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Linearization of Finite Elasticity with Surface Tension

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Abstract
We propose models in nonlinear elasticity for nonsimple materials that include surface energy terms. Additionally, we also discuss living surface loads on the boundary. We establish corresponding linearized models and show their relationship to the original ones by means of Γ-convergence.

Keywords Hyperelasticity · Linear elasticity · Interface measure · Variational methods · Gamma convergence

Mathematics Subject Classification 74G25 · 49J45

1 Introduction

Surfaces of elastic bodies exhibit properties that are different from those associated with the bulk. This behavior is caused either by the fact that the boundary of the material is exposed to fatigue, chemical processes, coating, etc., thus obviously resulting in very different properties in comparatively thin boundary layers, or due to the fact that the atomic bonds are broken at the surface of the body. These effects can be phenomenologically modeled in terms of boundaries equipped with their own stored energy density and they have been well studied in the literature. Rational continuum mechanics approach to elastic surfaces has started with the work of Gurtin and

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Murdoch (1975); Gurtin (1986) and of Steigmann and Ogden (1997). General energies for modeling body–environment interactions and surface loading are discussed in Podio-Guidugli (1988); Podio-Guidugli and Caffarelli (1990). More recently, Šilhavý introduced a concept of interfacial quasiconvexity and polyconvexity (Šilhavý 2010a, b, 2011) extending (Fonseca 1989) to establish the existence of minimizers for problems with multi-well bulk energy density for simple materials, i.e., depending only on the first deformation gradient. Javili and Steinmann (2009, 2010) designed finite-element methods to model surface elasticity. Thermomechanical approach to interfacial and surface energetics of materials is reviewed in Javili et al. (2013); Steinmann and Häsner (2005). Surface–substrate interactions for shells are described in Šilhavý (2013) and a phase-field modeling approach to the problem can be found, e.g., in Levitas and Warren (2016).

Rigorous derivation of linear models from nonlinear elasticity by Γ-convergence (Dal Maso 1993) has started by a pioneering work (Dal Maso et al. 2002) and then various finer results appeared (Agostiniani et al. 2012; Maddalena et al. 2019a, b; Mainini and Percivale 2020, 2021, 2022; Mainini et al. 2022; Maor and Mora 2021) that derive linear elasticity under various constraints as, e.g., incompressibility, or no Dirichlet boundary conditions, i.e., pure traction problems. We also refer to Mora and Riva (2023) where the authors performed the linearization procedure for a pressure load, i.e., living load (a follower force) that depends on the deformation.

The study (Casado Dias et al. 2025) explores the macroscopic elastic properties of elastomers with spherical cavities filled with pressurized liquid, considering the effects of surface tension. It starts by linearizing a fully nonlinear model and progresses with analyzing how the presence of multiple liquid-filled cavities enhances the elastic behavior in the linearized model.

Our contribution focuses on the linearization of nonlinear elasticity models with surface energy terms. In order to fix ideas, we perform the analysis on the model functional $\mathcal{G}$ below that penalizes local changes of the surface area in the deformed configuration in analogy with the incompressibility constraint in the bulk. The main results are given in Theorem 2.1 and Theorem 2.2, stating the convergence of minimizers of nonlinear models to the unique minimizer of the associated linearized models, in case of Dirichlet boundary conditions and in case of only Neumann boundary conditions, respectively. In the remainder of this section, we introduce the necessary notation and the functionals. The main results are then stated in Sect. 2 and Sects. 3 and 4 are devoted to proofs.

Modeling of Elastic Surface Energy

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded open connected Lipschitz set, representing the reference configuration of a hyperelastic body. Let us introduce the energy functional depending on the deformation field $y : \Omega \to \mathbb{R}^d$

$$\mathcal{G}(y) := \int_\Omega \Big(W(\nabla y(x)) + H(\nabla^2 y(x)) \Big)\, dx - \overline{\mathcal{L}}(y(x) - x) + \gamma \, \| \, |\mathrm{cof}\, \nabla y\, \mathbf{n}| - 1 \, \|^q_{L^q(\partial\Omega)} \, .$$

The global energy includes the stored elastic energy of a nonsimple material, as it depends on second gradient (Toupin 1962, 1964). Here, $\overline{\mathcal{L}}$ is a dead loading functional, accounting for the work of given external force fields. It is a linear functional acting on the displacement field $y(x) - x$. Moreover, local changes in the surface measure of the boundary of Ω are penalized, as $|\mathrm{cof}\, \nabla y\ \mathbf{n}|$ represents the density of the surface area element of the deformed configuration: the energy includes the $L^q(\partial\Omega)$ distance, $q \geq 1$, to its reference value corresponding to $y(x) = x$, being $\gamma > 0$ a surface tension coefficient. Note that this term is a penalization of the "inextensibility" constraint $|\mathrm{cof}\, \nabla y\, \mathbf{n}| = 1$ a.e. on $\partial\Omega$ which is an analogue of the incompressibility constraint $\det \nabla y = 1$ a.e. in Ω.

