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Synchronization of Generalized Lorenz Systems in a Loop

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The conditions for the synchronization of three interconnected generalized Lorenz systems are given. The interconnection topology contains a loop. It is shown that, under certain conditions on the strength of the coupling of the systems, the full synchronization of all three systems is guaranteed. The results are illustrated by an example.

Keywords: Generalized Lorenz system; synchronization; cyclic network.

1. Introduction

The purpose of this paper is to address the synchronization in the network represented by the directed graph composed of nodes being possibly chaotic systems while its edges represent a transmission of a scalar signal generated by the edge source node injected to the right-hand side of the target node. More specifically, the systems placed in the nodes are copies of the so-called Generalized Lorenz System (GLS) [Čelikovský & Chen, 2021].

The main contribution of this paper is the rigorous mathematical analysis and proof of such a synchronization even in the so-called **cyclic networks** represented by graphs with cycles. Recently, the synchronization of the networks composed from copies of the GLS was completely and rigorously addressed in [Čelikovský *et al.*, 2023] for the **acyclic case**, i.e. the respective graph has a **tree topology**. To the best of our knowledge, the only known theoretical result containing local synchronization rigorous proof was presented in [Čelikovský *et al.*, 2013] for a pair of GLS with bi-directional connection that can be viewed as the simplest and the smallest cyclic network, see Fig. 1. For the synchronization related to more general systems than GLS, see, e.g. [Rulkov *et al.*, 1995; Rehák & Lynnyk, 2021b; Zhang *et al.*, 2023] and references within there. Mathematical problems related to the synchronization of large and complex networks can often be addressed by similar methods and techniques as in the case of multiagent systems [Wu *et al.*, 2023; Rehák & Lynnyk, 2019b, 2020; Rehák & Lynnyk, 2023] and/or large-scale systems [Rehák & Lynnyk, 2019a, 2021a].

The study of complex networks and their synchronization has been quite intensive during the

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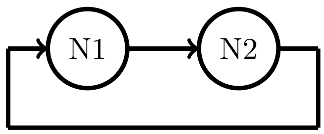

Fig. 1. The simplest and smallest cyclic network consisting of two nodes.

previous several decades. The basic framework of the respective field and its motivation by various applications are nicely described in [Chen *et al.*, 2014]. As far as synchronization is concerned, one can distinguish several kinds of synchronization: identical synchronization [Fujisaka & Yamada, 1983; Pikovsky, 1984], generalized synchronization [Rulkov *et al.*, 1995; Kocarev & Parlitz, 1996], phase synchronization [Rosenblum *et al.*, 1996], lag synchronization [Rosenblum *et al.*, 1997], projective synchronization [Mainieri & Rehacek, 1999], to name a few. The types of achievable synchronization are determined by many factors, e.g. by the quality of the communication channel between nodes, by the presence of delays or quantization of the transmitted signals.

This paper deals with the identical synchronization of systems interconnected in the network, which is represented by some graphs that have nodes and edges. The dynamical systems are placed in the nodes, while the synchronizing connection is represented by the edges. The identical synchronization has been intensively studied in the literature; see, e.g. [Bao & Cao, 2016; Zhu *et al.*, 2008] and references therein. In the case of the identical synchronization, the goal is to achieve equal values of the states in all nodes. This is conducted through the inter-node communication. Note also that this type of synchronization can be achieved under rather restrictive conditions — namely, the nodes to be synchronized must be identical, including values of their parameters. In case of a mismatch of parameters in structurally identical systems, only the generalized synchronization can be achieved, which means that a general convenient mapping between the states of the nodes is established. The so-called **interconnection topology** between systems to be synchronized crucially affects the possible synchronization and its viability. The graph theory is a suitable and usual tool for the description of this topology.

As already noted, the purpose of this paper is to extend the results obtained in [Čelikovský *et al.*, 2023] for the acyclic networks composed of GLS [Čelikovský & Chen, 2021] to the pilot case of the noncyclic networks — the cycle of three nodes with GLS inside them. This result, in a sense, generalizes [Čelikovský *et al.*, 2013], presenting a cycle of two bi-directionally connected GLS. Note that these two results open the avenue to prove the synchronization of any general network represented by the graph containing only these two types of cyclic subgraphs in a rigorous way. More specifically, the conditions for synchronizing connections (in fact, the output injections from one GLS into another) guarantee synchronization even in the presence of a loop in the interconnection topology. This result constitutes the main contribution of this paper.

The rest of this paper is organized as follows. The GLS is defined in Sec. 2. Also, the most important facts are summarized here. The following section contains the formulation of the main result — Theorem 3.1. This theorem states that, under some assumptions on the strength of the coupling in the loop, the system can be synchronized. However, this is under the assumption of boundedness of trajectories of all systems — the leader as well as the followers. This is a nontrivial assumption in the case of an interconnection topology containing loops. The results are illustrated by an example in Sec. 4. Then, the conclusions follow. The appendices contain the proof of Theorem 3.1. The proof of the boundedness of trajectories of the interconnected Lorenz system is given in Appendix B.

2. Generalized Lorenz System

GLS [Čelikovský & Vaněček, 1994; Čelikovský & Chen, 2002] unifies into a single five-parametric family the classical Lorenz system and Chen system, as well as some other more particular classes of the systems, see [Čelikovský & Chen, 2021] for a comprehensive survey and classification of all these systems, including detailed proofs of some canonical forms equivalence facilitating such a classification. The rest of this section briefly repeats some selected facts from [Čelikovský & Chen, 2021] as well as some results on equivalence to special forms allowing synchronization from [Lynnyk & Čelikovský, 2021; Čelikovský *et al.*, 2023]. All these facts will be used later on in this paper.

Definition 2.1. The GLS is the dynamical system having a three-dimensional state $x = [x_1, x_2, x_3]^\top$ and described by the following system of the ordinary differential equations:

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{pmatrix} = \begin{pmatrix}a_{11} & a_{12} & 0\\ a_{21} & a_{22} & 0\\ 0 & 0 & \lambda\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} + x_1\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}, \tag{1}$$

where a_{11}, a_{12}, a_{21}, a_{22}, and λ are real parameters such that

$$\begin{gathered} a_{11}a_{22} - a_{12}a_{21} < 0,\\ a_{11} + a_{22} < 0, \quad \lambda < 0. \end{gathered} \tag{2}$$

The following theorem presents the transformation of GLS into its equivalent form that will be useful for the analysis later on.

Theorem 2.1 [Lynnyk & Čelikovský, 2021; Čelikovský *et al.*, 2023]. *Consider system (1) and assume $a_{12} \neq 0$. Then, system (1) with its state is equivalent to*

$$\begin{pmatrix}\dot{w}_1\\ \dot{w}_2\\ \dot{w}_3\end{pmatrix} = \begin{pmatrix}a_{11} & a_{12} & 0\\ a_{21} & a_{22} & 0\\ 0 & 0 & \lambda\end{pmatrix}\begin{pmatrix}w_1\\ w_2\\ w_3\end{pmatrix} + \begin{pmatrix}0\\ -w_1w_3 - K'w_1^3\\ Kw_1^2\end{pmatrix}, \tag{3}$$

$$K = \frac{\lambda - 2a_{11}}{2a_{12}},$$

$$K' = \frac{1}{2a_{12}}.$$

Moreover, the corresponding state transformation converting (1) into (3) reads

$$w_1 = x_1, \quad w_2 = x_2, \quad w_3 = x_3 - \frac{x_1^2}{2a_{12}} \tag{4}$$

while the following transformation is its inverse:

$$x_1 = w_1, \quad x_2 = w_2, \quad x_3 = w_3 + \frac{w_1^2}{2a_{12}}. \tag{5}$$

The following lemma shows how (3) can be used to synchronize a pair of GLS.

