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Synchronization of Multi-Agent Systems Composed of Second-Order Underactuated Agents

Branislav Rehák †, Anna Lynnyk † and Volodymyr Lynnyk *,†

The Czech Academy of Sciences, Institute of Information Theory and Automation, Pod Vodárenskou věží 4, 182 00 Praha, Czech Republic; rehakb@utia.cas.cz (B.R.); lynnyk.a@utia.cas.cz (A.L.)
* Correspondence: voldemar@utia.cas.cz
† These authors contributed equally to this work.

Abstract: The consensus problem of a multi-agent system with nonlinear second-order underactuated agents is addressed. The essence of the approach can be outlined as follows: the output is redesigned first so that the agents attain the minimum-phase property. The second step is to apply the exact feedback linearization to the agents. This transformation divides their dynamics into a linear observable part and a non-observable part. It is shown that consensus of the linearizable parts of the agents implies consensus of the entire multi-agent system. To achieve the consensus of the original system, the inverse transformation of the exact feedback linearization is applied. However, its application causes changes in the dynamics of the multi-agent system; a way to mitigate this effect is proposed. Two examples are presented to illustrate the efficiency of the proposed synchronization algorithm. These examples demonstrate that the synchronization error decreases faster when the proposed method is applied. This holds not only for the states constituting the linearizable dynamics but also for the hidden internal dynamics.

Keywords: nonlinear multi-agent systems; underactuated systems; robust control

MSC: 93A16; 93C10

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1. Introduction

1.1. State of the Art

A multi-agent system is a system composed of several autonomous units, so-called agents, that are controlled to achieve a common goal (usually reaching synchronization). To accomplish this goal, the agents can communicate; nonetheless, inter-agent communication is restricted. To be specific, one agent can communicate with a limited number of other agents only; for details, see, e.g., [1,2]. This theoretical setting has numerous applications in biology, technology, physics, etc. This is why this problem belongs to the central problems of recent control theory.

There are two basic tasks to be accomplished by the multi-agent system: leader-following synchronization and consensus synchronization. The *leader-following* synchronization problem is characterized by the presence of an agent (so-called leader agent) that does not receive any information from other agents while the other agents should replicate the behavior of the leader agent. (Another term for this kind of synchronization is *master-slave synchronization* [3]). In the *consensus synchronization* problem, all agents can receive information; the goal is to force the agents to converge to a common trajectory, usually described by the average dynamics of all agents [4]. This paper focuses on the consensus problem.

The synchronization of linear multi-agent systems has been studied in numerous papers. This applies even to systems with delays in the communication channels, uncertainties, etc. On the other hand, there are significantly fewer results concerning nonlinear multi-agent systems, despite the practical importance of these systems. Several approaches have been developed to handle nonlinearities in the agents. The Young inequality was used to estimate the nonlinear terms in [5,6], among others. Similarly, in [7], Qian designed the synchronizing control based on the Jacobians of the nonlinear terms. Nonetheless, these approaches do not allow for precise matching of nonlinearity, inevitably leading to rather conservative results. Another method is based on game theory [8], yielding an iterative algorithm, while backstepping is also often used to design a synchronizing control for nonlinear multi-agent systems (see, e.g., [9–13], among others). This algorithm leads to a control law that is robust against disturbances. However, the design process is rather cumbersome. A passivity-based approach to the synchronization problem with delayed control signals was adopted in [14]; its application for synchronizing a group of mobile robots was also described.

Recently, exact feedback linearization-based algorithms for synchronizing nonlinear multi-agent systems have emerged. Synchronization algorithms using this nonlinear transformation can handle nonlinearity more precisely than those based on its linear approximation, leading to less conservative results. Moreover, many nonlinear systems encountered in practice exhibit nontrivial zero dynamics. Thus, the proposed algorithms have to effectively deal with such systems. The synchronization problem for nonlinear multi-agent systems was solved for systems with exponentially stable zero dynamics (*minimum-phase*) in [15]; these results were extended to multi-agent systems with delayed control signals in [16]. The requirement for the agents to be minimum-phase systems is reiterated in this paper. The design method described in these papers can be briefly outlined as follows: the exact feedback linearization is applied to all agents first; after this transformation, one finds the synchronizing control for the linearized agents. The control is then transformed into the original coordinates in the next step, yielding a nonlinear synchronizing control law. If hidden dynamics are present, it is shown that they are also synchronized by this control law, provided that these dynamics are exponentially stable. As shown in [17], this approach can be extended to systems with output feedback.

Underactuated systems are systems with fewer actuators than degrees of freedom. These systems constitute a very practically important class of nonlinear systems (see, e.g., [18–20], among others. Let us mention that the two-link manipulator [21], three-link manipulator [22], the pendulum on a cart, robot manipulators [23], the translational oscillator with a rotating actuator (TORA) [24], quadrotors [25], and several kinds of underwater [26] and water-surface vehicles [27] fall within this category, among other systems. On the other hand, these systems can generate rather complex behavior. Hence, their control is rather challenging [28]. A common feature of these systems is the presence of nontrivial zero dynamics, which often fail to be exponentially stable. Several approaches to their control have been proposed, including sliding-mode control [29], backstepping [30], and adaptive control [31]. Fuzzy control for fractional-order underactuated systems was adopted in [32], where stability and resilient control strategies were designed for this class of systems.

From the (inevitably incomplete) list of applications of underactuated systems, it follows that the synchronization problem of a group of underactuated systems also has strong practical importance, namely that this problem naturally arises, e.g., in the coordination or formation control of water-surface vehicles (see [33,34] or [35], among others). Let us also note that synchronizing a group of unmanned autonomous surface vehicles under cyber attacks was investigated in [36]. The synchronization problem of underwater autonomous vehicles, which also falls within this category, has been studied in numerous papers, such as in [37], and generalized in [38,39]. The control of swarms of unmanned underwater vehicles was presented in detail in [40], together with a presentation of neural-based control algorithms for these systems. Moreover, a thorough presentation of issues encountered when controlling swarms of underwater vehicles can also be found in [41]. The area of controlling underactuated robotics is rapidly developing (see [42] or [19], to name a few).

To find an effective way to control design for underactuated systems, the so-called collocated feedback method described in [43] seems to be a suitable approach. To apply this method, the procedure presented in [18] and [44] for the redefinition of the output is conducted (the so-called collocated output is designed). This ensures that the zero dynamics become asymptotically stable with this new output. As shown in [45], the stabilization of the linearized part of the system suffices for the stabilization of the entire system for minimum-phase systems. Hence, it seems that this method can be combined with the algorithms for synchronizing nonlinear multi-agent systems based on exact feedback linearization.

1.2. Purpose of this Paper

The purpose of this paper is to present a synchronization algorithm for a multi-agent system composed of underactuated agents with two degrees of freedom, based on collocated output design and the subsequent application of exact feedback linearization. The problem is studied in detail, including recovering the original average dynamics (the average dynamics are equivalent to the dynamics of a single agent). The solution to the problem of synchronization of underactuated systems via exact feedback linearization without affecting the average dynamics is the main contribution of this paper. To the best of our knowledge, this problem has not been investigated so far.

The advantage of the proposed approach is that it matches the nonlinearity precisely through the application of exact feedback linearization, which, consequently, leads to faster synchronization. A comparison with an existing method is given in the Examples section. The method proposed here has numerous practical applications, e.g., more efficient synchronization of robotic arms and swarms of surface, underwater, or aerial vehicles, among others, as these systems fall into the category of underactuated systems.

In other words, this paper aims to present how the combination of output design, exact feedback linearization, the linear synchronization algorithm, and, finally, inverse exact feedback linearization yields an efficient method for nonlinear agents' synchronization.

