Testing Axial Symmetry by Means of Directional Quantile Regression Coefficients

Miroslav Šiman

Abstract Testing axial symmetry given the axial direction has recently attracted considerable attention not only because of its direct practical applications but also because of its wide implications for testing exchangeability, independence, goodness-of-fit, or equality of scale. The contribution extends the family of recently developed tests of axial symmetry with new members that are based on coefficient estimates in directional quantile regression. The proposed testing tools are especially suitable for the situations not covered well by available competitors, i.e., in the linear regression context or when regression rank scores are not available. The performance of the new tests in such settings is illustrated with a few representative simulation experiments.

1 Introduction

This contribution focuses on nonparametric testing of the hypothesis that a stochastic vector $\mathbf{Y} \in \mathbb{R}^m$, $m \geq 2$, is symmetric around a line in a given (unit) direction $\mathbf{u} \in \mathcal{S}^{m-1}$, which means, from the mathematical standpoint, that

$$\mathcal{L}\{\mathbf{Y} - \mathrm{E}\mathbf{Y}\} = \mathcal{L}\{\mathbf{M}(\mathbf{Y} - \mathrm{E}\mathbf{Y})\}$$

for $\mathbf{M} = 2\mathbf{u}\mathbf{u}^\top - \mathbf{I}$. That kind of symmetry will be termed axial symmetry hereinafter.

If $\mathbf{Y}$ is accompanied with a covariate vector $\mathbf{Z}$, then it makes sense to speak about the symmetry of the conditional distribution $\mathcal{L}(\mathbf{Y}|\mathbf{Z})$. It is characterized with the invariance property:

$$\mathcal{L}\{\mathbf{Y} - \mathrm{E}(\mathbf{Y}|\mathbf{Z})|\mathbf{Z}\} = \mathcal{L}\{\mathbf{M}(\mathbf{Y} - \mathrm{E}(\mathbf{Y}|\mathbf{Z}))|\mathbf{Z}\}$$

M. Šiman (✉)
The Czech Academy of Sciences, Institute of Information Theory and Automation, Prague, Czech Republic
e-mail: siman@utia.cas.cz

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2024

M. Barigozzi et al. (eds.), *Recent Advances in Econometrics and Statistics*,
https://doi.org/10.1007/978-3-031-61853-6_13

for $\mathbf{M} = 2\mathbf{u}\mathbf{u}^\top - \mathbf{I}$ and called conditional axial symmetry for short. It is tested in regression settings.

The literature already knows a few nonparametric tests of symmetry about a particular or special line in the bivariate case when the axial and half-space symmetries coincide; see, e.g., Hollander (1971), Modarres (2008) and Rao and Raghunath (2012). Unfortunately, the general multivariate case has been rather neglected until recently, especially if one does not count the somewhat related tests of conditional central symmetry of Su (2006) and its references. And although there can never exist multiple parallel axes of symmetry, the natural idea of specifying only the axial direction (and thus avoiding the necessity to know the axis itself or to center the data first) seems also quite recent, originating probably only in the articles mentioned in the next paragraph.

The recent contributions to testing axial symmetry in spaces of arbitrary dimension can be summarized as follows:

(a) Kalina (2021) proposed some permutation tests of the hypothesis without desirable invariance properties.
(b) Hudecová and Šiman (2021b) came up with some powerful parametric, nonparametric, permutation, and asymptotic naturally invariant tests of the hypothesis, but under quite stringent distributional assumptions.
(c) Hudecová and Šiman (2021a) introduced naturally invariant asymptotic tests of the hypothesis even in a general regression setup, but either weak or, in the other case, badly sized, slow to compute and using both very restrictive distributional assumptions and complex asymptotic distributions intractable in large dimensions.
(d) Hudecová and Šiman (2023) proposed powerful tests with mild assumptions, good small-sample performance, and simple asymptotic distributions, but only in the non-regression case and still perhaps somewhat slow to compute for extremely large datasets.

In particular, the tests of Kalina (2021) used scatter matrix estimators, the tests of Hudecová and Šiman (2021b) employed canonical or rank correlations, and the tests of Hudecová and Šiman (2021a) and Hudecová and Šiman (2023) were respectively based on the rank scores and integrated rank scores resulting from the directional quantile regression of Hallin et al. (2010).

To sum up, the tests of Hudecová and Šiman (2023) already appear very satisfactory for the non-regression case, although they are known to be consistent only in the class of elliptically symmetric distributions and inconsistent against some rather special alternatives also illustrated in Hudecová and Šiman (2023). Their main problem may be that they are based on the quantile regression rank scores that may be sometimes slow or even impossible to compute with the statistical software at hand. They also exist only for the non-regression case. Unfortunately, the only tests so far proposed for the linear regression case are those of Hudecová and Šiman (2021a) which are still far from ideal. And they also use the rank scores.

Consequently, this contribution deals with a general linear regression case. It still employs the directional quantile regression of Hallin et al. (2010), which has already proved advantageous, but it avoids the use of any rank scores and the problems associated with their computation by using directly the directional quantile regression coefficient estimators. Although the corresponding quantile regression task involves response-dependent and stochastic regressors, the general asymptotic results needed for the testing are already available in this time; see Angrist et al. (2006) and Chernozhukov et al. (2022). They can be further simplified, thanks to the axial symmetry assumption, as is shown here in the non-regression case. All the presented tests involve kernel estimators, but it does not matter too much for dimension m up to five or so because they are proposed for large datasets anyway.

The `quantreg` package (Koenker, 2015) for `R` (R Core Team, 2021) produces the rank scores only when the modified Barrodale and Roberts simplex-type algorithm (`br`) is used whose computational complexity typically depends on the number of observations quadratically. However, the quantile regression coefficients alone can be computed with many other algorithms included in the package, and some of them are asymptotically much faster. For example, the computational complexity of some interior point methods with preprocessing often grows with the number of observations only linearly. This explains why avoiding rank scores and using only quantile regression coefficients may have significant speed benefits. For instance, the computational times of the most suitable algorithms (such as `conquer` or `pfnb`) included in the `quantreg` package were observed even one thousand times faster than those of the `br` algorithm for one million regression observations of small dimension. See Chapter 6 of Koenker (2005) for precise statements and further discussion regarding the computational aspects and complexities.

Axial symmetry naturally occurs in the world around us. Therefore, its tests may be useful. Furthermore, Hudecová and Šiman (2021a) and Hudecová and Šiman (2021b) explain and demonstrate by means of both simulated and real data examples how the tests of axial symmetry may be used even for testing certain exchangeability, independence, goodness-of-fit, and equality-of-scale hypotheses.

The outline of the paper is as follows. Section 2 introduces necessary notation, summarizes directional quantile regression to be used for the tests, and reviews relevant results. Section 3 presents the information about the general test statistics and their distributions. Section 4 simplifies the results in the non-regression case, Sect. 5 mainly applies the tests to simulated data, and Sect. 6 concludes with some remarks and comments.

2 Definitions and Notation

Let $\mathbf{Y} = (Y^{(1)}, \dots, Y^{(m)})^\top \in \mathbb{R}^m$ and $\mathbf{X} = (1, X^{(2)}, \dots, X^{(p)})^\top = (1, \mathbf{Z}^\top)^\top \in \mathbb{R}^p$ be the random vectors of responses and regressors, always satisfying

Assumption 1 The joint probability distribution $\mathcal{L}$ of $(\mathbf{Y}^\top, \mathbf{Z}^\top)^\top$ is absolutely continuous with finite expectation, cumulative distribution function F, and probability density function f that is continuous, bounded, and positive in the interior of a connected support.

In the non-regression case, $\mathbf{X}$ turns to 1, $\mathbf{Z}$ simply disappears from all the definitions, and the distributions conditional on $\mathbf{Z}$ or $\mathbf{X}$ change to the unconditional ones.

Consider also the null hypothesis of conditional axial symmetry.

$$H_0^S(\mathbf{u}): \ \mathcal{L}(\mathbf{Y}|\mathbf{Z}) \text{ is axially symmetric around a line in a given direction } \mathbf{u} \in \mathcal{S}^{m-1}$$

dealt with in this article.