In the linearization process, we view y as a perturbation of the identity map so that we write it as $y(x) = x + \varepsilon v(x)$ for a suitable rescaled displacement v and we set $\mathcal{L} = \overline{\mathcal{L}}/\varepsilon$, which reflects the scaling of external forces. Assuming that H and W are frame indifferent, that H is convex positively p-homogeneous for some suitable $p > 1$, and that W is minimized at the identity with $W(\mathbf{I}) = DW(\mathbf{I}) = 0$, i.e., that identity is the natural state of the body Ω, we consider the rescaled nonlinear global energy

$$\begin{aligned}\mathcal{G}_\varepsilon(v) := \frac{1}{\varepsilon^2}\int_\Omega W(\mathbf{I} + \varepsilon\nabla v)\, dx + \frac{1}{\varepsilon^p}\int_\Omega H(\varepsilon\nabla^2 v) - \mathcal{L}(v) \\ + \frac{\gamma}{\varepsilon^q}\, \|\, |\mathrm{cof}\,(\mathbf{I} + \varepsilon\nabla v)\, \mathbf{n}| - 1\, \|^q_{L^q(\partial\Omega)}.\end{aligned} \tag{1.1}$$

We notice that as $\varepsilon \to 0$

$$W(\mathbf{I} + \varepsilon\nabla v) = \frac{\varepsilon^2}{2}\, \mathbb{E}(v)\, \nabla^2 W(\mathbf{I})\, \mathbb{E}(v) + o(\varepsilon^2),$$

so that the second-order term in the Taylor expansion of W produces the standard quadratic potential of linear elasticity (being $\nabla^2 W(\mathbf{I})$ the fourth-order elastic tensor), acting by frame indifference only on the symmetric part of the gradient

$$\mathbb{E}(v) := \frac{\nabla v + (\nabla v)^T}{2}.$$

On the other hand, by the formula $\mathrm{cof}\ \mathbf{F} = (\det\ \mathbf{F})\ \mathbf{F}^{-T}$ we have

$$\begin{aligned}\mathrm{cof}\,(\mathbf{I} + \varepsilon\nabla v) &= \det(\mathbf{I} + \varepsilon\nabla v)\,(\mathbf{I} + \varepsilon(\nabla v)^T)^{-1} \\ &= (1 + \varepsilon\, \mathrm{div}\, v + o(\varepsilon))\,(\mathbf{I} - \varepsilon(\nabla v)^T + o(\varepsilon)) = \mathbf{I} + \varepsilon\mathbb{A}(v) + o(\varepsilon),\end{aligned} \tag{1.2}$$

where we have introduced the divergence free tensor

$$\mathbb{A}(v) = \mathbf{I}\, \mathrm{div}\, v - (\nabla v)^T,$$

corresponding to the linearization of the cofactor matrix. As a consequence,

$$|\mathrm{cof}\,(\mathbf{I} + \varepsilon\nabla v)\mathbf{n}| = \sqrt{1 + 2\varepsilon\, \mathbb{A}(v)\, \mathbf{n}\cdot\mathbf{n} + o(\varepsilon)} = 1 + \varepsilon\, \mathbb{A}(v)\mathbf{n}\cdot\mathbf{n} + o(\varepsilon), \tag{1.3}$$

where we notice that $\mathbb{A}(v)\mathbf{n}\cdot\mathbf{n}$ is the tangential divergence of v on $\partial\Omega$.Therefore in the limit as $\varepsilon\to 0$, we obtain the geometrically linearized functional

$$\mathcal{G}_*(v):=\frac{1}{2}\int_\Omega \mathbb{E}(v)\,D^2W(\mathbf{I})\,\mathbb{E}(v)\,dx+\int_\Omega H(\nabla^2 v)\,dx-\mathcal{L}(v)+\gamma\int_{\partial\Omega}|\mathbb{A}(v)\mathbf{n}\cdot\mathbf{n}|^q\,dS \tag{1.4}$$

as the pointwise limit of functionals $\mathcal{G}_h$ for every smooth enough v. We notice that the stored elastic energy and the surface tension term of functional $\mathcal{G}$ are frame indifferent, and indeed their counterparts in functional $\mathcal{G}_*$ depend on ∇v only through its symmetric part $\mathbb{E}(v)$, so that as expected they are invariant by addition of an infinitesimal rigid displacement. We have indeed $\nabla^2(c+\mathbf{W}\mathbf{x})=0$ and $\mathbb{A}(c+\mathbf{W}\mathbf{x})\mathbf{n}\cdot\mathbf{n}=\ \mathbf{W}\mathbf{n}\cdot\mathbf{n}=0$ for every $c\in\mathbb{R}^d$ and every $\mathbf{W}\in\mathbb{R}^{d\times d}_{\rm skew}$. We stress that the natural choice $H(\nabla^2 v)=|\nabla^2 v|^2$ and $q=2$, yielding a quadratic stored elastic energy in (1.4) and a quadratic surface energy term, is possible within our theory in the physical cases $d=2,3$, as our main results will be given under the restriction $p\geq dq/(q+1)$.

Alternative Examples of Surface Energies

Let us next introduce possible alternative models, for which we will prove the validity of the same convergence results. We can consider a surface energy term that penalizes differences in total surface areas between the reference and deformed configurations, e.g.,

$$\begin{aligned}\mathcal{F}(y):=&\int_\Omega (W(\nabla y)+H(\nabla^2 y))\,dx-\overline{\mathcal{L}}(y(x)-x)\\ &+\gamma\left|\int_{\partial\Omega}|\mathrm{cof}\,\nabla y(\sigma)\,\mathbf{n}(\sigma)|\,dS(\sigma)-|\partial\Omega|\right|^q.\end{aligned}$$

The change-of-variables formula for surface integrals indicates that the last term is equal to $\gamma\,||\partial\Omega^y|-|\partial\Omega||^q$, where $\partial\Omega^y$ denotes the boundary of the deformed configuration $y(\Omega)$. Here, $\gamma>0$ is again a physical parameter. The associated rescaled energies are given by

$$\begin{aligned}\mathcal{F}_\varepsilon(v):=&\frac{1}{\varepsilon^2}\int_\Omega W(\mathbf{I}+\varepsilon\nabla v)\,dx+\frac{1}{\varepsilon^p}\int_\Omega H(\varepsilon\nabla^2 v)-\mathcal{L}(v)\\ &+\frac{\gamma}{\varepsilon^q}\left|\int_{\partial\Omega}|\mathrm{cof}\,(\mathbf{I}+\varepsilon\nabla v)\ \mathbf{n}|\,dS-|\partial\Omega|\right|^q.\end{aligned} \tag{1.5}$$