1.3. Outline of this Paper

The necessary terms from graph theory are summarized in the next section. Then, the problem investigated in this paper is presented. The following sections contain the necessary material to deal with multi-agent systems with underactuated agents, namely the output redesign method and the problem of synchronizing a linear uncertain multi-agent system. As these sections contain previously obtained results, they are rather brief. Then, the synchronization of the internal dynamics is proved. As the control designed so far changes the average dynamics of the multi-agent system, a modification of this control law is proposed so that the average dynamics become equivalent to the dynamics of a single agent. The illustrative examples and the conclusions follow.

1.4. Notations

The notations used in this paper include the following:

- The symbol 0 denotes the zero matrix; the symbol I_μ denotes the μ-dimensional identity matrix (the dimension of the zero matrix is always clear from the context).
- The symbol $\|.\|$ stands for the quadratic norm.
- The blocks below the diagonal are replaced with an asterisk for symmetric matrices: $\begin{pmatrix} a & b \\ b^T & c \end{pmatrix} = \begin{pmatrix} a & b \\ * & c \end{pmatrix}$.
- The symbol $\otimes$ denotes the Kronecker product.
- If P is a symmetric matrix, then the matrix inequality $P > 0$ means that the matrix P is positive definite.
- The time dependence of functions is not explicitly expressed: if v is a function of time, then $v(t)$ is abbreviated as v.

2. Graph Theory

Graph theory is a common tool for describing the information exchange between agents in the entire multi-agent system. Since the required notions of graph theory can be found in the literature (such as [2] or others), only the most important facts are presented here.

The agents are denoted by integers from the set $\mathbf{N} = \{1, \dots, N\}$. Then, define the set $\mathbb{E} \subset \mathbf{N} \times \mathbf{N}$ as follows: let $(i, j) \in \mathbb{E}$ if and only if the agent i sends information to the agent j. The *directed graph* describing the topology of the agent network is then defined as $\mathbf{G} = (\mathbf{N}, \mathbb{E})$. If for all $i, j \in \{1, \dots, N\}$ such that $(i, j) \in \mathbb{E}$ holds $(j, i) \in \mathbb{E}$, then we say the graph $\mathbf{G}$ is *undirected*.

The $N \times N$-dimensional *adjacency matrix* $J = (J_{ij})$ is defined as $J_{ij} = 1$ if $(i, j) \in \mathbb{E}$; otherwise, $J_{ij} = 0$.

Assumption 1. *The following are assumed:*

i *The interconnection topology contains no loops. This implies that $(i, i) \notin \mathbb{E}$ and, consequently, $J_{ii} = 0$ for every $i = 1, \dots, N$.*

ii *The communication topology is undirected; hence, for all $i, j \in \{1, \dots, N\}$ such that $J_{ij} = 1$, it also holds that $J_{ji} = 1$.*

Generalizing the proposed results to directed communication topologies is rather technical; hence, we focus on multi-agent systems with topologies described by undirected graphs.

The graph $\mathbf{G}$ is said to *contain a spanning tree* if the following condition holds: for every $i, j \in \mathbf{N}$, we can find a path from node i to node j.

Let us introduce the *Laplacian matrix* $L \in \mathbb{R}^{N \times N}$ by $L_{ij} = J_{ij}$ for $i \neq j$ and $L_{ii} = -\sum_{j=1}^{N} J_{ij}$, $i = 1, \dots, N$.

The paper [2] and others show that if the interconnection topology is described by the graph $\mathbf{G}$ containing a spanning tree, then the matrix L has one simple zero eigenvalue that corresponds to the eigenvector $\mathbf{1} = (1, \dots, 1)^T \in \mathbb{R}^N$, while the other eigenvalues are real and negative. Let us denote the nonzero eigenvalues as $d_1, \dots, d_{N-1}$. Let $\bar{d}$ be the nonzero eigenvalue with the largest absolute value and $\underline{d}$ be the nonzero eigenvalue with the smallest absolute value of the matrix L.

Assumption 2. *The interconnection topology of the multi-agent system studied in the following section is described by a graph that contains a spanning tree.*

3. Problem Setting

The multi-agent system considered here comprises N identical agents of the fourth order; each agent is an underactuated system. The generalized coordinates and velocities of the ith agent are denoted as $(q_i, \dot{q}_i) \in \mathbb{R}^2 \times \mathbb{R}^2$. Also, let

$$M : \mathbb{R}^2 \to \mathbb{R}^{2\times 2},\ C : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^{2\times 2}, G, D : \mathbb{R}^2 \to \mathbb{R}^2. \tag{1}$$

The matrix M is called *the mass matrix*. The ith agent is defined as

$$M(q_i)\ddot{q}_i + C(q_i, \dot{q}_i)\dot{q}_i + G(q_i) + D(\dot{q}_i) = (0,\ I_2)^T u_i, \tag{2}$$

(For a more detailed treatment of underactuated systems, see [18]).

Define the average of the state of all agents

$$\bar{q} = \frac{1}{N} \sum_{i=1}^{N} (q_i^T, \dot{q}_i^T)^T. \tag{3}$$

The multi-agent system composed of agents in (2) is said to achieve consensus if

$$\lim_{t\to\infty} \sum_{i=1}^{N} \|(q_i, \dot{q}_i)^T - \bar{q}\| = 0. \tag{4}$$

The goal is to design a smooth function $\mathbf{k} : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^4 \to \mathbb{R}$ so that the control law given as

$$u_i = \mathbf{k}\Big(q_i, \dot{q}_i, \sum_{j=1}^{N} J_{ji}((q_j, \dot{q}_j) - (q_i, \dot{q}_i))\Big) \tag{5}$$

guarantees consensus of the multi-agent system described in (2) under the assumption that all agents have initial conditions $(q_i(0), \dot{q}_i(0)) \in \mathcal{U}$, where $\mathcal{U}$ is some neighborhood of the origin in $\mathbb{R}^4$.

4. Collocated Feedback and Output Redesign

As shown in [16], if the agents are not exactly linearizable systems, their synchronization can still be achieved, provided that the internal dynamics are asymptotically stable. This condition is not satisfied in many real-world systems. However, [18] presented a method for recovering this property through a suitable output redesign for nonminimum-phase systems. For the sake of clarity, this method is briefly described here.

In line with [18] and [43], the agents are assumed to admit partial feedback linearization. To be precise, we make the assumptions listed below.

Assumption 3. *The following are assumed:*

i *For every $q \in \mathbb{R}^2$, the matrix function M is symmetric, and $M(q) > 0$.*

ii *Let us divide matrix M as $M = \begin{pmatrix} m_{11}(q) & m_{12}(q) \\ * & m_{22}(q) \end{pmatrix}$ with $m_{11}(q) \in \mathbb{R}$, $m_{12}(q) \in \mathbb{R}$ and $m_{22}(q) \in \mathbb{R}$. Then, we assume that $m_{12}(q) \neq 0$ for any $q \in \mathbb{R}^2$.*

First, divide the vector q_i as follows: $q_i = (\tilde{q}_i, q_i^\circ)^T$ with $\tilde{q}_i \in \mathbb{R}$, $q_i^\circ \in \mathbb{R}$. Next, choose $y_i = \tilde{q}_i$ as the output. Then, after some manipulations, one can infer (see [43,44]) that there exist functions $\phi' : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$, $\Phi, \Psi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, and a neighborhood $\mathcal{U}$ of zero such that

$$\Psi(r, s) \neq 0 \text{ for any } r, s \in \mathcal{U}, \tag{6}$$

and, with the transformed input $\tilde{u}_i$ defined as

$$\tilde{u}_i = \Phi(q_i, \dot{q}_i) + \Psi(q_i, \dot{q}_i) u_i \tag{7}$$

one has

$$\ddot{\tilde{q}}_i = \tilde{u}_i, \tag{8}$$

$$\ddot{q}_i^\circ = \phi'(\tilde{q}_i, \dot{\tilde{q}}_i, q_i^\circ, \dot{q}_i^\circ, \tilde{u}_i). \tag{9}$$

Remark 1. *The above formulas were obtained by defining the fictitious output as $y_i = \tilde{q}_i$ and conducting exact feedback linearization on the system* (2).

