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Stochastic hyperplane-based ranks and their use in multivariate portmanteau tests

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ABSTRACT

The article proposes and justifies an optimal rank-based portmanteau test of multivariate elliptical strict white noise against multivariate serial dependence. It is based on new stochastic hyperplane-based ranks that are simpler and easier to compute than other usable hyperplane-based competitors and still share with them many good properties such as their distribution-free nature, affine invariance, efficiency, robustness and weak moment assumptions. The finite-sample performance of the portmanteau test is illustrated empirically in a small Monte Carlo simulation study.

1. Introduction

Tests based on signs and ranks are popular in statistics, including multivariate statistics, because they do not require restrictive distributional assumptions. As there exists no natural ordering of points in multivariate spaces, there have been many proposals of multivariate ranks and signs including component-wise ranks and signs [1], spatial ranks and signs [2,3], Oja ranks and signs [4], the ranks of pseudo-Mahalanobis distances [5], the recently proposed signs and ranks based on measure transportation [6,7], numerous variants of data depth [8] and other interesting concepts including [9].

Rank-based procedures already play an important role in testing randomness against serial dependence in univariate settings; see, *e.g.*, [10–13]. Multivariate generalizations of rank tests of randomness have also been considered in [14–16], for example. Genest and Rémillard [17] derived nonparametric tests of independence and randomness by means of the empirical copula process. Hallin and Liu [18] proposed a portmanteau test based on the promising measure-transportation ranks that is suitable for general but low-dimensional time series due to the curse of dimensionality. Oja and Paindaveine [19], inspired by Hallin and Paindaveine [20], proposed a portmanteau test statistic for elliptical strict white noise that uses hyperplane-based ranks and signs, introduced in [19,21,22]. They appear useful for elliptical distributions thanks to their simplicity, clear geometric interpretation, natural invariance, weak moment assumptions, complete avoidance of any shape matrix estimators and robustness to both radial and angular outliers [19]. Unfortunately, the original concepts from [19] are too computationally demanding to be applicable beyond dimension two or three because their calculation is based on combinatorial counts, namely on ordinary/complete interdirections (leading to signs) and ordinary/complete symmetrized lift-interdirections (leading to ranks).

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Recently, a couple of remedies to this problem have been proposed. First, Hudecová et al. [23] suggested to use incomplete interdirections and incomplete symmetrized lift-interdirections in the definitions of ranks and signs and thus to reduce the computational workload by means of incomplete U-statistics. The resulting signs are then reasonably quick to compute but the resulting ranks still suffer from the curse of dimensionality and become computationally intractable in $\mathbb{R}^p$ for large p due to the symmetrization involved. Therefore, Hudecová and Šiman [24] proposed to use randomized lift-interdirections, *i.e.*, to replace 2^p hyperplanes with a single one, which made the computation of ranks feasible in any dimension but still required as many random sign vectors as hyperplanes.

This article goes a few steps further in the simplification process and introduces two new and simpler hyperplane-based rank concepts, each of them requiring only a single uniformly distributed sign vector. It is also proved that the use of the new ranks and incomplete interdirections in the portmanteau test of [20] or [19] does not asymptotically change its distribution and optimal properties. Therefore, the portmanteau test with the new ranks and incomplete interdirections can be applied easily even to large-dimensional time series.

The main contribution of this article is thus twofold: (a) defining two new simplified concepts of hyperplane-based ranks and (b) proving that these new ranks (or their predecessors) are usable for testing serial independence of multi-dimensional time series in a rank-based portmanteau test.

Section 2 introduces necessary notation, terminology and definitions. Section 3 then investigates the new ranks, and Section 4 explores their use with complete and incomplete interdirections in the canonical portmanteau test of [20]. The computational side of the new ranks is illustrated in Section 5. The theoretical results are confirmed empirically in Section 6, which contains a small simulation study. A real-data example is also included in Section 7. Finally, Section 8 comments on the achievements and collects concluding remarks. The proofs of all assertions are relegated to the technical appendix.

2. Interdirections and lift-interdirections

The Euclidean norm of a vector $\boldsymbol{x}$ will be denoted as $\|\boldsymbol{x}\| = \sqrt{\boldsymbol{x}^\top \boldsymbol{x}}$. Let $\boldsymbol{X}$ be a p-dimensional random vector with continuous elliptical distribution $\mathcal{E}_{\boldsymbol{\theta},\boldsymbol{\Sigma},f}$ with median vector $\boldsymbol{\theta} \in \mathbb{R}^p$ and positive definite scatter matrix $\boldsymbol{\Sigma} \in \mathbb{R}^{p\times p}$; see [25, Chapter 2]. Then the density of $\boldsymbol{X}$ is proportional to (denoted $\propto$)

$$f\left(\sqrt{(\boldsymbol{x}-\boldsymbol{\theta})^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\theta})}\right), \quad \boldsymbol{x} \in \mathbb{R}^p, \tag{1}$$

where $f : [0,\infty) \to [0,\infty)$ is a function such that $\int_0^\infty z^{p-1} f(z)dz < \infty$, and the random variable $\|\boldsymbol{\Sigma}^{-1/2}(\boldsymbol{X}-\boldsymbol{\theta})\|$ has cumulative distribution function F_r with corresponding density $f_r(z) \propto z^{p-1} f(z) I[z > 0]$. An elliptical distribution with $\boldsymbol{\theta} = \boldsymbol{0}$ and $\boldsymbol{\Sigma} = \mathbf{I}_p$ (identity matrix) is called spherical.

Let $\mathcal{X}_n = \{\boldsymbol{X}_t\}_{t=1}^n, n > p$, be a sequence of random vectors such that each $\boldsymbol{X}_t$ has the same elliptical distribution as $\boldsymbol{X}$. Then $\mathcal{X}_n$ is called a strict white noise if its elements of $\{\boldsymbol{X}_t\}_{t=1}^n$ are mutually independent and $\boldsymbol{\theta} = \boldsymbol{0}$.

The class of elliptical distributions naturally induces the (oracle) Mahalanobis ranks and signs. The Mahalanobis rank R_t^M of the observation $\boldsymbol{X}_t$ is defined as the rank of the squared distance $d_t^2 = (\boldsymbol{X}_t - \boldsymbol{\theta})^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{X}_t - \boldsymbol{\theta})$, while the spatial sign of $\boldsymbol{X}_t$ is defined as the unit vector $\boldsymbol{U}_t = \boldsymbol{\Sigma}^{-1/2}(\boldsymbol{X}_t - \boldsymbol{\theta})/\|\boldsymbol{\Sigma}^{-1/2}(\boldsymbol{X}_t - \boldsymbol{\theta})\|$. If the parameters $\boldsymbol{\theta}$ and $\boldsymbol{\Sigma}$ are unknown, they have to be replaced with consistent estimates, which leads to the so-called pseudo-Mahalanobis ranks and signs that have been used in various multivariate nonparametric problems; see [16] for a discussion and further references.

It is well known that the estimator of the shape matrix $\boldsymbol{\Sigma}$ influences the properties of the pseudo-Mahalanobis ranks and signs and that its optimal choice and implementation may be a tricky question. If the test statistic depends only on the scalar products of the signs, as in Section 4, then it is sufficient to estimate consistently only the angles between $\boldsymbol{U}_t$ and $\boldsymbol{U}_s$ for $t \neq s$ or their cosines. Randles [22] and Oja and Paindaveine [19] introduced hyperplane-based ranks and signs that avoid the estimation of $\boldsymbol{\Sigma}$. They are based on certain counts of data-based hyperplanes.

If $\mathcal{X}_n$ is a spherically distributed strict white noise, then the angle

$$\alpha(\boldsymbol{y}_1, \boldsymbol{y}_2) = \arccos\left(\frac{\boldsymbol{y}_1^\top \boldsymbol{y}_2}{\|\boldsymbol{y}_1\|\|\boldsymbol{y}_2\|}\right)$$

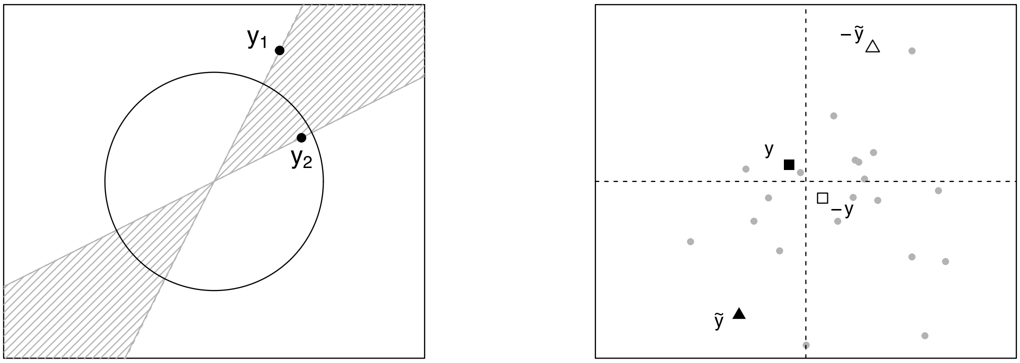
between any two $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathbb{R}^p$ can be consistently estimated by the π-multiple of the relative frequency of hyperplanes going through the origin and $p-1$ random vectors from $\mathcal{X}_n$ that separate $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$. The situation is clear for $p = 2$, see Fig. 1(a), but the claim is valid also for any $p > 2$; see [5,22].

Similarly, it can be shown that the number of hyperplanes going through p observations from spherically distributed $\mathcal{X}_n$ and separating $\boldsymbol{y}$ and $-\boldsymbol{y}$ tends to a monotonically increasing function of the norm of $\boldsymbol{y} \in \mathbb{R}^p$, which is indicated in Fig. 1(b). Consequently, the ranks of such numbers are tied to the Mahalanobis ranks of the corresponding points induced by the spherical distribution, *i.e.*, to the ranks of the norms of those points in this case.

Both the hyperplane-based notions can even be extended to the elliptical white noise thanks to the affine equivariance of the hyperplanes. The concepts and statements are formalized in the following paragraphs.