Hallin et al. (2010) introduced the concept of directional quantile regression which is, for the particular direction $\mathbf{u}$, based on the minimization problem:

$$\min_{(\mathbf{a}^\top, \mathbf{c}^\top)^\top \in \mathbb{R}^{m+p-1}} \mathrm{E}\rho_\tau(\mathbf{u}^\top \mathbf{Y} - \mathbf{a}^\top \mathbf{X} - \mathbf{c}^\top \boldsymbol{\Gamma}_\mathbf{u}^\top \mathbf{Y}) \tag{1}$$

leading to the τ-dependent and $\mathbf{u}$-dependent directional regression quantile coefficient vector $(\mathbf{a}_{\tau,\mathbf{u}}^\top, \mathbf{c}_{\tau,\mathbf{u}}^\top)^\top$ minimizing (1). Here $\tau \in (0, 1)$ denotes the quantile level, $\boldsymbol{\Gamma}_\mathbf{u} \in \mathbb{R}^{m\times(m-1)}$ complements $\mathbf{u}$ to an orthonormal matrix, and

$$\rho_\tau(t) = t(\tau - \mathrm{I}(t < 0)) = \max\{(\tau - 1)t, \tau t\}$$

stands for the quantile check function from the L_1 definition of ordinary or regression quantiles (Koenker, 2005). For simplicity, define also the vector $\mathbf{W} = (\mathbf{X}^\top, \mathbf{Y}^\top \boldsymbol{\Gamma}_\mathbf{u})^\top$ and residuum

$$r(\tau) := \mathbf{u}^\top \mathbf{Y} - (\mathbf{a}_{\tau,\mathbf{u}}^\top, \mathbf{c}_{\tau,\mathbf{u}}^\top)\mathbf{W}.$$

In other words, (1) stands for the quantile regression problem with response $\mathbf{u}^\top \mathbf{Y}$ (the projection of $\mathbf{Y}$ to the direction $\mathbf{u}$) and regressor vector $\mathbf{W}$ including $\mathbf{X}$ and the projection of $\mathbf{Y}$ to the orthogonal complement of $\mathbf{u}$.

Hudecová and Šiman (2021a) proved useful connections between $H_0^S(\mathbf{u})$ and the directional regression quantile coefficient vector $\mathbf{c}_{\tau,\mathbf{u}}$ obtained for the same $\mathbf{u}$. In particular, they showed in the non-regression multivariate case that $H_0^S(\mathbf{u})$ always implies

$$H_0(\mathbf{u}): \ \mathbf{c}_{\tau,\mathbf{u}} = \mathbf{0} \text{ for all } \tau \in (0, 1)$$

and that the implication changes to the equivalence for all elliptical distributions where, moreover, $\mathbf{c}_{\tau,\mathbf{u}}$ never depends on τ. Furthermore, they demonstrated that the link between $H_0^S(\mathbf{u})$ and $H_0(\mathbf{u})$ remains preserved even in certain linear regression models such as in the common-scale linear regression model:

$$\mathbf{Y} = \mathbf{F}\mathbf{X} + (\mathbf{d}^\top \mathbf{X})\boldsymbol{\eta} \tag{2}$$

with parametric matrix $\mathbf{F}$, parametric vector $\mathbf{d} \neq \mathbf{0}$, and a centered absolutely continuous error term $\boldsymbol{\eta} \in \mathbb{R}^m$ independent of $\mathbf{X}$. This representative linear model will always be considered here in the regression case.

In other words, $H_0^S(\mathbf{u})$ implies $H_0(\mathbf{u})$ for any $\mathbf{u}$, $\|\mathbf{u}\| = 1$, even in the regression model (2) assumed here, and the reverse is true if $\boldsymbol{\varepsilon}$ is elliptically distributed.

Consequently, the rest of the article focuses on testing if $\mathbf{c}_{\tau,\mathbf{u}} = \mathbf{0}$, which uses some general results for quantile regression with stochastic regressors, already published in the statistical literature. The assumption of $H_0^S(\mathbf{u})$ may lead to simplifications in some special cases; see Sect. 4.

Consider also the sample case with n independent copies $(\mathbf{Y}_i^\top, \mathbf{X}_i^\top)^\top$ and $\mathbf{W}_i$ of their population counterparts, $i = 1, \ldots, n$. Then the expectation in (1) is taken with respect to the empirical measure, all the sample characteristics are denoted with $\widehat{\ }$ and

$$\widehat{r}_i(\tau) = \mathbf{u}^\top \mathbf{Y}_i - (\widehat{\mathbf{a}}_{\tau,\mathbf{u}}^\top, \widehat{\mathbf{c}}_{\tau,\mathbf{u}}^\top)\mathbf{W}_i,\ i = 1, \ldots, n.$$

The task is to test $H_0^S(\mathbf{u})$ by means of $\widehat{\mathbf{c}}_{\tau,\mathbf{u}}$. Fortunately, (1) corresponds to the generally misspecified ordinary quantile regression model with stochastic regressor $\mathbf{W}$ and scalar response $\mathbf{u}^\top\mathbf{Y}$, and the asymptotic theory for its sample coefficient estimators and their linear functions has already been developed in Angrist et al. (2006); see also Proposition 1 of Chernozhukov et al. (2022). Those results useful for the purpose of this contribution are summarized in the next section.

3 Regression Case

Choose $\boldsymbol{\Gamma}_\mathbf{u}$ and fix $\varepsilon \in (0, 0.5)$. Suppose that Assumption 1 holds together with

Assumption 2 Assume that:

1. there exists $\delta > 0$ such that

$$\mathrm{E}\|\mathbf{Y}\|^{2+\delta} < \infty \quad \text{and} \quad \mathrm{E}\|\mathbf{Z}\|^{2+\delta} < \infty, \tag{3}$$

2. the conditional density $f_{\mathbf{u}^\top\mathbf{Y}|\mathbf{W}}(v|\mathbf{w})$ of $\mathbf{u}^\top\mathbf{Y}$ given $\mathbf{W} = \mathbf{w}$ is bounded and uniformly continuous in $(v, \mathbf{w}^\top)^\top$ over the support of $(\mathbf{u}^\top\mathbf{Y}, \mathbf{W}^\top)^\top$,
3. the Jacobian matrix

$$\mathbf{J}(\tau) := \mathrm{E}\big[f_{\mathbf{u}^\top\mathbf{Y}|\mathbf{W}}\big((\mathbf{a}_{\tau,\mathbf{u}}^\top, \mathbf{c}_{\tau,\mathbf{u}}^\top)\mathbf{W}|\mathbf{W}\big)\mathbf{W}\mathbf{W}^\top\big] \tag{4}$$

is positive definite for all $\tau \in [\varepsilon, 1-\varepsilon]$, and its minimal eigenvalue is bounded away from zero uniformly on that interval.

Then, one can state the following result.

Proposition 1 (Angrist et al. (2006); Chernozhukov et al. (2022)) *In the sample case, if Assumption 1 and Assumption 2 hold, then the τ-indexed process*

$$\sqrt{n}\mathbf{J}(\tau)\left[\begin{pmatrix}\widehat{\mathbf{a}}_{\tau,\mathbf{u}}\\ \widehat{\mathbf{c}}_{\tau,\mathbf{u}}\end{pmatrix}-\begin{pmatrix}\mathbf{a}_{\tau,\mathbf{u}}\\ \mathbf{c}_{\tau,\mathbf{u}}\end{pmatrix}\right],\quad \tau\in[\varepsilon,1-\varepsilon], \tag{5}$$

converges in distribution to a zero-mean vector Gaussian process $\{\mathbf{G}(\tau), \tau \in [\varepsilon, 1-\varepsilon]\}$ with covariance matrix function $\mathbf{\Sigma}(\tau,\tau') = \mathrm{E}\big[\mathbf{G}(\tau)\mathbf{G}(\tau')^\top\big]$ given by

$$\mathbf{\Sigma}(\tau,\tau') = \mathrm{E}\big([\tau-\mathrm{I}\{r(\tau)<0\}][\tau'-\mathrm{I}\{r(\tau')<0\}]\mathbf{W}\mathbf{W}^\top\big),\ \ \tau,\tau'\in[\varepsilon,1-\varepsilon],$$

which simplifies to

$$\mathbf{\Sigma}_0(\tau,\tau') = [\min\{\tau,\tau'\}-\tau\tau']\mathrm{E}\big[\mathbf{W}\mathbf{W}^\top\big]$$

if all the conditional τ-quantiles ($\tau \in [\varepsilon, 1-\varepsilon]$) of $\mathbf{u}^\top\mathbf{Y}$ given $\mathbf{W}$ are linear in $\mathbf{W}$ almost surely.

See Theorem 3 of Angrist et al. (2006) for the proof and Chernozhukov et al. (2022) and its Proposition 1 for a restatement.

Practical applications of Proposition 1 often require some uniformly consistent estimators of $\mathbf{J}(\tau)$ (in $[\varepsilon, 1-\varepsilon]$) and $\mathbf{\Sigma}(\tau,\tau')$ (in $[\varepsilon, 1-\varepsilon]^2$) such as those used by Chernozhukov et al. (2022) and Angrist et al. (2006) and provided by the next proposition.