Lemma 2.1 [Čelikovský *et al.*, 2023]. *Let (3) be given. Define*

$$\begin{pmatrix}\dot{\hat{w}}_1\\ \dot{\hat{w}}_2\\ \dot{\hat{w}}_3\end{pmatrix} = \begin{pmatrix}a_{11} & a_{12} & 0\\ a_{21} & a_{22} & 0\\ 0 & 0 & \lambda\end{pmatrix}\begin{pmatrix}\hat{w}_1\\ \hat{w}_2\\ \hat{w}_3\end{pmatrix} + \begin{pmatrix}l_1\\ l_2\\ 0\end{pmatrix}(w_1 - \hat{w}_1) + \begin{pmatrix}0\\ -w_1\hat{w}_3 - K'w_1^3\\ Kw_1^2\end{pmatrix}. \tag{6}$$

Suppose there exists a constant $W > 0$ so that, for solution of (3), the inequality $|w_1(t)| \leq W$, $\forall\, t \geq 0$ holds. Let also

$$\begin{gathered} a_{11} + a_{22} - l_1 < 0,\\ a_{11}a_{22} - a_{12}a_{21} - l_1a_{22} + l_2a_{12} > 0. \end{gathered} \tag{7}$$

Moreover, suppose $\lambda < 0$. Then, there exist constants $M > 0$, $m > 0$, $\forall\, t \in \mathbb{R}^+$ such that

$$\hat{w} := \begin{pmatrix}\hat{w}_1\\ \hat{w}_2\\ \hat{w}_3\end{pmatrix}, \quad w := \begin{pmatrix}w_1\\ w_2\\ w_3\end{pmatrix},$$

$$\|\hat{w}(t) - w(t)\| \leq M\exp(-mt)\|\hat{w}(0) - w(0)\|. \tag{8}$$

Lemma 2.1 served in [Čelikovský *et al.*, 2023] as a cornerstone tool to prove the generalized synchronization in the networks having a general number of nodes containing the copies of GLSs, albeit under the condition that the respective interconnection topology is acyclic (see [Čelikovský *et al.*, 2023], Theorem 3.2).

3. Interconnection of Systems and Its Synchronization

Let $a \in [0, 1)$. In the following text, we study the following interconnection of three systems: the system denoted by **L** is the leader and the remaining systems, **F1** and **F2,** are called followers.

The novelty is an investigation of the synchronization of chaotic systems under a topology that allows for feedback, see Fig. 2.

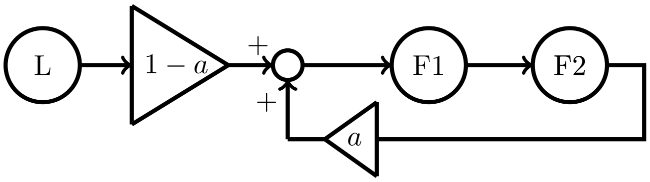

Fig. 2. Connection of systems.

Assume the transformation (4) was performed on all systems. Let the state of the leader be denoted by $w = (w_1, w_2, w_3)^T$, of the follower **F1** by $\hat{w} = (\hat{w}_1, \hat{w}_2, \hat{w}_3)^T$ and of the follower **F2** by $\widetilde{w} = (\widetilde{w}_1, \widetilde{w}_2, \widetilde{w}_3)^T$.

Let $a \in [0, 1)$. To facilitate the notation in the subsequent text, define

$$\xi_a = a\widetilde{w}_1 + (1-a)w_1. \tag{9}$$

Note that $a = 0$ corresponds to the case $\xi_a = w_1$; hence, there is no feedback from system **F2** to **F1**. This means the system has a tree topology. For this case, Čelikovský *et al.* [2023] prove the synchronization of the followers with the leader under any initial conditions.

The leader system is defined by (3). On the other hand, the follower **F1** obeys the equation (after transformation (4))

$$\begin{pmatrix} \dot{\hat{w}}_1 \\ \dot{\hat{w}}_2 \\ \dot{\hat{w}}_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \hat{w}_3 \end{pmatrix} + \begin{pmatrix} l_1 \\ l_2 \\ 0 \end{pmatrix} (\xi_a - \hat{w}_1) + \begin{pmatrix} 0 \\ -\xi_a \hat{w}_3 - K'\xi_a^3 \\ K\xi_a^2 \end{pmatrix}, \tag{10}$$

while **F2** obeys

$$\begin{pmatrix} \dot{\widetilde{w}}_1 \\ \dot{\widetilde{w}}_2 \\ \dot{\widetilde{w}}_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} \widetilde{w}_1 \\ \widetilde{w}_2 \\ \widetilde{w}_3 \end{pmatrix} + \begin{pmatrix} l_1 \\ l_2 \\ 0 \end{pmatrix} (\hat{w}_1 - \widetilde{w}_1) + \begin{pmatrix} 0 \\ -\hat{w}_1 \widetilde{w}_3 - K'\hat{w}_1^3 \\ K\hat{w}_1^2 \end{pmatrix}. \tag{11}$$

To investigate the convergence of the followers to the state of the leader, let us define the errors

$$\hat{e} = \hat{w} - w, \tag{12}$$

$$\tilde{e} = \widetilde{w} - \hat{w}. \tag{13}$$

Define also

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad C = (1, 0, 0), \quad L = \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}.$$

Then, the derivative of $\tilde{e}$ holds

$$\dot{\tilde{e}} = \begin{pmatrix} A - LC & 0 \\ 0 & \lambda \end{pmatrix} \tilde{e} + \begin{pmatrix} 0 \\ \hat{w}_1 \widetilde{w}_3 - \xi_a \hat{w}_3 - K'\hat{w}_1^3 + K'\xi_a^3 \\ K\hat{w}_1^2 - K\xi_a^2 \end{pmatrix}. \tag{14}$$

The following equality is useful in the subsequent text:

$$\hat{w}_1 - \xi_a = (1-a)\hat{e} - a\tilde{e}.$$

Then, after some manipulations, (14) turns into

$$\begin{aligned} \dot{\tilde{e}} = {} & \begin{pmatrix} A - LC & 0 \\ 0 & \lambda \end{pmatrix} \tilde{e} + \begin{pmatrix} 0 \\ \hat{w}_1 \tilde{e}_3 \\ 0 \end{pmatrix} - a\tilde{e}_1 \begin{pmatrix} 0 \\ \hat{w}_3 - K'(\hat{w}_1^2 + \hat{w}_1\xi_a + \xi_a^2) \\ K(\hat{w}_1 + \xi_a) \end{pmatrix} \\ & + (1-a)\hat{e}_1 \begin{pmatrix} 0 \\ -\hat{w}_3 + K'(\hat{w}_1^2 + \hat{w}_1\xi_a + \xi_a^2) \\ K(\hat{w}_1 + \xi_a) \end{pmatrix}. \end{aligned} \tag{15}$$

The derivative of $\hat{e}$ holds

$$\dot{\hat{e}} = \dot{\hat{w}} - \dot{w} = \begin{pmatrix} A & 0 \\ 0 & \lambda \end{pmatrix} \hat{e} + \begin{pmatrix} l_1 \\ l_2 \\ 0 \end{pmatrix} (a\widetilde{w} + (1-a)w - \hat{w}) + \begin{pmatrix} 0 \\ \xi_a \hat{w}_3 - w_1 w_3 - K'\xi_a^3 + K' w_1^3 \\ K\xi_a^2 - K w_1^2 \end{pmatrix}. \tag{16}$$

This can be rewritten as

$$\dot{\hat{e}} = (1-a) \begin{pmatrix} (A-LC) & 0 \\ 0 & \lambda \end{pmatrix} \hat{e} + a \begin{pmatrix} (A-LC) & 0 \\ 0 & \lambda \end{pmatrix} \tilde{e} + (1-a) \begin{pmatrix} 0 \\ w_1 \hat{e}_3 \\ 0 \end{pmatrix} + a \begin{pmatrix} 0 \\ (\tilde{e}_1 - \hat{e}_1)\hat{w}_3 - K'(\tilde{e}_1 + \hat{e}_1)(\xi_a^2 + \xi_a w_1 + w_1^2) \\ K(\tilde{e}_1 + \hat{e}_1)(\xi_a + w_1) \end{pmatrix}. \tag{17}$$

The procedure of proving the existence of a solution that converges to zero for the system $(\hat{e}^T, \tilde{e}^T)^T$ is conducted along the following lines:

(1) Existence of a solution for the system composed of the first two terms in (15) and first and third terms in (17) together with its convergence to zero at infinity is proved.
(2) The terms multiplied by a are added to the equation; convergence of its solution is again proved.
(3) Finally, the term multiplied by $(1-a)$ is considered.