The design of the synchronization algorithm that follows requires exponential stability of the zero dynamics, which is defined as the following autonomous system:

$$\ddot{q}_i^\circ = \phi'(0, 0, q_i^\circ, \dot{q}_i^\circ, 0). \tag{10}$$

However, in many cases, the zero dynamics of underactuated systems are not exponentially stable if $y_i = \tilde{q}_i$.

A remedy can be found in [18] or [44], where a procedure to modify the output to ensure the desired stability of the zero dynamics is presented.

The authors of [18] thoroughly described the construction of the output. In [44], a simplified version for systems with two degrees of freedom was presented. For the sake of completeness, the algorithm is summarized here:

1. Choose the output $\mathbf{y}_i = \tilde{q}_i$ and determine the functions Φ and Ψ as mentioned above.
2. Determine the zero dynamics of the system with the output $\mathbf{y}_i$.

3. If the zero dynamics are not asymptotically stable, choose a sufficiently smooth function $\varphi : \mathbb{R} \to \mathbb{R}$ and redefine the output $\mathbf{y}_i$ as $\mathbf{y}_i = \tilde{q}_i + \varphi(q_i^\circ)$.
4. Redefine the functions Φ and, as well as the input $\tilde{u}_i$, by conducting exact feedback linearization on the system (2) with the redefined output. If necessary, redefine the set $\mathcal{U}$ so that condition (6) holds.
5. As the choice of the new output alters the internal dynamics, redefine the function ϕ' and find the new zero dynamics using (10).
6. If the zero dynamics are exponentially asymptotically stable, stop.
7. Go to Step 3.

To select the function φ, a specific description of the controlled system is necessary. Hence, no specific advice can be given here. Nevertheless, [44] illustrated this procedure through several practical examples.

As noted above, the resulting output is defined as

$$\mathbf{y}_i = \tilde{q}_i + \varphi(q_i^\circ) \tag{11}$$

for a suitable function $\varphi : \mathbb{R} \to \mathbb{R}$ so that the zero dynamics are asymptotically stable—this feature will be useful in the sequel. The function φ is identical for all agents.

As a result, one obtains the functions Φ, Ψ, ϕ', $\tilde{u}_i$, and a neighborhood of zero $\mathcal{U}$ such that relations analogous to (7)–(9) hold, and moreover, system (10) is exponentially stable.

5. Synchronization of the Linearized External Dynamics

The problem of synchronizing the linear parts of the agents is briefly presented in this section. Since one can use various methods to robustly synchronize linearized multi-agent systems, the method used in this paper, based on the results in [46], can be freely replaced with another method. The important requirement is to guarantee the validity of inequality (20) (see below). In the context of this paper, this section presents rather technical material and should be understood as an indication of a possible way to derive this quality.

In this section and the subsequent section, the following assumption is made.

Assumption 4. *The output of the agents has been redesigned so that the system has exponentially stable zero dynamics.*

Let

$$\zeta_i = (\tilde{q}_i, \dot{\tilde{q}}_i)^T, \quad \zeta = \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_N \end{pmatrix}, \quad \tilde{u} = \begin{pmatrix} \tilde{u}_1 \\ \vdots \\ \tilde{u}_N \end{pmatrix}$$

and

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

as

$$\dot{\zeta} = (I_N \otimes A)\zeta + (I_N \otimes B)\tilde{u}. \tag{12}$$

The control signal $\tilde{u}_i$ is assumed to attain the form (with the matrix $K \in \mathbb{R}^{1\times 2}$ to be determined)

$$\tilde{u}_i = K \sum_{j=1}^{N} J_{ji}(\zeta_j - \zeta_i). \tag{13}$$

Then, one can rewrite Equation (12) as

$$\dot{\zeta} = (I \otimes A)\zeta + (L \otimes BK)\zeta. \tag{14}$$

As demonstrated in [47] and others, system (14) is asymptotically stable if the matrices

$$A + \underline{d}BK, \quad A + \bar{d}BK \tag{15}$$

are Hurwitz (scalars $\underline{d}, \bar{d}$ are defined in Section 2).

Remark 2. *The method to determine the matrix K is described in [47,48] and others. The main idea is to introduce the so-called disagreement vector e as follows: let* $\overline{\zeta} = \frac{1}{N}\sum_{i=1}^{N}\zeta_i$, $e_i = \zeta_i - \overline{\zeta}$, *and finally,* $e = (e_1^T, \ldots, e_N^T)^T$. *Then, the synchronization of the system (12) is equivalent to the problem of stabilizing the disagreement dynamics* ($\lim_{t\to\infty}\|e(t)\| = 0$)*. Moreover, it can be shown that the convergence of the disagreement dynamics is exponential.*

The generalization of the multi-agent system (14) to systems with Lipschitz-type uncertainty, introduced in [46], will be useful in the subsequent sections; hence, it is mentioned here. The version presented in this section is tailored to the application of the synchronization of the observable part of underactuated systems. Specifically, we present a result concerning the synchronization of the system (14) with an additional uncertain term.

For the case of the multi-agent system with external inputs $\xi_i \in \mathbb{R}^2$, $i = 1, \ldots, N$, one can derive the lemma given below (see [46]).

Lemma 1. *Let us consider that Assumptions 1, 2, and 4 hold, and suppose there exists a function* $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ *that satisfies the following Lipschitz condition: there exist scalars* $\alpha_1 > 0, \alpha_2 > 0$ *such that for all* $z_1, z_2, y_1, y_2 \in \mathbb{R}^2$, *the following holds:*

$$\|f(z_1, y_1) - f(z_2, y_2)\| \leq \alpha_1\|z_1 - z_2\| + \alpha_2\|y_1 - y_2\|. \tag{16}$$

Let $\xi_i \in \mathbb{R}^2$, $i = 1, \ldots, N$, $\xi = (\xi_1^T, \ldots, \xi_N^T)^T$, *and*

$$\mathcal{F}(\zeta, \xi) = \begin{pmatrix} f(\zeta_1, \xi_1) \\ \vdots \\ f(\zeta_N, \xi_N) \end{pmatrix}.$$

Moreover, let $E \in \mathbb{R}^{2\times 1}$, *and the multi-agent system be defined as*

$$\dot{\zeta} = (I \otimes A)\zeta + (L \otimes BK)\zeta + (I \otimes E)\mathcal{F}(\zeta, \xi) \tag{17}$$

with $\xi \in \mathbb{R}^2$. *Let there exist a* 2×2*-dimensional matrix* $P = P^T$, $P > 0$, *and a scalar* $\tau > 0$ *such that the following LMI holds*

$$\begin{pmatrix} A^T P + PA - \tau BB^T + (\alpha_1^2 + \alpha_2^2)EE^T & P \\ P & -I_2 \end{pmatrix} > 0. \tag{18}$$

Let

$$K = -\frac{\tau}{2\underline{d}}B^T P. \tag{19}$$

Then, there exist positive constants c_1, c_1' *such that, with* $\bar{\xi} = \frac{1}{N}\sum_{i=1}^{N}\xi_i$ *and* $V = e^T(I \otimes P)e$, *the following holds:*

$$\dot{V}_1(e) < -c_1 V_1(e) + c_1'\|\xi - \bar{\xi}\|^2. \tag{20}$$

Having conducted the design of the control gain matrix K, one can see that (7) and (13) imply

$$u_i = \Psi^{-1}(q_i, \dot{q}_i)\Big(K\textstyle\sum_{j=1}^{N} J_{ji}((\dot{\tilde{q}}_j, \tilde{q}_j)^T - (\dot{\tilde{q}}_i, \tilde{q}_i)) - \Phi(q_i, \dot{q}_i)\Big), \tag{21}$$

which is how the function **k** in (5) is expressed.