Taking into account (1.2) and (1.3), we see that the as $\varepsilon\to 0$

$$\left|\int_{\partial\Omega}|\mathrm{cof}\,(\mathbf{I}+\varepsilon\nabla v)\,\mathbf{n}|\,dS-|\partial\Omega|\right|=\varepsilon\left|\int_{\partial\Omega}\mathbb{A}(v)\mathbf{n}\cdot\mathbf{n}\,dS\right|+o(\varepsilon),$$

so that the formal linearized functional obtained as $\varepsilon\to 0$ is

$$\mathcal{F}_*(v) := \frac{1}{2}\int_\Omega \mathbb{E}(v)\,\nabla^2 W(\mathbf{I})\,\mathbb{E}(v)\,dx + \int_\Omega H(\nabla^2 v)\,dx - \mathcal{L}(v)$$
$$+\gamma\left|\int_{\partial\Omega} \mathbb{A}(v)\,\mathbf{n}\cdot\mathbf{n}\,dS\right|^q. \tag{1.6}$$

We notice that the divergence theorem implies, assuming enough smoothness of $\partial\Omega$ and introducing a suitable extension of the normal $\mathbf{n}$ to the whole of Ω,

$$\int_{\partial\Omega} \mathbb{A}(v)\,\mathbf{n}\cdot\mathbf{n}\,dS = \int_{\partial\Omega} \mathbb{A}(v)^T\mathbf{n}\cdot\mathbf{n}\,dS = \int_\Omega \operatorname{div}(\mathbb{A}(v)^T\mathbf{n})\,dx = \int_\Omega \mathbb{A}(v) : \nabla\mathbf{n}\,dx,$$

having used the fact that $\mathbb{A}(v)$ is divergence free. Therefore, functional $\mathcal{F}_*$ can also be rewritten as

$$\mathcal{F}_*(v) = \frac{1}{2}\int_\Omega \mathbb{E}(v)\,\nabla^2 W(\mathbf{I})\,\mathbb{E}(v)\,dx + \int_\Omega H(\nabla^2 v)\,dx - \mathcal{L}(v) + \gamma\left|\int_\Omega \mathbb{A}(v) : \nabla\mathbf{n}\,dx\right|^q.$$

We can also consider a model featuring surface loading

$$\mathcal{I}(y) := \int_\Omega (W(\nabla y) + H(\nabla^2 y))\,dx - \overline{\mathcal{L}}(y(x) - x)$$
$$+\gamma\int_{\partial\Omega} |(\operatorname{cof}\nabla y(\sigma) - \mathbf{I})\,\mathbf{n}(\sigma)|^q\,dS(\sigma).$$

The last term features the $L^q(\partial\Omega;\mathbb{R}^3)$ distance between the normal vector $\mathbf{n}^y = \operatorname{cof}\nabla y\,\mathbf{n}$ in the deformed configuration and the undeformed one corresponding to $y(x) = x$. This restricts energetically favorable deformations in a stronger way as it penalizes local changes not only in the area of the boundary but also in the orientation of the boundary, and indeed the last term pays energy also for rigid motions of the reference configuration. Consequently, this surface term is minimized if the deformation locally preserves the area of the boundary and if the deformed and undeformed normals at $x \in \partial\Omega$ and in $y(x) \in \partial\Omega^y$ are parallel. It can be seen as a living (i.e., deformation-dependent) load. Again we shall introduce the rescaled energies

$$\mathcal{I}_\varepsilon(v) : = \frac{1}{\varepsilon^2}\int_\Omega W(\mathbf{I} + \varepsilon\nabla v)\,dx + \frac{1}{\varepsilon^p}\int_\Omega H(\varepsilon\nabla^2 v) - \mathcal{L}(v)$$
$$+\frac{\gamma}{\varepsilon^q}\int_{\partial\Omega} |(\operatorname{cof}(\mathbf{I} + \varepsilon\nabla v) - \mathbf{I})\,\mathbf{n}|^q\ dS.$$

Thanks to (1.2), we see that the formal limiting functional as $\varepsilon \to 0$ is

$$\mathcal{I}_*(v) = \frac{1}{2}\int_\Omega \mathbb{E}(v)\,\nabla^2 W(\mathbf{I})\,\mathbb{E}(v)\,dx + \int_\Omega H(\nabla^2 v)\,dx - \mathcal{L}(v) + \gamma\int_{\partial\Omega} |\mathbb{A}(v)\mathbf{n}|^q\,dS.$$

The main results about convergence of minimizer of functionals $\mathcal{G}_\varepsilon$ with Dirichlet boundary conditions, that we shall state in the next section, carry over for functionals $\mathcal{F}_\varepsilon$ and $\mathcal{I}_\varepsilon$ (we refer to Corollary 2.3 below). On the other hand, when considering the Neumann problem, an interesting difference will arise in the treatment of functional $\mathcal{I}_\varepsilon$.