For a future purpose, let us define $\lambda' = \max\{\mathrm{Re}\ \mathrm{eig}(A - LC)\}$.

Note that $\lambda' < 0$.

Assumption 3.1. $\lambda' \neq \lambda$, $(1-a)^2\lambda \neq \lambda'$, $\lambda \neq (1+a)\lambda'$.

The following theorem constitutes the main result of the paper.

Theorem 3.1. *Consider three GLSs* (1) *connected as in Fig. 3. Let Assumption 3.1 hold. Assume all trajectories of the interconnection of the systems are bounded. Then, there exists* $a^* \in (0,1)$ *so that, for all* $a \in [0, a^*)$ *and all initial conditions* $(w(0), \hat{w}(0), \widetilde{w}(0)) \in \mathbb{R}^3$, *the interconnected system is synchronized.*

Proof. See Appendix A. ■

Remark 3.1. The assumption about the boundedness of the trajectories is satisfied in the case of tree topology (that means, $a = 0$). On the other hand, it is not easy to see the validity of this assumption for interconnection topologies with loops. The following section thus presents conditions guaranteeing the boundedness of trajectories of interconnected Lorenz systems even for some $a > 0$.

The conditions guaranteeing boundedness of the trajectories of the feedback interconnection of the Lorenz systems can be found in Appendix B.

4. Example

In this example, the synchronization of three coupled Lorenz systems is presented. The Lorenz systems were chosen as in [Čelikovský *et al.*, 2023]. To be precise, the constants describing them were

$$a_{11} = -10, \quad a_{12} = 10, \quad a_{21} = 28, \quad a_{22} = -1$$

while the coupling constant $a = 0.01$ and $L = (1, 70)$. From Figs. 3–5, it can be seen that the states are synchronized — Fig. 3 shows the state w_1 of all three systems; analogously, Figs. 4 and 5 depict the states, w_2 and w_3, respectively. The meaning of the lines in all figures is the same: the solid line represents the leader system, the dashed line illustrates the follower **F1**, and the dash-dot line stands for the follower **F2** (see also the scheme in

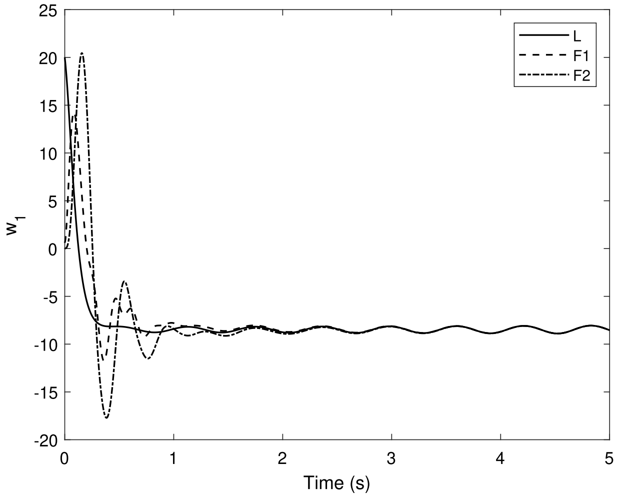

Fig. 3. States w_1.

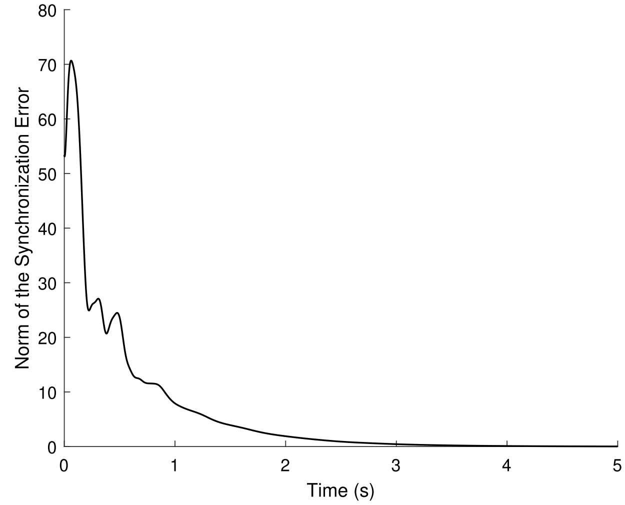

Fig. 6. Synchronization error.

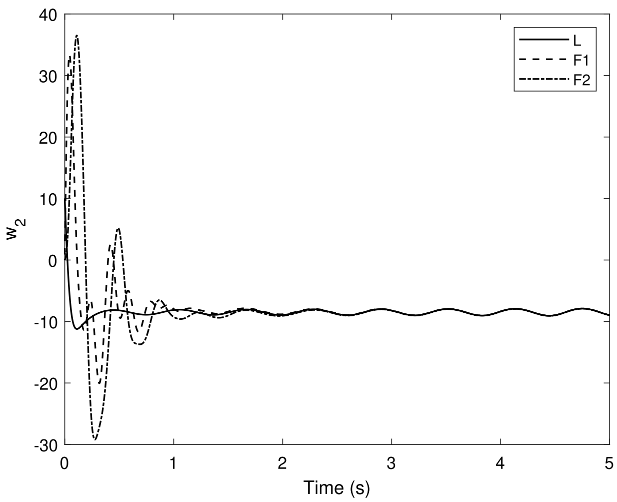

Fig. 4. States w_2.

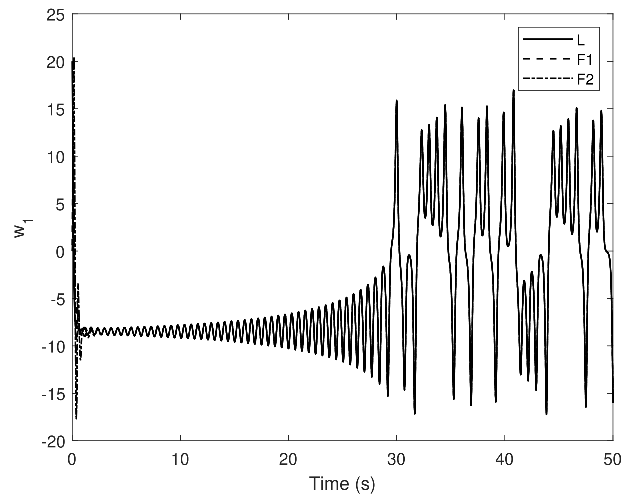

Fig. 7. States w_1.

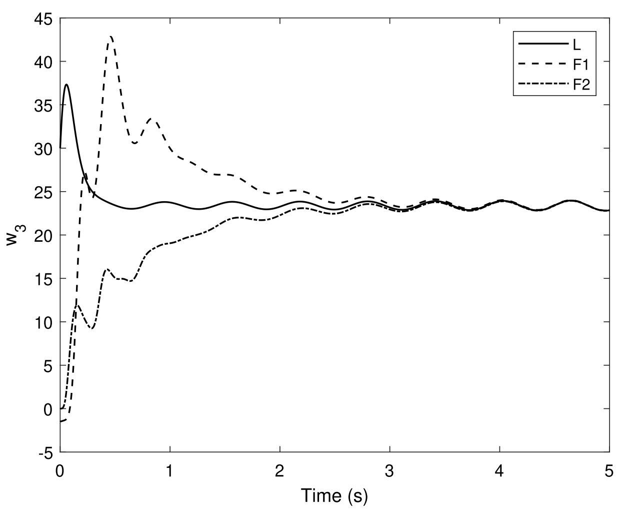

Fig. 5. States w_3.