It is important to realize that the control signals u_i may also depend on the state of the internal dynamics q_i° since the functions Φ and Ψ depend on the entire state q_i and its derivative.

6. Synchronization of the Internal Dynamics

First, for the sake of conciseness of the notations, define vectors $\chi_i = (q_i^\circ, \dot{q}_i^\circ)$, $i = 1, \ldots, N$, $\bar{\chi} = \frac{1}{N}\sum_{i=1}^N \chi_i$, and let $\bar{\tilde{u}} = \frac{1}{N}\sum_{i=1}^N \tilde{u}_i$, where the function $\tilde{u}_i$ is defined in (13).

As a consequence of Assumption 4, there exist the matrix U and a function $\Delta' : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying

$$U_1 = \frac{\partial \phi'}{\partial \chi_{1,1}}(0,0,0,0,0), \quad U_2 = \frac{\partial \phi'}{\partial \chi_{1,2}}(0,0,0,0,0),$$
$$U = \begin{pmatrix} 0 & 1 \\ U_1 & U_2 \end{pmatrix} \tag{22}$$

and

$$\Delta'(\tilde{q}_i, \dot{\tilde{q}}_i, q_i^\circ, \dot{q}_i^\circ, \tilde{u}_i) = \phi'(\tilde{q}_i, \dot{\tilde{q}}_i, q_i^\circ, \dot{q}_i^\circ, \tilde{u}_i) - (U_1, U_2)\begin{pmatrix} q_i^\circ \\ \dot{q}_i^\circ \end{pmatrix}. \tag{23}$$

Note that the function Δ' vanishes at the origin together with its derivatives.

The exponential stability of the zero dynamics implies that for any $\alpha > 0$, there exists a positive definite matrix $R \in \mathbb{R}^{2\times 2}$ such that

$$U^T R + RU = -\alpha I. \tag{24}$$

Let us define the functions $\phi, \Delta : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ as

$$\phi(\zeta_i, \chi_i, \tilde{u}_i) = \begin{pmatrix} 0 \\ \phi'(\zeta_i, \chi_i, \tilde{u}_i) \end{pmatrix}, \Delta(\zeta_i, \chi_i, \tilde{u}_i) = \begin{pmatrix} 0 \\ \Delta'(\zeta_i, \chi_i, \tilde{u}_i) \end{pmatrix}.$$

In the sequel, we will investigate the differences $\chi_i - \bar{\chi}$ for $i \in 1, \ldots, N$:

$$\dot{\chi}_i - \dot{\bar{\chi}} = \phi(\zeta_i, \chi_i, \tilde{u}_i) - \phi(\bar{\zeta}, \bar{\chi}, \bar{\tilde{u}}). \tag{25}$$

With the help of the function Δ, one can reformulate expression (25) as

$$\dot{\chi}_i - \dot{\bar{\chi}} = U(\chi_i - \bar{\chi}) + \Delta(\zeta_i, \chi_i, \tilde{u}_i) - \Delta(\bar{\zeta}, \bar{\chi}, \bar{\tilde{u}}), \tag{26}$$

which, with (24), yields

$$\frac{d}{dt}(\chi_i - \bar{\chi})^T R(\chi_i - \bar{\chi}) < -\alpha(\chi_i - \bar{\chi})^T(\chi_i - \bar{\chi}) + 2(\chi_i - \bar{\chi})^T R(\Delta(\zeta_i, \chi_i, \tilde{u}_i) - \Delta(\bar{\zeta}, \bar{\chi}, \bar{\tilde{u}})). \tag{27}$$

Then, the Young inequality implies the validity of the following estimate for every $\beta > 0$:

$$|2(\chi_i - \bar{\chi})^T R(\Delta(\zeta_i, \chi_i, \tilde{u}_i) - \Delta(\bar{\zeta}, \bar{\chi}, \bar{\tilde{u}}))|$$
$$\leq \beta \|(\chi_i - \bar{\chi})\|^2 + \frac{1}{\beta}\|\Delta(\zeta_i, \chi_i, \tilde{u}_i) - \Delta(\bar{\zeta}, \bar{\chi}, \bar{\tilde{u}})\|^2. \tag{28}$$

Moreover, observe that

$$\Delta(\zeta_i, \chi_i, \tilde{u}_i) - \Delta(\bar{\zeta}, \bar{\chi}, \bar{\tilde{u}}) = \Delta(\zeta_i, \chi_i, \tilde{u}_i) - \Delta(\bar{\zeta}, \chi_i, \tilde{u}_i) + \Delta(\bar{\zeta}, \chi_i, \tilde{u}_i) - \Delta(\bar{\zeta}, \bar{\chi}, \tilde{u}_i)$$
$$+ \Delta(\bar{\zeta}, \bar{\chi}, \tilde{u}_i) - \Delta(\bar{\zeta}, \bar{\chi}, \bar{\tilde{u}}).$$

Let us suppose that $\mu > 1$. Then, there exists a neighborhood of the origin $\mathcal{U}_\mu \subset \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}$ such that if $\zeta_a, \zeta_b, \chi_a, \chi_b, v_a, v_b \in \mathcal{U}_\mu$, then

$$\|\Delta(\zeta_a, \chi_a, v_a) - \Delta(\zeta_b, \chi_a, v_a)\|^2 \leq \mu\|\zeta_a - \zeta_b\|^2,$$
$$\|\Delta(\zeta_b, \chi_a, v_a) - \Delta(\zeta_b, \chi_b, v_a)\|^2 \leq \mu\|\chi_a - \chi_b\|^2,$$
$$\|\Delta(\zeta_b, \chi_b, v_a) - \Delta(\zeta_b, \chi_b, v_b)\|^2 \leq \mu\|v_a - v_b\|^2.$$

Hence,

$$
\begin{aligned}
&|2(\chi_i - \bar{\chi})^T R(\Delta(\zeta_i, \chi_i, \tilde{u}_i) - \Delta(\bar{\zeta}, \bar{\chi}, \bar{\tilde{u}}))| \\
\leq &(\beta + \frac{\mu}{\beta})\|\chi_i - \bar{\chi}\|^2 + \frac{\mu}{\beta}(\|\zeta_i - \bar{\zeta}\|^2 + \|\tilde{u}_i - \bar{\tilde{u}}\|^2).
\end{aligned} \tag{29}
$$

As the function Δ contains only higher-order terms, a choice of constants β, μ such that

$$
\alpha - \beta - \frac{\mu}{\beta} > 0 \tag{30}
$$

is always possible (at the cost of taking a smaller set $\mathcal{U}_\mu$). Choosing these parameters such that (30) is satisfied allows us to infer the following expression from (27)

$$
\frac{d}{dt}(\chi_i - \bar{\chi})^T R(\chi_i - \bar{\chi}) \leq -(\alpha - \beta - \frac{\mu}{\beta})\|\chi_i - \bar{\chi}\|^2 + \frac{\mu}{\beta}\|\zeta_i - \bar{\zeta}\|^2 + \frac{\mu}{\beta}\|\tilde{u}_i - \bar{\tilde{u}}\|^2. \tag{31}
$$