Any ordered k-tuple $\boldsymbol{q}(n,k) = (q_1,\ldots,q_k)$ of distinct integer indices $1 \le q_1 < \cdots < q_k \le n$ can be associated with the subsample $\mathcal{X}^{\boldsymbol{q}(n,k)} = \{\boldsymbol{X}_{q_1},\ldots,\boldsymbol{X}_{q_k}\}$. The set of all $\binom{n}{k}$ possible k-tuples $\boldsymbol{q}(n,k)$ will be denoted by $\mathcal{Q}(n,k)$. Any subsample $\mathcal{X}^{\boldsymbol{q}(n,p-1)}$ almost surely defines the hyperplane $H^{\boldsymbol{q}(n,p-1)} \subset \mathbb{R}^p$ containing all of its observations and the origin, namely:

$$H^{\boldsymbol{q}(n,p-1)} = \{\boldsymbol{x} : \det\left(\mathbb{M}^{\boldsymbol{q}(n,p-1)}(\boldsymbol{x})\right) = \boldsymbol{d}^{\boldsymbol{q}(n,p-1)^\top}\boldsymbol{x} = 0\},$$



(a) On the interdirections (b) On the lift-interdirections

Fig. 1. Panel (a) shows that the angle between $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ is equal to the π-multiple of the probability that a spherically-distributed random observation lies in the highlighted area, which is the probability that a hyperplane going through the origin and the random point separates the two points $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$. This idea extends to dimension $p > 2$. Panel (b) illustrates the idea of lift-interdirections for a simulated random sample with $\boldsymbol{\theta} = \mathbf{0}$ and $n = 20$, which leads to 190 data-generated hyperplanes in total. The point $\boldsymbol{y}$ is close to the center and, therefore, there are only 51 hyperplanes passing through $p = 2$ observations that separate $\boldsymbol{y}$ and $-\boldsymbol{y}$. The number of such hyperplanes increases with the distance of the point from the center. For instance, $\tilde{\boldsymbol{y}}$ and $-\tilde{\boldsymbol{y}}$ are already separated by 161 of the hyperplanes.

where $\boldsymbol{d}^{\boldsymbol{q}(n,p-1)} = (d_1^{\boldsymbol{q}(n,p-1)}, \ldots, d_p^{\boldsymbol{q}(n,p-1)})^\top$ and $d_j^{\boldsymbol{q}(n,p-1)}$, $j \in \{1, \ldots, p\}$, is the cofactor of the jth element in the last column of the matrix

$$\mathbb{M}^{\boldsymbol{q}(n,p-1)}(\boldsymbol{x}) = (\boldsymbol{X}_{q_1}, \boldsymbol{X}_{q_2}, \ldots, \boldsymbol{X}_{q_{p-1}}, \boldsymbol{x}).$$

Similarly, any subsample $\mathcal{X}^{\boldsymbol{q}(n,p)}$ almost surely defines the hyperplane $H^{\boldsymbol{q}(n,p)} \subset \mathbb{R}^p$ containing its observations, namely:

$$H^{\boldsymbol{q}(n,p)} = \{\boldsymbol{x} : \det\left(\mathbb{M}^{\boldsymbol{q}(n,p)}(\boldsymbol{x})\right) = d_0^{\boldsymbol{q}(n,p)} + \boldsymbol{d}^{\boldsymbol{q}(n,p)^\top}\boldsymbol{x} = 0\},$$

where $\boldsymbol{d}^{\boldsymbol{q}(n,p)} = (d_1^{\boldsymbol{q}(n,p)}, \ldots, d_p^{\boldsymbol{q}(n,p)})^\top$ and $d_j^{\boldsymbol{q}(n,p)}$, $j \in \{0, \ldots, p\}$, is the cofactor of the $(j+1)$th element in the last column of the matrix

$$\mathbb{M}^{\boldsymbol{q}(n,p)}(\boldsymbol{x}) = \left((1, \boldsymbol{X}_{q_1}^\top)^\top, (1, \boldsymbol{X}_{q_2}^\top)^\top, \ldots, (1, \boldsymbol{X}_{q_p}^\top)^\top, (1, \boldsymbol{x}^\top)^\top\right).$$

The signs

$$S^{\boldsymbol{q}(n,p-1)}(\boldsymbol{x}) = \operatorname{sign}(\boldsymbol{d}^{\boldsymbol{q}(n,p-1)^\top}\boldsymbol{x}), \quad S^{\boldsymbol{q}(n,p)}(\boldsymbol{x}) = \operatorname{sign}(d_0^{\boldsymbol{q}(n,p)} + \boldsymbol{d}^{\boldsymbol{q}(n,p)^\top}\boldsymbol{x})$$

then indicate the position of $\boldsymbol{x} \in \mathbb{R}^p$ with respect to $H^{\boldsymbol{q}(n,p-1)}$ and $H^{\boldsymbol{q}(n,p)}$, respectively.

Any couple of points $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathbb{R}^p$ may be associated with two fundamental affine-invariant hyperplane-based counts, namely interdirection

$$C_{\boldsymbol{y}_1,\boldsymbol{y}_2} = C_{\boldsymbol{y}_1,\boldsymbol{y}_2}(\mathcal{X}_n) = \sum_{\boldsymbol{q}\in\mathcal{Q}_C} \left(1 - S^{\boldsymbol{q}}(\boldsymbol{y}_1)S^{\boldsymbol{q}}(\boldsymbol{y}_2)\right)/2, \quad \mathcal{Q}_C \subset \mathcal{Q}(n, p-1),$$

which is the number of hyperplanes in $\mathbb{R}^p$ passing through the origin and $p-1$ observations and separating $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$, and lift-interdirection

$$L_{\boldsymbol{y}_1,\boldsymbol{y}_2} = L_{\boldsymbol{y}_1,\boldsymbol{y}_2}(\mathcal{X}_n) = \sum_{\boldsymbol{q}\in\mathcal{Q}_L} \left(1 - S^{\boldsymbol{q}}(\boldsymbol{y}_1)S^{\boldsymbol{q}}(\boldsymbol{y}_2)\right)/2, \quad \mathcal{Q}_L \subset \mathcal{Q}(n, p),$$

which is the number of hyperplanes in $\mathbb{R}^p$ passing through p observations and separating $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$. Any point $\boldsymbol{y} \in \mathbb{R}^p$ (with its reflection $-\boldsymbol{y}$) also gives rise to a symmetrized lift-interdirection

$$\underline{L}_{\boldsymbol{y}} = \underline{L}_{\boldsymbol{y}}(\mathcal{X}_n) = \frac{1}{2}\sum_{\boldsymbol{q}\in\mathcal{Q}_L}\sum_{\boldsymbol{s}\in\mathcal{S}(\boldsymbol{q})}\left(1 - \operatorname{sign}(d_{0\boldsymbol{s}}^{\boldsymbol{q}(n,p)} + \boldsymbol{d}_{\boldsymbol{s}}^{\boldsymbol{q}(n,p)^\top}\boldsymbol{y})\operatorname{sign}(d_{0\boldsymbol{s}}^{\boldsymbol{q}(n,p)} - \boldsymbol{d}_{\boldsymbol{s}}^{\boldsymbol{q}(n,p)^\top}\boldsymbol{y})\right) \tag{2}$$

where $(d_{0\boldsymbol{s}}^{\boldsymbol{q}(n,p)}, \boldsymbol{d}_{\boldsymbol{s}}^{\boldsymbol{q}(n,p)^\top})^\top$ is the vector of cofactors of the last column of matrix

$$\mathbb{M}_{\boldsymbol{s}}^{\boldsymbol{q}(n,p)}(\boldsymbol{x}) = ((1, s_1\boldsymbol{X}_{q_1}^\top)^\top, (1, s_2\boldsymbol{X}_{q_2}^\top)^\top, \ldots, (1, s_p\boldsymbol{X}_{q_p}^\top)^\top, (1, \boldsymbol{x}^\top)^\top),$$

$\mathcal{S}(\boldsymbol{q}) \subset \{-1, 1\}^p$ for any $\boldsymbol{q} \in \mathcal{Q}_L$, and $\{-1, 1\}^p$ is the set of size 2^p consisting of all p-dimensional vectors $\boldsymbol{s} = (s_1, \ldots, s_p)^\top$ with individual coordinates equal to either 1 or -1. Roughly speaking, the symmetrized lift-interdirection counts all the hyperplanes in

$\mathbb{R}^p$ that separate $\boldsymbol{y}$ and $-\boldsymbol{y}$ and pass through p distinct random points where each of the points is either an original observation or its reflection. The sets $\mathcal{Q}_C$ and $\mathcal{Q}_L$ are called design sets.

These hyperplane-based characteristics proved useful for statistical inference regarding elliptical distributions with $\boldsymbol{\theta} = \boldsymbol{0}$ when

$$\pi C_{\boldsymbol{X}_t,\boldsymbol{X}_s}(\mathcal{X}_n)/|\mathcal{Q}_C|$$

is a consistent estimator of the angle $\alpha(\boldsymbol{U}_t, \boldsymbol{U}_s)$ between $\boldsymbol{U}_t$ and $\boldsymbol{U}_s$, and when the normalized ranks $R_t/(n+1)$ of $L_{\boldsymbol{X}_t,-\boldsymbol{X}_t}(\mathcal{X}_n)$ or $\underline{L}_{\boldsymbol{X}_t}(\mathcal{X}_n)$ in corresponding samples of the particular quantities are asymptotically equivalent to the normalized (oracle) Mahalanobis ranks $R_t^M/(n+1)$.

The ordinary/complete concepts of [19,22] correspond to $\mathcal{Q}_C = \mathcal{Q}(n, p-1)$, $\mathcal{Q}_L = \mathcal{Q}(n, p)$, and $\mathcal{S}(\boldsymbol{q}) = \{-1, 1\}^p$, $\boldsymbol{q} \in \mathcal{Q}_L$. Then $C_{\boldsymbol{y}_1,\boldsymbol{y}_2}(\mathcal{X}_n)$ (resp. $L_{\boldsymbol{y}_1,\boldsymbol{y}_2}(\mathcal{X}_n)$) counts all the hyperplanes separating $\boldsymbol{y}_1$ from $\boldsymbol{y}_2$ that pass through the origin and $p-1$ observations (resp. through p observations). Furthermore, $\underline{L}_{\boldsymbol{y}}(\mathcal{X}_n)$ is then a symmetrized version of $L_{\boldsymbol{y},-\boldsymbol{y}}(\mathcal{X}_n)$ that is desirably invariant with respect to the reflections of the observations around the origin.

