell} b^\alpha_{i_{s_\alpha+1},\dots,i_{s_\alpha+d_\alpha}}\right)\left(\prod_{\beta=1}^{k}\delta_{i_{m_\beta},i_{n_\beta}}\right)\bigotimes_{\gamma=1}^{\ell}\left(e_{i_{s_\gamma+\lambda_\gamma+1}}\otimes\cdots\otimes e_{i_{s_\gamma+d_\gamma}}\right).\end{aligned}$$

Define, for $\alpha\in\{1,\dots,\ell\}$,

$$\varrho^\alpha(i_1,\dots,i_{\lambda_\alpha},j_1,\dots,j_{\lambda_\alpha})=\sum_{u_{\lambda_\alpha+1},\dots,u_{d_\alpha}} a^\alpha_{i_1,\dots,i_{\lambda_\alpha},u_{\lambda_\alpha+1},\dots,u_{d_\alpha}} b^\alpha_{j_1,\dots,j_{\lambda_\alpha},u_{\lambda_\alpha+1},\dots,u_{d_\alpha}}$$

if $\lambda_\alpha\in\{1,d_\alpha-1\}$ (and if $d_\alpha>1$) and with obvious modifications if $\lambda_\alpha\in\{0,d_\alpha\}$. Then, with the convention that a product over the empty set equals 1, we have

$$\begin{aligned}&\langle R^V(A_1,\dots,A_\ell), R^V(B_1,\dots,B_\ell)\rangle_{H^{\otimes(N-2|V|)}}\\ &= \sum_i\sum_j\left(\prod_{\alpha=1}^{\ell} a^\alpha_{i_{s_\alpha+1},\dots,i_{s_\alpha+d_\alpha}} b^\alpha_{j_{s_\alpha+1},\dots,j_{s_\alpha+d_\alpha}}\prod_{c=\lambda_\alpha+1}^{d_\alpha}\delta_{i_{s_\alpha+c},j_{s_\alpha+c}}\right)\left(\prod_{\beta=1}^{k}\delta_{i_{m_\beta},i_{n_\beta}}\delta_{j_{m_\beta},j_{n_\beta}}\right)\\ &= \sum_{\substack{i_{s_1+1},\dots,i_{s_1+\lambda_1}\\ \dots\\ i_{s_\ell+1},\dots,i_{s_\ell+\lambda_\ell}}}\ \sum_{\substack{j_{s_1+1},\dots,j_{s_1+\lambda_1}\\ \dots\\ j_{s_\ell+1},\dots,j_{s_\ell+\lambda_\ell}}}\left(\prod_{\alpha=1}^{\ell}\varrho^\alpha_{i_{s_\alpha+1},\dots,i_{s_\alpha+\lambda_\alpha},j_{s_\alpha+1},\dots,j_{s_\alpha+\lambda_\alpha}}\right)\left(\prod_{\beta=1}^{k}\delta_{i_{m_\beta},i_{n_\beta}}\delta_{j_{m_\beta},j_{n_\beta}}\right)\\ &= \sum_{\substack{i_{m_1},j_{m_1},\dots,i_{m_k},j_{m_k}\\ i_{n_1},j_{n_1},\dots,i_{n_k},j_{n_k}}}\left(\prod_{\alpha=1}^{\ell}\varrho^\alpha_{i_{s_\alpha+1},\dots,i_{s_\alpha+\lambda_\alpha},j_{s_\alpha+1},\dots,j_{s_\alpha+\lambda_\alpha}}\right)\left(\prod_{\beta=1}^{k}\delta_{i_{m_\beta},i_{n_\beta}}\delta_{j_{m_\beta},j_{n_\beta}}\right)\end{aligned}$$

since

$$\{s_\alpha+l:\ 1\le l\le\lambda_\alpha,\ 1\le\alpha\le\ell\}=\{m_1,\dots,m_k,n_1,\dots,n_k\}.$$

Let $s_{\alpha_v^1} + l_v^1 = m_v$, $s_{\alpha_v^2} + l_v^2 = n_v$ be the assignment. Then $\alpha_v^1 \neq \alpha_v^2$ by (F2) and each of the $4k$ variables $i_{m_1}, j_{m_1}, \dots, i_{m_k}, j_{m_k}, i_{n_1}, j_{n_1}, \dots, i_{n_k}, j_{n_k}$ appears only once in the formula above, so we can apply the Cauchy-Schwarz inequality in the form

$$\sum_{\mu \in M} \sum_{\nu \in M} f_\mu g_\nu \delta_{\mu,\nu} \le \|f\|_{\ell_2(M)} \|g\|_{\ell_2(M)} \tag{18}$$

successively on the variables $(i_{m_1}, i_{n_1}), (j_{m_1}, j_{n_1}), \dots, (i_{m_k}, i_{n_k}), (j_{m_k}, j_{n_k})$. Eventually, after the last application of (18), we obtain

$$\begin{aligned}\langle R^V(A_1, \dots, A_\ell), R^V(B_1, \dots, B_\ell)\rangle_{H^{\otimes(N-2|V|)}} &\le \|\varrho^1\|_{\ell_2} \dots \|\varrho^\ell\|_{\ell_2} \\ &= \|R^{V_1}(A_1, B_1)\|_{H^{\otimes d_1}} \dots \|R^{V_\ell}(A_\ell, B_\ell)\|_{H^{\otimes d_\ell}}.\end{aligned}$$

If $V \neq \emptyset$ but is general, we reorganize by a permutation π to have the form from the above step and use (14) to get

$$\begin{aligned}&\langle R^V(A_1, \dots, A_\ell), R^V(B_1, \dots, B_\ell)\rangle_{H^{\otimes(N-2|V|)}} \\ &\quad = \langle R^{V^\pi}(P_{\pi_1^{-1}} A_1, \dots, P_{\pi_\ell^{-1}} A_\ell), R^{V^\pi}(P_{\pi_1^{-1}} B_1, \dots, P_{\pi_\ell^{-1}} B_\ell)\rangle_{H^{\otimes(N-2|V^\pi|)}} \\ &\quad \le \|R^{V_1^\pi}(P_{\pi_1^{-1}} A_1, P_{\pi_1^{-1}} B_1)\|_{H^{\otimes(N-2|V_1^\pi|)}} \dots \|R^{V_\ell^\pi}(P_{\pi_\ell^{-1}} A_\ell, P_{\pi_\ell^{-1}} B_\ell)\|_{H^{\otimes(N-2|V_\ell^\pi|)}} \\ &\quad = \|R^{(V_1)^{\pi_1'}}(P_{\pi_1^{-1}} A_1, P_{\pi_1^{-1}} B_1)\|_{H^{\otimes(N-2|(V_1)^{\pi_1'}|)}} \dots \|R^{(V_\ell)^{\pi_\ell'}}(P_{\pi_\ell^{-1}} A_\ell, P_{\pi_\ell^{-1}} B_\ell)\|_{H^{\otimes(N-2|(V_\ell)^{\pi_\ell'}|)}} \\ &\quad = \|R^{V_1}(A_1, B_1)\|_{H^{\otimes(N-2|V_1|)}} \dots \|R^{V_\ell}(A_\ell, B_\ell)\|_{H^{\otimes(N-2|V_\ell|)}}.\end{aligned}$$

since $V_*^{\pi,(j)} = \pi_j[V_*^{(j)}]$ and $V_j^\pi = (V_j)^{\pi_j'}$ where

$$\pi_j' = (\pi_j(1), \dots, \pi_j(d_j), d_j + \pi_j(1), \dots, d_j + \pi_j(d_j))$$

for every $j \in \{1, \dots, \ell\}$. □

5.4 Expansion Formula

Let W be an H-isonormal Gaussian process and $W_n : H^{\otimes n} \to L^2(\Omega)$, $n \in \mathbb{N}_0$, be the divergence operators as in Section 3. We adopt the following