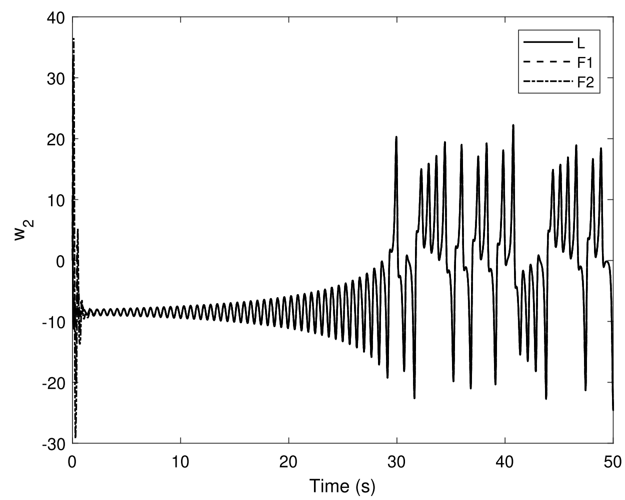

Fig. 8. States w_2.

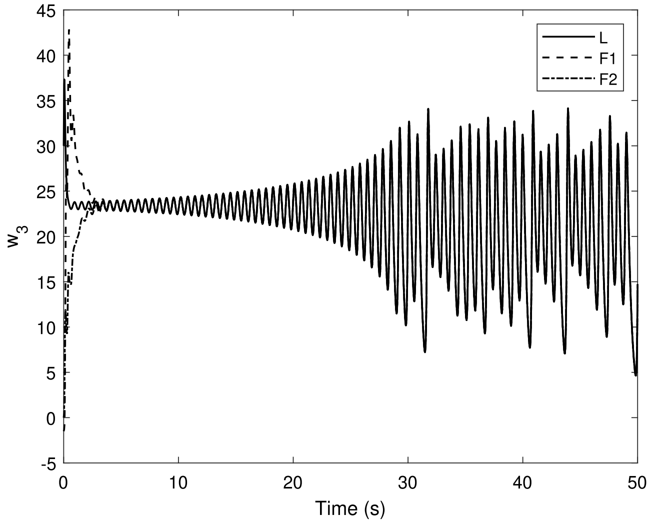

Fig. 9. States w_3.

Fig. 3). Moreover, the norm of the synchronization error can be seen in Fig. 6. From this figure, one sees that the synchronization error converges to zero. All the aforementioned figures illustrate the behavior of the interconnected Lorenz systems on the interval $(0, 5)$s. The behavior of the interconnected systems on a longer time interval, namely $(0, 50)$s, is shown in Figs. 7–9. The meaning of the lines is identical as in Figs. 3–5. Moreover, one can see the norm of the error on this longer interval in Fig. 10. The last four figures illustrate the fact that the synchronization of the interconnected chaotic systems is not violated even under the chaotic behavior of the leader.

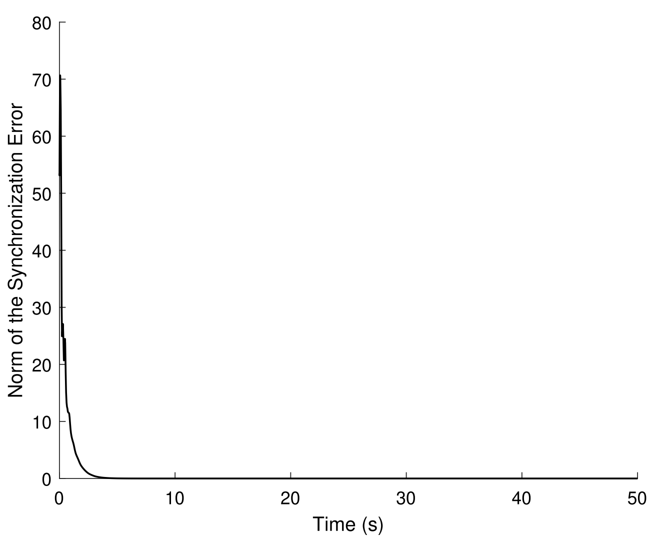

Fig. 10. Synchronization error.

5. Conclusions

In this paper, sufficient conditions for the synchronization of coupled GLSs were given. The interconnection topology contains a loop. It was shown that coupling strengths of all interconnections can be found so that synchronization is achieved. Moreover, for the case of the coupled Lorenz systems, it was shown that even if their interconnection contains a loop, the trajectories of all such nodes are bounded, and consequently, the synchronization of all nodes of Lorenz systems is possible for sufficiently small coupling strengths. The results were illustrated by an example.

The interconnection of three GLSs was studied; however, in principle, the number of coupled systems can be extended to many, perhaps with slight technical modifications. Moreover, in future research, a generalization of the results to the networks with a more complex interconnection topology will be investigated. Another direction of further research is the investigation of the generalized synchronization of complex networks with a loop. In this case, the requirement for equal values of parameters across the nodes will be relaxed.

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Appendices

Appendix A

Proof of Theorem 3.1

Three auxiliary lemmas are formulated and proved. Then, the proof of Theorem 3.1 follows directly as a consequence of these lemmas.

Lemma A.1. *Assume there exists a constant* $M > 0$ *so that* $\|w_1(t)\| + \|\hat{w}_1(t)\| \le M$ *for all* $t \ge 0$. *Consider the system*

$$\dot{z}_1 = (1-a)(A - LC)z_1 + (0, w_1 z_2)^T, \tag{A.1}$$

$$\dot{z}_2 = (1-a)\lambda z_2, \tag{A.2}$$

$$\dot{z}_3 = (A - LC)z_3 + (0, \hat{w}_1 z_4)^T, \tag{A.3}$$

$$\dot{z}_4 = \lambda z_4 \tag{A.4}$$

with initial conditions $z_1(0) \in \mathbb{R}^2$, $z_2(0) \in \mathbb{R}$, $z_3(0) \in \mathbb{R}^2$, $z_4(0) \in \mathbb{R}$. *Then, there exist constants* $K > 0$, $\bar{\lambda} \in (\lambda', 0)$ *so that*

$$\begin{aligned} \|(z_3(t), z_4(t))^T\| &\le K e^{\bar{\lambda} t} \|(z_3(0), z_4(0))^T\|, \\ \|(z_1(t), z_2(t))^T\| &\le K e^{\bar{\lambda} t} \|(z_1(0), z_2(0))^T\|. \end{aligned} \tag{A.5}$$

Proof. The proof is conducted along similar lines as the proof of Lemma 2.4 in [Čelikovský *et al.*, 2023]. First, note that $z_4(t) = z_4(0)e^{\lambda t}$ for $t \ge 0$. Then, for z_3, the following holds:

$$z_3(t) = e^{(A-LC)t} z_3(0) + \int_0^t e^{(A-LC)(t-s)} \times (0, \hat{w}_1(s) z_4(s)) ds. \tag{A.6}$$

First, note that there exists a constant $\varkappa > 0$ so that for all $t \geq 0$, the following holds:

$$\begin{aligned} \|e^{(A-LC)t}\| &\leq \varkappa e^{\lambda' t}, \\ \|e^{(1-a)(A-LC)t}\| &\leq \varkappa e^{\lambda' t}. \end{aligned} \tag{A.7}$$