2 Main Results

Let $d \geq 2$. The strain energy density $W : \mathbb{R}^{d\times d} \to [0, +\infty]$ appearing in (1.1) is a frame indifferent function that is assumed to be minimized at rotations, and sufficiently smooth around rotations. Summarizing, the basic assumptions satisfied by W are the following

$$\begin{aligned}
&W : \mathbb{R}^{d\times d} \to [0, +\infty] \text{ is continuous },\\
&W(\mathbf{R}\,\mathbf{F}) = W(\mathbf{F}) \text{ for every } \mathbf{R} \in SO(d) \text{ and every } \mathbf{F} \in \mathbb{R}^{d\times d},\\
&W(\mathbf{F}) \geq W(\mathbf{I}) = 0 \text{ for every } \mathbf{F} \in \mathbb{R}^{d\times d},\\
&W \in C^2(\mathcal{U}) \text{ for some suitable open neighborhood } \mathcal{U} \text{ of } SO(d) \text{ in } \mathbb{R}^{d\times d},
\end{aligned} \tag{2.1}$$

where $SO(d)$ denotes the special orthogonal group. A consequence of the smoothness of W around rotations is therefore $DW(\mathbf{R}) = 0$ for every $\mathbf{R} \in SO(d)$. Continuity of W means that whenever $F_k \to F$ in $\mathbb{R}^{d\times d}$ then $W(F_k) \to W(F)$ as $k \to \infty$. Moreover, W is assumed to satisfy the following convexity property at the identity:

$$\text{there exists } C > 0 \text{ such that } \quad \mathbf{F}^T\, D^2W(\mathbf{I})\, \mathbf{F} \geq C|\mathrm{sym}\ \mathbf{F}|^2 \quad \text{for every } \mathbf{F} \in \mathbb{R}^{d\times d}, \tag{2.2}$$

where sym $\mathbf{F}$ denotes the symmetric part of $\mathbf{F}$ and $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^{d\times d}$, i.e., $|\mathbf{G}|^2 = \mathrm{tr}\,(\mathbf{G}^T\mathbf{G})$. Here, D^2W denotes the Hessian of W, and $D^2W(\mathbf{I})$ is the fourth-order elasticity tensor, appearing in the quadratic potential acting on the infinitesimal strain tensor in the linearized energy $\mathcal{G}_*$ from (1.4). Another standard coercivity condition that we shall use is the following:

$$\text{there exists } \bar{C} > 0 \text{ such that } \quad W(\mathbf{F}) \geq \bar{C}\,\mathrm{dist}^2(\mathbf{F}, SO(d)) \quad \text{for every } \mathbf{F} \in \mathbb{R}^{d\times d}, \tag{2.3}$$

where $\mathrm{dist}(\mathbf{F}, SO(d)) := \inf\{|\mathbf{F} - \mathbf{R}| : \mathbf{R} \in SO(d)\}$. We remark that if W satisfies (2.1), then (2.3) is stronger than (2.2). Indeed, since for every $\mathbf{G} \in \mathbb{R}^{d\times d}$ with positive determinant there holds $\mathrm{dist}(\mathbf{G}, SO(d)) = |\sqrt{\mathbf{G}^T\mathbf{G}} - \mathbf{I}|$, we see that if (2.3) holds then

$$\begin{aligned}
\frac{1}{2}\mathbf{F}^T\, D^2W(\mathbf{I})\,\mathbf{F} &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}W(\mathbf{I}+\varepsilon\mathbf{F}) \geq \bar{C}\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\,\mathrm{dist}^2(\mathbf{I}+\varepsilon\mathbf{F}, SO(d))\\
&= \bar{C}\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\,|\sqrt{(\mathbf{I}+\varepsilon\mathbf{F}^T)(\mathbf{I}+\varepsilon\mathbf{F})} - \mathbf{I}|^2 = \bar{C}|\mathrm{sym}\,\mathbf{F}|^2
\end{aligned}$$

for every $\mathbf{F} \in \mathbb{R}^{d\times d}$. We also remark that the natural conditions $\det \mathbf{F} \le 0 \Rightarrow W(\mathbf{F}) = +\infty$ and $W(\mathbf{F}) \to +\infty$ if $\det \mathbf{F} \to 0^+$ are compatible with all the above assumptions, but not necessary for the theory.

We will further assume that, for suitable $p > 1$, the function H appearing in the second gradient term of (1.1) satisfies the following:

$$\begin{aligned}
&H : \mathbb{R}^{d\times d\times d} \to \mathbb{R} \text{ is a convex positively } p\text{-homogeneous function,}\\
&\text{there exist } C_0 > 0, C_1 > 0 \quad \text{s.t. } \forall \mathbf{B} \in \mathbb{R}^{d\times d\times d} \;\; C_0|\mathbf{B}|^p \le H(\mathbf{B}) \le C_1(1 + |\mathbf{B}|^p),\\
&H(\mathbf{RB}) = H(\mathbf{B}) \text{ for every } \mathbf{B} \in \mathbb{R}^{d\times d\times d} \text{ and every } \mathbf{R} \in SO(d).
\end{aligned} \tag{2.4}$$

Here, the product between $\mathbf{R}$ and $\mathbf{B}$ is defined as $(\mathbf{RB})_{imn} = \mathbf{R}_{ik}\mathbf{B}_{kmn}$ for all $i, m, n \in \{1, \ldots, d\}$.