The sets $\mathcal{Q}(n, p-1)$ and $\mathcal{Q}(n, p)$ generate too many hyperplanes and effectively prohibit the computation of original characteristics in multidimensional spaces. Nevertheless, one can consider only some hyperplanes and still obtain meaningful results. This option leads to incomplete modifications with possibly random $\mathcal{Q}_C \subset \mathcal{Q}(n, p-1)$, $\mathcal{Q}_L \subset \mathcal{Q}(n, p)$, and $\mathcal{S}(\boldsymbol{q}) = \{-1, 1\}^p$, $\boldsymbol{q} \in \mathcal{Q}_L$. They have been analyzed by means of the theory of incomplete U-statistics in [23]. Unfortunately, the use of all sign vectors in the symmetrization still makes the computation of incomplete $\underline{L}_{\boldsymbol{y}}(\mathcal{X}_n)$ too demanding for large p because the number of all sign vectors grows with p exponentially. As a possible remedy, Hudecová and Šiman [24] proposed to consider randomized lift-interdirections, corresponding to $\mathcal{Q}_L \subset \mathcal{Q}(n, p)$ and $\mathcal{S}(\boldsymbol{q}) = \{\boldsymbol{S}_q\}$, $\boldsymbol{S}_q \in \{-1, 1\}^p$, where $\boldsymbol{S}_q$ is chosen for each $\boldsymbol{q}$ independently of $\boldsymbol{q}$, $\mathcal{X}_n$, $\mathcal{Q}_C$ and $\mathcal{Q}_L$ at random from the uniform distribution on $\{-1, 1\}^p$. They are not U-statistics any more and use only one random hyperplane instead of 2^p ones, which speeds their computation.

Proposition 1 summarizes some relevant and useful results established in [19,22–24]. It shows that the hyperplane-based ranks and signs have the desirable properties even if the design sets $\mathcal{Q}(n, p-1)$ and $\mathcal{Q}(n, p)$ are substantially reduced in size.

Proposition 1. *Let $\mathcal{X}_n = \{\boldsymbol{X}_1, \dots, \boldsymbol{X}_n\}$ be a p-dimensional strict white noise from elliptical distribution $\mathcal{E}_{\boldsymbol{0},\boldsymbol{\Sigma},f}$, and assume that the design sets $\mathcal{Q}_C$ and $\mathcal{Q}_L$ are independent, and also independent of $\mathcal{X}_n$. Let $R_1, \dots, R_n$ be the ranks of $L_{\boldsymbol{X}_t,-\boldsymbol{X}_t}(\mathcal{X}_n)$ or $\underline{L}_{\boldsymbol{X}_t}(\mathcal{X}_n)$, $t \in \{1, \dots, n\}$.*

(i) *Let $\mathcal{Q}_C = \mathcal{Q}(n, p-1)$, $\mathcal{Q}_L = \mathcal{Q}(n, p)$, and $\mathcal{S}(\boldsymbol{q}) = \{-1, 1\}^p$. Then*
$$\pi \frac{C_{\boldsymbol{y}_1,\boldsymbol{y}_2}(\mathcal{X}_n)}{|\mathcal{Q}_C|} = \alpha(\boldsymbol{y}_1, \boldsymbol{y}_2) + o_P(1), \quad \frac{R_t}{n+1} = \frac{R_t^M}{n+1} + o_P(1) \tag{3}$$
as $n \to \infty$.

(ii) *Let $\mathcal{Q}_C$ be sampled from $\mathcal{Q}(n, p-1)$ randomly with or without replacement such that $|\mathcal{Q}_C|/n \to \infty$, let $\mathcal{Q}_L$ be sampled from $\mathcal{Q}(n, p)$ randomly with or without replacement such that $|\mathcal{Q}_L|/n \to \infty$, and let $\mathcal{S}(\boldsymbol{q}) = \{-1, 1\}^p$ or $\mathcal{S}(\boldsymbol{q}) = \{\boldsymbol{S}_q\}$, $\boldsymbol{q} \in \mathcal{Q}_L$, where $\boldsymbol{S}_q$ is randomly sampled from the uniform distribution on $\{-1, 1\}^p$ independently of $\boldsymbol{q}$ and $\mathcal{X}_n$. Then (3) holds for $n \to \infty$.*

Unfortunately, even the randomized lift-interdirections need as many random sign vectors as hyperplanes, which can be inconvenient in high dimensions.

3. New stochastic lift-interdirections and ranks

This article proposes two conceptually and computationally simpler alternatives to incomplete and randomized lift-interdirections:

1. *Semi-randomized lift-interdirections*, defined as

$$\widetilde{L}_{\boldsymbol{y}}^{s} = \frac{1}{2}\sum\nolimits_{\boldsymbol{q}\in\mathcal{Q}_L} \left(1 - \operatorname{sign}(d_{0s}^{q(n,p)} + \boldsymbol{d}_s^{q(n,p)^\top}\boldsymbol{y}) \operatorname{sign}(d_{0s}^{q(n,p)} - \boldsymbol{d}_s^{q(n,p)^\top}\boldsymbol{y})\right),$$

where $\mathcal{Q}_L = \mathcal{Q}(n, p)$ or $\mathcal{Q}_L$ is sampled from $\mathcal{Q}(n, p)$ randomly with or without replacement and $\boldsymbol{s}$ is uniformly distributed on $\{-1, 1\}^p$. In other words, $\widetilde{L}_{\boldsymbol{y}}^{s}$ is defined as (2) but with $\mathcal{S}(\boldsymbol{q}) = \{\boldsymbol{s}\}$, where $\boldsymbol{s}$ is uniformly distributed on $\{-1, 1\}^p$ and common to all $\boldsymbol{q} \in \mathcal{Q}_L$.

2. *Reflective lift-interdirections* $\bar{L}_{\boldsymbol{y}}^{s}$, defined as complete or incomplete lift-interdirections computed from sample $\mathcal{X}_n^s = \{s_1\boldsymbol{X}_1, \dots, s_n\boldsymbol{X}_n\}$, where $\boldsymbol{s} = (s_1, \dots, s_n)^\top \in \{-1, 1\}^n$ is a uniformly distributed random vector on $\{-1, 1\}^n$.

It is apparent that both the semi-randomized and reflective lift-interdirections further reduce the computational burden. For example, the original ranks from [19], based on the symmetrized lift-interdirections, could be computed in a reasonable time only for data dimension p up to three or so, but the ranks based on these two new concepts of lift-interdirections may be computed even for $p \sim 1000$ if n is reasonably small. It turns out that even the normalized ranks of $\widetilde{L}_{\boldsymbol{X}_t}^{s}$ as well as the normalized ranks of $\bar{L}_{\boldsymbol{X}_t}^{s}$ are asymptotically equivalent to the normalized Mahalanobis ranks $R_t^M/(n+1)$ and that they can be used in the sign-and-rank portmanteau test of [20] without changing its asymptotic behavior or optimal properties.

Proposition 2. *Let $\mathcal{X}_n = \{\boldsymbol{X}_1, \dots, \boldsymbol{X}_n\}$ be a p-dimensional strict white noise from elliptical distribution $\mathcal{E}_{\boldsymbol{0},\boldsymbol{\Sigma},f}$ that is independent both of the random sign vector $\boldsymbol{s}$ used in the definition of reflective or semi-randomized lift-interdirections and of the design set $\mathcal{Q}_L$ chosen equal to $\mathcal{Q}(n, p)$ or randomly with or without replacement, $|\mathcal{Q}_L|/n \to \infty$. If $\widehat{\underline{R}}_t$, $t \in \{1, \dots, n\}$, are the ranks of reflective or semi-randomized lift-interdirections, then, as $n \to \infty$,*

$$\frac{\widehat{\underline{R}}_t}{n+1} = \frac{R_t^M}{n+1} + o_P(1). \tag{4}$$

The proof of Proposition 2 is relegated to Appendix.

4. Portmanteau test

Testing for serial independence is one of the basic problems in time series analysis. The benchmark tool for that is the multivariate Ljung–Box portmanteau test, proposed by Hosking [26], based on the test statistic

$$S_{LB} = n(n+2)\sum_{j=1}^{m}\frac{1}{n-j}\,\mathrm{Tr}[\widehat{\mathbb{C}}_j\widehat{\mathbb{C}}_0^{-1}\widehat{\mathbb{C}}_j^{\top}\widehat{\mathbb{C}}_0^{-1}],$$

where

$$\widehat{\mathbb{C}}_j = \frac{1}{n}\sum_{t=j+1}^{n}(\boldsymbol{X}_t-\bar{\boldsymbol{X}})(\boldsymbol{X}_{t-j}-\bar{\boldsymbol{X}})^{\top},\ j\in\{0,\dots,m\},$$

are the sample autocovariance matrices and $m \geq 1$ is a pre-specified integer, the so-called threshold parameter. The test statistic S_{LB} is asymptotically $\chi^2_{p^2m}$ distributed under the null hypothesis $\mathcal{H}_0 : \{\boldsymbol{X}_t\}$ is a strict white noise, if the required moments of the noise are finite. The null hypothesis is rejected at significance level α when S_{LB} exceeds the $(1-\alpha)$-quantile of the $\chi^2_{p^2m}$ distribution.

Hallin and Paindaveine [20] investigated the problem of testing $\mathcal{H}_0$, where $\boldsymbol{X}_t$ follows an elliptical distribution $\mathcal{E}_{\boldsymbol{0},\boldsymbol{\Sigma},f}$ and f satisfies the following assumption.

Assumption 1. The function f from (1) satisfies $\int_0^\infty z^{p+1}f(z)dz < \infty$, and $\sqrt{f}$ admits a weak derivative $(\sqrt{f})'$ such that $\int_0^\infty[(\sqrt{f})'(z)]^2z^{p-1}dz < \infty$.

They showed that a test based on

$$\sum_{j=1}^{m}\frac{1}{n-j}\sum_{t,s=j+1}^{n}\varphi_f(d_t)\varphi_f(d_s)d_{t-j}d_{s-j}\boldsymbol{U}_t^{\top}\boldsymbol{U}_s\boldsymbol{U}_{t-j}^{\top}\boldsymbol{U}_{s-j},$$

with $\varphi_f = -2(\sqrt{f})'/\sqrt{f}$, is optimal for testing the null hypothesis against a nontrivial VARMA(q_1,q_2) model with $\max(q_1,q_2)=m$ and error distribution $\mathcal{E}_{\boldsymbol{0},\boldsymbol{\Sigma},f}$

This suggests that sign-and-rank variants of the test will have to use some score function(s) and replace d_t, d_s, d_{t-j}, d_{s-j}, $\boldsymbol{U}_t^{\top}\boldsymbol{U}_s$ and $\boldsymbol{U}_{t-j}^{\top}\boldsymbol{U}_{s-j}$ with some counterparts based on the signs and ranks.

Oja and Paindaveine [19] proposed such a portmanteau test for elliptical strict white noise based on ordinary/complete interdirections and ordinary/complete symmetrized lift-interdirections. However, the computational complexity makes the test applicable only to low-dimensional data. The following proposition generalizes the test to use other types of interdirections and lift-interdirections, including those fast to compute in any dimension.