This implies

$$\begin{aligned} \|z_3(t)\| &\leq \varkappa e^{\lambda' t}\|z_3(0)\| + \int_0^t e^{\lambda'(t-s)} M|z_4(0)|e^{\lambda s} ds \\ &= \varkappa e^{\lambda' t}\|z_3(0)\| + \frac{1}{|\lambda - \lambda'|} 2M|z_4(0)| \\ &\quad \times e^{\max(\lambda,\lambda')t} \\ &\leq \max\left(\varkappa, \frac{2M}{|\lambda-\lambda'|}\right) e^{\max(\lambda,\lambda')t} \\ &\quad \times (\|z_3(0)\| + |z_4(0)|). \end{aligned}$$

The estimate for the pair z_1, z_2 is derived analogously. For function z_2, the following holds:

$$z_2(t) = e^{(1-a)\lambda t} z_2(0). \tag{A.8}$$

Finally,

$$\begin{aligned} z_1(t) &= e^{(1-a)(A-LC)t} z_1(0) + \int_0^t e^{(1-a)(A-LC)(t-s)} \\ &\quad \times (0, w_1(s) z_2(s))^T ds. \end{aligned} \tag{A.9}$$

This yields

$$\begin{aligned} \|z_1(t)\| &\leq \|e^{(1-a)(A-LC)t} z_1(0)\| \\ &\quad + \left\| \int_0^t e^{(1-a)(A-LC)(t-s)} \right. \\ &\quad \left. \times (0, w_1(s) z_2(s))^T ds \right\| \\ &\leq \varkappa e^{\lambda' t}\|z_1(0)\| + \int_0^t \|e^{(1-a)(A-LC)(t-s)} \\ &\quad \times (0, w_1(s) z_2(s))^T\| ds \\ &\leq \max\left(\varkappa, \frac{2M}{(1-a)|\lambda-\lambda'|}\right) e^{(1-a)\max(\lambda,\lambda')t} \\ &\quad \times (\|z_1(0)\| + |z_2(0)|). \end{aligned}$$

Then, with $K = \max(\varkappa, \frac{2M}{(1-a)|\lambda-\lambda'|})$, $\bar{\lambda} = (1-a)\max(\lambda, \lambda')$, the claim of the lemma holds. ■

Lemma A.2. *Let $\varphi : [0, \infty) \to \mathbb{R}^2$ and $\psi : [0, \infty) \to \mathbb{R}$ be continuous functions. Suppose there exists a constant $M > 0$ so that $\|\varphi(t)\| \leq M$, $|\psi(t)| \leq M$ for all $t \geq 0$. Consider the system*

$$\begin{aligned} \dot{y}_1 &= (1-a)(A-LC)y_1 + a(A-LC)y_3 \\ &\quad + (0, w_1 y_2)^T, \end{aligned} \tag{A.10}$$

$$\dot{y}_2 = (1-a)\lambda y_2 + a\lambda y_4, \tag{A.11}$$

$$\begin{aligned} \dot{y}_3 &= (A-LC)y_3 + (0, \hat{w}_1 y_4)^T \\ &\quad + \varphi(t) y_1, \end{aligned} \tag{A.12}$$

$$\dot{y}_4 = \lambda y_4 + \psi(t) y_1 \tag{A.13}$$

with initial conditions $y_1(0) \in \mathbb{R}^2$, $y_2(0) \in \mathbb{R}$, $y_3(0) \in \mathbb{R}^2$, $y_4(0) \in \mathbb{R}$. Then, there exist constants $a^* \in (0,1)$ so that for each $a \in [0, a^*)$, a solution of system (A.10)–(A.13) exists. Moreover, there exist constants $K' > 0$, $\mu \in (\bar{\lambda}, 0)$ so that

$$\begin{aligned} \|(y_3(t), y_4(t))^T\| &\leq K' e^{\mu t}\|(y_3(0), y_4(0))^T\|, \\ \|(y_1(t), y_2(t))^T\| &\leq K' e^{\mu t}\|(y_1(0), y_2(0))^T\|. \end{aligned} \tag{A.14}$$

Proof. Assume $y_{1,0} \in \mathbb{R}^2$, $y_{2,0} \in \mathbb{R}$, $y_{3,0} \in \mathbb{R}^2$, $y_{4,0} \in \mathbb{R}$. Then, for $k \in \mathbb{N}$, define the sequence of the following differential equations:

Let $(y_1^{(0)}, y_2^{(0)}, y_3^{(0)}, y_4^{(0)})^T = (0,0,0,0)$, $(y_1^{(1)}, y_2^{(1)}, y_3^{(1)}, y_4^{(1)})^T$ solve Eqs. (A.1)–(A.4) with initial conditions $(y_1^{(1)}(0), y_2^{(1)}(0), y_3^{(1)}(0), y_4^{(1)}(0))^T = (y_{1,0}, y_{2,0}, y_{3,0}, y_{4,0})^T$. Then, thanks to Lemma A.1, $\|y^{(1)} - y^{(0)}\| \leq K\|(y_{1,0}, y_{2,0}, y_{3,0}, y_{4,0})^T\| e^{\bar{\lambda} t}$.

For $k \geq 1$, let

$$\begin{aligned} \dot{y}_1^{(k)} &= (1-a)(A-LC)y_1^{(k)} + a(A-LC)y_3^{(k-1)} \\ &\quad + (0, w_1 y_2^{(k)})^T, \end{aligned} \tag{A.15}$$

$$\dot{y}_2^{(k)} = (1-a)\lambda y_2^{(k)} + a\lambda y_4^{(k-1)}, \tag{A.16}$$

$$\begin{aligned} \dot{y}_3^{(k)} &= (A-LC)y_3^{(k)} + (0, \hat{w}_1 y_4^{(k)})^T \\ &\quad + \varphi(t) y_1^{(k)}, \end{aligned} \tag{A.17}$$

$$\dot{y}_4^{(k)} = \lambda y_4^{(k)} + \psi(t) y_1^{(k)} \tag{A.18}$$

with initial conditions $(y_1^{(k)}(0), y_2^{(k)}(0), y_3^{(k)}(0), y_4^{(k)}(0))^T = (y_{1,0}, y_{2,0}, y_{3,0}, y_{4,0})^T$.

Take $\mu \in (\bar{\lambda}, 0)$. We now aim to show that there exists a continuous function $\omega : [0, \Omega) \to [0, \infty)$ so that $\omega(0) = 0$ and

$$\|(y_1^{(k+1)}(t), y_2^{(k+1)}(t), y_3^{(k+1)}(t), y_4^{(k+1)}(t))^T - (y_1^{(k)}(t), y_2^{(k)}(t), y_3^{(k)}(t), y_4^{(k)}(t))^T\| \leq K(\omega(a))^k e^{\mu t}. \tag{A.19}$$

To prove this statement, let us start with Eq. (A.16). The difference $y_2^{(k+1)}(t) - y_2^{(k)}(t)$ is

$$y_2^{(k+1)}(t) - y_2^{(k)}(t) = a\int_0^t e^{(1-a)\lambda(t-s)}(y_4^{(k)}(s) - y_4^{(k-1)}(s))ds. \tag{A.20}$$

Let $K_2(a) = \frac{1}{\mu-(1-a)|\lambda|}$. Then,

$$\begin{aligned}|y_2^{(k+1)}(t) - y_2^{(k)}(t)| &\leq a\frac{1}{\mu-(1-a)|\lambda|}K(\omega(a))^k(e^{\mu t} - e^{(1-a)\lambda t}) \\ &\leq \frac{a}{\mu-(1-a)|\lambda|}K(\omega(a))^k e^{\mu t} = aK_2(a)K(\omega(a))^k e^{\mu t}. \end{aligned}\tag{A.21}$$