Notation 5.7 Let $\ell \ge 2$ be an integer, let $d_1, \dots, d_\ell$ be positive integers, let $N = d_1 + \dots + d_\ell$, and decompose $\{1, \dots, N\}$ to subsequent intervals $I_1, \dots, I_\ell$ of lengths $d_1, \dots, d_\ell$, respectively. We denote by $\mathcal{E}_{I_1,\dots,I_\ell}$ the system of all admissible sets of pairs V for the the intervals $I_1, \dots, I_\ell$ (Definition 5.2) and by $\mathcal{E}_{I_1,\dots,I_\ell}^k$ its subsystem consisting of V such that $|V| = k$.

Remark 5.8 We have the following estimate of the cardinality

$$|\mathcal{E}_{I_1,\dots,I_\ell}^k| \le \frac{N!}{2^k k!(N-2k)!}; \tag{19}$$

see, e.g., [12, Chapter 1.5, page 16].

Theorem 5.9 *Let $\ell \geq 2$ be an integer, let $d_1, \dots, d_\ell$ be positive integers, let $N = d_1 + \dots + d_\ell$, and decompose $\{1, \dots, N\}$ to subsequent intervals $I_1, \dots, I_\ell$ of lengths $d_1, \dots, d_\ell$ respectively. Then*

$$W_{d_1}(A_1) \dots W_{d_\ell}(A_\ell) = \sum_{V \in \mathcal{E}_{I_1, \dots, I_\ell}} W_{N-2|V|}(R^V(A_1, \dots, A_\ell))$$

holds for every $A_1 \in H^{\otimes d_1}, \dots, A_\ell \in H^{\otimes d_\ell}$.

Remark 5.10 In [12, Theorem 7.33], the theorem is proved in a different language. Below, we provide a proof based on the tensor calculus.

Proof of Theorem 5.9 We have that

$$\begin{aligned}
&W_{d_1}(\underbrace{h_1 \otimes \dots \otimes h_{d_1}}_{A_1}) W_{d_2}(\underbrace{h_{d_1+1} \otimes \dots \otimes h_{d_1+d_2}}_{A_2}) \\
&\quad = \sum_{k=0}^{d_1 \wedge d_2} k! \binom{d_1}{k} \binom{d_2}{k} W_{d_1+d_2-2k}(R^{V_k}(h_1 \tilde{\otimes} \dots \tilde{\otimes} h_{d_1}, h_{d_1+1} \tilde{\otimes} \dots \tilde{\otimes} h_{d_1+d_2})) \\
&\quad = \sum_{\pi \in P_{I_1}} \sum_{\sigma \in P_{I_2}} \sum_{k=0}^{d_1 \wedge d_2} \frac{k!}{d_1! d_2!} \binom{d_1}{k} \binom{d_2}{k} W_{d_1+d_2-2k}(R^{V_k}(h_{\pi_1} \otimes \dots \otimes h_{\pi_{d_1}}, h_{\sigma_1} \otimes \dots \otimes h_{\sigma_{d_2}})) \\
&\quad = \sum_{\pi \in P_{I_1}} \sum_{\sigma \in P_{I_2}} \sum_{k=0}^{d_1 \wedge d_2} \frac{k!}{d_1! d_2!} \binom{d_1}{k} \binom{d_2}{k} W_{d_1+d_2-2k}(R^{\{\{\pi_1, \sigma_1\}, \dots, \{\pi_k, \sigma_k\}\}}(A_1, A_2)) \\
&\quad = \sum_{k=0}^{d_1 \wedge d_2} \sum_{V \in \mathcal{E}^k_{I_1, I_2}} W_{d_1+d_2-2k}(R^V(A_1, A_2)) \\
&\quad = \sum_{V \in \mathcal{E}_{I_1, I_2}} W_{d_1+d_2-2|V|}(R^V(A_1, A_2))
\end{aligned}$$

holds almost surely where $V_k = \{(1, d_1 + 1), \dots, (k, d_1 + k)\}$, P_I denotes the set of permutations on a set I, and $\tilde{\otimes}$ denotes the symmetric tensor product. The first equality follows by, e.g., [20, Proposition 1.1.3], the third equality follows from the fact that $W_n(A) = W_n(B)$ for every $A, B \in H^{\otimes n}$ such that B is a permutation of A, and the fourth equality follows from the fact that

$$|\{(\pi, \sigma) \in P_{I_1} \times P_{I_2} : \{\{\pi_1, \sigma_1\}, \dots, \{\pi_k, \sigma_k\}\} = V\}| = k!(d_1 - k)!(d_2 - k)!$$

holds for every $V \in \mathcal{E}^k_{I_1, I_2}$. We finish the proof by induction on ℓ for

$$A_1 = h_1 \otimes \dots \otimes h_{d_1}, \quad A_2 = h_{d_1+1} \otimes \dots \otimes h_{d_1+d_2}, \quad A_3 = h_{d_1+d_2+1} \otimes \dots \otimes h_{d_1+d_2+d_3}, \quad \dots$$

By induction hypothesis, it holds, almost surely, that

$$W_{d_1}(A_1) \dots W_{d_\ell}(A_\ell) = \sum_{V \in \mathcal{E}_{I_1, \dots, I_{\ell-1}}} W_{N-d_\ell-2|V|}(R^V(A_1, \dots, A_{\ell-1})) W_{d_\ell}(A_\ell)$$

and we must distinguish two cases. If $2|V| < N - d_\ell$, then, by Lemma 5.3,

$$W_{N-d_\ell-2|V|}(R^V(A_1, \dots, A_{\ell-1})) W_{d_\ell}(A_\ell) = \sum_{U \in \mathcal{E}_{I_1, \dots, I_\ell}(V)} W_{N-2|U|}(R^U(A_1, \dots, A_\ell)) \quad (20)$$

almost surely where $\mathcal{E}_{I_1,\dots,I_\ell}(V)$ contains all those elements U of $\mathcal{E}_{I_1,\dots,I_\ell}$ that complete V by pairs $\{m,n\}$ for which m or n belongs to I_ℓ. More rigorously, U must satisfy

$$U \in \mathcal{E}_{I_1,\dots,I_\ell}, \quad V \subseteq U, \quad \text{and} \quad U \setminus V \in \mathcal{E}_{I_1\cup\dots\cup I_{\ell-1},I_\ell}.$$