Proposition 2 (Angrist et al. (2006); Chernozhukov et al. (2022)) *If $\mathrm{E}\|\mathbf{Y}\|^4 < \infty$, $\mathrm{E}\|\mathbf{Z}\|^4 < \infty$, and $\varepsilon \le \tau, \tau' \le 1-\varepsilon$, then*

$$\widehat{\mathbf{\Sigma}}(\tau,\tau') := \frac{1}{n}\sum_{i=1}^{n}[\tau-\mathrm{I}\{\widehat{r}_i(\tau)\le 0\}][\tau'-\mathrm{I}\{\widehat{r}_i(\tau')\le 0\}]\mathbf{W}_i\mathbf{W}_i^\top$$

and

$$\widehat{\mathbf{J}}(\tau) := (nh_n)^{-1}\sum_{i=1}^{n}\phi\big(\widehat{r}_i(\tau)/h_n\big)\mathbf{W}_i\mathbf{W}_i^\top$$

are uniformly consistent estimators of $\mathbf{\Sigma}(\tau,\tau')$ and $\mathbf{J}(\tau)$ where ϕ stands for the density of the standard normal distribution and the bandwidth h_n satisfies $h_n \to 0$ and $nh_n^2 \to \infty$ as $n \to \infty$.

Here, the adapted Bofinger bandwidth

$$h_n = n^{-1/5}\left(\frac{4.5\phi^4\big(\Phi^{-1}(\tau)\big)}{\big(2\Phi^{-1}(\tau)^2+1\big)^2}\right)^{1/5}$$

of Koenker (1994) is preferred because of certain optimality discussed ibidem. The symbol ϕ again represents the density of the standard normal distribution, and Φ is the corresponding standard normal cumulative distribution function.

Consider

$$\mathbf{R} := (\mathbf{O}_{(m-1)\times p}\ \mathbf{I}_{m-1})$$

and

$$\mathbf{V}(\tau, \tau') := \mathbf{R}\mathbf{J}(\tau)^{-1}\mathbf{\Sigma}(\tau, \tau')\mathbf{J}(\tau')^{-1}\mathbf{R}^\top \in \mathbb{R}^{(m-1)\times(m-1)}.$$

Proposition 1 with (5) implies

$$\{\sqrt{n}\,\widehat{\mathbf{c}}_{\tau,\mathbf{u}}\}_{\tau\in[\varepsilon,1-\varepsilon]} \xrightarrow{D} \left\{\mathbf{R}\mathbf{J}(\tau)^{-1}\mathbf{G}(\tau)\right\}_{\tau\in[\varepsilon,1-\varepsilon]}, \tag{6}$$

resp.

$$\{\sqrt{n}\,\mathbf{V}(\tau, \tau)^{-1/2}\widehat{\mathbf{c}}_{\tau,\mathbf{u}}\}_{\tau\in[\varepsilon,1-\varepsilon]} \xrightarrow{D} \left\{\mathbf{V}(\tau, \tau)^{-1/2}\mathbf{R}\mathbf{J}(\tau)^{-1}\mathbf{G}(\tau)\right\}_{\tau\in[\varepsilon,1-\varepsilon]}. \tag{7}$$

If $H_0^S(\mathbf{u})$ holds and $\varepsilon \le \tau_1 < \cdots < \tau_k \le 1-\varepsilon$ for some positive integer k, then $\mathbf{c}_{\tau,\mathbf{u}} = \mathbf{0}$ and $\sqrt{n}(\widehat{\mathbf{c}}_{\tau_1,\mathbf{u}}^\top, \ldots, \widehat{\mathbf{c}}_{\tau_k,\mathbf{u}}^\top)^\top$ has asymptotically $(m-1)k$ dimensional zero-mean normal distribution with block variance matrix $\mathbf{S} = \left(\mathbf{V}(\tau_i, \tau_j)\right)_{i,j=1}^k$, thanks to (6). Therefore,

$$T_{\chi^2} := n(\widehat{\mathbf{c}}_{\tau_1,\mathbf{u}}^\top, \ldots, \widehat{\mathbf{c}}_{\tau_k,\mathbf{u}}^\top)\widehat{\mathbf{S}}^{-1}(\widehat{\mathbf{c}}_{\tau_1,\mathbf{u}}^\top, \ldots, \widehat{\mathbf{c}}_{\tau_k,\mathbf{u}}^\top)^\top \xrightarrow{D} \chi^2_{(m-1)k} \tag{8}$$

as $n \to \infty$ for any consistent estimator $\widehat{\mathbf{S}}$ of $\mathbf{S}$ such as that based on

$$\widehat{\mathbf{V}}(\tau, \tau') := \mathbf{R}\widehat{\mathbf{J}}(\tau)^{-1}\widehat{\mathbf{\Sigma}}(\tau, \tau')\widehat{\mathbf{J}}(\tau')^{-1}\mathbf{R}^\top,$$

with $\widehat{\mathbf{J}}(\tau)$ and $\widehat{\mathbf{\Sigma}}(\tau, \tau')$ given by Proposition 2.

Similarly,

$$T_C := \sup_{\tau\in[\varepsilon,1-\varepsilon]} g\left(\sqrt{n}\widehat{\mathbf{V}}(\tau, \tau)^{-1/2}\widehat{\mathbf{c}}_{\tau,\mathbf{u}}\right) \xrightarrow{D} \sup_{\tau\in[\varepsilon,1-\varepsilon]} g\left(\mathbf{V}(\tau, \tau)^{-1/2}\mathbf{R}\mathbf{J}(\tau)^{-1}\mathbf{G}(\tau)\right) \tag{9}$$

for any continuous function $g : \mathbb{R}^{m-1} \to \mathbb{R}$, thanks to (6) and (7), which can be used for testing $H_0^S(\mathbf{u})$ by means of subsampling (with the aforementioned matrix estimators) that is described in Section 3 of Angrist et al. (2006). The number of subsamples N has to satisfy $N \to \infty$ and $N/n \to 0$ as $n \to \infty$.

In particular, $g(\mathbf{b}) = \max_j |b_j| \equiv \max_j |\mathbf{b}|_j$ (for $\mathbf{b} = (b_1, \ldots, b_{m-1})^\top$) leads to the Kolmogorov-type statistic:

$$T_{C,1} := \sup_{\tau \in [\varepsilon, 1-\varepsilon]} \max_j \left| \sqrt{n} \widehat{\mathbf{V}}(\tau, \tau)^{-1/2} \widehat{\mathbf{c}}_{\tau,\mathbf{u}} \right|_j \tag{10}$$

and $g(\mathbf{b}) = \mathbf{b}^\top \mathbf{b}$ results in

$$T_{C,2} := \sup_{\tau \in [\varepsilon, 1-\varepsilon]} n \widehat{\mathbf{c}}_{\tau,\mathbf{u}}^\top \widehat{\mathbf{V}}(\tau, \tau)^{-1} \widehat{\mathbf{c}}_{\tau,\mathbf{u}}.$$

4 Non-regression Case

In the non-regression case, $\mathbf{X} = 1$ and f is the density of $\mathbf{Y}$ alone. Assume $H_0^S(\mathbf{u})$, $\tau \in (0, 1)$, and $\mathrm{E}\mathbf{Y} = \mathbf{0}$ for simplification, without any loss of generality. Then, $\mathbf{a}_{\tau,\mathbf{u}}$ is the τ-quantile of $\mathbf{u}^\top \mathbf{Y}$, i.e., $\mathbf{a}_{\tau,\mathbf{u}} = F^{-1}_{\mathbf{u}^\top \mathbf{Y}}(\tau)$ where $F_{\mathbf{u}^\top \mathbf{Y}}$ is the cumulative distribution function of $\mathbf{u}^\top \mathbf{Y}$, with corresponding density $f_{\mathbf{u}^\top \mathbf{Y}}$. Let $f_{\mathbf{u}^\top \mathbf{Y}|\boldsymbol{\Gamma}_\mathbf{u}^\top \mathbf{Y}}(v|\mathbf{w})$ be the conditional density of $\mathbf{u}^\top \mathbf{Y}$ at point v given $\boldsymbol{\Gamma}_\mathbf{u}^\top \mathbf{Y} = \mathbf{w}$.