Now, let us focus on the difference $y_1^{(k+1)} - y_1^{(k)}$:

$$\begin{aligned}\|y_1^{(k+1)}(t) - y_1^{(k)}(t)\| &\leq \int_0^t \|e^{(1-a)(A-LC)(t-s)}\|(a\|A-LC\|\|y_3^{(k)}(s) - y_3^{(k-1)}(s)\| \\ &\quad + M|y_2^{(k+1)}(s) - y_2^{(k)}(s)|)ds \\ &\leq \int_0^t e^{(1-a)\lambda'(t-s)}\left(a\|A-LC\|\|y_3^{(k)}(s) - y_3^{(k-1)}(s)\|\right. \\ &\quad \left. + \frac{M}{(1-a)|\lambda|}|y_2^{(k)}(s) - y_2^{(k-1)}(s)|\right)ds \\ &\leq \int_0^t e^{(1-a)\lambda'(t-s)}\left(a\|A-LC\| + \frac{M}{(1-a)|\lambda|}aK_2(a)\right)K(\omega(a))^k e^{\mu s}ds \\ &\leq \frac{1}{\mu-(1-a)\lambda'}\left(\|A-LC\| + \frac{M}{(1-a)|\lambda|}K_2(a)\right)aK(\omega(a))^k e^{\mu t}.\end{aligned}$$

Denote $K_1(a) = \frac{1}{\mu-(1-a)\lambda'}(\|A-LC\| + \frac{M}{(1-a)|\lambda|}\frac{1}{\mu-(1-a)|\lambda|})$. Then,

$$\|y_1^{(k+1)}(t) - y_1^{(k)}(t)\| \leq K_1(a)Ka(\omega(a))^k e^{\mu t}. \tag{A.22}$$

In the next step, the term $|y_4^{(k+1)}(t) - y_4^{(k)}(t)|$ is estimated as

$$y_4^{(k+1)}(t) - y_4^{(k)}(t) = \int_0^t e^{\lambda(t-s)}\psi(s)(y_1^{(k)}(s) - y_1^{(k-1)}(s))ds. \tag{A.23}$$

Define $K_4(a) = MK_1(a)\frac{1}{\mu-\lambda}$. Then,

$$\begin{aligned}|y_4^{(k+1)}(t) - y_4^{(k)}(t)| &\leq \int_0^t e^{\lambda(t-s)}|\psi(s)||y_1^{(k)}(s) - y_1^{(k-1)}(s)|ds \\ &\leq \int_0^t e^{\lambda(t-s)}MK_1(a)Ka(\omega(a))^k e^{\mu s}ds\end{aligned}$$

$$\leq MK_1(a)Ka(\omega(a))^k\frac{1}{\mu-\lambda}|e^{\mu t}-e^{\lambda t}|$$

$$\leq MK_1(a)aK(\omega(a))^k\frac{1}{\mu-\lambda}e^{\mu t}=K_4(a)aK(\omega(a))^k e^{\mu t}.$$

Finally,

$$\begin{aligned}\|y_3^{(k+1)}(t)-y_3^{(k)}(t)\| &\leq \int_0^t \|e^{(A-LC)(t-s)}\|(|y_4^{(k+1)}(s)-y_4^{(k)}(s)| \\ &\quad + \|\varphi(s)\|\|y_1^{(k)}(s)-y_1^{(k-1)}(s)\|)ds \\ &\leq \int_0^t e^{\lambda'(t-s)}\left(MK_4(a)\frac{1}{\mu-\lambda}+MK_1(a)\right)a(\omega(a))^k e^{\mu s}ds \\ &\leq \frac{1}{\mu-\lambda'}\left(MK_4(a)\frac{1}{\mu-\lambda}+K_1(a)\right)aK(\omega(a))^k e^{\mu t}.\end{aligned}$$

Let

$$K_3(a)=\frac{1}{\mu-\lambda'}\left(MK_4(a)\frac{1}{\mu-\lambda}+K_1(a)\right).$$

Then,

$$\|y_3^{(k+1)}(t)-y_3^{(k)}(t)\|\leq K_3(a)aK(\omega(a))^k. \tag{A.24}$$

Finally, define $\omega(a)=a\max(K_1(a),K_2(a),K_3(a),K_4(a))$. Then,

$$\|(y_1^{(k+1)}(t),y_2^{(k+1)}(t),y_3^{(k+1)}(t),y_4^{(k+1)}(t))^T\|\leq K(\omega(a))^{k+1}. \tag{A.25}$$

Since ω is a non-negative continuous function and $\omega(0)=0$, there exists $a^*\in(0,1)$ so that for all $a\in[0,a^*)$, $\omega(a)<1$ holds. Hence, the sequence $y_i^{(k)}(t)$ converges to $y_i(t)$ for all $i=1,\ldots,4$. Moreover, $\lim_{t\to\infty}y_i(t)=0$.

Finally, the Dominated Convergence Theorem yields

$$\int_0^t e^{(1-a)(A-LC)(t-s)}a(A-LC)y_3^{(k)}(s)ds\to\int_0^t e^{(1-a)(A-LC)(t-s)}a(A-LC)y_3(s)ds,$$

$$\int_0^t e^{(1-a)(A-LC)(t-s)}(0,w_1(s)y_2^{(k)}(s))^T ds\to\int_0^t e^{(1-a)(A-LC)(t-s)}(0,w_1(s)y_2(s))^T ds,$$

$$\int_0^t e^{(1-a)\lambda(t-s)}ay_4^{(k)}(s)ds\to\int_0^t e^{(1-a)\lambda(t-s)}ay_4(s)ds,$$

$$\begin{aligned}\int_0^t e^{(A-LC)(t-s)}((0,\hat{w}_1(s)y_4^{(k)}(s))^T+\varphi(s)y_1^{(k)}(s))ds &\to \int_0^t e^{(A-LC)(t-s)}((0,\hat{w}_1(s)y_4(s))^T \\ &\quad +\varphi(s)y_1(s))ds,\end{aligned}$$

$$\int_0^t e^{\lambda(t-s)}\psi(s)y_1^{(k)}(s)ds\to\int_0^t e^{\lambda(t-s)}\psi(s)y_1(s)ds.$$

Thus, functions y_i are the solutions of system (A.10)–(A.13). ■

Lemma A.3. *Let* $\varphi : [0,\infty) \to \mathbb{R}^2$ *and* $\psi : [0,\infty) \to \mathbb{R}$, $\chi_i : [0,\infty) \to \mathbb{R}^{2\times 6}$, $i = 1,2,3,4$ *be continuous functions. Suppose there exists a constant* $M > 0$ *so that* $\|\varphi(t)\| \le M$, $|\psi(t)| \le M$, $\|\chi_1(t)\| + \|\chi_2(t)\| + \|\chi_3(t)\| + \|\chi_4(t)\| \le M$ *for all* $t \ge 0$. *Consider the system*

$$\dot{v}_1 = (1-a)(A-LC)v_1 + a(A-LC)v_3 + (0, w_1 v_2)^T + a\chi_1(t)v, \tag{A.26}$$

$$\dot{v}_2 = (1-a)\lambda v_2 + a\lambda v_4 + a\chi_2(t)v, \tag{A.27}$$

$$\dot{v}_3 = (A-LC)v_3 + (0, \hat{w}_1 v_4)^T + \varphi(t)v_1 + a\chi_3(t)v, \tag{A.28}$$

$$\dot{v}_4 = \lambda v_4 + \psi(t)v_1 + a\chi_4(t)v \tag{A.29}$$

with initial conditions $v_1(0) \in \mathbb{R}^2$, $v_2(0) \in \mathbb{R}$, $v_3(0) \in \mathbb{R}^2$, $v_4(0) \in \mathbb{R}$. *Then, there exist constants* $a^* \in (0,1)$ *so that for each* $a \in [0, a^*)$, *a solution of system (A.10)–(A.13) exists. Moreover, there exist constants* $K' > 0$, $\mu \in ((1-a)\lambda, 0))$ *so that*

$$\begin{aligned} &\|(v_3(t), v_4(t))^T\| \le K' e^{\mu t}\|(v_3(0), v_4(0))^T, \\ &\|(v_1(t), v_2(t))^T\| \le K' e^{\mu t}\|(v_1(0), v_2(0))^T\|. \end{aligned} \tag{A.30}$$

Proof. If $\chi_i = 0$ for all $i = 1, \ldots, 4$, then there is $a^* \in (0,1)$ so that for all $a \in [0, a^*)$ the solution of (A.26)–(A.29) exists according to Lemma A.2. Denote such a solution by v^* moreover, there is $\mu < 0$ so that for all $a \in [0, a^*)$, and all $t \ge 0$, $\|v^*(t)\| \le c_0\|v^*(0)\|e^{\mu t}$ for some $c_0 > 0$.