If $2|V| = N - d_\ell$, then (20) holds as well but Lemma 5.3 is not needed here as $\mathcal{E}_{I_1,\dots,I_\ell}(V) = \{V\}$. And indeed,

$$\begin{aligned} W_{N-d_\ell-2|V|}(R^V(A_1,\dots,A_{\ell-1}))W_{d_\ell}(A_\ell) &= R^V(A_1,\dots,A_{\ell-1})W_{d_\ell}(A_\ell) \\ &= W_{N-2|V|}(R^V(A_1,\dots,A_\ell)) \\ &= \sum_{U\in\mathcal{E}_{I_1,\dots,I_\ell}(V)} W_{N-2|U|}(R^U(A_1,\dots,A_\ell)) \end{aligned}$$

almost surely. Thus,

$$\begin{aligned} W_{d_1}(A_1)\dots W_{d_\ell}(A_\ell) &= \sum_{V\in\mathcal{E}_{I_1,\dots,I_{\ell-1}}} \sum_{U\in\mathcal{E}_{I_1,\dots,I_\ell}(V)} W_{N-2|U|}(R^U(A_1,\dots,A_\ell)) \\ &= \sum_{U\in\mathcal{E}_{I_1,\dots,I_\ell}} W_{N-2|U|}(R^U(A_1,\dots,A_\ell)) \end{aligned}$$

holds almost surely since $\{\mathcal{E}_{I_1,\dots,I_\ell}(V) : V \in \mathcal{E}_{I_1,\dots,I_{\ell-1}}\}$ is a partition of $\mathcal{E}_{I_1,\dots,I_\ell}$. □

6 Proofs

6.1 Proof of Theorem 3.3

Let $\ell \geq 2$ be an even integer and define $N = \ell n$. Then, according to Theorem 5.9,

$$[G(s+\delta) - G(s)]^\ell = \sum_{k=0}^{N/2} W_{N-2k}(d_{s,\delta}^{(k)}), \qquad s \in [0, T-\delta],$$

holds almost surely where

$$d_{s,\delta}^{(k)} = \sum_{V\in\mathcal{E}^k_{I_1,\dots,I_\ell}} R^V(A_{s,\delta},\dots,A_{s,\delta}), \qquad s \in [0, T-\delta],$$

for $I_j = \{(j-1)n+1,\dots,jn\}$, $j = 1,\dots,\ell$. By Lemma 5.4, assumption (G1), and inequality (19), there is the estimate

$$\|d_{s,\delta}^{(k)}\|_{H^{\otimes(N-2k)}} \leq \frac{N!\kappa^\ell}{2^k k!(N-2k)!}\delta^{\alpha\ell} \tag{21}$$

from which it follows that the function $s \mapsto d_{s,\delta}^{(k)}$ is integrable and if we define

$$Y_{\ell,\delta} = \|G(\cdot+\delta) - G(\cdot)\|_{L^\ell(0,T-\delta)}, \qquad \delta \in (0,T),$$

then

$$Y_{\ell,\delta}^\ell = \sum_{k=0}^{N/2} W_{N-2k}(d_\delta^{(k)}), \qquad \delta \in (0,T),$$

holds almost surely where

$$d_\delta^{(k)} = \int_0^{T-\delta} d_{s,\delta}^{(k)} \mathrm{d}s = \sum_{V \in \mathcal{E}^k_{I_1,\ldots,I_\ell}} \int_0^{T-\delta} R^V(A_{s,\delta},\ldots,A_{s,\delta})\mathrm{d}s, \qquad \delta \in (0,T).$$

Now, for $k = \frac{N}{2}$, we have

$$|d_\delta^{(\frac{N}{2})}| \leq \frac{N!}{2^{\frac{N}{2}}(\frac{N}{2})!} T\kappa^\ell \delta^{\alpha\ell} \tag{22}$$

by estimate (21), and for $k < \frac{N}{2}$, we have

$$\begin{aligned}
&\|d_\delta^{(k)}\|_{H^{\otimes(N-2k)}} \\
&\quad \leq \sum_{V \in \mathcal{E}^k_{I_1,\ldots,I_\ell}} \left\| \int_0^{T-\delta} R^V(A_{s,\delta},\ldots,A_{s,\delta})\, ds \right\|_{H^{\otimes(N-2k)}} \\
&\quad = \sum_{V \in \mathcal{E}^k_{I_1,\ldots,I_\ell}} \left(\int_0^{T-\delta}\int_0^{T-\delta} \langle R^V(A_{s,\delta},\ldots,A_{s,\delta}), R^V(A_{t,\delta},\ldots,A_{t,\delta})\rangle_{H^{\otimes(N-2k)}} \mathrm{d}s\mathrm{d}t \right)^{\frac{1}{2}} \\
&\quad \leq \sum_{V \in \mathcal{E}^k_{I_1,\ldots,I_\ell}} \left(\int_0^{T-\delta}\int_0^{T-\delta} \|R^{V_1}(A_{s,\delta},A_{t,\delta})\|_{H^{\otimes(2n-2|V_1|)}} \ldots \|R^{V_\ell}(A_{s,\delta},A_{t,\delta})\|_{H^{\otimes(2n-2|V_\ell|)}} \mathrm{d}s\mathrm{d}t \right)^{\frac{1}{2}}
\end{aligned}$$

by using Lemma 5.6. Now, we have in fact that $k \leq N-2$ as N is even which means that the set V_* defined by (12) contains at least two elements. It follows that either there are $i, i' \in \{1,\ldots,\ell\}$, $i \neq i'$, such that $V_i \neq \emptyset$ and $V_{i'} \neq \emptyset$ or there is $i \in \{1,\ldots,\ell\}$ such that V_i contains at least two elements. Consequently, the estimate

$$\|R^{V_1}(A_{s,\delta},A_{t,\delta})\|_{H^{\otimes(2n-2|V_1|)}} \ldots \|R^{V_\ell}(A_{s,\delta},A_{t,\delta})\|_{H^{\otimes(2n-2|V_\ell|)}} \leq 2n\kappa^{2\ell}\delta^{2\alpha\ell} C_{\delta,\delta}(s,t) \tag{23}$$

is obtained from (G1) by Lemma 5.4. Indeed, if there are $i, i' \in \{1,\ldots,\ell\}$, $i \neq i'$, such that $V_i \neq \emptyset$ and $V_{i'} \neq \emptyset$, then we can estimate

$$\begin{aligned}
&\underbrace{\|R^{V_1}(A_{s,\delta},A_{t,\delta})\|_{H^{\otimes(2n-2|V_1|)}}}_{\leq \|A_{s,\delta}\|_{H^{\otimes n}}\|A_{t,\delta}\|_{H^{\otimes n}}} \cdots \underbrace{\|R^{V_i}(A_{s,\delta},A_{t,\delta})\|_{H^{\otimes(2n-2|V_i|)}}}_{\leq \sum_{j=1}^n \|A_{s,\delta}\otimes_j A_{t,\delta}\|_{H^{\otimes(2n-2j)}}} \\
&\qquad \cdots \underbrace{\|R^{V_{i'}}(A_{s,\delta},A_{t,\delta})\|_{H^{\otimes(2n-2|V_{i'}|)}}}_{\leq \sum_{j=1}^n \|A_{s,\delta}\otimes_j A_{t,\delta}\|_{H^{\otimes(2n-2j)}}} \cdots \underbrace{\|R^{V_\ell}(A_{s,\delta},A_{t,\delta})\|_{H^{\otimes(2n-2|V_\ell|)}}}_{\leq \|A_{s,\delta}\|_{H^{\otimes n}}\|A_{t,\delta}\|_{H^{\otimes n}}},
\end{aligned}$$