Define the Lyapunov function V_2 as

$$
V_2 = \sum_{i,j=1}^{N} (\chi_i - \bar{\chi})^T R(\chi_i - \bar{\chi}).
$$

Using (31), one can see that there exists a constant $c_2 > 0$ satisfying

$$
\dot{V}_2 \leq -(\alpha - \beta - \frac{\mu}{\beta})c_2 V_2 + \frac{\mu}{\beta}\sum_{i,j=1}^{N}(\|\zeta_i - \bar{\zeta}\|^2 + \|\tilde{u}_i - \bar{\tilde{u}}\|^2). \tag{32}
$$

Due to the results from the previous section, the application of the control signal $\tilde{u}_i = K\sum_{j=1}^{N} J_{ji}(\zeta_i - \bar{\zeta})$ implies that $\zeta_i - \bar{\zeta} \to 0$. This, in turn, yields synchronization of the functions χ_i by the same control $\tilde{u}_i$, for $i = 1, \ldots, N$. Hence,

$$
\lim_{t\to\infty} \sum_{i=1}^{N} \|\chi_i - \bar{\chi}\| = 0. \tag{33}
$$

Theorem 1. *Consider the multi-agent system* (2) *with an interconnection topology satisfying Assumptions 1 and 2. Let Assumptions 3 and 4 hold. Suppose the linearized part is synchronized, condition* (30) *is valid, and the trajectories of all agents lie in the set $\mathcal{U} \cap \mathcal{U}_\mu$. Then, the consensus of the system* (2) *is guaranteed by the control law* (5)*.*

Proof. Assume the external dynamics are synchronized. This also implies synchronization of the control signals $\tilde{u}_i$. Thanks to the Vanishing Perturbation Theorem (see [45]), relation (32) guarantees $\chi_i \to \bar{\chi}$. □

Since exact feedback linearization is a diffeomorphism between two descriptions of one system, we immediately arrive at the corollary given below.

Corollary 1. *Under the assumptions of Theorem 1, the control law* (21) *guarantees synchronization of the multi-agent system* (2)*.*

7. Recovery of Dynamical Properties

The control law (21) guarantees achieving the consensus of the multi-agent system, at the cost of changing the dynamics of the agents caused by the application of exact feedback linearization. Specifically, the average dynamics (the dynamics that govern the evolution of $\bar{q}$) are equivalent to the linearized dynamics (8), (9). A change in the average dynamics is a rather undesirable feature—in many cases, it is required that the average external dynamics obey the equations describing one agent (with zero external input). Hence, the control law (21) must be modified to restore this property. In this section, such a control law is proposed with the help of robust control theory.

If consensus is achieved, then $(q_1^T, \dot{q}_1^T)^T = \cdots = (q_N^T, \dot{q}_N^T)^T = \bar{q}$, and for the transformed control signals, we have $\tilde{u}_1 = \cdots = \tilde{u}_N = 0$. Then, (7) implies that, after reaching consensus, the control signal fed to every agent is equal to (with a slight abuse of notation)

$$u_{consensus} = -\Psi^{-1}(\bar{q})\Phi(\bar{q}). \tag{34}$$

This control input acts upon all agents, even after reaching the consensus state, and modifies their dynamical properties since it does not (in general) equal to zero.

On the other hand, subtracting (34) from the control signal (21) is not possible since the quantity $\bar{q}$ is not known to the agents (unless in the special case when the complete graph describes the interconnection topology). Still, for future purposes, consider the (practically non-realizable) case when the control of all agents is defined for some matrix K as

$$\begin{aligned} u_i =& \Psi^{-1}(q_i, \dot{q}_i)\Big(K\textstyle\sum_{j=1}^N J_{ji}((\dot{\tilde{q}}_j, \tilde{q}_j)^T - (\dot{\tilde{q}}_i, \tilde{q}_i)) - \Phi(q_i, \dot{q}_i)\Big) - u_{consensus} \\ =& \Psi^{-1}(q_i, \dot{q}_i)\Big(K\textstyle\sum_{j=1}^N J_{ji}((\dot{\tilde{q}}_j, \tilde{q}_j)^T - (\dot{\tilde{q}}_i, \tilde{q}_i)) - \Phi(q_i, \dot{q}_i)\Big) - \Psi^{-1}(\bar{q})\Phi(\bar{q}). \end{aligned} \tag{35}$$

Remark 3. *Since $u_{consensus}$ is independent of i, adding this term to all agents' inputs does not modify the synchronization of the multi-agent system.*

In other words, when consensus is established, the control input (35) guarantees that the average dynamics are governed by the same law as the dynamics of a single agent.

To design a practically realizable control law, the term $\Psi^{-1}(\bar{q})\Phi(\bar{q})$ will be replaced with its approximation obtained only from values directly available to agent i; this procedure is described in the subsequent text. As will be seen, some uncertainty appears in this process; thus, the matrix K must be designed with this feature in mind.

Denote by N_i the number of nodes that send information directly to the ith agent (meaning $N_i = \sum_{j=1}^N J_{ji}$). Moreover, define

$$\begin{aligned} \tilde{q}_i' =& \tfrac{1}{N_i+1}(\tilde{q}_i + \textstyle\sum_{j=1, j\neq i}^N J_{ji}\tilde{q}_j), q_i^{\circ\prime} = \tfrac{1}{N_i+1}(q_i^\circ + \sum_{j=1, j\neq i}^N J_{ji}q_j^\circ), \\ \dot{\tilde{q}}_i' =& \tfrac{1}{N_i+1}(\dot{\tilde{q}}_i + \textstyle\sum_{j=1, j\neq i}^N J_{ji}\dot{\tilde{q}}_j), \dot{q}_i^{\circ\prime} = \tfrac{1}{N_i+1}(\dot{q}_i^\circ + \sum_{j=1, j\neq i}^N J_{ji}\dot{q}_j^\circ). \end{aligned}$$

The quantity $\tilde{q}_i'$ is realizable as it is an average of the values of $\tilde{q}_j$ from those agents that send information directly to agent i. Thus, $\tilde{q}_i'$ can be regarded as the approximation of the first component of $\bar{q}$ based on the information available to agent i. First, define the function $\tilde{f} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ as

$$\tilde{f}(\zeta_i, \chi_i) = \Psi^{-1}(q_i, \dot{q}_i)\Phi(q_i, \dot{q}_i) \tag{36}$$

where ζ and χ were defined in the previous sections. We also introduce the vectors $\zeta_i' = (\tilde{q}_i', \dot{\tilde{q}}_i')$, $\chi_i' = (q_i^{\circ\prime}, \dot{q}_i^{\circ\prime})$.

Assumption 5. *There exist scalars $\alpha_1 > 0$ and $\alpha_2 > 0$ such that, for any $z_1, z_1, z_3, z_4 \in \mathbb{R}^2$, the function $\tilde{f}$ satisfies inequality (16).*

The function $\tilde{f}(\zeta_i', \chi_i')$ can be regarded as the realizable approximation of the term $\Psi^{-1}(\bar{q}, \dot{\bar{q}})\Phi(\bar{q}, \dot{\bar{q}})$. With the function $\tilde{f}$, one can modify the control signal defined in (35) so that the following physically realizable control signal is fed into the ith agent:

$$u_i = \Psi^{-1}(q_i, \dot{q}_i)\Big(K\textstyle\sum_{j=1}^N J_{ji}((\dot{\tilde{q}}_j, \tilde{q}_j)^T - (\dot{\tilde{q}}_i, \tilde{q}_i)) - \Phi(q_i, \dot{q}_i)\Big) - \tilde{f}(\zeta_i', \chi_i'). \tag{37}$$

Note that if $q_1 = \cdots = q_N$ and $\dot{q}_1 = \cdots = \dot{q}_N$, then $u_i = 0$.