Then,

$$\mathrm{E} f_{\mathbf{u}^\top \mathbf{Y}|\boldsymbol{\Gamma}_\mathbf{u}^\top \mathbf{Y}}(v|\boldsymbol{\Gamma}_\mathbf{u}^\top \mathbf{Y}) = f_{\mathbf{u}^\top \mathbf{Y}}(v)$$

and

$$\mathrm{E} f_{\mathbf{u}^\top \mathbf{Y}|\boldsymbol{\Gamma}_\mathbf{u}^\top \mathbf{Y}}\big(F^{-1}_{\mathbf{u}^\top \mathbf{Y}}(\tau)|\boldsymbol{\Gamma}_\mathbf{u}^\top \mathbf{Y}\big) = f_{\mathbf{u}^\top \mathbf{Y}}\big(F^{-1}_{\mathbf{u}^\top \mathbf{Y}}(\tau)\big) =: 1/s(\tau)$$

for the sparsity function $s(\tau)$ known from the quantile regression theory.

The assumed hypothesis of axial symmetry implies that vectors $(\mathbf{u}^\top \mathbf{Y}, \mathbf{Y}^\top \boldsymbol{\Gamma}_\mathbf{u})^\top$ and $(\mathbf{u}^\top \mathbf{Y}, -\mathbf{Y}^\top \boldsymbol{\Gamma}_\mathbf{u})^\top$ are equally distributed (which $H_0(\mathbf{u})$ cannot guarantee only by itself). Therefore, $\mathrm{E}g(\mathbf{u}^\top \mathbf{Y}, \boldsymbol{\Gamma}_\mathbf{u}^\top \mathbf{Y}) = \mathbf{0}$ for any function g odd in the second argument: $g(v, \mathbf{w}) = -g(v, -\mathbf{w})$. In particular,

$$\mathrm{E}\big[f_{\mathbf{u}^\top \mathbf{Y}|\boldsymbol{\Gamma}_\mathbf{u}^\top \mathbf{Y}}\big(F^{-1}_{\mathbf{u}^\top \mathbf{Y}}(\tau)|\boldsymbol{\Gamma}_\mathbf{u}^\top \mathbf{Y}\big)\boldsymbol{\Gamma}_\mathbf{u}^\top \mathbf{Y}\big] = \mathbf{0}.$$

Therefore,

$$\mathbf{J}(\tau) = \begin{pmatrix} f_{\mathbf{u}^\top \mathbf{Y}}\left(F^{-1}_{\mathbf{u}^\top \mathbf{Y}}(\tau)\right) & \mathbf{0}^\top \\ \mathbf{0} & \mathbf{H}(\tau) \end{pmatrix}$$

where

$$\mathbf{H}(\tau) = \mathrm{E}\big[f_{\mathbf{u}^\top \mathbf{Y}|\boldsymbol{\Gamma}_\mathbf{u}^\top \mathbf{Y}}\left(F^{-1}_{\mathbf{u}^\top \mathbf{Y}}(\tau)|\boldsymbol{\Gamma}_\mathbf{u}^\top \mathbf{Y}\right) \boldsymbol{\Gamma}_\mathbf{u}^\top \mathbf{Y}\mathbf{Y}^\top \boldsymbol{\Gamma}_\mathbf{u}\big].$$

It follows from (5) that

$$\sqrt{n}\mathbf{H}(\tau)\widehat{\mathbf{c}}_{\tau,\mathbf{u}} \xrightarrow{D} \mathbf{G}_2(\tau), \tag{11}$$

where $\mathbf{G}_2(\tau)$ is a zero-mean Gaussian process with the covariance function

$$\mathbf{\Sigma}_{22}(\tau,\tau') = \tau\tau'\mathbf{\Gamma}_\mathbf{u}^\top \operatorname{Var}\mathbf{Y}\mathbf{\Gamma}_\mathbf{u} - \tau'\mathbf{C}_\tau - \tau\mathbf{C}_{\tau'} + \mathbf{C}_{\min\{\tau,\tau'\}},\ \varepsilon \le \tau,\tau' \le 1-\varepsilon,$$

where

$$\mathbf{C}_\tau = \mathrm{E}\big[\mathrm{I}\{\mathbf{u}^\top\mathbf{Y} < F^{-1}_{\mathbf{u}^\top\mathbf{Y}}(\tau)\}\mathbf{\Gamma}_\mathbf{u}^\top\mathbf{Y}\mathbf{Y}^\top\mathbf{\Gamma}_\mathbf{u}\big],$$

cf. Proposition 2(2) in Hudecová and Šiman (2021a). Then also

$$\mathbf{V}(\tau,\tau') = \mathbf{H}(\tau)^{-1}\mathbf{\Sigma}_{22}(\tau,\tau')\mathbf{H}(\tau')^{-1}.$$

If moreover $\mathbf{u}^\top\mathbf{Y}$ is independent of $\mathbf{\Gamma}_\mathbf{u}^\top\mathbf{Y}$, then

$$\mathbf{\Sigma}_{22}(\tau,\tau') = [\min\{\tau,\tau'\} - \tau\tau']\mathbf{\Gamma}_\mathbf{u}^\top \operatorname{Var}\mathbf{Y}\mathbf{\Gamma}_\mathbf{u}$$

and $\mathbf{G}_2(\tau) = [\mathbf{\Gamma}_\mathbf{u}^\top \operatorname{Var}\mathbf{Y}\mathbf{\Gamma}_\mathbf{u}]^{1/2}\mathbf{B}(\tau)$, where $\{\mathbf{B}(\tau), \tau \in [\varepsilon, 1-\varepsilon]\}$ is the $m-1$ dimensional Brownian bridge. This also implies that

$$\mathbf{V}(\tau,\tau') = (\min\{\tau,\tau'\} - \tau\tau')s(\tau)s(\tau')\big(\mathbf{\Gamma}_\mathbf{u}^\top \operatorname{Var}\mathbf{Y}\mathbf{\Gamma}_\mathbf{u}\big)^{-1},$$

and that T_C of (9), for $g(\mathbf{b}) = \mathbf{b}^\top\mathbf{b}$, turns into

$$\widetilde{T}_B := \sup_{\tau\in[\varepsilon,1-\varepsilon]} \frac{n}{\tau(1-\tau)s^2(\tau)}\widehat{\mathbf{c}}^\top_{\tau,\mathbf{u}}\left[\mathbf{\Gamma}_\mathbf{u}^\top \operatorname{Var}(\mathbf{Y})\mathbf{\Gamma}_\mathbf{u}\right]\widehat{\mathbf{c}}_{\tau,\mathbf{u}}, \tag{12}$$

which converges, for $n \to \infty$, in distribution to

$$\sup_{\tau\in[\varepsilon,1-\varepsilon]} \frac{\|\mathbf{B}(\tau)\|^2}{\tau(1-\tau)} = \sup_{\tau\in[\varepsilon,1-\varepsilon]} Q^2(\tau)$$

where $Q(\tau) = \|\mathbf{B}(\tau)\|[\tau(1-\tau)]^{-1/2}$ is the well-known Bessel process, and the critical values of $\sup_{\tau\in[\varepsilon,1-\varepsilon]} Q^2(\tau)$ are known and tabulated (Andrews, 1993; Estrella, 2003). The convergence remains valid even if $\operatorname{Var}(\mathbf{Y})/s^2(\tau)$ is replaced with a uniformly consistent estimator.

Note that $\widetilde{T}_B$ virtually coincides with the Wald-type test statistic of Theorem 2 in Koenker and Machado (1999), formulated for correctly specified quantile regression models with deterministic regressors and already supported by common statistical software.

Evidently, $\widetilde{T}_B$ is invariant with respect to the choice of $\boldsymbol{\Gamma}_{\mathbf{u}}$, rotation, and certain scale transformations:

$$\begin{aligned}\widetilde{T}_B(\mathbf{u}, \boldsymbol{\Gamma}_{\mathbf{u}}, \mathbf{Y}) &= \widetilde{T}_B(\mathbf{u}, \boldsymbol{\Gamma}_{\mathbf{u}}\boldsymbol{\Delta}, \mathbf{Y}) \\ &= \widetilde{T}_B(\mathbf{A}\mathbf{u}, \mathbf{A}\boldsymbol{\Gamma}_{\mathbf{u}}, \mathbf{A}\mathbf{Y}) = \widetilde{T}_B(\mathbf{u}, \boldsymbol{\Gamma}_{\mathbf{u}}, (\mathbf{u}|\boldsymbol{\Gamma}_{\mathbf{u}})\mathbf{D}(\mathbf{u}|\boldsymbol{\Gamma}_{\mathbf{u}})^\top\mathbf{Y})\end{aligned}$$

for any rotational (i.e., orthonormal) matrices $\boldsymbol{\Delta} \in \mathbb{R}^{(m-1)\times(m-1)}$ and $\mathbf{A} \in \mathbb{R}^{m\times m}$ and for any regular matrix $\mathbf{D} = \operatorname{diag}(d_{11}, \ldots, d_{mm})$ with $d_{11} > 0$. The shift invariance of $\widetilde{T}_B$ is also evident, although one should work with $\mathbf{Y} - \mathrm{E}\mathbf{Y}$ rather than with $\mathrm{E}\mathbf{Y} = \mathbf{0}$ to prove that rigorously. All the good invariance properties of $\widetilde{T}_B$ would remain untouched if Var $(\mathbf{Y})$ were replaced with its sample counterpart. They usually get lost by using an estimator of $s^2(\tau)$.