and if there is $i \in \{1,\ldots,\ell\}$ such that $|V_i| \geq 2$, we can estimate

$$\underbrace{\|R^{V_1}(A_{s,\delta},A_{t,\delta})\|_{H^{\otimes(2n-2|V_1|)}}}_{\leq \|A_{s,\delta}\|_{H^{\otimes n}}\|A_{t,\delta}\|_{H^{\otimes n}}} \cdots \underbrace{\|R^{V_i}(A_{s,\delta},A_{t,\delta})\|_{H^{\otimes(2n-2|V_i|)}}}_{\leq \sum_{j=2}^n \|A_{s,\delta}\otimes_j A_{t,\delta}\|_{H^{\otimes(2n-2j)}}} \cdots \underbrace{\|R^{V_\ell}(A_{s,\delta},A_{t,\delta})\|_{H^{\otimes(2n-2|V_\ell|)}}}_{\leq \|A_{s,\delta}\|_{H^{\otimes n}}\|A_{t,\delta}\|_{H^{\otimes n}}}$$

by using Lemma 5.4 in both cases. Taking both these possibilities into account, the estimate

$$
\begin{aligned}
&\|R^{V_1}(A_{s,\delta}, A_{t,\delta})\|_{H^{\otimes(2n-2|V_1|)}} \dots \|R^{V_\ell}(A_{s,\delta}, A_{t,\delta})\|_{H^{\otimes(2n-2|V_\ell|)}} \\
&\leq \|A_{s,\delta}\|^{\ell-2}_{H^{\otimes n}} \|A_{t,\delta}\|^{\ell-2}_{H^{\otimes n}} \left(\sum_{j=1}^{n} \|A_{s,\delta} \otimes_j A_{t,\delta}\|_{H^{\otimes(2n-2j)}} \right)^2 \\
&+ \|A_{s,\delta}\|^{\ell-1}_{H^{\otimes n}} \|A_{t,\delta}\|^{\ell-1}_{H^{\otimes n}} \left(\sum_{j=2}^{n} \|A_{s,\delta} \otimes_j A_{t,\delta}\|_{H^{\otimes(2n-2j)}} \right)
\end{aligned} \tag{24}
$$

is obtained. Now, it follows by a second use of Lemma 5.4 that the square of the sum on the right-hand side of the above inequality can be estimated by

$$
2\|A_{s,\delta} \otimes_1 A_{t,\delta}\|^2_{H^{\otimes(2n-2)}} + 2(n-1)\|A_{s,\delta}\|_{H^{\otimes n}} \|A_{t,\delta}\|_{H^{\otimes n}} \sum_{j=2}^{n} \|A_{s,\delta} \otimes_j A_{t,\delta}\|_{H^{\otimes(2n-2j)}}.
$$

Inserting this estimate into inequality (24) yields

$$
\begin{aligned}
&\|R^{V_1}(A_{s,\delta}, A_{t,\delta})\|_{H^{\otimes(2n-2|V_1|)}} \dots \|R^{V_\ell}(A_{s,\delta}, A_{t,\delta})\|_{H^{\otimes(2n-2|V_\ell|)}} \\
&\leq 2\|A_{s,\delta}\|^{\ell-2}_{H^{\otimes n}} \|A_{t,\delta}\|^{\ell-2}_{H^{\otimes n}} \|A_{s,\delta} \otimes_1 A_{t,\delta}\|^2_{H^{\otimes(2n-2j)}} \\
&+(2n-1)\|A_{s,\delta}\|^{\ell-1}_{H^{\otimes n}} \|A_{t,\delta}\|^{\ell-1}_{H^{\otimes n}} \sum_{j=2}^{n} \|A_{s,\delta} \otimes_j A_{t,\delta}\|_{H^{\otimes(2n-2j)}}
\end{aligned}
$$

from which inequality (23) is obtained by using assumption (G1). Now, inequality (23) yields, together with (19), the estimate

$$
\|d_\delta^{(k)}\|_{H^{\otimes(N-2k)}} \leq \frac{N!}{2^k k!(N-2k)!} (2n)^{\frac{1}{2}} \kappa^\ell \delta^{\alpha\ell} [F(\delta,\delta)]^{\frac{1}{2}}.
$$

Hence,

$$
\begin{aligned}
\mathbb{E}\,(Y^\ell_{\ell,\delta} - \mathbb{E}\,Y^\ell_{\ell,\delta})^2 &= \mathbb{E}\left[\sum_{k=0}^{\frac{N}{2}-1} W_{N-2k}(d_\delta^{(k)}) \right]^2 \\
&= \sum_{k=0}^{\frac{N}{2}-1} \mathbb{E}\left[W_{N-2k}(d_\delta^{(k)}) \right]^2 \\
&\leq \sum_{k=0}^{\frac{N}{2}-1} (N-2k)! \|d_\delta^{(k)}\|^2_{H^{\otimes(N-2k)}} \\
&\leq 2n\kappa^{2\ell}\delta^{2\alpha\ell} N! F(\delta,\delta) \sum_{k=0}^{\frac{N}{2}-1} \frac{(2k)!}{2^{2k}(k!)^2} \binom{N}{2k} \\
&\leq 2n\kappa^{2\ell}\delta^{2\alpha\ell} (\ell n)! F(\delta,\delta) 2^{\ell n}
\end{aligned} \tag{25}
$$

since

$$
\sup_{k\geq 0} \frac{(2k)!}{2^{2k}(k!)^2} = 1. \tag{26}
$$

Now, if $q \in (2,\infty)$, then we have

$$
\mathbb{E}\,|Y^\ell_{\ell,\delta} - \mathbb{E}\,Y^\ell_{\ell,\delta}|^q \leq N^{\frac{q}{2}} (q-1)^{\frac{qN}{2}} [\mathbb{E}(Y^\ell_{\ell,\delta} - \mathbb{E}\,Y^\ell_{\ell,\delta})^2]^{\frac{q}{2}} \tag{27}
$$

by the equivalence of moments on a finite Wiener chaos [18, Corollary 2.8.14]. Thus the estimate

$$\mathbb{E}\,|Y^{\ell}_{\ell,\delta} - \mathbb{E}\,Y^{\ell}_{\ell,\delta}|^q \le (\ell n)^{\frac{q}{2}}(q-1)^{\frac{q\ell n}{2}}(2n)^{\frac{q}{2}}\kappa^{\ell q}\delta^{\alpha\ell q}[(\ell n)!]^{\frac{q}{2}}2^{\frac{\ell n q}{2}}[F(\delta,\delta)]^{\frac{q}{2}}$$

is obtained from (25). Let us define $\delta_j = 2^{-j}$ in the range $\{j : j \ge j_0\} = \{j : \delta_j < T\}$ and

$$\boldsymbol{C} = \left[\sum_{j=j_0}^{\infty}\sum_{\ell\in 2\mathbb{N}}\left(\frac{|Y^{\ell}_{\ell,\delta_j} - \mathbb{E}\,Y^{\ell}_{\ell,\delta_j}|}{(2\kappa)^{\ell}(q-1)^{\frac{\ell n}{2}}\delta_j^{\alpha\ell}\sqrt{2^{\ell n}(\ell n)(\ell n)!}}\right)^q\right]^{\frac{1}{q}}.$$