Proposition 3. *Let $\mathcal{X}_n = \{\boldsymbol{X}_1,\dots,\boldsymbol{X}_n\}$ be a p-dimensional strict white noise from elliptical distribution $\mathcal{E}_{\boldsymbol{0},\boldsymbol{\Sigma},f}$ and consider its (possibly incomplete) interdirections $C_{\boldsymbol{X}_t,\boldsymbol{X}_s}(\mathcal{X}_n)$ and one of the following types of incomplete lift-interdirections: symmetrized, randomized, semi-randomized or reflective. Let the design sets $\mathcal{Q}_L$, $\mathcal{Q}_C$ and the sign vectors involved in the definition of lift-interdirections be mutually independent, independent of $\mathcal{X}_n$, and satisfying the conditions of Proposition 1 (ii) for $\mathcal{Q}_C$ and the conditions of Proposition 1 (ii) or Proposition 2 for the rest. Let $\widehat{\underline{R}}_t$, $t\in\{1,\dots,n\}$, be the ranks of lift-interdirections and let $a_{\boldsymbol{X}_t,\boldsymbol{X}_s} = \pi C_{\boldsymbol{X}_t,\boldsymbol{X}_s}(\mathcal{X}_n)/|\mathcal{Q}_C|$. Let $K_1 : (0,\infty)\to\mathbb{R}$ and $K_2 : (0,\infty)\to\mathbb{R}$ be two continuous score functions that satisfy*

$$\frac{1}{n}\sum_{i=1}^{n}\left|K_j\left(\frac{i}{n+1}\right)\right|^{2+\delta}\to\int_0^1\left|K_j(u)\right|^{2+\delta}du<\infty,\quad j=1,2,$$

for some $\delta > 0$. Then, for a fixed threshold $m\in\mathbb{N}$ and for $n\to\infty$,

$$\begin{aligned}S_P =&\frac{p^2}{\mathrm{E}\,K_1^2(V)\,\mathrm{E}\,K_2^2(V)}\sum_{i=1}^{m}\frac{1}{n-i}\sum_{s,t=i+1}^{n}K_1\left(\frac{\widehat{\underline{R}}_s}{n+1}\right)K_1\left(\frac{\widehat{\underline{R}}_t}{n+1}\right)\\&\times K_2\left(\frac{\widehat{\underline{R}}_{s-i}}{n+1}\right)K_2\left(\frac{\widehat{\underline{R}}_{t-i}}{n+1}\right)\cos(a_{\boldsymbol{X}_s,\boldsymbol{X}_t})\cos(a_{\boldsymbol{X}_{s-i},\boldsymbol{X}_{t-i}})\rightsquigarrow\chi^2_{p^2m}\end{aligned}\qquad(5)$$

where V is uniformly distributed on $[0,1]$ and $\rightsquigarrow$ stands for the convergence in distribution.

The proof of Proposition 3 can be found in Appendix.

According to Proposition 3, the stochastic sign-vector occurring in the definition of the (complete or incomplete) semi-randomized or reflective lift-interdirections may be the same for all the lift-interdirections employed, as it is in all the empirical examples below.

The portmanteau test statistic of (5) from Proposition 3 is analogous to the statistics considered in [20] or [19]. The difference is that this time the test allows for various versions of hyperplane-based ranks and interdirections, including the rank versions presented in this article, which makes the test applicable even to data with fixed dimension $p \gg 2$.

It follows directly from [20] and the proof of Proposition 3, namely from the zero limit of (A.1) in probability, that the test based on S_P remains asymptotically optimal if the scores are tailored to the underlying elliptical density (1):

Table 1
The table lists average times (in seconds) needed for ranking n observations (uniformly distributed in $[0,1]^p$) by means of reflective, semi-randomized and randomized incomplete lift-interdirections for various data dimensions p and sample sizes n. The design size $N_H = |Q_L|$ was set to $5n$.

Average times (in seconds) based on 10,000 replications and $p = 20$							
ILI $/n$	$n = 25$	$n = 50$	$n = 100$	$n = 200$	$n = 400$	$n = 800$	$n = 1600$
reflective	0.010	0.020	0.045	0.120	0.272	0.749	2.407
semi-randomized	0.010	0.020	0.045	0.120	0.273	0.748	2.394
randomized	0.010	0.022	0.048	0.125	0.287	0.771	2.435
Average times (in seconds) based on 10,000 replications and $n = 1000$							
ILI$/p$	$p = 2$	$p = 4$	$p = 8$	$p = 16$	$p = 32$	$p = 64$	$p = 128$
reflective	0.946	0.974	1.001	1.038	1.145	1.536	3.294
semi-randomized	0.945	0.973	1.000	1.038	1.145	1.539	3.296
randomized	0.972	0.998	1.017	1.063	1.170	1.551	3.356

Proposition 4. *Let $\mathcal{X}_n$, $\hat{\underline{R}}_t$, and $C_{\mathbf{X}_t,\mathbf{X}_s}(\mathcal{X}_n)$ be as in Proposition 3, let f satisfy Assumption 1 and be positive almost everywhere, and consider the cumulative distribution function F_r corresponding to the radial density f_r. If $K_1 = -(2(\sqrt{f})'/\sqrt{f}) \circ F_r^{-1}$, $K_2 = F_r^{-1}$, and the assumptions of Proposition 3 hold, then the test based on (5) which rejects the null hypothesis for $S_P > \chi^2_{p^2\ m,1-\alpha}$ is locally asymptotically maximin at asymptotic level α for $\mathcal{H}_0$ with elliptical strict white noise against alternatives that $\{\mathbf{X}_t\}$ follows a nontrivial VARMA(q_1, q_2) model with $\max(q_1, q_2) = m$ and with a strict white noise from an elliptical distribution $\mathcal{E}_{\mathbf{0},\cdot,f}$ (i.e., with arbitrary scatter matrix).*

As f is typically unknown in practice, some standard score functions are used as K_1 and K_2. Different choices lead to different test statistics. In particular, constant unit scores lead to the multivariate portmanteau sign test without any ranks at all, working even for more general noise distributions with elliptical directions. Furthermore, if one chooses the van der Waerden score functions $K_j = \sqrt{G^{-1}}$, $j \in \{1, 2\}$, where G^{-1} stands for the quantile function of the χ^2_p distribution with p degrees of freedom, then it follows from Proposition 4 that the resulting test based on S_P is locally uniformly no worse than its Gaussian counterpart. It is easy to see that the asymptotic relative efficiencies provided in [20] remain valid even for S_P.

As far as the choice of the threshold parameter m is concerned, one could apply the recommendations formulated for the univariate or multivariate Ljung–Box test. As a rule of thumb, we recommend to choose m fixed and as small as possible to capture the significantly nonzero autocorrelation structure expected under the alternatives. As Proposition 4 implies, $m = \max(q_1, q_2)$ is optimal for VARMA(q_1, q_2) alternatives. This is in line with the recommendations contained in the literature. The behavior of the test based on (5) in dependence on m is also explored in the simulation study in Section 6.

5. Speed comparison

The newly introduced hyperplane-based ranks may be found useful in various contexts because they are fast to compute even for quite large dimensions and sample sizes. The speed of their computation is illustrated in Table 1 with the computational times recorded by a work notebook (AMD Ryzen 7, 64 GB RAM, 64 bit Win 11). It shows the average computational times of reflective, semi-randomized and randomized hyperplane-based ranks for uniformly distributed data when p is fixed and n is growing or when n is fixed and p is growing. The ranks are based on reflective, semi-randomized and randomized incomplete lift-interdirections using $N_H = |Q_L| = 5n$ hyperplanes. The complete symmetrized lift-interdirections are not involved in Table 1, because their computation is practically impossible in almost all of the settings considered. The computational times obtained for reflective and semi-randomized ranks are virtually the same and marginally but uniformly smaller than those observed for the randomized ranks.

The average computational times grow with n slightly faster than linearly (which can be expected because of the ranking involved) and with p slower than linearly if p and n are reasonably small. The steep increase in the computational times for high values of n (> 1000) or p (> 100) might be partly caused by the limitations of the notebook and the particular software implementation in R [27]. The ranks can be computed in a reasonable time even for $n = 10{,}000$ and $p = 1000$. In principle, the computation could be parallelized, which could further fasten it substantially. There is also a trade-off between the computational time and the choice of N_H, *i.e.*, the finite-sample quality of the Mahalanobis ranks' approximation. That is to say that the times needed to compute the incomplete lift-interdirections are roughly proportionate to the number N_H of considered hyperplanes. The asymptotic theory requires only $N_H/n \to \infty$ for fixed p.

6. Simulation study

This section considers the portmanteau test of (5) with the sign, Wilcoxon (*i.e.*, linear) or van der Waerden score functions $K_1 = K_2$, and compares it to the benchmark Ljung–Box test. The behavior of the tests is explored under the null hypothesis as well as under some alternatives in the form of a vector autoregressive model (VAR).

The simulation experiments employ time series lengths from $n = 100$ to $n = 1000$, time series dimensions from $p = 2$ to $p = 40$, threshold parameters from $m = 1$ to $m = 50$, and two innovation distribution families (multivariate Gaussian, multivariate Student) with two different choices for the matrix parameter $\mathbf{\Sigma}$ ($\mathrm{vec}(\mathbf{\Sigma}) = (1, 0.5, 0.5, 2)^\top$ for $p = 2$ and $\mathbf{\Sigma} = \mathbf{I}_p$ for $p > 2$) that stands for the covariance matrix in the Gaussian family and for the scale matrix in the Student family of distributions. The data were simulated