Adding the term $\tilde{f}(\zeta_i', \chi_i')$ is equivalent to adding the same term to the transformed control $\tilde{u}_i$. Hence, this control is replaced with

$$\hat{u}_i = \tilde{u}_i - \tilde{f}(\zeta_i', \chi_i'). \tag{38}$$

Let $\zeta = (\zeta_1^T, \ldots, \zeta_N^T)^T$, $\chi = (\xi_1^T, \ldots, \chi_N^T)^T$, and

$$\mathcal{F} = \begin{pmatrix} \tilde{f}(\zeta_1', \chi_1') - \tilde{f}(\bar{\zeta}, \bar{\chi}) \\ \vdots \\ \tilde{f}(\zeta_N', \chi_N') - \tilde{f}(\bar{\zeta}, \bar{\chi}) \end{pmatrix}. \tag{39}$$

Then, the linearized part of the multi-agent system can be compactly expressed in the form

$$\dot{\zeta} = (I \otimes A)\zeta + (L \otimes BK)\zeta - \mathcal{F}(\zeta, \chi) - \mathbf{1} \otimes u_{consensus}. \tag{40}$$

Hence, replacing the non-realizable term $u_{consensus}$ with the term $f(\zeta_i', \chi_i')$ in (35) corresponds to the appearance of the uncertain term $\mathcal{F}(\zeta, \chi)$ and thus introduces an additional requirement for the robustness of the synchronizing control.

Let $e_i^\circ = \chi_i - \bar{\chi}$, $e^\circ = ({e_1^\circ}^T, \ldots, {e_N^\circ}^T)^T$, and define the function V as

$$V(e, e^\circ) = V_1(e) + V_2(e^\circ). \tag{41}$$

Since the last term in (40) is equal for all agents, it does not affect the disagreement vectors. Assume the control gain K was designed using the method described in Corollary 1. Then, (20) and (32) yield

$$\dot{V} \leq -c_1 V_1(e) + c_1' \|e^\circ\|^2 - (\alpha - \beta - \tfrac{\mu}{\beta}) c_2 V_2(e^\circ) + \tfrac{\mu}{\beta}(1 + \|K\|^2)\|e\|^2. \tag{42}$$

Theorem 2. *Consider the multi-agent system (2) with an interconnection topology satisfying Assumptions 1 and 2. Assume the agents are systems with two degrees of freedom and an underactuation degree of one such that Assumptions 3 and 4 hold. Moreover, suppose their outputs are chosen so that the agents are minimum-phase systems. Let the assumptions of Lemma 1 hold so that the positive constants c_1, c_1', c_2, μ, and β satisfy*

$$\begin{array}{rcl} 0 & > & (\alpha - \beta - \frac{\mu}{\beta})c_2 - \frac{c_1'}{\lambda_{\min}(R)}, \\ 0 & > & c_1 - \frac{\mu}{\beta \lambda_{\min}(P)}(1 + \|K\|^2). \end{array} \tag{43}$$

Let the control signal be equal to (37). Assume that the trajectories of all agents lie in the set $\mathcal{U} \cap \mathcal{U}_\mu$. Then, the consensus of the system (2) is guaranteed, and the vector $(q, \dot{q})$ obeys the dynamics described by (2) with zero input.

Proof. Conditions (43) imply that $\dot{V} < 0$ on $\mathcal{U} \cap \mathcal{U}_\mu - \{0\}$; hence, synchronization is achieved. Moreover, as noted above, the average dynamics are equivalent to the dynamics of a single uncontrolled agent. □

Remark 4. *The method for output redesign might be difficult to apply, as it requires an in-depth knowledge of the controlled system. In contrast, the application of exact feedback linearization is quite straightforward. On the other hand, the zero dynamics must be asymptotically stable; if matrix R is such that the constant α is too small, then one may experience difficulties in satisfying inequality (30). However, such limitations are often encountered in control theory; thus, their presence is not perceived as a serious drawback.*

8. Examples

8.1. Coupled Inverted Pendulums on a Cart

For the agents, the inverted pendulums on the cart are chosen. The system is thoroughly described in [18], and its synchronization is briefly described in [49]. As the agents are equal, we omit the subscript i in the following.

The single inverted pendulum on a cart is described by the equations

$$\ddot{x} = \frac{1}{M + m + m\cos^2\theta}(u - ml\sin\theta\dot{\theta}^2 + mg\sin\theta\cos\theta),$$
$$\ddot{\theta} = \frac{1}{M + m + m\cos^2\theta}\Big(\frac{u}{l}\cos\theta - m\sin\theta\cos\theta\dot{\theta}^2 + (M+m)\frac{g}{l}\sin\theta\Big),$$

where the meanings of the variables are as follows: x is the cart's position and θ is the pendulum's angle. The other constants are as follows: l is the length of the pendulum, g is the gravitational acceleration, m is the mass of the cart, and M is the mass of the pendulum ($M \ll m$). The variable u is the external force acting upon the cart, which is the control input. The application of exact feedback linearization (with output $y = x$ first) yields

$$\ddot{x} = \tilde{u}_1, \tag{44}$$
$$\ddot{\theta} = \frac{g}{l}\sin\theta - \frac{\cos\theta}{l}\tilde{u}_1 \tag{45}$$

for

$$\tilde{u}_1 = \frac{1}{\frac{M}{m} + 1 + \cos^2\theta}\Big(\frac{u}{m} - l\sin\theta\dot{\theta}^2 + g\sin\theta\cos\theta\Big), \tag{46}$$

where the fraction $\frac{M}{m}$ is negligibly small.

As the zero dynamics are not stable for this choice of output, [18] suggested redefining the output as follows:

$$y = x + k_1\sin\theta + k_2\int_0^t \sin\theta(\tau)d\tau \tag{47}$$

with suitably chosen constants k_1 and k_2. In this example, $k_1 = 2$ and $k_2 = 20$ were chosen.

With the help of (47), one can design a stabilizing control law $\tilde{u}$, which is a function of y and $\dot{y}$. Let the signal $\tilde{u}$ be given by (for some real constants k_3, k_4)

$$\tilde{u} = k_3(\dot{x} + k_1\dot{\theta}\cos\theta + k_2\sin\theta) + k_4(x + k_1\sin\theta + k_2\int_0^t \sin\theta(s)ds).$$

As shown in [18], the control signal $\tilde{u}_1$ fed into the transformed system and the designed control input $\tilde{u}$ are related by

$$\tilde{u}_1 = \frac{1}{\frac{k_1}{l}\cos^2\theta - 1}\Big(\tilde{u} + k_1\sin\theta(\frac{g}{l}\cos\theta - \dot{\theta}^2) + k_2\dot{\theta}\cos\theta\Big). \tag{48}$$

This equation, together with (46), yields the relation between u and $\tilde{u}$.

For simulations, we set $l = 0.0475$ m. The values of m and M are not required. The algorithm described in Section 5 yields $k_3 = 7.27$ and $k_4 = 13.07$. If used for only one system, this control stabilizes not only the pendulum's vertical position but also the cart's position.

Remark 5. *Note that [18] derived a slightly different control action, as the authors aimed to enlarge the region of attraction. However, for the sake of simplicity, the linear controller in the form* $v = k_3 y + k_4\dot{y}$ *was used here.*

The multi-agent system composed of six identical agents described above, along with the control presented here, was interconnected into the loop (see Figure 1). Moreover, a sinusoidal disturbance was added to agent 1.

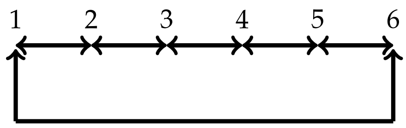

Figure 1. Interconnection of the agents.