Then $[\mathbb{E}\,\boldsymbol{C}^q]^{\frac{1}{q}} \lesssim_n [\sum_{j=j_0}^{\infty} F(\delta_j,\delta_j)]^{\frac{1}{2}}$ where the sum on the right-hand side is finite by (G3). Consequently, the constant $\boldsymbol{C}$ is almost surely finite and the inequality

$$|Y^{\ell}_{\ell,\delta_j} - \mathbb{E}\,Y^{\ell}_{\ell,\delta_j}| \le \boldsymbol{C}(2\kappa)^{\ell}\delta_j^{\alpha\ell}(q-1)^{\frac{\ell n}{2}}\sqrt{2^{\ell n}(\ell n)(\ell n)!} \tag{28}$$

holds for every $j \ge j_0$ and every even $\ell \ge 2$ almost surely. Moreover, there is the almost sure convergence

$$\lim_{j\to\infty}\delta_j^{-\alpha\ell}|Y^{\ell}_{\ell,\delta_j} - \mathbb{E}\,Y^{\ell}_{\ell,\delta_j}| = 0 \tag{29}$$

for every even $\ell \ge 2$. Now

$$\mathbb{E}\,Y^{\ell}_{\ell,\delta_j} = d_{\delta_j}^{(\frac{N}{2})} \le \frac{(\ell n)!}{2^{\frac{\ell n}{2}}(\frac{\ell n}{2})!}T\kappa^{\ell}\delta_j^{\alpha\ell} \le T\kappa^{\ell}\delta_j^{\alpha\ell}\sqrt{(\ell n)!}$$

by (22) and (26) which, together with (28), yields that

$$Y_{\ell,\delta_j} \lesssim_{n,\kappa} (\boldsymbol{C} + T + 1)\,\delta_j^{\alpha}(q-1)^{\frac{n}{2}}\ell^{\frac{n}{2}} \tag{30}$$

holds for every $j \ge j_0$, every even $\ell \ge 2$, and every $q > 2$ almost surely. By interpolation,

$$\|G(\cdot+\delta_j) - G(\cdot)\|_{L^p(0,T-\delta_j)} \lesssim_{n,\kappa} (\boldsymbol{C} + T + 1)\,\delta_j^{\alpha}(q-1)^{\frac{n}{2}}p^{\frac{n}{2}} \tag{31}$$

is obtained for every $j \ge j_0$, every $p \ge 1$, and every $q > 2$ almost surely and, consequently,

$$[G]_{B^{\alpha}_{\Phi_{2/n},\infty}(0,T)} \lesssim_{n,\kappa} (\boldsymbol{C} + T + 1)(q-1)^{\frac{n}{2}}$$

holds for every $q > 2$ almost surely and

$$\left[\mathbb{E}[G]^q_{B^{\alpha}_{\Phi_{2/n},\infty}(0,T)}\right]^{\frac{1}{q}} \lesssim_{n,\kappa,T,F} (q-1)^{\frac{n}{2}}$$

holds for every $q > 2$. The first assertion then follows from the above estimate by noticing that $G(0) \in L^{\Phi_{2/n}}(\Omega)$ holds by [18, Corollary 2.8.14] and by appealing to [25, Corollary 26] and Remark 2.1. In order to prove the second assertion of the theorem, assume that process G additionally satisfies condition (G2). Note first that we have

$$\mathbb{E}\,|G(s+\delta_j) - G(s)|^{\ell} = \mathbb{E}\,|W_n(A_{s,\delta_j})|^{\ell} \ge \left(\mathbb{E}\,|W_n(A_{s,\delta_j})|^2\right)^{\frac{\ell}{2}} = (n!)^{\frac{\ell}{2}}\|A_{s,\delta_j}\|^{\ell}_{H^{\otimes n}}$$

for every even $\ell \geq 2$ by Jensen's inequality. By using this estimate, Fatou's lemma, and assumption (G2) successively, we obtain that

$$\liminf_{j\to\infty} \delta_j^{-\alpha\ell}\,\mathbb{E}\,Y_{\ell,\delta_j}^{\ell} \geq (n!)^{\frac{\ell}{2}} \liminf_{j\to\infty} \int_0^{T-\delta_j} \delta_j^{-\alpha\ell} \|A_{s,\delta_j}\|_{H^{\otimes n}}^{\ell} \mathrm{d}s \geq T(\kappa')^{\ell}(n!)^{\frac{\ell}{2}} > 0$$

holds for every even $\ell \geq 2$ from which it follows, by appealing to (29), that also

$$\liminf_{j\to\infty} \delta_j^{-\alpha\ell} Y_{\ell,\delta_j}^{\ell} \geq T(\kappa')^{\ell}(n!)^{\frac{\ell}{2}} > 0$$

holds for every even $\ell \geq 2$ almost surely and this yields that

$$\liminf_{j\to\infty} \delta_j^{-\alpha} Y_{\ell,\delta_j} \geq T^{\frac{1}{\ell}} \kappa' \sqrt{n!} > 0 \tag{32}$$

holds for every even $\ell \geq 2$ almost surely. By interpolation, the inequality

$$Y_{2,\delta_j} \lesssim Y_{1,\delta_j}^{\frac{1}{3}} Y_{4,\delta_j}^{\frac{2}{3}}$$

holds almost surely and, by using estimates (30) and (32), we obtain that

$$\liminf_{j\to\infty} \delta_j^{-\alpha} Y_{1,\delta_j} > 0$$

holds almost surely. As a consequence, we have for $q \geq 1$ that

$$[G]_{B^{\alpha}_{1,q}(0,T)} = \left(\sum_{j\geq j_0} \delta_j^{-\alpha q} \|G(\cdot + \delta_j) - G(\cdot)\|_{L^1(0,T-\delta_j)}^q \right)^{\frac{1}{q}}$$

diverges almost surely so that the paths of G do not belong to the space $B^{\alpha}_{1,q}(0,T)$. The second assertion of the theorem follows by the embedding of Besov spaces from [26, Theorem 3.3.1].