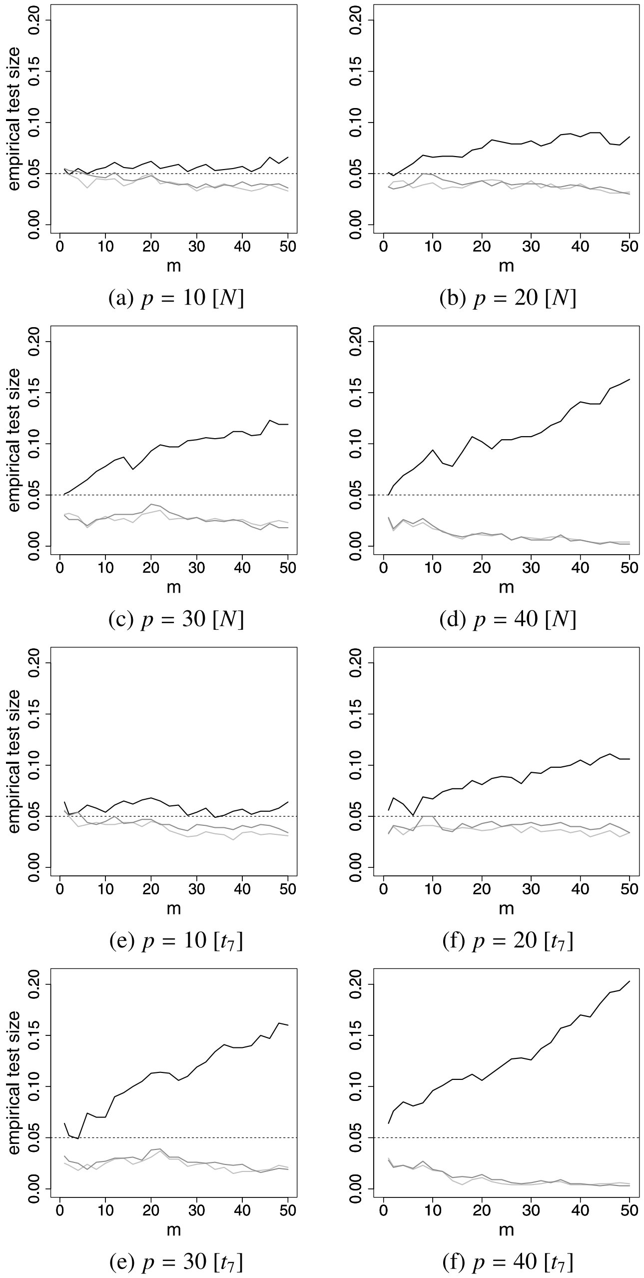


Fig. 2. Size comparison I. Size comparison of the Ljung–Box (parametric) portmanteau test (black) and the new sign-and-rank portmanteau tests with sign (dark gray) or van der Waerden (light gray) scores, at significance level $\alpha = 0.05$ and for various threshold parameters m, when applied to $n = 1000$ independent observations of dimension p from the standard normal distribution or from the canonical Student distribution with 7 degrees of freedom. The dimension and the distribution are indicated below each plot.

from two VAR models, VAR(1): $\boldsymbol{X}_t = \boldsymbol{\Psi}\boldsymbol{X}_{t-1} + \boldsymbol{\varepsilon}_t$ and VAR(2): $\boldsymbol{X}_t = \boldsymbol{\Psi}_1\boldsymbol{X}_{t-1} + \boldsymbol{\Psi}_2\boldsymbol{X}_{t-2} + \boldsymbol{\varepsilon}_t$, where $\boldsymbol{\Psi} = a\mathbf{I}_p$, $\boldsymbol{\Psi}_1 = \boldsymbol{\Psi}_2 = \frac{1}{2}a\mathbf{I}_p$, and a is a single scalar parameter. In other words, both the (stationary) VAR models use very simple diagonal VAR matrices that are scalar multiples of the identity matrix, and both the models degenerate to the strict white noise $\boldsymbol{\varepsilon}_t$ for $a = 0$. This simple setting makes it possible to display the test power as a function of a, *i.e.*, as a function of the departure from the null hypothesis. More complicated VAR(1) and VAR(2) (stationary) models would likely lead to the same conclusions.

The design sets $\mathcal{Q}_C$ and $\mathcal{Q}_L$ are chosen independently with replacement such that $|\mathcal{Q}_C| = |\mathcal{Q}_L| = 5n$, with the exception of Fig. 3 where also $|\mathcal{Q}_C| = |\mathcal{Q}_L| = 20n$ is considered (but with no significant improvement).

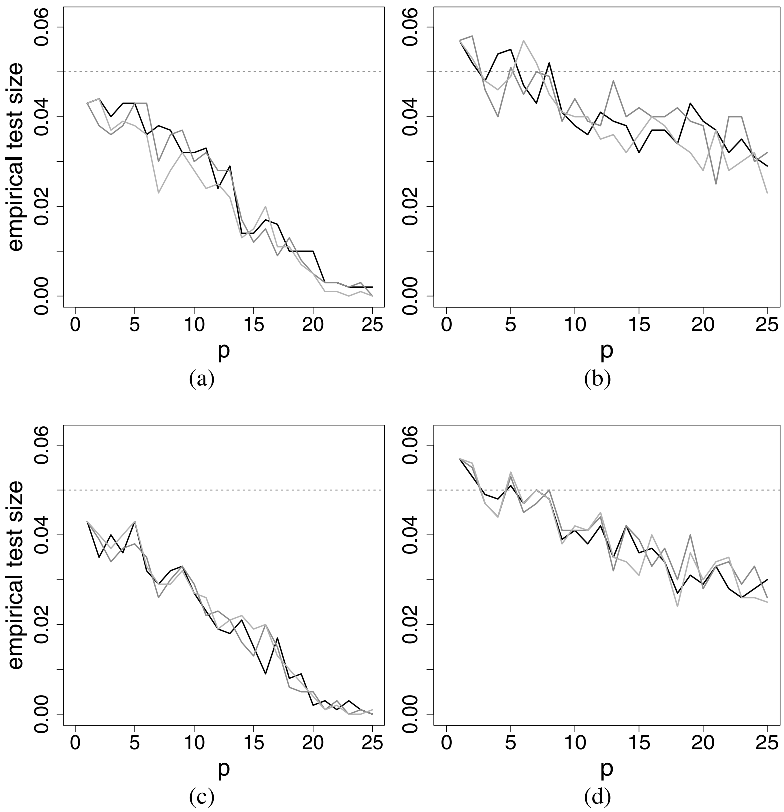


Fig. 3. Size comparison II. Size comparison regarding the new sign-and-rank portmanteau test with the Wilcoxon scores and $m = 1$, at significance level $\alpha = 0.05$ and for various dimension p, when applied to $n = 100$ (left) or $n = 250$ (right) observations coming from the multivariate standard normal distribution. The ranks of the test are based on three kinds of incomplete lift-interdirections: randomized [24] (light gray), semi-randomized (dark gray) and reflective (black). The hyperplane-based signs and ranks use $5n$ (top) or $20n$ (bottom) hyperplanes.

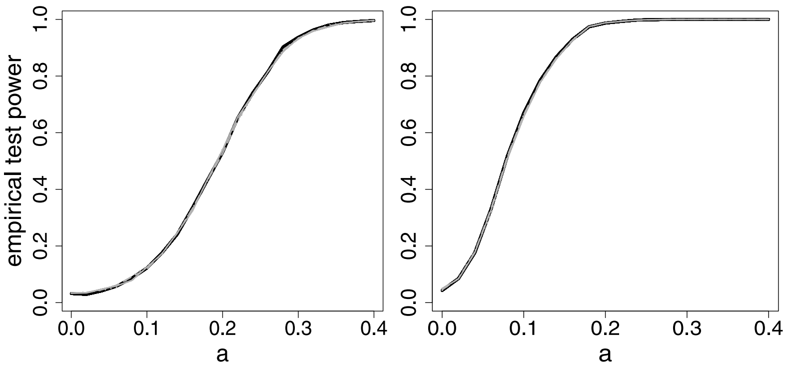


Fig. 4. Power comparison. The empirical power of the new sign-and-rank portmanteau test with the Gaussian (van der Waerden) scores and $m = 1$, at significance level $\alpha = 0.05$, when applied to time series of length $n = 100$ generated by the VAR(1) autoregressive model $\boldsymbol{X}_t = a\boldsymbol{X}_{t-1} + \epsilon_t$, where ϵ_t are independent and identically distributed zero-centered error terms coming from the bivariate Gaussian distribution with covariance matrix $\boldsymbol{\Sigma}$ (left) or from the bivariate Student t_1 (or Cauchy) distribution with scale matrix $\boldsymbol{\Sigma}$ (right) and $\mathrm{vec}(\boldsymbol{\Sigma}) = (1, 0.5, 0.5, 2)^{\mathsf{T}}$ in both cases. The ranks of the test are based on four kinds of lift-interdirections: ordinary/complete symmetrized [23] (thick black), incomplete randomized [24] (dotted gray), incomplete semi-randomized (dashed gray) and incomplete reflective (solid gray). The ranks based on ordinary/complete symmetrized lift-interdirections were used with the signs based on ordinary/complete interdirections for benchmarking. The various types of incomplete lift-interdirections were used with incomplete interdirections, with $5n$ hyperplanes in all cases. All the power curves in each plot virtually coincide.

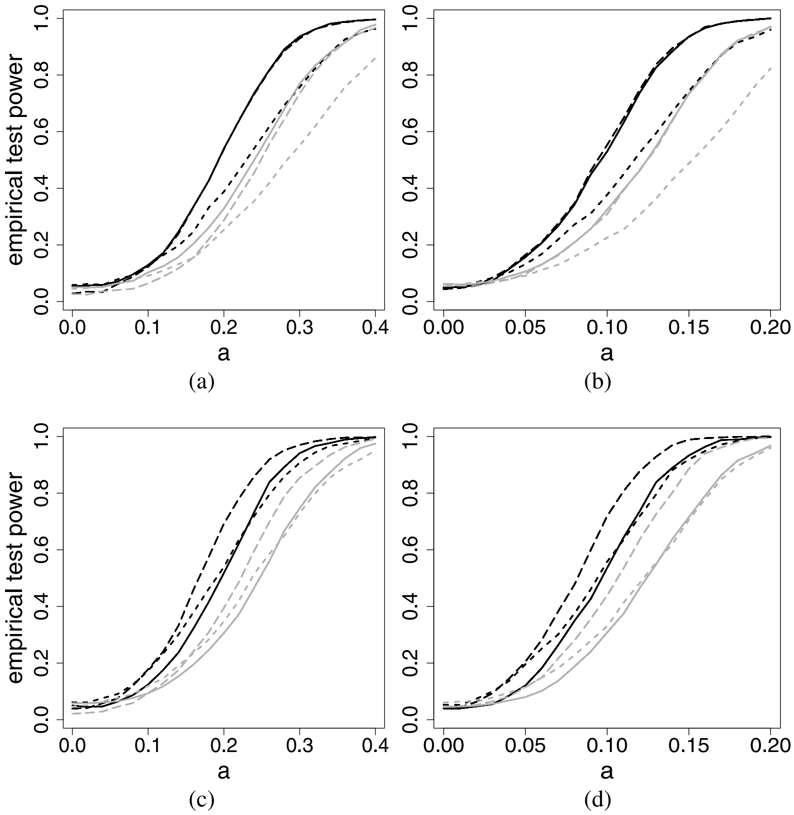


Fig. 5. Portmanteau tests I. Power comparison of the Ljung–Box (parametric) portmanteau test (solid) and the new sign-and-rank portmanteau tests with sign (short-dashed) or van der Waerden (long-dashed) scores and with threshold parameter $m = 1$ (black) and $m = 4$ (gray) when applied to time series of length $n = 100$ (left) and $n = 400$ (right) generated by the VAR(1) autoregressive model $\boldsymbol{X}_t = a\boldsymbol{X}_{t-1} + \epsilon_t$, where ϵ_t are independent and identically distributed zero-mean error terms coming from the bivariate Gaussian distribution with covariance matrix $\boldsymbol{\Sigma}$ (top) or from the bivariate Student t_3 distribution with scale matrix $\boldsymbol{\Sigma}$ (bottom) and $\text{vec}(\boldsymbol{\Sigma}) = (1, 0.5, 0.5, 2)^{\top}$ in all cases.