5 Illustrations

This section illustrates the usefulness of the introduced tests for testing axial symmetry in large datasets with $n = 10{,}000$, 20,000, 40,000, or 700,000 observations. It uses representative (conditional) multivariate distributions (uniform, normal, and Student t_7) in spaces of dimension $m = 2, 3, 4$, and 5 and in the general linear regression setup with $p = 1, 5$, or 10 regressors.

For the considered numbers of observations, the test size has never been generally acceptable for dimension m higher than five owing to the nonparametric estimation involved. This is also one of the reasons for using only large sample sizes (the other is that only then the computation of quantile regression rank scores may become substantially slower than that of quantile regression coefficients).

All the computations have been conducted in the software environment `R` (R Core Team, 2021) with the aid of the `quantreg` package (Koenker, 2015). The p-values regarding the Bessel process have been obtained by means of the algorithm published in Estrella (2003).

After the investigated tests are specified, the figures and associated simulation experiments are described in detail, all using the null hypothesis $H_0^S(\mathbf{u})$ for $\mathbf{u} = (\cos(\alpha), \sin(\alpha), \mathbf{0}^\top)^\top \in \mathcal{S}^{m-1}$ where $\alpha \in [0, \pi/90]$ or $\alpha \in [0, \pi/360]$ is considered in radians and only $\alpha = 0$ makes the null hypothesis satisfied in all the experiments considered. Then this section indicates possible computational benefits.

The test T_{χ^2} is used with $k = 3$, which is in line with the recommendations given in Hudecová and Šiman (2021a). In particular, $\tau_1 = 0.2$, $\tau_2 = 0.5$ and $\tau_3 = 0.8$.

The test T_S employs the statistic $T_{C,1}$ of (10), also recommended in Angrist et al. (2006), but with the supremum changed to the maximum over two values of τ, namely:

$$T_S := \max_{\tau\in\{0.2,0.8\}} \max_{j=1,\ldots,m-1} \left|\sqrt{n}\widehat{\mathbf{V}}(\tau,\tau)^{-1/2}\widehat{\mathbf{c}}_{\tau,\mathbf{u}}\right|_j$$

for the sake of computational simplicity. It does not have any standard asymptotic distribution. In principle, one could also consider maxima over larger sets of equidistributed τ's or other metrics for measuring the distance of the τ-process $\sqrt{n}\widehat{\mathbf{V}}(\tau,\tau)^{-1/2}\widehat{\mathbf{c}}_{\tau,\mathbf{u}}$ from zero, for example.

The test T_B is based on $\widetilde{T}_B$ of (12), but with estimated population quantities and with the supremum over $[\varepsilon, 1-\varepsilon]$ replaced with the maximum over 101 equidistant points from $[0.1, 0.9]$ including the end points.

The behavior of the tests is illustrated by means of the averages and/or empirical distribution functions of sample p-values based on 1 000 independent simulations, in line with Hudecová and Šiman (2021a), Hudecová and Šiman (2021b), and Hudecová and Šiman (2023).

Figure 1 illuminates the behavior of the test T_B in the non-regression case with $n = 10{,}000$ observations $(Y_1, 2Y_2, \ldots, mY_m)^\top$ of dimension $m = 3$ or $m = 4$ for the axial direction $\mathbf{u} = (\cos(\alpha), \sin(\alpha), \mathbf{0}^\top)^\top$, $\alpha \in [0, \pi/90]$, where $(Y_1, Y_2, \ldots, Y_m)^\top$ comes from the multivariate uniform distribution on $[-1, 1]^m$ or from the multivariate standard normal distribution.

Figures 2, 3, and 4 show the performance of T_{χ^2} in the non-regression case with $n = 10{,}000$, 20,000, or 40,000 observations $(Y_1, 2Y_2, \ldots, mY_m)^\top$ of dimension $m = 3$, $m = 4$, or $m = 5$ for the axial direction $\mathbf{u} = (\cos(\alpha), \sin(\alpha), \mathbf{0}^\top)^\top$ for $\alpha \in [0, \pi/90]$ where $(Y_1, Y_2, \ldots, Y_m)^\top$ comes from the multivariate uniform distribution on $[-1, 1]^m$, multivariate standard normal distribution, or multivariate canonical Student distribution t_7 with 7 degrees of freedom.

Figure 5 analogously compares the behavior of T_{χ^2} in the non-regression ($p = 1$) and regression ($p = 10$) case with $n = 20{,}000$ observations of dimension $m = 3, 4$, or 5 from the linear regression model $(Y_1, Y_2, \ldots, Y_m)^\top = \mathbf{BX} + (\varepsilon_1, 2\varepsilon_2, \ldots, m\varepsilon_m)^\top$ where $\mathbf{B} = \mathbf{1}_m\mathbf{1}_p^\top \in \mathbb{R}^{m\times p}$ is the matrix of ones, $\mathbf{X} = 1$ or $\mathbf{X} = (1, \mathbf{Z}^\top)^\top \in \mathbb{R}^p$, $p = 10$, with multivariate standard normal $\mathbf{Z}$ independent of $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m)^\top$ whose distribution is multivariate canonical t_7, multivariate standard normal or multivariate uniform. Apparently, the test performance does not deteriorate in the presence of a moderate number of regressors, which is not surprising in the quantile regression context. This also explains why Figs. 2, 3, and 4 focus only on the non-regression case.

Figure 6 illustrates the performance of the test T_S when applied to $n = 700{,}000$ observations from the linear regression model of Fig. 5 but this time with $p = 5$ and only in the most important cases of $m = 2$ or $m = 3$. The p-values regarding T_S have been determined by means of re-centered subsampling, described in Angrist et al. (2006). Their averages and empirical distribution functions are based on 1,000 independent simulations, each using 1,000 subsamples of length 1,000. The number of subsamples could be increased easily in real data examples without numerous replications.

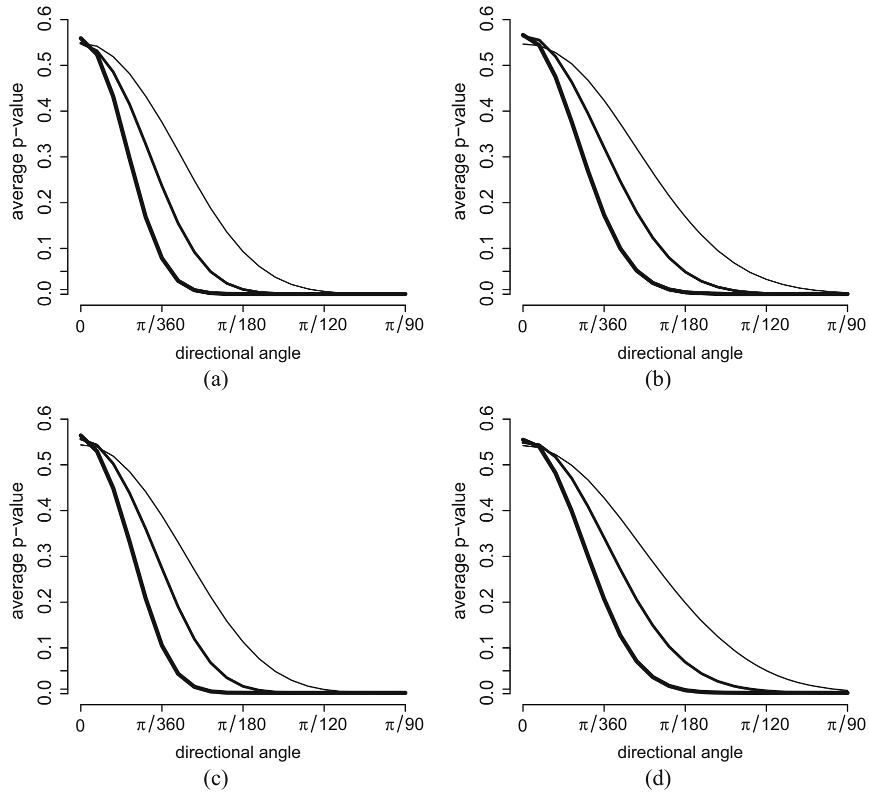

Fig. 1 The T_B test. The figure shows the averages of sample p-values coming from the test T_B of axial symmetry around a line in direction $\mathbf{u} = (\cos(\alpha), \sin(\alpha), \mathbf{0}^\top)^\top$ for $\alpha \in [0, \pi/90]$. The plots have been obtained from 1,000 independent samples of dimension $m = 3$ (top) or $m = 4$ (bottom) containing $n = 10{,}000$ (thin), $n = 20{,}000$ (normal) or $n = 40{,}000$ (thick) independent observations $(Y_1, 2Y_2, \ldots, mY_m)^\top$ where the distribution of $(Y_1, Y_2, \ldots, Y_m)^\top$ is multivariate uniform on $[-1, 1]^m$ (left) or multivariate standard normal (right). The null hypothesis is satisfied for $\alpha = 0$