The dashed red line illustrates the position of the first cart in Figure 2. The solid black line shows the position of the cart of the third agent, while the dotted blue line represents the fifth agent. The meanings of the lines are the same in Figure 3, where the angle θ is depicted. Similarly, the control signals of these agents are shown in Figure 4. The norm of the synchronization error for the entire network is depicted in Figure 5, which shows the norm of the synchronization error for the entire multi-agent system. One can see that the collocated feedback stabilizes the zero dynamics (in the case of the inverted pendulum, the angle θ), while the variable x of all agents is synchronized.

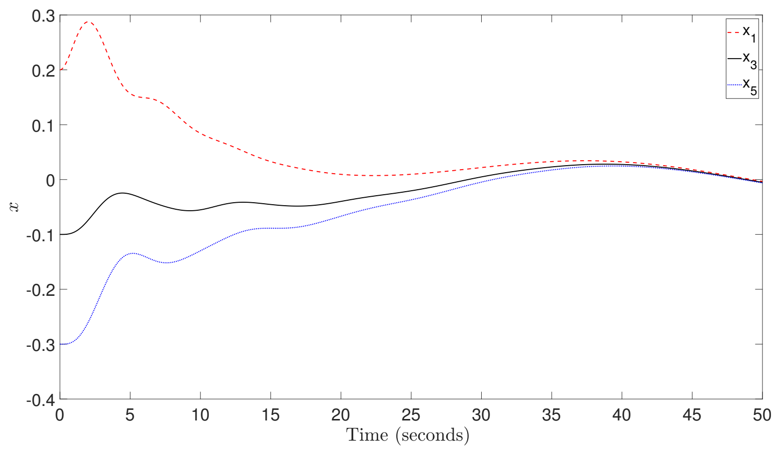

Figure 2. The position of the cart of agents 1, 3, and 5.

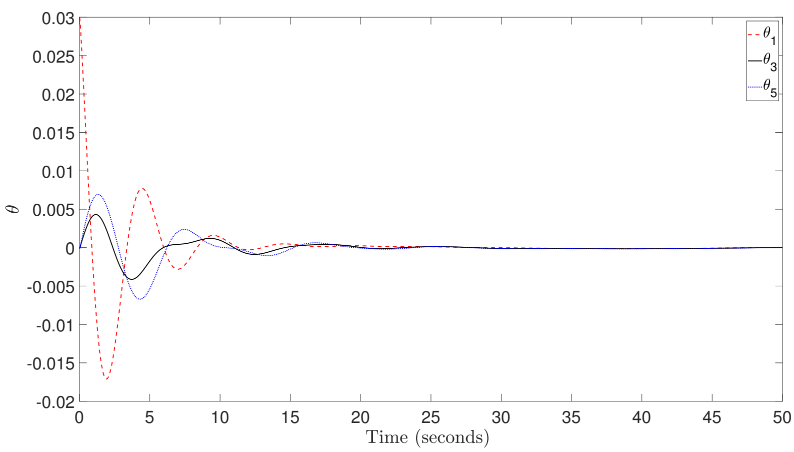

Figure 3. The angle of the pendulum of agents 1, 3, and 5.

The choice of the output ensures that not only the agent's state x—the positions—are synchronized, but it also naturally stabilizes the angle at $\theta = 0$—that is, the upright position of the pendulum—if no interactions between agents are active. This, in turn, guarantees synchronization of the angle of the pendulums.

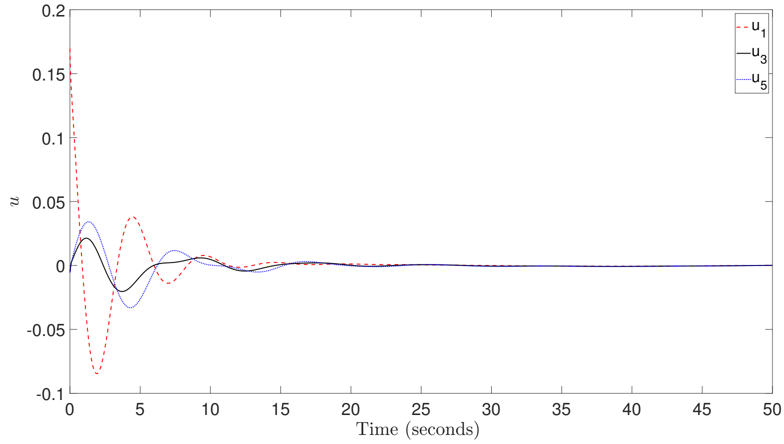

Figure 4. The control of agents 1, 3, and 5.

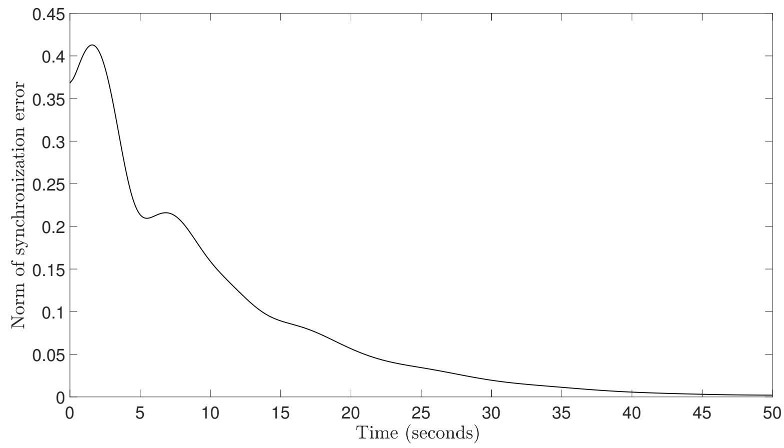

Figure 5. Norm of the synchronization error.

8.2. Coupled van der Pol and Linear Oscillators

In this example, the multi-agent system composed of nine interconnected van der Pol and linear oscillators is considered. The interconnection topology is again circular.

One agent is described by the equations

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = u \tag{49}$$

$$\ddot{z} + z = u. \tag{50}$$

with $\mu = 2$.

Proceeding as in [18], taking $y = x$ as the output does not lead to asymptotically stable zero dynamics. Thus, we modify the output as

$$y = x + kz \tag{51}$$

where k is a constant that will be determined later. Using $\zeta_1 = y$, $\zeta_2 = \dot{y}$, $\chi_1 = z$, $\chi_2 = \dot{z}$, and $v = -x + \mu(1 - x^2)\dot{x} - kz + (1 + k)u$, the agent's description can then be transformed into

$$\begin{aligned}
\dot{\zeta}_1 &= \zeta_2, \\
\dot{\zeta}_2 &= v, \\
\dot{\chi}_1 &= \chi_2, \\
\dot{\chi}_2 &= -\chi_1 + \tfrac{1}{1+k}\Big(v + \zeta_1 + \mu(1 - (\zeta_1 - (k\chi_1)^2)(\zeta_2 - k\chi_1)\Big).
\end{aligned}$$

Thus, the zero dynamics read

$$\begin{aligned}\dot{\chi}_1 &= \chi_2, \\ \dot{\chi}_2 &= -\chi_1 + \frac{k\mu}{1+k}\left(1-(k\chi_1)^2\right)\chi_2.\end{aligned}$$

Proposition 1. *Assume that $k \in (-1,0)$. The origin is an asymptotically stable equilibrium, and the set $D = \{(\chi_1, \chi_2) \in \mathbb{R}^2 \mid \chi_1^2 + \chi_2^2 \leq \frac{1}{k^2}\}$ is a subset of its domain of attraction.*

Proof. Taking the Lyapunov function $V = \frac{1}{2}(\chi_1^2 + \chi_2^2)$, one can infer that $\dot{V} \leq 0$. Since the set D is positively invariant and the set $\{(0,0)\}$ is the only invariant subset of $D \cap \{(\chi_1, \chi_2) \mid \dot{V}(\chi_1, \chi_2) = 0\}$, the origin is asymptotically stable due to LaSalle's invariance principle. □

Consider a multi-agent system composed of nine systems (49), (50), with $\mu = 2$ and $k = -\frac{1}{8}$. Then, $K = (-1.35,\ -3.85)$, and the eigenvalues of the closed loop are $-0.926 \pm 1.22j$.