The presented plots show empirical sizes and powers of the tests for significance level $\alpha = 0.05$ computed from 1000 independent replications. As the two new rank concepts lead to almost identical results, only those based on incomplete semi-randomized lift-interdirections are used in the pictures, except for Figs. 3 and 4 included for comparison. A web page `https://www.karlin.mff.cuni.cz/~hudecova/research/example.html` provides an implementation of the test in R.

The empirical size is investigated in Figs. 2 and 3. If the sample size is $n = 1000$, then the sign-and-rank tests are virtually never liberal. They seem correctly sized for dimension p up to $p = 20$ or so and conservative for higher values of p, and the size very slightly decreases with m. In contrast, the parametric benchmark is virtually never conservative, its size is acceptable only up to $p = 10$ or so and further increases with m, especially for $p > 10$. These observations are valid at least for the random samples coming from the canonical multivariate normal and multivariate t_7 distributions. In other words, the newly presented sign-and-rank tests clearly surpass the benchmark in terms of test size in the special cases considered.

Fig. 3 illustrates with normally distributed data how the empirical size of the sign-and-rank test with Wilcoxon's scores and three types of hyperplane-based ranks decreases with p for $m = 1$ and small sample sizes $n = 100$ and $n = 250$. It also shows that there is hardly any difference in using $N_H = 5n$ and $N_H = 20n$ hyperplanes. The type of hyperplane-based ranks also apparently does not matter.

Fig. 4 confirms that the type of hyperplane-based ranks does not affect the test power, which is always virtually the same as for the portmanteau test of [19] using the ordinary/complete interdirections and symmetrized lift-interdirections. It also shows that the nonparametric portmanteau test is valid even when its parametric counterpart is not, namely for time series that are driven by heavy-tailed noise and not even weakly stationary.

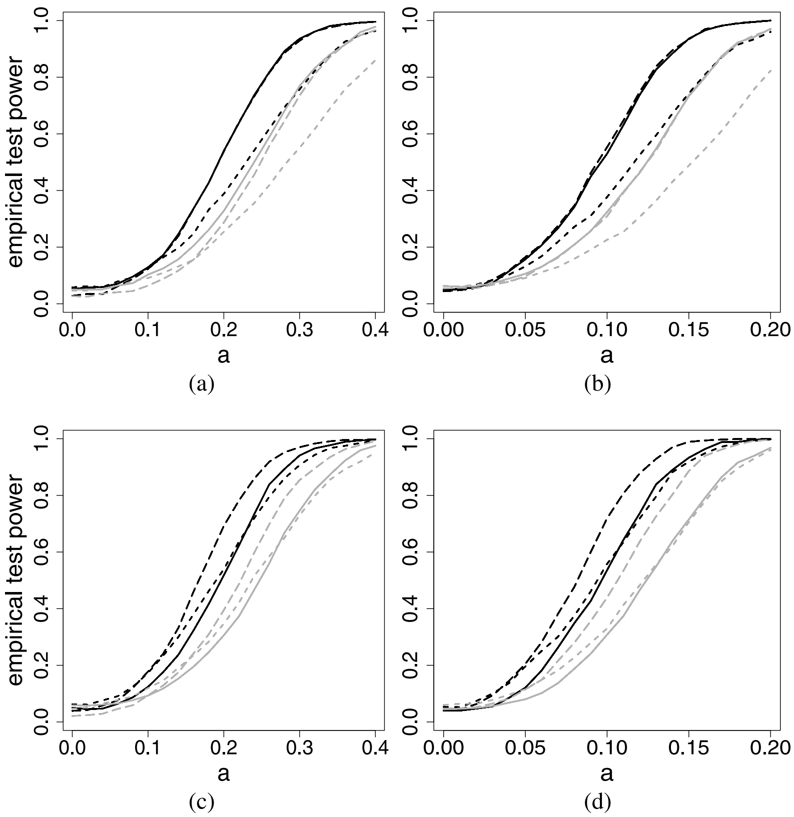


Fig. 6. Portmanteau tests II. Power comparison of the Ljung–Box (parametric) portmanteau test (solid) and the new sign-and-rank portmanteau tests with sign (short-dashed) or van der Waerden (long-dashed) scores and with threshold parameter $m = 1$ (black) and $m = 4$ (gray) when applied to time series of length $n = 100$ (left) and $n = 400$ (right) generated by the VAR(2) autoregressive model $\boldsymbol{X}_t = 0.5a\boldsymbol{X}_{t-1} + 0.5a\boldsymbol{X}_{t-2} + \boldsymbol{\epsilon}_t$, where $\boldsymbol{\epsilon}_t$ are independent and identically distributed zero-mean error terms coming from the bivariate Gaussian distribution with covariance matrix $\boldsymbol{\Sigma}$ (top) or from the bivariate Student t_3 distribution with scale matrix $\boldsymbol{\Sigma}$ (bottom) and $\mathrm{vec}(\boldsymbol{\Sigma}) = (1, 0.5, 0.5, 2)^{\top}$ in all cases.

The results, shown in Figs. 5, 6, and 7, are in good agreement with the theory. All the tests appear (roughly) correctly sized and weaker for $m = 4$ than for $m = 1$. The van der Waerden variant of the sign-and-rank test seems to be as powerful as the benchmark in the Gaussian case and strictly more powerful otherwise. The sign variant is generally less powerful than the van der Waerden one, and it considerably outperforms the benchmark in terms of test power only in the heavy-tailed case.

Note that, given the same design sets, the semi-randomized or reflective lift-interdirections are substantially faster to compute than the symmetrized lift-interdirections of [23] but only marginally faster to compute than the randomized ones of [24].

7. Data example

As a modest concrete real data illustration, consider daily returns of 14 public firms from 1990 to 2004, appearing also in [28]. The dataset is available in the `erer` [29] `R` package. The data form a 14-dimensional time series of 3747 observations. The benchmark as well as the sign-and-rank tests with the Wilcoxon, van der Waerden and sign scores reject the null hypothesis of elliptical strict white noise with p-value technically zero for any $m \in \{1, 3, 5, 10\}$.

8. Concluding remarks

This article introduces two new hyperplane-based multivariate rank concepts and justifies their use in an optimal, distribution-free, affine invariant and robust portmanteau test with weak assumptions.

The incomplete reflective and semi-randomized ranks appear as simple and fast to compute as possible, *i.e.*, as ideal hyperplane-based ranks for multivariate data, especially for data dimension higher than two or three. The sign-and-rank portmanteau test also seems very promising, especially when the identical van der Waerden scores are employed and when the number of observations is relatively large in comparison to the degrees of freedom of the null test χ^2 distribution.

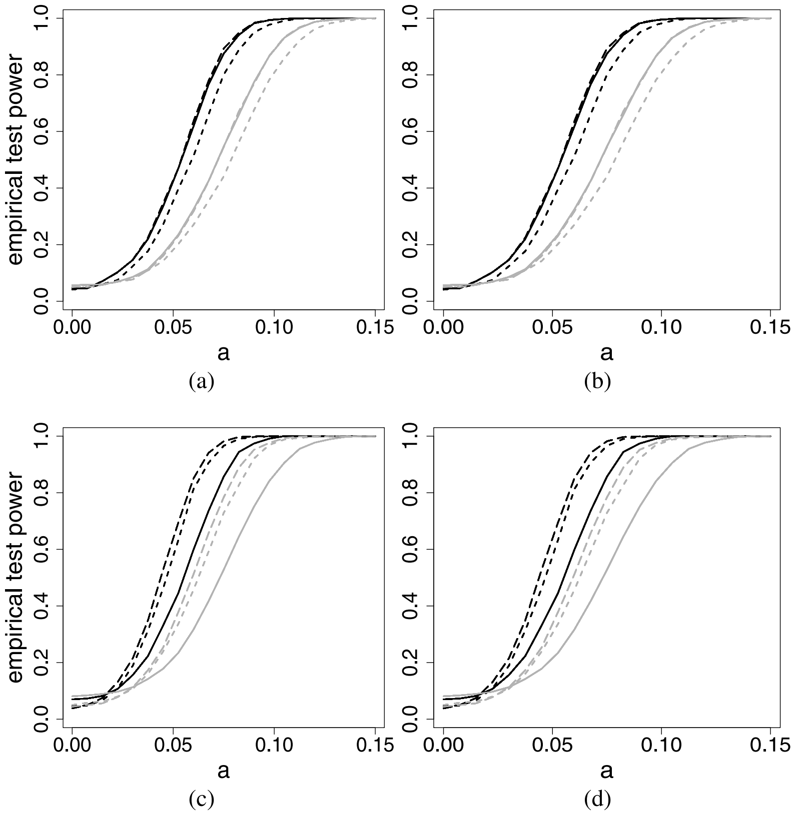


Fig. 7. Portmanteau tests III. Power comparison of the Ljung–Box (parametric) portmanteau test (solid) and the new sign-and-rank portmanteau tests with sign (short-dashed) or van der Waerden (long-dashed) scores and with threshold parameter $m = 1$ (black) and $m = 4$ (gray) when applied to five-dimensional time series of length $n = 1000$ generated by the VAR(1) model (left) or the VAR(2) model (right) used in the previous pictures but now with canonical Gaussian (top) or canonical Student t_3 (bottom) errors.

CRediT authorship contribution statement

Šárka Hudecová: Conceptualization, Methodology, Resources, Validation, Visualization, Writing. **Miroslav Šiman:** Conceptualization, Methodology, Software, Writing.

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Appendix. Proofs

If not stated otherwise, the expectations will be taken over everything stochastic, including the uniformly distributed random design sets and vectors used in the definitions of ranks and interdirections. Their independence from $\mathcal{X}_n$ greatly simplifies the proofs.