The load functional $\mathcal{L}$ appearing in $\mathcal{G}_\varepsilon$ and in $\mathcal{G}_*$ is assumed to be a linear continuous functional on $W^{2,p}(\Omega; \mathbb{R}^d)$, where $p > 1$ is the exponent that appears in (2.4), so that there exists a constant $C_{\mathcal{L}} > 0$ such that

$$|\mathcal{L}(v)| \le C_{\mathcal{L}} \|v\|_{W^{2,p}(\Omega;\mathbb{R}^d)} \quad \forall\, v \in W^{2,p}(\Omega; \mathbb{R}^d). \tag{2.5}$$

Let us start by considering the Dirichlet problem. Let Γ denote a closed subset of $\partial\Omega$ such that $S(\Gamma) = \mathcal{H}^{d-1}(\Gamma) > 0$. For $k \in \mathbb{N}$ and $p > 1$, let $W^{k,p}_\Gamma(\Omega; \mathbb{R}^d)$ denote the $W^{k,p}(\Omega; \mathbb{R}^d)$ completion of the space of restrictions to Ω of $C^\infty_c(\mathbb{R}^d \setminus \Gamma; \mathbb{R}^d)$ functions. Taking advantage of the homogeneous Dirichlet boundary condition on Γ, for $p > 1$ we have the following Poincaré inequality: there is $K > 0$ such that

$$\|v\|_{W^{2,p}(\Omega;\mathbb{R}^d)} \le K \|\nabla^2 v\|_{L^p(\Omega;\mathbb{R}^{d\times d\times d})} \quad \forall\, v \in W^{2,p}_\Gamma(\Omega; \mathbb{R}^d). \tag{2.6}$$

The following is the main result about convergence of minimizers of the Dirichlet problem for functionals $\mathcal{G}_\varepsilon$. For $p > d/2$ we let $\mathcal{G}_\varepsilon : W^{2,p}_\Gamma(\Omega; \mathbb{R}^d) \to \mathbb{R} \cup \{+\infty\}$ be defined by (1.1) and $\mathcal{G}_* : W^{2,p}_\Gamma(\Omega; \mathbb{R}^d) \to \mathbb{R}$ be defined by (1.4). Existence of minimizers over $W^{2,p}_\Gamma(\Omega; \mathbb{R}^d)$ for functional $\mathcal{G}_\varepsilon$ (for fixed ε) and functional $\mathcal{G}_*$ will be preliminarily proved in Sect. 3.

Theorem 2.1 *Let $q \ge 1$ and $p \ge dq/(q + 1)$. If $d = 2$ and $q = 1$, assume in addition that $p > 1$. Suppose that $\mathcal{L}$ is a bounded linear functional on $W^{2,p}(\Omega; \mathbb{R}^d)$, that W satisfies (2.1) and (2.2), and that (2.4) holds. Let $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0, 1)$ be a vanishing sequence and let $(v_j)_{j\in\mathbb{N}} \subset W^{2,p}_\Gamma(\Omega; \mathbb{R}^d)$ be a sequence of minimizers for functionals $\mathcal{G}_{\varepsilon_j}$ over $W^{2,p}_\Gamma(\Omega; \mathbb{R}^d)$. Then, the sequence $(v_j)_{j\in\mathbb{N}}$ is weakly converging in $W^{2,p}(\Omega; \mathbb{R}^d)$ to the unique solution to the problem*

$$\min\left\{\mathcal{G}_*(v) : v \in W^{2,p}_\Gamma(\Omega; \mathbb{R}^d)\right\}.$$

When considering the pure traction problem, it is natural to assume that external loads are equilibrated, i.e., with null resultant and momentum:

$$\mathcal{L}(a+\mathbf{W}\mathbf{x})=0 \quad \text{for every } a\in\mathbb{R}^d \text{ and every skew-symmetric matrix } \mathbf{W}\in\mathbb{R}^{d\times d}. \tag{2.7}$$

A consequence of (2.7) is that, by invoking Korn and Poincaré inequalities, if $\mathcal{L}$ is a bounded linear functional over $W^{2,r}(\Omega;\mathbb{R}^d)$ for some $r>1$, thus satisfying (2.5) with $p=r$, and if $u\in W^{2,r}(\Omega;\mathbb{R}^d)$, we have for some suitable $a\in\mathbb{R}^d$ and some suitable skew-symmetric $\mathbf{W}\in\mathbb{R}^{d\times d}$

$$\begin{aligned}|\mathcal{L}(u)| &= |\mathcal{L}(u-\mathbf{W}\mathbf{x}-a)|\\ &\le C_{\mathcal{L}}\left(\|u-\mathbf{W}\mathbf{x}-a\|_{L^r(\Omega;\mathbb{R}^d)}+\|\nabla u-\mathbf{W}\|_{L^r(\Omega;\mathbb{R}^{d\times d})}+\|\nabla^2 u\|_{L^r(\Omega;\mathbb{R}^{d\times d\times d})}\right)\\ &\le C_{\mathcal{L}}\left((1+\mathfrak{c})\mathfrak{K}\,\|\mathbb{E}(u)\|_{L^r(\Omega;\mathbb{R}^{d\times d})}+\|\nabla^2 u\|_{L^r(\Omega;\mathbb{R}^{d\times d\times d})}\right),\end{aligned} \tag{2.8}$$

where $\mathfrak{c}$ is the constant in Poincaré inequality and $\mathfrak{K}$ is the constant in second Korn inequality (Nitsche 1981), both depending on r and Ω only.

A further condition that proves to be crucial for the theory is

$$\mathcal{L}(\mathbf{R}\mathbf{x}-x)\le 0 \quad \text{for every } \mathbf{R}\in SO(d) \tag{2.9}$$

expressing the fact that external loads have an overall effect of traction (and not of compression) on Ω. This condition appears in Maddalena et al. (2019a, b); Mainini and Percivale (2020, 2021); Mainini et al. (2022). Following the same references, for load functionals that satisfy (2.9) we introduce the set

$$\mathcal{S}^0_{\mathcal{L}}:=\{\mathbf{R}\in SO(d):\mathcal{L}(\mathbf{R}\mathbf{x}-x)=0\},$$

which plays a crucial role in the linearization process. Indeed, if $\mathbf{R}_0\in SO(d)$ exists such that $\mathcal{L}(\mathbf{R}_0x-x)>0$, if $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ is any vanishing sequence, letting $\tilde v_j:=\frac{1}{\varepsilon_j}(\mathbf{R}_0x-x)$, since $\operatorname{cof}\mathbf{R}_0=\mathbf{R}_0$, $W(\mathbf{R}_0)=0$ and $\nabla^2(\mathbf{R}_0x-x)=0$, we see that

$$\mathcal{G}_{\varepsilon_j}(\tilde v_j)=-\frac{1}{\varepsilon_j}\mathcal{L}(\mathbf{R}_0x-x),$$

thus

$$\lim_{j\to+\infty}\inf_{C^\infty(\overline{\Omega};\mathbb{R}^d)}\mathcal{G}_{\varepsilon_j}=-\infty.$$