6.2 Proof of Theorem 3.3 with condition (G3′)

The proof follows the same strategy as the proof in Section 6.1. The difference comes in the estimate of $\|d_\delta^{(k)}\|_{H^{\otimes(N-2k)}}$ for $k < N/2$. In particular, let $J_\delta \in \mathbb{N}$ be such that $\lambda_\delta = \frac{T-\delta}{J_\delta} \geq \delta$. Then we have

$$
\begin{aligned}
&\|d_\delta^{(k)}\|_{H^{\otimes(N-2k)}}\\
&\quad\le \sum_{V\in\mathcal{E}^k_{I_1,\ldots,I_\ell}} \left\| \int_0^{T-\delta} R^V(A_{s,\delta},\ldots,A_{s,\delta})\mathrm{d}s \right\|_{H^{\otimes(N-2k)}}\\
&\quad= \sum_{V\in\mathcal{E}^k_{I_1,\ldots,I_\ell}} \left\| \lambda_\delta \sum_{j=1}^{J_\delta} \int_0^1 R^V(A_{(j-1)\lambda_\delta+s\lambda_\delta,\delta},\ldots,A_{(j-1)\lambda_\delta+s\lambda_\delta,\delta})\mathrm{d}s \right\|_{H^{\otimes(N-2k)}}\\
&\quad\le \sum_{V\in\mathcal{E}^k_{I_1,\ldots,I_\ell}} \lambda_\delta \int_0^1 \left\| \sum_{j=1}^{J_\delta} R^V(A_{(j-1)\lambda_\delta+s\lambda_\delta,\delta},\ldots,A_{(j-1)\lambda_\delta+s\lambda_\delta,\delta}) \right\|_{H^{\otimes(N-2k)}} \mathrm{d}s\\
&\quad\le \sum_{V\in\mathcal{E}^k_{I_1,\ldots,I_\ell}} \lambda_\delta \int_0^1 \Bigg(\sum_{m=1}^{J_\delta}\sum_{m'=1}^{J_\delta} |\langle R^V(A_{(m-1)\lambda_\delta+s\lambda_\delta,\delta},\ldots,A_{(m-1)\lambda_\delta+s\lambda_\delta,\delta}),\\
&\qquad\qquad R^V(A_{(m'-1)\lambda_\delta+s\lambda_\delta,\delta},\ldots,A_{(m'-1)\lambda_\delta+s\lambda_\delta,\delta})\rangle_{H^{\otimes(N-2k)}}| \Bigg)^{\frac12}\mathrm{d}s\\
&\quad\le \sum_{V\in\mathcal{E}^k_{I_1,\ldots,I_\ell}} \lambda_\delta \int_0^1 \left(\sum_{m=1}^{J_\delta}\sum_{m'=1}^{J_\delta}\prod_{i=1}^{\ell} \|R^{V_i}(A_{(m-1)\lambda_\delta+s\lambda_\delta,\delta},A_{(m'-1)\lambda_\delta+s\lambda_\delta,\delta})\|_{H^{\otimes(2n-2|V_i|)}} \right)^{\frac12}\mathrm{d}s
\end{aligned}
$$

by using Lemma 5.6. Now, we have in fact that $k\le N-2$ as N is even which means that the set V_* defined by (12) contains at least two elements. It follows that either there are $i,i'\in\{1,\ldots,\ell\}$, $i\ne i'$, such that $V_i\ne\emptyset$ and $V_{i'}\ne\emptyset$ or there is $i\in\{1,\ldots,\ell\}$ such that V_i contains at least two elements. Consequently, the estimate

$$
\prod_{i=1}^{\ell}\|R^{V_i}(A_{(m-1)\lambda_\delta+s\lambda_\delta,\delta},A_{(m'-1)\lambda_\delta+s\lambda_\delta,\delta})\|_{H^{\otimes(2n-2|V_i|)}}\lesssim \kappa^{2\ell}\delta^{2\alpha\ell}K(\delta,|m-m'|\lambda_\delta)
$$

for $m,m'\in\{1,\ldots,J_\delta\}$, $m\ne m'$, is obtained from assumption (G1) and the assumption that

$$
C_{s,s}(x,y)\le K(s,|x-y|)
$$

holds for $x,y\ge0$ and $s>0$ such that $x+s\le T$, $y+s\le T$, and $(x,x+s)\cap(y,y+s)=\emptyset$ by Lemma 5.4. This, together with (19), yields

$$
\|d_\delta^{(k)}\|_{H^{\otimes(N-2k)}}\lesssim \frac{N!\kappa^\ell}{2^k k!(N-2k)!}\delta^{\alpha\ell}[\tilde F(\delta)]^{\frac12}
$$

where

$$
\tilde F(\delta)=\lambda_\delta^2\left[n\kappa^4 J_\delta+\sum_{\substack{m,m'=1\\ m\ne m'}}^{J_\delta}K(\delta,|m-m'|\lambda_\delta)\right]
$$

and hence

$$
\mathbb{E}(Y^\ell_{\ell,\delta}-\mathbb{E}Y^\ell_{\ell,\delta})^2\lesssim \kappa^{2\ell}\delta^{2\alpha\ell}(\ell n)!2^{\ell n}\tilde F(\delta).
$$

as in (25). It now follows for $q>2$ by (27) that

$$
\mathbb{E}\,|Y^\delta_{\ell,\delta}-\mathbb{E}\,Y^\ell_{\ell,\delta}|^q\lesssim(\ell n)^{\frac q2}(q-1)^{\frac{q\ell n}{2}}\kappa^{\ell q}\delta^{\alpha\ell q}[(\ell n)!]^{\frac q2}2^{\frac{\ell n q}{2}}[\tilde F(\delta)]^{\frac q2},
$$

and if we define $\delta_j = T2^{-j}$, $J_{\delta_j} = 2^j - 1$ (so that $\lambda_{\delta_j} = \delta_j$), and

$$\boldsymbol{C} = \left[\sum_{j \in \mathbb{N}} \sum_{\ell \in 2\mathbb{N}} \left(\frac{|Y^{\ell}_{\ell,\delta_j} - \mathbb{E}Y^{\ell}_{\ell,\delta_j}|}{(2\kappa)^{\ell}(q-1)^{\frac{\ell n}{2}} \delta_j^{\alpha\ell} \sqrt{2^{\ell n}(\ell n)(\ell n)!}} \right)^q \right]^{\frac{1}{q}},$$

then $[\mathbb{E}\boldsymbol{C}_q^q]^{\frac{1}{q}} \lesssim [\sum_{j\in\mathbb{N}} \tilde{F}(\delta_j)]^{\frac{1}{2}}$ where the finiteness of the sum follows from the assumption

$$\sum_{j=1}^{\infty} \delta_j^2 \sum_{\substack{m,m'=1 \\ m \neq m'}}^{J_{\delta_j}} K(\delta_j, |m - m'|\delta_j) < \infty.$$

The rest of the proof follows as in Section 6.1 (with $j_0 = 1$). □

6.3 Proof of Theorem 3.9

We shall use the same notation as in the proof of Theorem 3.3 in Section 6.1. In there, the upper bounds are already obtained under assumption (G3). (Indeed, the map $\delta \mapsto \|G(\cdot + \delta) - G(\cdot)\|_{L^p(0,T-\delta)}$ is sub-additive and lower semi-continuous so that it follows from (31) that $\|G(\cdot + \delta) - G(\cdot)\|_{L^p(0,T-\delta)} \lesssim_\omega \delta^\alpha p^{\frac{n}{2}}$ holds for every $\delta \in (0, T)$ and $p \geq 1$ almost surely.) To obtain the lower bounds, it suffices to consider the case $p = 1$. The lower bound for this case follows from (32) by interpolation once it is shown that the process

$$f_G^{\ell,\alpha}(r) = r^{-\alpha\ell}[Y^{\ell}_{\ell,r} - \mathbb{E}Y^{\ell}_{\ell,r}], \qquad r \in [0, T),$$

where ℓ is a positive even integer, has a continuous version. To this end, let ℓ be a positive even integer, define $N = \ell n$, and let k be an integer such that $0 \leq k < \frac{N}{2}$. Let also $0 < s < t < T$. Proceeding similarly as in the proof of Theorem 3.3, we obtain

$$\mathbb{E}[f_G^{\ell,\alpha}(t) - f_G^{\ell,\alpha}(s)]^2 \leq \sum_{k=0}^{\frac{N}{2}-1} (N-2k)! \|t^{-\alpha\ell} d_t^{(k)} - s^{-\alpha\ell} d_s^{(k)}\|^2_{H^{\otimes(N-2k)}}. \tag{33}$$