Without any loss of generality, it is possible to assume only spherical distributions with $\boldsymbol{\theta} = \mathbf{0}$, $\boldsymbol{\Sigma} = \mathbf{I}_p$, and distances $d_t = \|\boldsymbol{X}_t\|$ with ranks R_t^M. The standardized vectors $\boldsymbol{U}_t = d_t^{-1}\boldsymbol{X}_t$, $t \in \{1, \dots, n\}$, then form a strict white noise on the unit sphere in $\mathbb{R}^p$.

Proof of Proposition 2. The assertion (4) for incomplete reflective lift-interdirections $\bar{L}^s_{\boldsymbol{X}_t}$ follows directly from the analogous claim for incomplete lift-interdirections of [23] because all the randomized samples $\mathcal{X}^s$ are equidistributed and because the Mahalanobis ranks of $\boldsymbol{X}_t$ and $-\boldsymbol{X}_t$, $t \in \{1, \dots, n\}$, are by definition the same.

As far as the ranks of incomplete semi-randomized lift-interdirections are considered, the assertion (4) follows for a fixed $\boldsymbol{x} \in \mathbb{R}^p$ and design size $m = |\mathcal{Q}_L|$ from

$$\mathrm{E}\left(\frac{1}{m}\widetilde{L}^s_{\boldsymbol{x}}\right) = \frac{1}{m2^p}\sum_{\boldsymbol{s}\in\{-1,1\}^p}\mathrm{E}(\widetilde{L}^s_{\boldsymbol{x}}|\boldsymbol{s}) = \frac{1}{m2^p|\{\mathcal{Q}_L\}|}\sum_{\mathcal{Q}_L\in\{\mathcal{Q}_L\}}\sum_{\boldsymbol{s}\in\{-1,1\}^p}\mathrm{E}(\widetilde{L}^s_{\boldsymbol{x}}|\boldsymbol{s},\mathcal{Q}_L) = \ell(\boldsymbol{x}),$$

where $\ell(\boldsymbol{x})$ is the theoretical lift-interdirection from Proposition 1 of [19], and from

$$\operatorname{var}\left(\frac{1}{m}\widetilde{L}_{\boldsymbol{x}}^{s}\right) = \operatorname{E}\operatorname{var}\left(\frac{1}{m}\widetilde{L}_{\boldsymbol{x}}^{s}|\boldsymbol{s}\right) + \operatorname{var}\left(\frac{1}{m}\operatorname{E}(\widetilde{L}_{\boldsymbol{x}}^{s}|\boldsymbol{s})\right) = O(n^{-1}).$$

This holds because $\widetilde{L}_{\boldsymbol{x}}^{s}/m$, for fixed $\boldsymbol{s}$, is a U-statistic with bounded kernel and mean $\ell(\boldsymbol{x})$ equal to that of properly normalized incomplete symmetrized lift-interdirection of [23]. Therefore, $\widetilde{L}_{\boldsymbol{x}}^{s}/m \xrightarrow{L_2} \ell(\boldsymbol{x})$, and the rest is like in [19]. □

Lemma 1. *Consider spherically distributed vector $\boldsymbol{X}_t = (X_{t1}, \dots, X_{tp})^\top$ and define $B_t = \operatorname{sgn}(X_{t1})$ and $\mathbf{Y}_t = B_t\boldsymbol{X}_t$, $d_t = \|\boldsymbol{X}_t\| = \|\mathbf{Y}_t\|$ and $\boldsymbol{U}_t = \boldsymbol{X}_t/d_t = \mathbf{Y}_t B_t/d_t$. Let $\mathcal{Y}_n$ consist of $\mathbf{Y}_1, \dots, \mathbf{Y}_n$ and, analogously, let $\mathcal{B}_n$ be the sample of $B_1, \dots, B_n$.*

(i) *Then B_t is independent of $\mathbf{Y}_t$, and B_t, $t \in \{1, \dots, n\}$, are mutually independent and identically distributed with $\mathsf{P}(B_t = 1) = \mathsf{P}(B_t = -1) = 1/2$, $t \in \{1, \dots, n\}$. Furthermore, both the d_t's and their ranks R_t^M's are functions of $\mathcal{Y}_n$ only.*

(ii) *Assume random entities $\mathcal{R}_1$ and $\mathcal{R}_2$ such that $\mathcal{B}_n$, $\mathcal{Y}_n$, $\mathcal{R}_1$ and $\mathcal{R}_2$ are independent. Consider functions $h_0(\mathcal{X}_n, \mathcal{R}_1)$ and $h_1(\mathcal{X}_n, \mathcal{R}_2)$ such that $\operatorname{E}_{\mathcal{R}_1} h_0(\mathcal{X}_n, \mathcal{R}_1) = \operatorname{E}_{\mathcal{R}_1} h_0(\mathcal{Y}_n, \mathcal{R}_1)$, $h_1(\mathcal{X}_n, \mathcal{R}_2) = h_2(\mathcal{B}_n)h_3(\mathcal{Y}_n, \mathcal{R}_2)$, and $\operatorname{E} h_2(\mathcal{B}_n) = 0$. Then,*

$$\operatorname{E}\left[h_0(\mathcal{X}_n, \mathcal{R}_1)h_1(\mathcal{X}_n, \mathcal{R}_2)\right] = 0.$$

(iii) *Let the R_t's be some ranks of the $\boldsymbol{X}_t$'s, $t \in \{1, \dots, n\}$, exchangeable under $\mathcal{H}_0$. Assume $\alpha > 0$, and consider a product function g such that $g(r_1, \dots, r_k) = \prod_{i=1}^k K_i(r_i/(n+1))$ for functions K_i satisfying*

$$\frac{1}{n+1}\sum_{i=1}^{n}\left|K_i\left(\frac{i}{n+1}\right)\right|^{\alpha} \to \operatorname{E}|K_i(V)|^{\alpha} < \infty, \quad i \in \{1, \dots, k\},$$

as $n \to \infty$ for V uniformly distributed on $[0, 1]$. Then, as $n \to \infty$,

$$\operatorname{E}|g(R_1, \dots, R_k)|^{\alpha} \to \prod_{i=1}^{k}\operatorname{E}|K_i(V)|^{\alpha} < \infty.$$

Proof. For (i), see, *e.g.*, [5]. Statement (ii) follows from

$$\begin{aligned}\operatorname{E}\left[h_0(\mathcal{X}_n, \mathcal{R}_1)h_1(\mathcal{X}_n, \mathcal{R}_2)\right] &= \operatorname{E}_{\mathcal{X}_n,\mathcal{R}_2}[h_1(\mathcal{X}_n, \mathcal{R}_2)\operatorname{E}_{\mathcal{R}_1} h_0(\mathcal{X}_n, \mathcal{R}_1)] = \operatorname{E}_{\mathcal{Y}_n,\mathcal{B}_n,\mathcal{R}_2}\left[h_3(\mathcal{Y}_n, \mathcal{R}_2)h_2(\mathcal{B}_n)\operatorname{E}_{\mathcal{R}_1} h_0(\mathcal{Y}_n, \mathcal{R}_1)\right]\\ &= \operatorname{E}_{\mathcal{Y}_n,\mathcal{R}_2}\left[h_3(\mathcal{Y}_n, \mathcal{R}_2)\operatorname{E}_{\mathcal{R}_1} h_0(\mathcal{Y}_n, \mathcal{R}_1)\right]\left[\operatorname{E}_{\mathcal{B}_n} h_2(\mathcal{B}_n)\right] = 0.\end{aligned}$$

Finally, let the conditions of (iii) be satisfied. Then

$$p_k = P(R_1 = i_1, R_2 = i_2, \dots, R_k = i_k) = \frac{1}{n(n-1)\cdots(n-k+1)}$$

for any k-tuple $(i_1, \dots, i_k)$ of distinct indices from $\{1, \dots, n\}$. Consequently,

$$\begin{aligned}\operatorname{E}|g(R_1, \dots, R_k)|^{\alpha} &= p_k\sum_{i_1=1}^{n}\cdots\sum_{i_k=1}^{n}|g(i_1, \dots, i_k)|^{\alpha}\cdot \mathrm{I}[i_1, \dots, i_k \text{ are distinct}]\\ &\le p_k(n+1)^k\prod_{i=1}^{k}\left\{\frac{1}{n+1}\sum_{j=1}^{n}\left|K_i\left(\frac{j}{n+1}\right)\right|^{\alpha}\right\} \to \operatorname{E}|K_1(V)|^{\alpha}\cdots\operatorname{E}|K_k(V)|^{\alpha} < \infty,\end{aligned}$$

which concludes the argument. □

Lemma 1 here generalizes both Lemma 1 of [24] and Lemma 4 of [20]. It is applied several times in the next proof: $h_0(\mathcal{X}_n, \mathcal{R}_1)$ is always a function of ranks and $\mathcal{R}_1$ consists of $\mathcal{Q}_L$ and possibly also of the sign vector(s) defining the ranks, $h_1(\mathcal{X}_n, \mathcal{R}_2)$ is a function of directions $\boldsymbol{U}_t$ and/or interdirections and $\mathcal{R}_2$ may include $\mathcal{Q}_C$, and h_3 is a function of the signs B_t's. Note that $\cos(a_{\boldsymbol{X}_t,\boldsymbol{X}_s}) = B_t B_s\cos(a_{\mathbf{Y}_t,\mathbf{Y}_s})$, $t, s \in \{1, \dots, n\}$.