Therefore, (2.9) is a necessary condition for avoiding the nonlinear energies being unbounded from below as the parameter ε goes to zero. Moreover, under the conditions

(2.7) and (2.9), it turns out (see Theorem 2.2 below) that the actual limit functional is

$$\overline{\mathcal{G}}(u) : = \frac{1}{2}\int_{\Omega} \mathbb{E}(u)\, \nabla^2 W(\mathbf{I})\, \mathbb{E}(u)\, dx \\ + \int_{\Omega} H(\nabla^2 v)\, dx - \max_{\mathbf{R}\in\mathcal{S}^0_{\mathcal{L}}} \mathcal{L}(\mathbf{R}v) + \gamma \int_{\partial\Omega} |\mathbb{A}(v)\mathbf{n}\cdot\mathbf{n}|^q\, dS.$$

We may notice that under a stronger condition on external loads, i.e., under the additional assumption $\mathcal{S}^0_{\mathcal{L}} \equiv \{\mathbf{I}\}$ (meaning that external loads do not have any axis of equilibrium), then $\overline{\mathcal{G}} \equiv \mathcal{G}_*$. See Mainini and Percivale (2021); Mainini et al. (2022) for a thorough discussion on this topic in linearized models with no surface tension effect.

We have the following second main result. It will be proven in Sect. 4, after having proven the existence of minimizers over $W^{2,p}(\Omega;\mathbb{R}^d)$, in the pure traction problem, for functional $\mathcal{G}_\varepsilon$ (for fixed ε) and for functional $\overline{\mathcal{G}}$.

Theorem 2.2 *Let $q \ge 1$, $p \ge dq/(q+1)$. If $d = 2$ and $q = 1$, assume in addition that $p > 1$. Let W satisfy (2.1) and (2.3) and let H satisfy (2.4). Let $\mathcal{L}$ be a bounded linear functional over $W^{2,2\wedge p}(\Omega;\mathbb{R}^d)$ that satisfies (2.7) and (2.9). Let $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ be a vanishing sequence. If $(v_j)_{j\in\mathbb{N}} \subset W^{2,p}(\Omega,\mathbb{R}^d)$ is a sequence of minimizers of $\mathcal{G}_{\varepsilon_j}$ over $W^{2,p}(\Omega;\mathbb{R}^d)$,*

then there exists a sequence $(\boldsymbol{R}_j)_{j\in\mathbb{N}} \subset SO(d)$ such that, by defining

$$u_j(x) := \boldsymbol{R}_j^T v_j(x) + \frac{1}{\varepsilon_j}(\boldsymbol{R}_j^T x - x),$$

in the limit as $j \to +\infty$, along a suitable (not relabeled) subsequence, there holds

$$\nabla u_j \to \nabla u_* \ \ \text{weakly in } W^{1,p}(\Omega;\mathbb{R}^{d\times d}),$$

where $u_* \in W^{2,p}(\Omega,\mathbb{R}^d)$ is a minimizer of $\overline{\mathcal{G}}$ over $W^{2,p}(\Omega,\mathbb{R}^d)$, and

$$\mathcal{G}_{\varepsilon_j}(v_j) \to \overline{\mathcal{G}}(u_*), \qquad \min_{W^{2,p}(\Omega,\mathbb{R}^d)} \mathcal{G}_{\varepsilon_j} \to \min_{W^{2,p}(\Omega,\mathbb{R}^d)} \overline{\mathcal{G}}. \tag{2.10}$$

In order to obtain compactness in Theorem 2.2, we shall make use of the Firesecke–James–Müller rigidity inequality (Friesecke et al. 2002), stating that there exists a constant $C_\Omega > 0$ (only depending on Ω) such that for every $\varphi \in W^{1,2}(\Omega;\mathbb{R}^d)$ there is $\mathbf{R} \in SO(d)$ such that

$$\int_{\Omega} |\nabla\varphi - \mathbf{R}|^2 \le C_\Omega \int_{\Omega} \mathrm{dist}^2(\nabla\varphi, SO(d)). \tag{2.11}$$

In fact, as we shall see from the proof the sequence $(\mathbf{R}_j)_{j\in\mathbb{N}}$ in Theorem 2.2 can be chosen as a sequence of rotations for which (2.11) holds with $\varphi = v_j$ for every $j \in \mathbb{N}$.

We remark that our main converge results hold true for other model energies that we have introduced in Sect. 1. Indeed, defining $\overline{\mathcal{F}}$ by replacing the term $\mathcal{L}(v)$ with the term $\max_{\mathbf{R}\in\mathcal{S}^0_{\mathcal{L}}} \mathcal{L}(\mathbf{R}v)$ in functional $\mathcal{F}_*$ we have

Corollary 2.3 *The same statement of* Theorem 2.1 *holds true if we replace functionals $\mathcal{G}_\varepsilon$ and functional $\mathcal{G}_*$ therein with functionals $\mathcal{F}_\varepsilon$ (resp. $\mathcal{I}_\varepsilon$) and functional $\mathcal{F}_*$ (resp. $\mathcal{I}_*$). Moreover, the same statement of* Theorem 2.2 *holds true if we replace functionals $\mathcal{G}_\varepsilon$ and functional $\overline{\mathcal{G}}$ therein with functionals $\mathcal{F}_\varepsilon$ and functional $\overline{\mathcal{F}}$.*

We conclude with the following result that shows a remarkable difference in the limiting behavior of functionals $\mathcal{I}_\varepsilon$. The surface live load term prevents indeed rigid rotations.