The norm of the difference can be estimated as

$$\begin{aligned} &\|t^{-\alpha\ell}\, d_t^{(k)} - s^{-\alpha\ell} d_s^{(k)}\|_{H^{\otimes(N-2k)}} \\ &\quad \leq \sum_{V \in \mathcal{E}^k_{I_1,\ldots,I_\ell}} \left\| \int_0^{T-t} t^{-\alpha\ell} R^V(A_{x,t}, \ldots, A_{x,t}) \mathrm{d}x - \int_0^{T-s} s^{-\alpha\ell} R^V(A_{x,s}, \ldots, A_{x,s}) \mathrm{d}x \right\|_{H^{\otimes(N-2k)}} \end{aligned}$$

and, upon denoting

$$\boldsymbol{A}_{z,r} = \underbrace{(A_{z,r}, \ldots, A_{z,r})}_{\ell\times}$$

for $r \in (0, T)$ and $z \in (0, T - r)$ for simplicity, the chain continues as

$$\begin{aligned} &\leq \sum_{V \in \mathcal{E}^k_{I_1,\ldots,I_\ell}} \left\{ \left\| \int_0^{T-t} [t^{-\alpha\ell} R^V(\boldsymbol{A}_{x,t}) - s^{-\alpha\ell} R^V(\boldsymbol{A}_{x,s})] \mathrm{d}x \right\|_{H^{\otimes(N-2k)}} \right. \\ &\qquad \left. + \left\| \int_{T-t}^{T-s} s^{-\alpha\ell} R^V(\boldsymbol{A}_{x,s}) \mathrm{d}x \right\|_{H^{\otimes(N-2k)}} \right\}. \end{aligned} \tag{34}$$

Denote the first and the second term in the sum by I_1 and I_2, respectively. We have

$$I_2 \leq s^{-\alpha\ell}(t-s) \sup_{x\in(T-t,T-s)} \|\boldsymbol{A}_{x,s}\|_{H^{\otimes(N-2k)}} \leq s^{-\alpha\ell}(t-s) \sup_{x\in(T-t,T-s)} \|A_{x,s}\|^{\ell}_{H^{\otimes n}} \leq \kappa^{\ell}(t-s)$$

by using Lemma 5.4 and assumption (G1) successively. The focus is on term I_1 now. Write

$$\begin{aligned} I_1^2 \;=\; & t^{-\alpha\ell}\int_0^{T-t}\int_0^{T-t} \langle R^V(\boldsymbol{A}_{x,t}), R^V(\boldsymbol{A}_{y,t})\rangle_{H^{\otimes(N-2k)}}\,\mathrm{d}x\mathrm{d}y \\ & -2(st)^{-\alpha\ell}\int_0^{T-t}\int_0^{T-t} \langle R^V(\boldsymbol{A}_{x,t}), R^V(\boldsymbol{A}_{y,s})\rangle_{H^{\otimes(N-2k)}}\,\mathrm{d}x\mathrm{d}y \\ & +s^{-2\alpha\ell}\int_0^{T-t}\int_0^{T-t} \langle R^V(\boldsymbol{A}_{x,s}), R^V(\boldsymbol{A}_{y,s})\rangle_{H^{\otimes(N-2k)}}\,\mathrm{d}x\mathrm{d}y \end{aligned} \tag{35}$$

Denote $\lambda = \frac{s}{t}$ for simplicity and assume for now that $\frac{t}{2} \leq s < t$. From (35), we also have

$$\begin{aligned} I_1^2 \;=\; & t^{-2\alpha\ell}(1-\lambda^{-\alpha\ell})\int_0^{T-t}\int_0^{T-t} \langle R^V(\boldsymbol{A}_{x,t}), R^V(\boldsymbol{A}_{y,t})\rangle_{H^{\otimes(N-2k)}}\,\mathrm{d}x\mathrm{d}y \\ & +t^{-2\alpha\ell}(\lambda^{-2\alpha\ell}-\lambda^{-\alpha\ell})\int_0^{T-t}\int_0^{T-t} \langle R^V(\boldsymbol{A}_{x,s}), R^V(\boldsymbol{A}_{y,s})\rangle_{H^{\otimes(N-2k)}}\,\mathrm{d}x\mathrm{d}y \\ & +t^{-2\alpha\ell}\lambda^{-\alpha\ell}\int_0^{T-t}\int_0^{T-t} \langle R^V(\boldsymbol{A}_{x,t})-R^V(\boldsymbol{A}_{x,s}), R^V(\boldsymbol{A}_{y,t})-R^V(\boldsymbol{A}_{y,s})\rangle_{H^{\otimes(N-2k)}}\,\mathrm{d}x\mathrm{d}y. \end{aligned}$$

Now, denote

$$\boldsymbol{A}^i_{z,r_1,r_2} = (A_{z,r_1},\ldots,A_{z,r_1},\underbrace{A_{z,r_1}-A_{z,r_2}}_{i^{\text{th}}\text{ position}},A_{z,r_2},\ldots,A_{z,r_2})$$

for $r_1, r_2 \in (0,T)$, $z \in (0, T-\max\{r_1,r_2\})$, $i \in \{1,\ldots,\ell\}$; and denote also its m^{th} element, $m \in \{1,\ldots,\ell\}$, by $[\boldsymbol{A}^i_{z,r_1,r_2}]_m$. Then for $x, y \in (0, T-t)$, we have

$$\begin{aligned} & \left|\langle R^V(\boldsymbol{A}_{x,t})-R^V(\boldsymbol{A}_{x,s}), R^V(\boldsymbol{A}_{y,t})-R^V(\boldsymbol{A}_{y,s})\rangle_{H^{\otimes(N-2k)}}\right| \\ & \quad\leq \sum_{i=1}^{\ell}\sum_{j=1}^{\ell}\left|\langle R^V(\boldsymbol{A}^i_{x,t,s}), R^V(\boldsymbol{A}^j_{y,t,s})\rangle_{H^{\otimes(N-2k)}}\right| \\ & \quad\leq \sum_{i=1}^{\ell}\sum_{j=1}^{\ell}\left\{\|[\boldsymbol{A}^i_{x,t,s}]_{\tilde{m}}\otimes_l[\boldsymbol{A}^j_{y,t,s}]_{\tilde{m}}\|_{H^{\otimes(2n-2l)}}\prod_{m\neq\tilde{m}}\left\|[\boldsymbol{A}^i_{x,t,s}]_m\right\|_{H^{\otimes n}}\|[\boldsymbol{A}^j_{y,t,s}]_m\|_{H^{\otimes n}}\right\} \end{aligned}$$