Consider the system with at least one nonzero function χ_i as a perturbation of the system with all $\chi_i = 0$. Hence, there exists $a^{**} \in (0, a^*)$ with the following property: for all $a \in [0, a^{**})$, the solution v of (A.26–A.29) exists. Moreover, there exists a constant $c > 0$, so that for all $t \ge 0$,

$$\|v(t)\| \le c\|v(0)\|e^{\frac{\mu}{2}t}. \tag{A.31}$$

■

Finally, the proof of Theorem 3.1 follows.

Proof. Define

$$\chi_1 = ((\tilde{e}_1 - \hat{e}_1)\hat{w}_3 - K'(\tilde{e}_1 + \hat{e}_1)(\xi_a^2 + \xi_a w_1 + w_1^2)),$$

$$\chi_2 = K(\tilde{e}_1 + \hat{e}_1)(\xi_a + w_1),$$

$$\chi_3 = \hat{w}_3 - K'(\hat{w}_1^2 + \hat{w}_1\xi_a + \xi_a^2),$$

$$\chi_4 = K(\hat{w}_1 + \xi_a),$$

$$\varphi = (1-a)\hat{e}_1(-\hat{w}_3 + K'(\hat{w}_1^2 + \hat{w}_1\xi_a + \xi_a^2)),$$

$$\psi = (1-a)\hat{e}_1 K(\hat{w}_1 + \xi_a).$$

This enables us to reformulate system (15, 17) into the form (A.26) to (A.29) with $(v_1, v_2)^T = \hat{e}$, $(v_3, v_4)^T = \tilde{e}$. Then, Lemma A.3 guarantees existence of a constant $a^{**} \in (0,1)$ so that for all $a \in [0, a^{**})$, there exist functions $\hat{e}$ satisfying (17) and $\tilde{e}$ satisfying (15) (both defined on $[0,\infty)$) such that

$$\lim_{t\to\infty} \hat{e} = 0, \quad \lim_{t\to\infty} \tilde{e} = 0. \tag{A.32}$$

Taking the definition of functions $\hat{e}$ and $\tilde{e}$, one can see that (A.32) is equivalent to the synchronization of the interconnected systems (2.1), (10), and (11). As transformations (4) and (5) are diffeomorphisms, this yields the synchronization of the systems **L, F1, F2** in the original coordinates. ■

Appendix B

Boundedness of Trajectories of Interconnected Lorenz Systems

It is a well-known result that the trajectories of the Lorenz system are bounded. Moreover, the requirement for bounded trajectories does not pose any major challenge for the systems with a tree-like topology, where the fact that the trajectories of the leader are bounded is sufficient [Čelikovský *et al.*, 2023]. The situation becomes more involved if the interconnection contains loops: it is not clear whether and under what conditions, the trajectories of the feedback interconnection of the three Lorenz systems considered in this paper are bounded. To be specific, one has to exclude any "resonance phenomena" when the circular interconnection of two followers might potentially cause the trajectories to diverge. On the other hand, the boundedness of trajectories is an essential assumption in the theory elaborated in this paper. Hence, conditions, under which the boundedness of trajectories of three coupled Lorenz systems are derived, are given in this Appendix.

The results of the previous section rely heavily on the assumption of boundedness of trajectories of the entire interconnection of chaotic systems.

In this section, it is demonstrated that, under suitable conditions, the trajectories of the interconnection of the Lorenz systems are bounded. Let

$\sigma > 0$, $r > 0$, $b > 0$, then the Lorenz system is defined as

$$\begin{aligned}\dot{\xi} &= -\sigma\xi + \sigma\eta,\\ \dot{\eta} &= r\xi - \eta - \xi\zeta,\\ \dot{\zeta} &= -b\zeta + \xi\eta.\end{aligned} \tag{B.1}$$

Proposition B.1. *Let $\alpha > 0$ be a constant. Define function $V : \mathbb{R}^3 \to [0,\infty)$ by*

$$V(\xi,\eta,\zeta) = \frac{1}{2}(\alpha\xi^2 + \eta^2 + (\zeta - r - \alpha r)^2). \tag{B.2}$$

Let $R > 0$ also be a constant. Define the set

$$\mathcal{V}(R) = \{(\xi,\eta,\zeta) \in \mathbb{R}^3 \,|\, V(\xi,\eta,\zeta) \le R\}. \tag{B.3}$$

Assume the constant R is large enough so that the following relation holds:

$$\left\{(\xi,\eta,\zeta) \in \mathbb{R}^3 \,|\, \alpha\sigma\xi^2 + \eta^2 + b\left(\zeta + \frac{1}{2}(r+\alpha\sigma)\right)^2 \le \frac{b}{4}(r+\alpha\sigma)^2\right\} \subset \mathcal{V}(R). \tag{B.4}$$

Then, the set $\mathcal{V}(R)$ is an invariant and attractive set for system (B.1).

This proposition is a slight generalization of the fact cited in [Barboza, 2018, Sec. 3.4.1].

Consider the interconnection of three Lorenz systems.

The interconnected triple of systems is governed by equations

$$\begin{aligned}\dot{x}_1 &= -\sigma x_1 + \sigma x_2,\\ \dot{x}_2 &= r x_1 - x_2 - x_1 x_3,\\ \dot{x}_3 &= -b x_3 + x_1 x_2,\\ \dot{\hat{x}}_1 &= -\sigma\hat{x}_1 + \sigma\hat{x}_2 + l_1(a\tilde{x}_1 + (1-a)x_1 - \hat{x}_1),\\ \dot{\hat{x}}_2 &= r\hat{x}_1 - \hat{x}_2 - \hat{x}_1\hat{x}_3 + l_2(a\tilde{x}_1 + (1-a)x_1 - \hat{x}_1),\\ \dot{\hat{x}}_3 &= -b\hat{x}_3 + \hat{x}_1\hat{x}_2,\\ \dot{\tilde{x}}_1 &= -\sigma\tilde{x}_1 + \sigma\tilde{x}_2 + l_1(\hat{x}_1 - \tilde{x}_1),\\ \dot{\tilde{x}}_2 &= r\tilde{x}_1 - \tilde{x}_2 - \tilde{x}_1\tilde{x}_3 + l_2(\hat{x}_1 - \tilde{x}_1),\\ \dot{\tilde{x}}_3 &= -b\tilde{x}_3 + \tilde{x}_1\tilde{x}_2.\end{aligned} \tag{B.5}$$

Lemma B.1. *Consider system (B.5). Assume $l_1 > 0$. Let there exist constants $\tilde{\alpha} > 0$, $\hat{\alpha} > 0$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\mu_1 > 0$, $\mu_2 > 0$, $\gamma > 0$ such that*

$$0 < \sigma + l_1 - \varepsilon_1 l_1, \tag{B.6}$$

$$1 > \varepsilon_2 l_2, \tag{B.7}$$

$$0 < \tilde{\alpha}(\sigma + l_1 - \varepsilon_1 l_1) - \frac{a l_1\hat{\alpha}}{\mu_1} - \frac{a l_2}{\mu_2}, \tag{B.8}$$

$$1 > \varepsilon_2 l_2, \tag{B.9}$$

$$0 < \hat{\alpha}(\sigma + l_1 - \mu_1 l_1) - \frac{l_1\tilde{\alpha}}{\varepsilon_1} - \frac{l_2}{\varepsilon_2}, \tag{B.10}$$

$$1 > \mu_2 l_2, \tag{B.11}$$

$$\gamma\sigma > (1-a)\left(\frac{l_1\hat{\alpha}}{\mu_1} + \frac{l_2}{\mu_2}\right). \tag{B.12}$$

Then, trajectories of system (B.5) are bounded.