Proof of Proposition 3. The proof proceeds similarly as in [20] with some necessary modifications. Define

$$S^0 = \frac{k^2}{\operatorname{E}K_1^2(V)\operatorname{E}K_2^2(V)}\sum_{i=1}^{m}\frac{1}{n-i}\sum_{s,t=i+1}^{n}K_1\left(F_r(d_s)\right)K_1\left(F_r(d_t)\right)K_2\left(F_r(d_{s-i})\right)K_2\left(F_r(d_{t-i})\right)\boldsymbol{U}_s^\top\boldsymbol{U}_t\boldsymbol{U}_{s-i}^\top\boldsymbol{U}_{t-i},$$

$$T_{1,i}^n = \frac{1}{n-i}\sum_{s,t=i+1}^{n}K_1\left(\frac{\widehat{\mathrm{R}}_s}{n+1}\right)K_1\left(\frac{\widehat{\mathrm{R}}_t}{n+1}\right)K_2\left(\frac{\widehat{\mathrm{R}}_{s-i}}{n+1}\right)K_2\left(\frac{\widehat{\mathrm{R}}_{t-i}}{n+1}\right)\left(\cos(a_{\boldsymbol{X}_s,\boldsymbol{X}_t})\cos(a_{\boldsymbol{X}_{s-i},\boldsymbol{X}_{t-i}}) - \boldsymbol{U}_s^\top\boldsymbol{U}_t\boldsymbol{U}_{s-i}^\top\boldsymbol{U}_{t-i}\right),$$

$$\begin{aligned}T_2^n = \sum_{i=1}^{m}\frac{1}{n-i}\sum_{s,t=i+1}^{n}\Bigg(&K_1\left(\frac{\widehat{\mathrm{R}}_s}{n+1}\right)K_1\left(\frac{\widehat{\mathrm{R}}_t}{n+1}\right)K_2\left(\frac{\widehat{\mathrm{R}}_{s-i}}{n+1}\right)K_2\left(\frac{\widehat{\mathrm{R}}_{t-i}}{n+1}\right)\\ &- K_1\left(F_r(d_s)\right)K_1\left(F_r(d_t)\right)K_2\left(F_r(d_{s-i})\right)K_2\left(F_r(d_{t-i})\right)\Bigg)\boldsymbol{U}_s^\top\boldsymbol{U}_t\boldsymbol{U}_{s-i}^\top\boldsymbol{U}_{t-i}.\end{aligned}$$

Furthermore, set

$$\mathbf{T}_n = (\mathbf{T}_{n,1}^\top, \dots, \mathbf{T}_{n,m}^\top)^\top, \quad \mathbf{S}_n = (\mathbf{S}_{n,1}^\top, \dots, \mathbf{S}_{n,m}^\top)^\top, \quad \hat{\mathbf{S}}_n = (\hat{\mathbf{S}}_{n,1}^\top, \dots, \hat{\mathbf{S}}_{n,m}^\top)^\top$$

for

$$\mathbf{T}_{n,i} = \frac{1}{\sqrt{n-i}} \sum_{s=i+1}^{n} \operatorname{vec}\left(K_1(F_r(d_s))K_2(F_r(d_{s-i}))\boldsymbol{U}_s\boldsymbol{U}_{s-i}^\top\right), \quad \mathbf{S}_{n,i} = \frac{1}{\sqrt{n-i}} \sum_{s=i+1}^{n} \operatorname{vec}\left(K_1\left(\frac{R_s^M}{n+1}\right)K_2\left(\frac{R_{s-i}^M}{n+1}\right)\boldsymbol{U}_s\boldsymbol{U}_{s-i}^\top\right),$$

$$\hat{\mathbf{S}}_{n,i} = \frac{1}{\sqrt{n-i}} \sum_{s=i+1}^{n} \operatorname{vec}\left(K_1\left(\frac{\hat{\mathrm{R}}_s}{n+1}\right)K_2\left(\frac{\hat{\mathrm{R}}_{s-i}}{n+1}\right)\boldsymbol{U}_s\boldsymbol{U}_{s-i}^\top\right).$$

Basically, the proof approximates S_P of (5) with $S^0 = \mathbf{T}_n^\top \Gamma^{-1}\mathbf{T}_n$,

$$\Gamma = \frac{1}{p^2}\,\mathrm{E}\,K_1^2(V)\,\mathrm{E}\,K_2^2(V)[\mathrm{I}_m \otimes (\mathrm{I}_p \otimes \mathrm{I}_p)],$$

whose asymptotic $\chi^2_{p^2m}$ distribution follows from the asymptotic distribution $\mathcal{N}(\mathbf{0}, \Gamma)$ of $\mathbf{T}_n$. The difference

$$S_P - S^0 = \frac{p^2}{\mathrm{E}\,K_1^2(V)\,\mathrm{E}\,K_2^2(V)}\left(\sum_{i=1}^{m} T_{1,i}^n + T_2^n\right) \tag{A.1}$$

has to be asymptotically negligible because T_2^n and all $T_{1,i}^n$, $i \in \{1, \dots, m\}$, converge to zero in probability for $n \to \infty$, which will be proved below.

The norm $\|\mathbf{S}_n - \mathbf{T}_n\|$ is the same as in [20]. Therefore,

$$\|\mathbf{S}_n - \mathbf{T}_n\| = o_{L^2}(1) \text{ and } K_1\left(\frac{R_s^M}{n+1}\right)K_2\left(\frac{R_{s-i}^M}{n+1}\right) - K_1(F_r(d_s))K_2(F_r(d_{s-i})) = o_{L^2}$$

as is proved there. Furthermore,

$$\mathrm{E}\,\|\hat{\mathbf{S}}_n - \mathbf{S}_n\|^2 = \sum_{i=1}^{m}\sum_{s=i+1}^{n}\frac{1}{n-i}\,\mathrm{E}\left[K_1\left(\frac{R_s^M}{n+1}\right)K_2\left(\frac{R_{s-i}^M}{n+1}\right) - K_1\left(\frac{\hat{\mathrm{R}}_s}{n+1}\right)K_2\left(\frac{\hat{\mathrm{R}}_{s-i}}{n+1}\right)\right]^2$$

(thanks to Lemma 1, see the details below) is also $o(1)$ because the squared differences in the summands are $o_P(1)$ and also uniformly integrable. Then $\mathrm{E}\,\|T_n\|^2 = m\,\mathrm{E}\,K_1^2(V)K_2^2(V) < \infty$, therefore $\mathrm{E}\,\|\hat{\mathbf{S}}_n\| < \infty$ and

$$\mathrm{E}\,|T_2^n| \le \sqrt{\mathrm{E}\,\|\hat{\mathbf{S}}_n + \mathbf{T}_n\|^2}\sqrt{\mathrm{E}\,\|\hat{\mathbf{S}}_n - \mathbf{T}_n\|^2} \to 0,$$

for $n \to \infty$, owing to the Cauchy–Schwarz inequality. Therefore, $T_2^n \to 0$ with $n \to \infty$ both in L_1 and in probability.

As for $\mathrm{E}\,\|T_{1,i}^n\|^2$, define

$$G_{s,t,i}(\mathcal{X}_n) = \left(\cos(a_{\boldsymbol{X}_s,\boldsymbol{X}_t})\cos(a_{\boldsymbol{X}_{s-i},\boldsymbol{X}_{t-i}}) - \boldsymbol{U}_s^\top\boldsymbol{U}_t\boldsymbol{U}_{s-i}^\top\boldsymbol{U}_{t-i}\right)$$

and observe that $G_{s,t,i}(\mathcal{X}_n) = 0$ for $s = t$. Thanks to Lemma 1 (ii) (see the details below)

$$\mathrm{E}\,\|T_{1,i}^n\|^2 = \frac{4}{(n-i)^2}\sum_{s,t=i+1,s<t}^{n}\mathrm{E}\,K_1^2\left(\frac{\hat{\mathrm{R}}_s}{n+1}\right)K_2^2\left(\frac{\hat{\mathrm{R}}_t}{n+1}\right)K_1^2\left(\frac{\hat{\mathrm{R}}_{s-i}}{n+1}\right)K_2^2\left(\frac{\hat{\mathrm{R}}_{t-i}}{n+1}\right)G_{s,t,i}^2$$

$$\le \frac{4}{(n-i)^2}\sum_{s,t=i+1,s<t}^{n}\left(\mathrm{E}\left|K_1\left(\frac{\hat{\mathrm{R}}_s}{n+1}\right)K_2\left(\frac{\hat{\mathrm{R}}_t}{n+1}\right)K_1\left(\frac{\hat{\mathrm{R}}_{s-i}}{n+1}\right)K_2\left(\frac{\hat{\mathrm{R}}_{t-i}}{n+1}\right)\right|^{2+\delta}\right)^{\frac{2}{2+\delta}}\left(\mathrm{E}\,|G_{s,t,i}|^{2(2+\delta)/\delta}\right)^{\delta/(2+\delta)}$$

for the particular $\delta > 0$ from the assumption owing to Hölder's inequality. Now the summands are products of two terms in the exponentiated parentheses: the first parenthesis is bounded for $s \neq t-i$ thanks to Lemma 1 (iii) and the second term converges to 0 in L_2 and in probability as $n \to \infty$ because of bounded $|G_{s,t,i}|$ and the consistency property of used interdirections. The case $s = t-i$ can be handled analogously, as in the proof of Lemma 3 in [20].

Lemma 1 is applied above in the same way as Lemma 4 in [20]: $h_0(\mathcal{X}_n, \mathcal{R}_1)$ is typically a function of ranks and $\mathcal{R}_1$ may reflect the randomness in their definition possibly hidden in the design set and/or sign vector(s). The lift-interdirections considered in Proposition 3 ensure that $\mathrm{E}_{\mathcal{R}_1}\, h_0(\mathcal{X}_n, \mathcal{R}_1) = \mathrm{E}_{\mathcal{R}_1}\, h_0(\mathcal{Y}_n, \mathcal{R}_1)$. Lemma 1 (ii) is then used with h_1, h_2, h_3, where for example

$$h_1(\mathcal{X}_n, \mathcal{R}_2) = \boldsymbol{U}_s^\top\boldsymbol{U}_t\boldsymbol{U}_{s-i}^\top\boldsymbol{U}_{t-i} = \boldsymbol{X}_s^\top\boldsymbol{X}_t\boldsymbol{X}_{s-i}^\top\boldsymbol{X}_{t-i}/(d_sd_td_{s-i}d_{t-i}),$$

$h_2(\mathcal{B}_n) = B_sB_tB_{s-i}B_{t-i}$ and $h_3(\mathcal{Y}_n, \mathcal{R}_2) = \mathbf{Y}_s^\top\mathbf{Y}_t\mathbf{Y}_{s-i}^\top\mathbf{Y}_{t-i}/(d_sd_td_{s-i}d_{t-i})$ (where the dependence on $\mathcal{R}_2$ is missing entirely), or

$$h_1(\mathcal{X}_n, \mathcal{R}_2) = G_{s,t,i}(\mathcal{X}_n)G_{\tilde{s},\tilde{t},i}(\mathcal{X}_n),$$

$h_2(\mathcal{B}_n) = B_sB_tB_{s-i}B_{t-i}B_{\tilde{s}}B_{\tilde{t}}B_{\tilde{s}-i}B_{\tilde{t}-i}$ and $h_3(\mathcal{Y}_n, \mathcal{R}_2) = G_{s,t,i}(\mathcal{Y}_n)G_{\tilde{s},\tilde{t},i}(\mathcal{Y}_n)$ where $\mathcal{R}_2$ may account for the random design of incomplete interdirections if they are used. □

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