The computational benefits of not using the `br` algorithm may be substantial. For example, if $n = 1{,}000{,}000$, $p = 1$, $m = 5$, the data are uniformly distributed and a normal computer (CPU: Intel Core i5-6600 3.30 GHz, RAM: 16 GB) is used, then the average (elapsed) times of computing three quantile regression coefficient vectors (for $\tau = 0.2$, 0.5 and 0.8) are 1.1 s for the `conquer` algorithm, 1.5 s for the `pfnb` algorithm but 1,505 s for the `br` one (all the algorithms are included in the `quantreg` package).

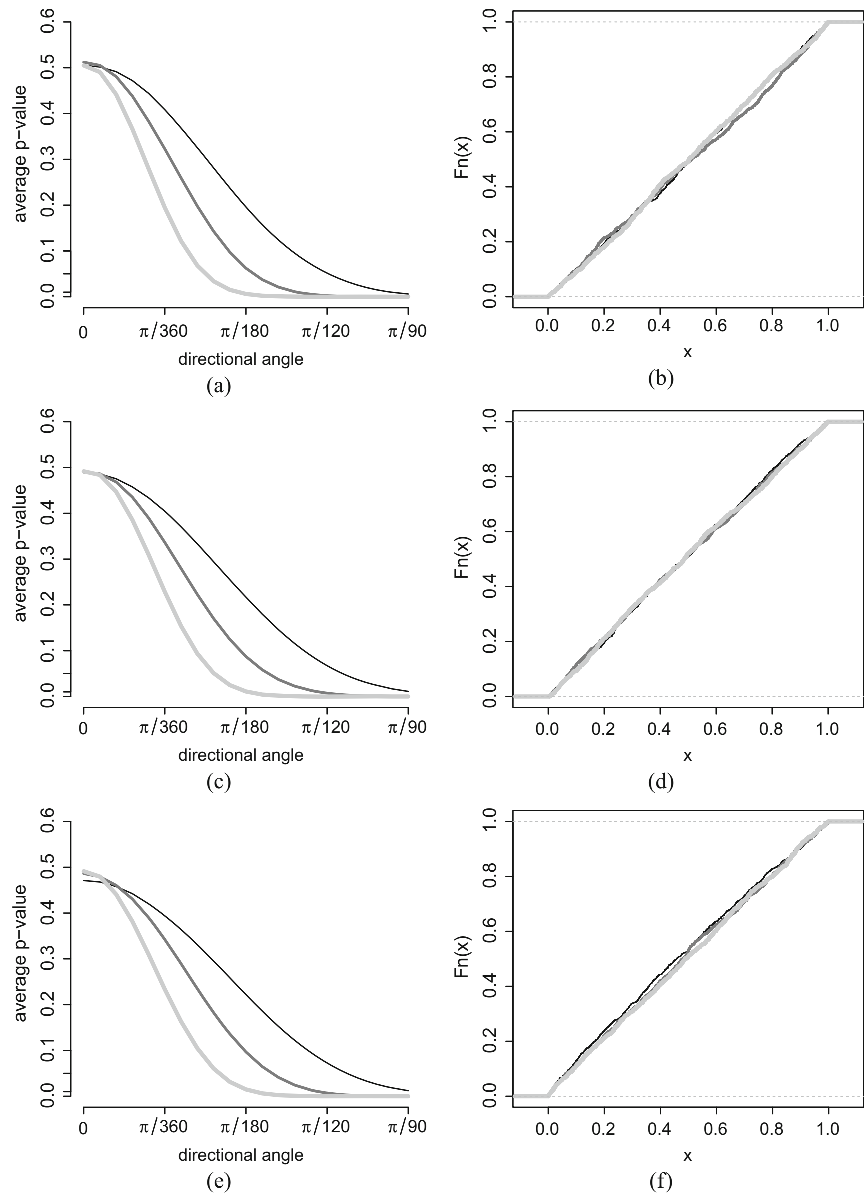

Fig. 2 The T_{χ^2} test and normally distributed data. The figure shows the averages (left) and empirical distribution functions (right) of sample p-values coming from the χ^2-test T_{χ^2} of axial symmetry around a line in direction $\mathbf{u} = (\cos(\alpha), \sin(\alpha), \mathbf{0}^\top)^\top$ for $\alpha \in [0, \pi/90]$. The plots have been obtained from 1,000 independent samples of dimension (**a**) $m = 3$, (**c**) $m = 4$, and (**e**) $m = 5$ containing $n = 10{,}000$ (thin black), $n = 20{,}000$ (normal dark gray) or $n = 40{,}000$ (thick light gray) independent observations $(Y_1, 2Y_2, \ldots, mY_m)^\top$ where the distribution of $(Y_1, Y_2, \ldots, Y_m)^\top$ is multivariate standard normal. The empirical distribution function is included for $m = 3$ in (**b**), for $m = 4$ in (**d**) and for $m = 5$ in (**f**). It corresponds to $\alpha = 0$ when the null hypothesis is satisfied

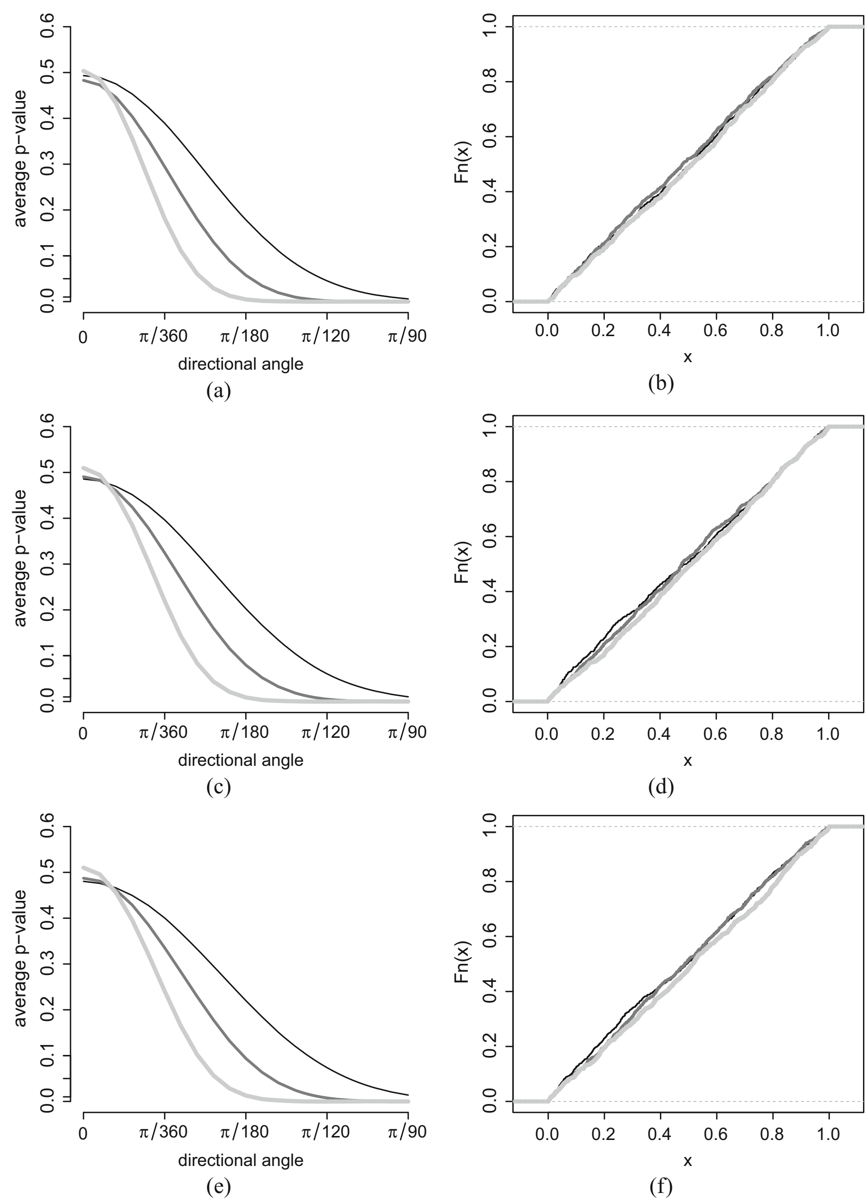

Fig. 3 The T_{χ^2} test and uniformly distributed data. This figure differs from Fig. 2 only in the employed distribution of $(Y_1, Y_2, \ldots, Y_m)^\top$ that is now multivariate uniform on $[-1, 1]^m$