Theorem 2.4 *Let $p \geq dq/(q+1)$. If $d = 2$ and $q = 1$, assume in addition that $p > 1$. Let W satisfy (2.1) and (2.3) and let H satisfy (2.4). Let $\mathcal{L}$ be a bounded linear functional over $W^{2,2\wedge p}(\Omega;\mathbb{R}^d)$ that satisfies (2.7) and (2.9). Let $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ be a vanishing sequence. If $(v_j)_{j\in\mathbb{N}} \subset W^{2,p}(\Omega,\mathbb{R}^d)$ is a sequence of minimizers of $\mathcal{I}_{\varepsilon_j}$ over $W^{2,p}(\Omega;\mathbb{R}^d)$,*

then in the limit as $j \to +\infty$, along a suitable (not relabeled) subsequence, there holds

$$\nabla v_j \to \nabla v_* \ \textit{ weakly in } W^{1,p}(\Omega;\mathbb{R}^{d\times d}),$$

where $v_* \in W^{2,p}(\Omega,\mathbb{R}^d)$ is a minimizer of $\mathcal{I}_*$ over $W^{2,p}(\Omega,\mathbb{R}^d)$, and

$$\mathcal{I}_{\varepsilon_j}(v_j) \to \mathcal{I}_*(v_*), \qquad \min_{W^{2,p}(\Omega,\mathbb{R}^d)} \mathcal{I}_{\varepsilon_j} \to \min_{W^{2,p}(\Omega,\mathbb{R}^d)} \mathcal{I}_*. \tag{2.12}$$

3 The Dirichlet Problem: Proof of Theorem 2.1

We first prove existence of minimizers for functional $\mathcal{G}_*$.

Lemma 3.1 *Let $p \geq 2d/(d+2)$ and $p \geq dq/(d-1+q)$. If $d = 2$ and $q = 1$, let $p > 1$. Let (2.1), (2.2) and (2.4) hold. Let $\mathcal{L}$ be a bounded linear functional over $W^{2,p}(\Omega;\mathbb{R}^d)$. There exists a unique solution to the problem*

$$\min\{\mathcal{G}_*(v) : v \in W^{2,p}_\Gamma(\Omega;\mathbb{R}^d)\}.$$

Proof The conditions $d > p \geq (2d/(d+2)) \vee (dq/(d-1+q))$ and $p > 1$ imply that $dp/(d-p) \geq 2$ and $(d-1)p/(d-p) \geq q$; therefore, we have the Sobolev embedding $W^{1,p}(\Omega;\mathbb{R}^d) \hookrightarrow L^2(\Omega;\mathbb{R}^d)$ and the Sobolev trace embedding $W^{1,p}(\Omega;\mathbb{R}^d) \hookrightarrow L^q(\partial\Omega)$. These embeddings also hold true if $p \geq d$. We deduce that $\mathbb{E}(v) \in L^2(\Omega;\mathbb{R}^{d\times d})$ and that $\mathbb{A}(v) \in L^q(\partial\Omega;\mathbb{R}^{d\times d})$ for every $v \in W^{2,p}(\Omega;\mathbb{R}^d)$. Thus, $\mathcal{G}_*(v)$ is finite for every $v \in W^{2,p}(\Omega;\mathbb{R}^d)$.

By (2.1), (2.4), (2.5) and (2.6), we get

$$\begin{aligned}\mathcal{G}_*(v) &\geq \int_\Omega H(\nabla^2 v) - C_{\mathcal{L}}\|v\|_{W^{2,p}(\Omega;\mathbb{R}^d)} \geq C_0 \int_\Omega |\nabla^2 v|^p - C_{\mathcal{L}}\|v\|_{W^{2,p}(\Omega;\mathbb{R}^d)}\\ &\geq \frac{C_0}{K^p}\|v\|^p_{W^{2,p}(\Omega;\mathbb{R}^d)} - C_{\mathcal{L}}\|v\|_{W^{2,p}(\Omega;\mathbb{R}^d)}\end{aligned} \tag{3.1}$$

and the right hand side is uniformly bounded from below on $W^{2,p}_\Gamma(\Omega;\mathbb{R}^d)$ because $p > 1$.

Let $(v_n)_{n\in\mathbb{N}} \subset W^{2,p}_\Gamma(\Omega;\mathbb{R}^d)$ be a minimizing sequence. The above estimate, along with (2.6) and $\mathcal{G}_*(0) = 0$, shows that such a sequence is bounded in $W^{2,p}(\Omega;\mathbb{R}^d)$, hence admitting a (not relabeled) weakly converging subsequence, the limit point being denoted by v. However, the term involving H in functional $\mathcal{G}_*$ is weakly lower semicontinuous over $W^{2,p}(\Omega;\mathbb{R}^d)$, thanks to (2.4), as well as the interfacial term, thanks to the Sobolev trace embedding $W^{1,p}(\Omega;\mathbb{R}^d) \hookrightarrow L^q(\partial\Omega;\mathbb{R}^d)$. Since $p > 1$ we also have the compactness of the Sobolev trace embedding $W^{1,p}(\Omega;\mathbb{R}^d) \hookrightarrow L^1(\partial\Omega;\mathbb{R}^d)$, which implies that $v \in W^{2,p}_\Gamma(\Omega;\mathbb{R}^d)$. On the other hand, $\mathcal{L}$ is weakly continuous over $W^{2,p}(\Omega;\mathbb{R}^d)$, and we have