Figure 6 shows the state x of the first agent (dashed red line), fourth agent (solid black line), and seventh agent (dotted blue line). Similarly, the control signals of these agents are shown in Figure 7. The norms of the synchronization error are depicted in Figures 8 and 9. To clearly illustrate the potential of the method, the magnitude of the initial conditions was increased, and the time interval of the simulation was extended. Moreover, these figures present a comparison with the same system controlled by a linear controller with the same gain matrix K. Specifically, the time evolution of the norm of the synchronization error $\sqrt{\sum_{i=1}^{9}(x_i - \bar{x})^2}$ is depicted in Figure 8. Analogously, the norm $\sqrt{\sum_{i=1}^{9}(z_i - \bar{z})^2}$ is shown in Figure 9. In both figures, the solid black line represents the norm of the synchronization error for a proposed nonlinear controller, and the red dashed line stands for the norm of the synchronization error when the multi-agent system was synchronized by the linear controller. One can see that the synchronization error converged to zero faster when the synchronization algorithm proposed here was applied. This is due to the fact that the nonlinearity was matched more precisely. Specifically, when the effect of the nonlinear terms was large, the control was less conservative compared to the control obtained by estimating the nonlinearities using the Lipschitz inequality combined with robust control methods.

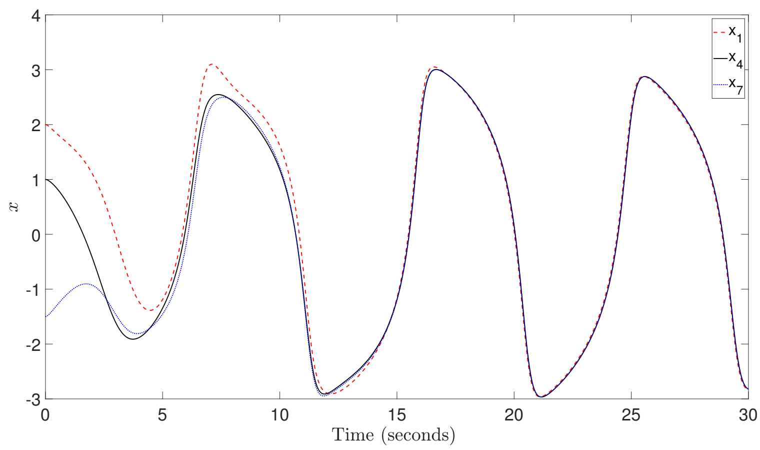

Figure 6. States x_1, x_4, and x_7.

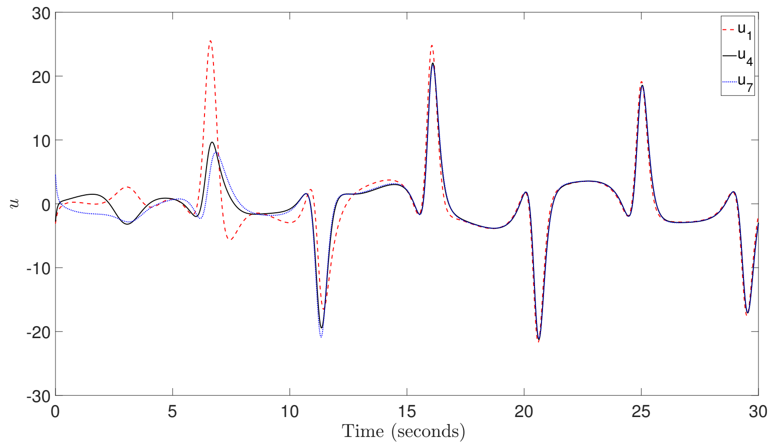

Figure 7. The control of agents 1, 4, and 7.

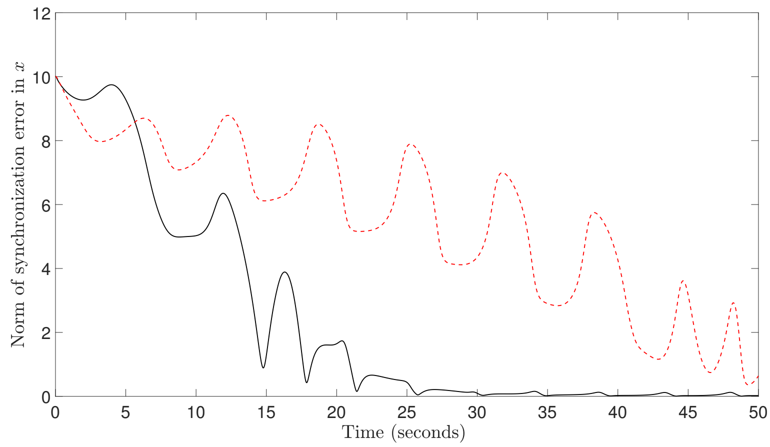

Figure 8. Norm of the synchronization error in x. The solid black line represents the norm of the synchronization error for the proposed nonlinear controller, while the red dashed line represents the norm of the synchronization error for the linear controller.

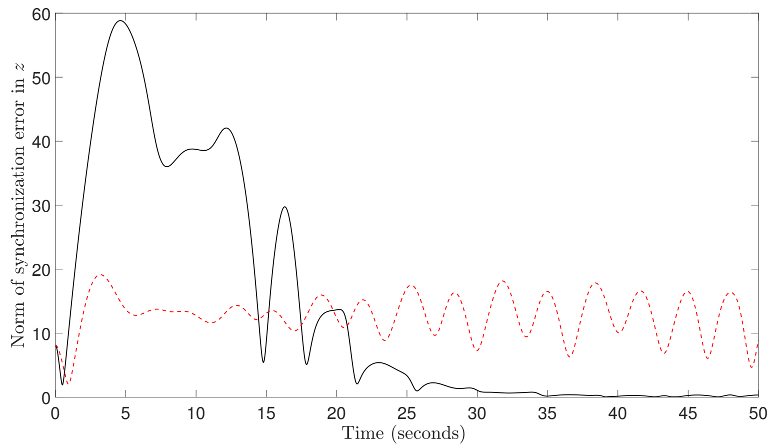

Figure 9. Norm of the synchronization error in z. The solid black line represents the norm of the synchronization error for the proposed nonlinear controller, while the red dashed line represents the norm of the synchronization error for the linear controller.

9. Conclusions and Future Work

The synchronization of nonlinear agents was investigated. The agents were not fully linearizable, but it was shown that synchronization of all states can be achieved.

This was accomplished by finding an auxiliary output in which the agents' systems are minimum-phase. Then, one group of states can be synchronized while the minimum-phase property ensures the synchronization of the remaining states. Two examples illustrated the potential of the proposed method: the synchronization error converges faster for a multi-agent system synchronized by the proposed controller compared to the case when the synchronizing controller is linear.

In the future, research will include systems with uncertainties, leading to a robust control law. Moreover, as the control of multi-agent systems is often subject to time delays, the effects of these time delays will also be studied, which will require the application of suitable Lyapunov–Krasovskii functionals. Last but not least, more general topologies will be investigated.

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