by Lemmas 5.4 and 5.6 since $V_{\tilde{m}} \neq \emptyset$ for at least one index $\tilde{m} \in \{1,\ldots,\ell\}$, i.e. $l \in \{1,\ldots,\ell\}$. Now, denote the product in the brackets above by $J^{i,j}_{t,s,\tilde{m}}(x,y)$ and define

$$\tilde{C}_{r_1,r_2}(z_1,z_2) = \sum_{j=1}^{n}\|A_{z_1,r_1}\otimes_j A_{z_2,r_2}\|_{H^{\otimes(2n-2j)}}$$

for $z_1, z_2 \geq 0$ and $r_1, r_2 > 0$ that satisfy $z_1 + r_1 \leq T$, $z_2 + r_2 \leq T$. If $\tilde{m}$ is, for example, such that $[\boldsymbol{A}^i_{x,t,s}]_{\tilde{m}} = A_{x,t}$ and $[\boldsymbol{A}^j_{y,t,s}]_{\tilde{m}} = A_{y,t}$, we have the estimate

$$\begin{aligned} J^{i,j}_{t,s,\tilde{m}}(x,y) &\leq \tilde{C}_{t,t}(x,y)\|A_{x,t}\|^{i-2}_{H^{\otimes n}}\|A_{x,s}\|^{\ell-i}_{H^{\otimes n}}\|A_{x,t}-A_{x,s}\|_{H^{\otimes n}} \\ &\qquad \cdot \|A_{y,t}\|^{j-2}_{H^{\otimes n}}\|A_{y,s}\|^{\ell-j}_{H^{\otimes n}}\|A_{y,t}-A_{y,s}\|_{H^{\otimes n}} \\ &\leq (\kappa^2 t^{2\alpha})^{\ell-1}(1-\lambda)^{2\alpha}\tilde{C}_{t,t}(x,y) \end{aligned}$$

by realizing that $A_{x,t} - A_{x,s} = A_{x+s,t-s}$ holds and using (G1). By similar arguments, the estimates

$$J^{i,j}_{t,s,\tilde{m}}(x,y) \leq (\kappa^2 t^{2\alpha})^{\ell-1} \begin{cases} (1-\lambda)^{2\alpha}\tilde{C}_{t,t}(x,y) \\ (1-\lambda)^{2\alpha}\tilde{C}_{t,s}(x,y) \\ (1-\lambda)^{2\alpha}\tilde{C}_{s,t}(x,y) \\ (1-\lambda)^{2\alpha}\lambda^{-2\alpha}\tilde{C}_{s,s}(x,y) \\ (1-\lambda)^{\alpha}\lambda^{\alpha}\tilde{C}_{t-s,t}(x+s,y) \\ (1-\lambda)^{\alpha}\tilde{C}_{t-s,s}(x+s,y) \\ (1-\lambda)^{\alpha}\lambda^{\alpha}\tilde{C}_{t,t-s}(x,y+s) \\ (1-\lambda)^{\alpha}\tilde{C}_{s,t-s}(x,y+s) \\ \tilde{C}_{t-s,t-s}(x+s,y+s) \end{cases}$$

depending on the precise value of $\tilde{m}$ can be obtained but in any case, we have the estimate

$$\int_0^{T-t}\int_0^{T-t} J^{i,j}_{t,s,\tilde{m}}(x,y)\mathrm{d}x\mathrm{d}y \lesssim t^{2\alpha\ell+\varepsilon}(1-\lambda)^{2\alpha}$$

because assumption (G4) implies

$$\int_0^{T-r_2}\int_0^{T-r_1} \tilde{C}_{r_1,r_2}(z_1,z_2)\mathrm{d}z_1\mathrm{d}z_2 \lesssim r_1^{\alpha+\frac{\varepsilon}{2}} r_2^{\alpha+\frac{\varepsilon}{2}}, \qquad r_1, r_2 \in (0,T).$$

Thus, the inequality

$$\int_0^{T-t}\int_0^{T-t} \left|\langle R^V(\boldsymbol{A}_{x,t}) - R^V(\boldsymbol{A}_{x,s}), R^V(\boldsymbol{A}_{y,t}) - R^V(\boldsymbol{A}_{y,s})\rangle_{H^{\otimes(N-2k)}}\right| \mathrm{d}x\mathrm{d}y \lesssim t^{2\alpha\ell+\varepsilon}(1-\lambda)^{2\alpha}$$

is shown. Similarly, the estimates

$$\begin{aligned} \int_0^{T-t}\int_0^{T-t} \left|\langle R^V(\boldsymbol{A}_{x,s}), R^V(\boldsymbol{A}_{y,s})\rangle_{H^{\otimes(N-2k)}}\right| \mathrm{d}x\mathrm{d}y &\lesssim s^{2\alpha\ell+\varepsilon} \\ \int_0^{T-t}\int_0^{T-t} \left|\langle R^V(\boldsymbol{A}_{x,t}), R^V(\boldsymbol{A}_{y,t})\rangle_{H^{\otimes(N-2k)}}\right| \mathrm{d}x\mathrm{d}y &\lesssim t^{2\alpha\ell+\varepsilon} \end{aligned}$$

are shown to hold by appealing to Lemma 5.4, Lemma 5.6, and assumptions (G1) and (G4). Consequently,

$$I_1^2 \lesssim t^{\varepsilon}(\lambda^{-\alpha\ell} - \lambda^{\alpha\ell}) + t^{\varepsilon}\lambda^{-\alpha\ell}(1-\lambda)^{2\alpha} \lesssim t^{\varepsilon}(1-\lambda)^{\min\{1,2\alpha\}}$$

as $\frac{t}{2} \leq s < t$. Now, if $0 < s < \frac{t}{2}$, proceeding as above via Lemmas 5.4 and 5.6, equality (35) yields

$$\begin{aligned} I_1^2 \leq t^{-2\alpha\ell}(\kappa^2 t^{2\alpha})^{\ell-1} \int_0^{T-t}\int_0^{T-t} \tilde{C}_{t,t}(x,y)\mathrm{d}x\mathrm{d}y \\ +2t^{-\alpha\ell}s^{-\alpha\ell}(\kappa^2 t^{\alpha}s^{\alpha})^{\ell-1} \int_0^{T-t}\int_0^{T-t} \tilde{C}_{t,s}(x,y)\mathrm{d}x\mathrm{d}y \\ +s^{-2\alpha\ell}(\kappa^2 s^{2\alpha})^{\ell-1} \int_0^{T-t}\int_0^{T-t} \tilde{C}_{s,s}(x,y)\mathrm{d}x\mathrm{d}y \end{aligned}$$

so that

$$I_1^2 \lesssim t^{\varepsilon} \lesssim t^{\varepsilon}(1-\lambda)$$

holds by (G1) and (G4). Thus we obtain

$$\|t^{-\alpha\ell}d_t^{(k)} - s^{-\alpha\ell}d_s^{(k)}\|_{H^{\otimes(N-2k)}} \lesssim t^{\frac{\varepsilon}{2}}(1-\lambda)^{\min\{\frac{1}{2},\alpha\}}$$

from (34). Consequently, it follows from (33) by Kolmogorov's continuity theorem that process $f_G^{\ell,\alpha}$ has a continuous version. □

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