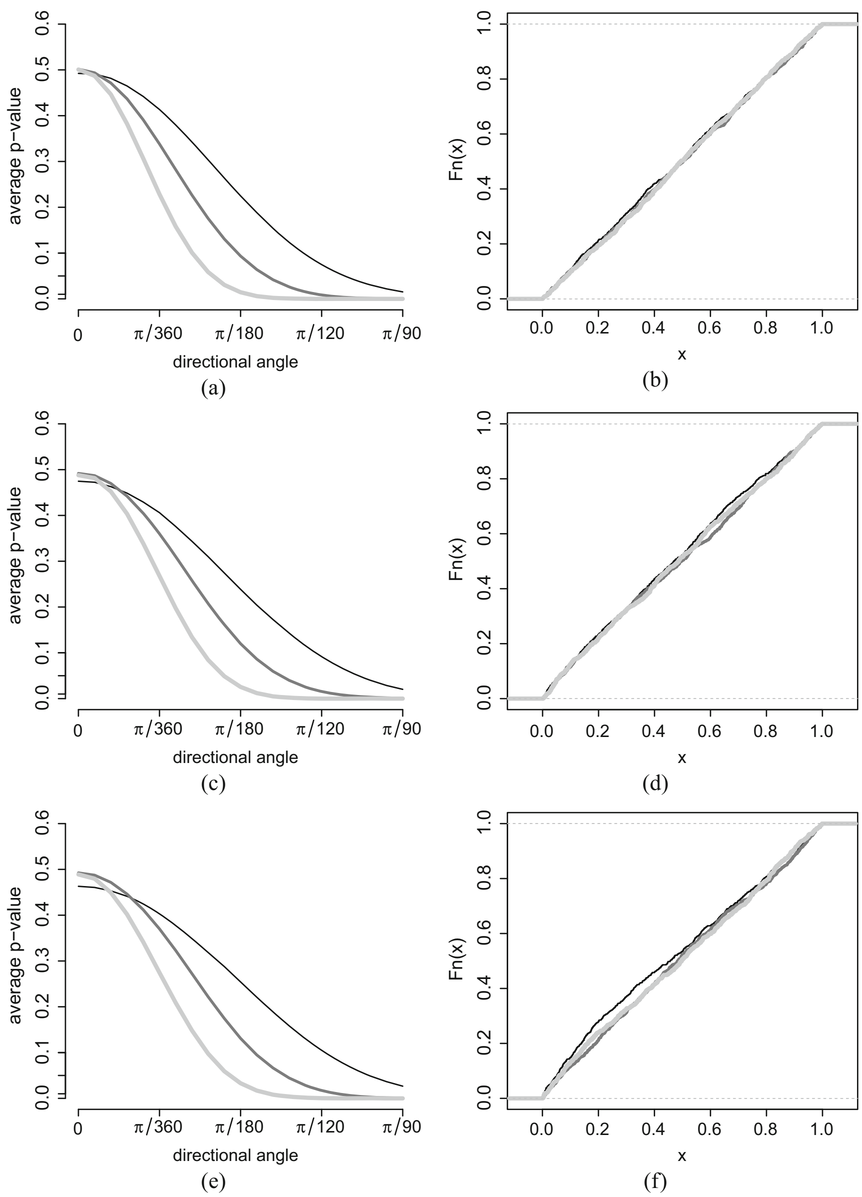

Fig. 4 The T_{χ^2} test and t_7 distributed data. This figure differs from Fig. 2 only in the employed distribution of $(Y_1, Y_2, \ldots, Y_m)^\top$ that is now multivariate Student with 7 degrees of freedom

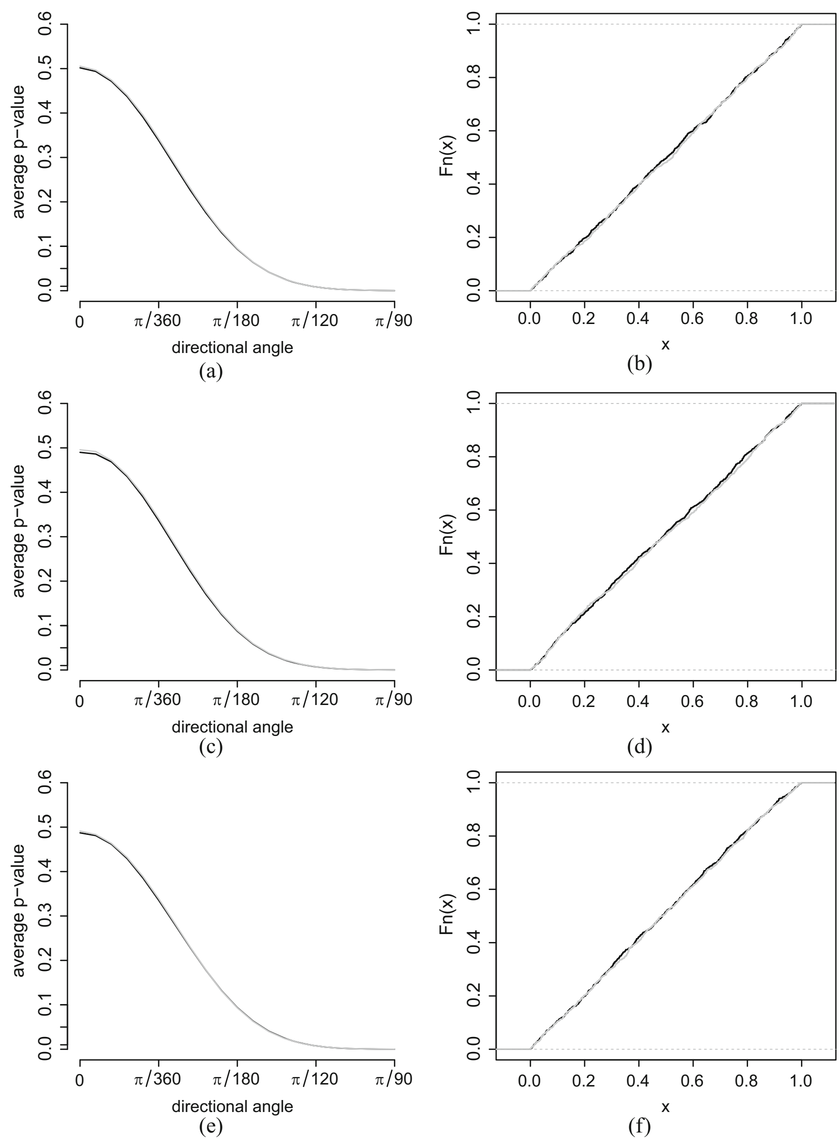

Fig. 5 The T_{χ^2} test in the regression case. The figure shows the averages (left) and empirical distribution functions (right) of sample p-values coming from the χ^2-test T_{χ^2} of axial symmetry around a line in direction $\mathbf{u} = (\cos(\alpha), \sin(\alpha), \mathbf{0}^\top)^\top \in \mathcal{S}^{m-1}$ for $\alpha \in [0, \pi/90]$, and for $\tau_1 = 0.2$, $\tau_2 = 0.5$ and $\tau_3 = 0.8$. The plots have been obtained from 1,000 independent samples of dimension $m = 3$ (top), $m = 4$ (center), and $m = 5$ (bottom) containing $n = 20{,}000$ observations from the linear regression model $(Y_1, Y_2, \ldots, Y_m)^\top = \mathbf{B}\boldsymbol{X} + (\varepsilon_1, 2\varepsilon_2, \ldots, m\varepsilon_m)^\top$ where $\mathbf{B} = \mathbf{1}_m \mathbf{1}_p^\top \in \mathbb{R}^{m\times p}$, $\boldsymbol{X} = 1$ (black) or $\boldsymbol{X} = (1, \mathbf{Z}^\top)^\top \in \mathbb{R}^{10}$ with multivariate standard normal $\mathbf{Z}$ (gray) independent of $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m)^\top$ whose distribution is multivariate canonical t_7 (top), multivariate standard normal (center) or multivariate uniform on $[-1, 1]^m$ (bottom). The empirical distribution functions correspond to $\alpha = 0$ when the null hypothesis is satisfied