Proof. Let

$$V = \frac{\gamma}{2}(x_1^2 + x_2^2 + (\hat{x}_3 - \sigma - r)^2),$$

$$\hat{V} = \frac{1}{2}(\hat{\alpha}\hat{x}_1^2 + \hat{x}_2^2 + (\hat{x}_3 - \sigma\hat{\alpha} - r + l_2)^2)$$

$$\tilde{V} = \frac{1}{2}(\tilde{\alpha}\tilde{x}_1^2 + \tilde{x}_2^2 + (\tilde{x}_3 - \sigma\tilde{\alpha} - r + l_2)^2).$$

Then, after some manipulations, one obtains the following relation for the derivative of $\tilde{V}$ along trajectories of (B.5):

$$\begin{aligned}\dot{\tilde{V}} = -\tilde{\alpha}(\sigma + l_1)\tilde{x}_1^2 + \tilde{\alpha}l_1\tilde{x}_1\hat{x}_1 - \tilde{x}_2^2 + l_2\tilde{x}_2\hat{x}_1\\ - b\tilde{x}_3^2 + b(\tilde{\alpha}\sigma + r - l_2)\tilde{x}_3.\end{aligned}$$

The Young inequality implies

$$\tilde{x}_1\hat{x}_1 \le \varepsilon_1\tilde{x}_1^2 + \frac{1}{\varepsilon_1}\hat{x}_1^2,$$

$$\tilde{x}_2\hat{x}_1 \le \varepsilon_2\tilde{x}_2^2 + \frac{1}{\varepsilon_2}\hat{x}_1^2.$$

Define $\tilde{A} = \frac{1}{2}(\tilde{\alpha}\sigma + r - l_2)$. Then,

$$\begin{aligned}\dot{\tilde{V}} = -\tilde{\alpha}(\sigma + l_1 - \varepsilon_1 l_1)\tilde{x}_1^2 - (1 - \varepsilon_2 l_2)\tilde{x}_2^2\\ - b(\tilde{x}_3 - \tilde{A})^2 + b\tilde{A}^2 + \left(\frac{l_1\tilde{\alpha}}{\varepsilon_1} + \frac{l_2}{\varepsilon_2}\right)\hat{x}_1^2.\end{aligned} \tag{B.13}$$

Using the similar reasoning, one obtains for the derivative of $\hat{V}$:

$$\begin{aligned}\dot{\hat{V}} &= -\hat{\alpha}(\sigma + l_1)\hat{x}_1^2 + \hat{\alpha} l_1 \hat{x}_1(a\tilde{x}_1 + (1-a)x_1)\\ &\quad - \hat{x}_2^2 + l_2\hat{x}_2(a\tilde{x}_1 + (1-a)x_1)\\ &\quad - b\hat{x}_3(\hat{x}_3 - r + l_2 - \hat{\alpha}\sigma).\end{aligned}$$

The terms containing variables from different parts of the interconnection are estimated as follows:

$$\tilde{x}_1\hat{x}_1 \le \mu_1\hat{x}_1^2 + \frac{1}{\mu_1}\tilde{x}_1^2,$$

$$x_1\hat{x}_1 \le \mu_1\hat{x}_1^2 + \frac{1}{\mu_1}x_1^2,$$

$$\tilde{x}_1\hat{x}_2 \le \mu_2\hat{x}_2^2 + \frac{1}{\mu_2}\tilde{x}_1^2,$$

$$x_1\hat{x}_2 \le \mu_2\hat{x}_2^2 + \frac{1}{\mu_2}x_1^2.$$

Let $\hat{A} = \frac{1}{2}(\hat{\alpha}\sigma + r - l_2)$. Then,

$$\begin{aligned}\dot{\hat{V}} &= -\hat{\alpha}(\sigma + l_1 - \mu_1 l_1)\hat{x}_1^2 - (-\mu_2 l_2)\hat{x}_2^2\\ &\quad - b(\hat{x}_3 - \hat{A}^2) + b\hat{A}^2 + \frac{a l_1\hat{\alpha}}{\mu_1}\tilde{x}_1^2\\ &\quad + \frac{(1-a)l_1\hat{\alpha}}{\mu_1}x_1^2 + \frac{a l_2}{\mu_2}\tilde{x}_1^2\\ &\quad + \frac{(1-a)l_2}{\mu_2}x_1^2. \end{aligned} \tag{B.14}$$

Finally,

$$\begin{aligned}\dot{V} &= -\gamma\left(\sigma x_1^2 + x_2^2 + b\left(x_3 - \frac{\sigma + r}{2}\right)^2\right)\\ &\quad + \gamma\left(\frac{\sigma + r}{2}\right)^2. \end{aligned} \tag{B.15}$$

Summing (B.13), (B.14), and (B.15) yields

$$\begin{aligned}&\dot{\tilde{V}} + \dot{\hat{V}} + \dot{V}\\ &\quad = -\left(\tilde{\alpha}(\sigma + l_1 - \varepsilon_1 l_1) - \frac{a l_1\hat{\alpha}}{\mu_1} - \frac{a l_2}{\mu_2}\right)\tilde{x}_1^2\\ &\qquad - (1 - \varepsilon_2 l_2)\tilde{x}_2^2 - b(\tilde{x}_3 - \tilde{A})^2 + b\tilde{A}^2\\ &\qquad - \left(\hat{\alpha}(\sigma + l_1 - \mu_1 l_1) - \frac{l_1\tilde{\alpha}}{\varepsilon_1} - \frac{l_2}{\varepsilon_2}\right)\hat{x}_1^2\\ &\qquad - (1 - \mu_2 l_2)\hat{x}_2^2 - b(\hat{x}_3 - \hat{A})^2 + b\hat{A}^2\\ &\qquad - \left(\gamma\sigma - (1-a)\left(\frac{l_1\hat{\alpha}}{\mu_1} + \frac{l_2}{\mu_2}\right)\right)x_1^2\\ &\qquad - \gamma\left(x_2^2 + b\left(x_3 - \frac{\sigma + r}{2}\right)^2\right)\\ &\qquad + \gamma\left(\frac{\sigma + r}{2}\right)^2. \end{aligned} \tag{B.16}$$

As a consequence, there exists a compact set $\Gamma \subset \mathbb{R}^9$ so that $\dot{\tilde{V}} + \dot{\hat{V}} + \dot{V} < 0$ in $\mathbb{R}^9 - \Gamma$.

Let $\mathcal{W}(R) \subset \mathbb{R}^9$ be defined as

$$\begin{aligned}\mathcal{W}(R) = \{&(x_1, x_2, x_3, \hat{x}_1, \hat{x}_2, \hat{x}_3, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3)\\ &\in \mathbb{R}^9 \mid V(x_1, x_2, x_3) + \hat{V}(\hat{x}_1, \hat{x}_2, \hat{x}_3)\\ &+ \tilde{V}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \le R\}. \end{aligned} \tag{B.17}$$

There exists $R^* > 0$ so that $\Gamma \subset \mathcal{W}(\mathcal{R}^*)$. Consequently, for any $\varepsilon > 0$, the set $\mathcal{W}(\mathcal{R}^* + \varepsilon)$ is an attractive invariant set of the interconnection of the Lorenz systems. ■