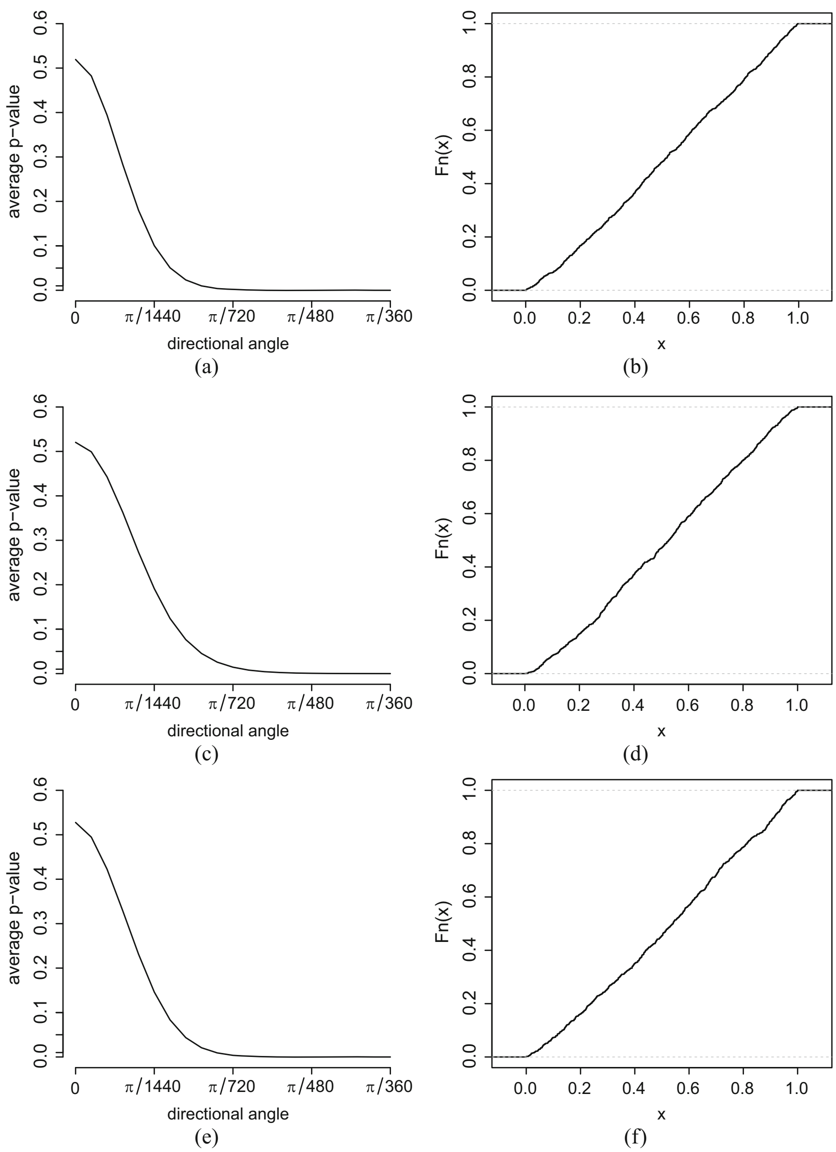

Fig. 6 The subsampling approach in the regression case. The figure shows the averages (left) and empirical distribution functions (right) of sample p-values coming from the test T_S of axial symmetry around a line in direction $\mathbf{u} = (\cos(\alpha), \sin(\alpha), \mathbf{0}^\top)^\top \in \mathcal{S}^{m-1}$ for $\alpha \in [0, \pi/360]$. The plots have been obtained from 1,000 simulation experiments, each using 1,000 independent subsamples of length 1,000 out of $n = 700{,}000$ observations from the linear regression model $(Y_1, \ldots, Y_m)^\top = \mathbf{B}\boldsymbol{X} + (\varepsilon_1, \ldots, m\varepsilon_m)^\top$ where $\mathbf{B} = \mathbf{1}_m\mathbf{1}_p^\top \in \mathbb{R}^{m \times p}$, the component $\mathbf{Z} \in \mathbb{R}^4$ of $\boldsymbol{X} = (1, \mathbf{Z}^\top)^\top$ is multivariate standard normal, and the distribution of $(\varepsilon_1, \ldots, \varepsilon_m)^\top$ is independent of $\mathbf{Z}$ and multivariate uniform on $[-1, 1]^2$ (top) or on $[-1, 1]^3$ (bottom), or bivariate canonical t_7 (center). The empirical distribution functions correspond to $\alpha = 0$ when the null hypothesis is satisfied

6 Concluding Remarks

The tests always assume that the axial direction is known, and they cannot be easily extended to the situations with an unknown or estimated axial direction due to their complex nonlinear dependence on the axial directional vector. Therefore, their usefulness for exploratory analysis is limited.

The performance of the test T_B is disappointing due to the puzzling small but persistent size issues present even for large and normally distributed samples. These problems have also been observed in other supremum-based tests in the quantile regression context; see, e.g., Hudecová and Šiman (2021a) and Koenker and Machado (1999).

Nevertheless, the introduced tests based on T_{χ^2} and T_C constitute a viable procedure for testing conditional axial symmetry in large datasets of small-dimensional vector observations where other options are not yet available, i.e., mainly in the linear regression context. In the multivariate case with $p = 1$, they may be recommended only if the rank scores are not available and if the assumptions of Hudecová and Šiman (2021b) are not satisfied. Otherwise the tests of Hudecová and Šiman (2021b) and Hudecová and Šiman (2023) should be preferred.

Acknowledgments The research of Miroslav Šiman was supported by the Czech Science Foundation project GA21-05325S. The author would like to thank the editor(s) and reviewers for valuable comments and to extraordinary and amazing Šárka Hudecová for her contribution, collaboration, and support.

References

Andrews, D. W. K. (1993). Tests for parameter instability and structural change with unknown change point. *Econometrica, 61*, 821–856.

Angrist, J., Chernozhukov, V., & Fernández-Val, I. (2006). Quantile regression under misspecification, with an application to the U.S. wage structure. *Econometrica, 74*, 539–563.

Chernozhukov, V., Fernández-Val, I., & Melly, B. (2022). Fast algorithms for the quantile regression process. *Empirical Economics, 62*, 7–33.

Estrella, A. (2003). Critical values and p values of Bessel process distributions: Computation and application to structural break tests. *Econometric Theory, 19*, 1128–1143.

Hallin, M., Paindaveine, D., & Šiman, M. (2010). Multivariate quantiles and multiple-output regression quantiles: From L_1 optimization to halfspace depth. *Annals of Statistics, 38*, 635–669.

Hollander, M. (1971). A nonparametric test for bivariate symmetry. *Biometrika, 58*, 203–212.

Hudecová, Š., & Šiman, M. (2021a). Testing axial symmetry by means of directional regression quantiles. *Electronic Journal of Statistics, 15*, 2690–2715.

Hudecová, Š., & Šiman, M. (2021b). Testing symmetry around a subspace. *Statistical Papers, 62*, 2491–2508.

Hudecová, Š., & Šiman, M. (2023). Testing axial symmetry by means of integrated rank scores. *Journal of Nonparametric Statistics, 35*, 474–490.

Kalina, J. (2021). Common multivariate estimators of location and scatter capture the symmetry of the underlying distribution. *Communications in Statistics - Simulation and Computation, 50*, 2845–2857.

Koenker, R. (1994). Confidence intervals for regression quantiles. In P. Mandl, & M. Hušková (Eds.), *Contributions to statistics*. Physica.

Koenker, R. (2005). *Quantile regression*. Cambridge University Press.

Koenker, R. (2015). `quantreg`*: Quantile Regression*. R package.

Koenker, R., & Machado, J. A. F. (1999). Goodness of fit and related inference processes for quantile regression. *Journal of the American Statistical Association, 94*, 1296–1310.

Modarres, R. (2008). Tests of bivariate exchangeability. *International Statistical Review, 76*, 203–213.

Rao, K. S. M., & Raghunath, M. (2012). A simple nonparametric test for bivariate symmetry about a line. *Journal of Statistical Planning and Inference, 142*, 430–444.

R Core Team (2021). *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing.

Su, L. (2006). A simple test for multivariate conditional symmetry. *Economics Letters, 93*, 374–378.