$$\liminf_{n\to+\infty} \int_{\mathbb{R}^d} \mathbb{E}(v_n)\, D^2W(\mathbf{I})\, \mathbb{E}(v_n) \geq \int_\Omega \mathbb{E}(v)\, D^2W(\mathbf{I})\, \mathbb{E}(v)$$

by the embedding $W^{1,p}(\Omega;\mathbb{R}^d) \hookrightarrow L^2(\Omega;\mathbb{R}^d)$ and by the weak $L^2(\Omega;\mathbb{R}^{d\times d})$ lower semicontinuity of the map $\mathbf{G} \mapsto \int_\Omega \mathbf{G}^T D^2W(\mathbf{I})\,\mathbf{G}$. Therefore, $\mathcal{G}_*$ is lower semicontinuous with respect to the weak $W^{2,p}(\Omega;\mathbb{R}^d)$ convergence. The result follows by the direct method of the calculus of variations. Uniqueness of the minimizer follows from strict convexity of $\mathcal{G}_*$. □

The following is a key lemma, providing the rigorous linearization of the interfacial term.

Lemma 3.2 *Let $p \geq dq/(q+1)$. Let $v \in W^{2,p}(\Omega;\mathbb{R}^d)$. Let $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ be a vanishing sequence and let $(v_j)_{j\in\mathbb{N}} \subset W^{2,p}(\Omega;\mathbb{R}^d)$ be a sequence such that ∇v_j weakly converge to ∇v in $W^{1,p}(\Omega;\mathbb{R}^{d\times d})$ as $j \to +\infty$. Then*

$$\liminf_{j\to+\infty} \frac{1}{\varepsilon_j^q} \int_{\partial\Omega} \big||\mathrm{cof}\,(\boldsymbol{I} + \varepsilon_j \nabla v_j)\,\boldsymbol{n}| - 1\big|^q \, dS \geq \int_{\partial\Omega} |\mathbb{A}(v)\boldsymbol{n}\cdot\boldsymbol{n}|^q \, dS. \tag{3.2}$$

If ∇v_j strongly converge to ∇v in $W^{1,p}(\Omega;\mathbb{R}^{d\times d})$ as $j \to +\infty$ we also have

$$\lim_{j\to+\infty} \frac{1}{\varepsilon_j^q} \int_{\partial\Omega} \big||\mathrm{cof}\,(\boldsymbol{I} + \varepsilon_j \nabla v_j)\,\boldsymbol{n}| - 1\big|^q \, dS = \int_{\partial\Omega} |\mathbb{A}(v)\boldsymbol{n}\cdot\boldsymbol{n}|^q \, dS. \tag{3.3}$$

Proof We have the Sobolev trace embedding $W^{1,p}(\Omega;\mathbb{R}^d) \hookrightarrow L^{(d-1)p/(d-p)}(\partial\Omega;\mathbb{R}^d)$ if $d > p$. Else if $p \geq d$, we have $W^{1,p}(\Omega;\mathbb{R}^d) \hookrightarrow L^r(\partial\Omega)$ for every $r \in [1,+\infty)$. Since $p \geq dq/(q+1)$, we have $(d-1)p/(d-p) \geq (d-1)q$.

Since the entries of cof $\mathbf{F}$ are polynomials of degree $d-1$ in the entries of $\mathbf{F}$, we deduce that cof $\nabla\psi \in L^q(\partial\Omega;\mathbb{R}^{d\times d})$ for every $\psi \in W^{2,p}(\Omega;\mathbb{R}^d)$. This shows that all the integrands appearing in (3.2) are in $L^1(\partial\Omega)$.

By the Cayley–Hamilton formula, we have

$$\operatorname{cof}(\mathbf{I}+\varepsilon_j\nabla v_j) = \mathbf{I} + \varepsilon_j\mathbb{A}(v_j) + \sum_{k=2}^{d-1}\varepsilon_j^k\,\mathbb{B}_k(v_j)$$

where $\mathbb{B}_k(v_j)$ is a matrix whose entries are polynomials of degree k in the entries of ∇v_j, and the sum is understood to be zero if $d=2$. Letting

$$\mathbb{B}(v_j) := \sum_{k=2}^{d-1}\varepsilon_j^{k-2}\mathbb{B}_k(v_j),$$

we get therefore

$$\operatorname{cof}(\mathbf{I}+\varepsilon_j\nabla v_j) = \mathbf{I} + \varepsilon_j\mathbb{A}(v_j) + \varepsilon_j^2\,\mathbb{B}(v_j). \tag{3.4}$$

We notice that

$$\begin{aligned} |\operatorname{cof}(\mathbf{I}+\varepsilon_j\nabla v_j)\,\mathbf{n}|^2 &= |\mathbf{n} + \varepsilon_j\mathbb{A}(v_j)\,\mathbf{n} + \varepsilon_j^2\,\mathbb{B}(v_j)\,\mathbf{n}|^2 \\ &= 1 + 2\varepsilon_j\mathbb{A}(v_j)\,\mathbf{n}\cdot\mathbf{n} + \varepsilon_j^2\,\mathbb{D}(v_j)\,\mathbf{n}\cdot\mathbf{n}, \end{aligned} \tag{3.5}$$

where

$$\mathbb{D}(v_j) := \mathbb{A}(v_j)^T\mathbb{A}(v_j) + 2\mathbb{B}(v_j) + 2\varepsilon_j\,\mathbb{B}(v_j)^T\,\mathbb{A}(v_j) + \varepsilon_j^2\,\mathbb{B}(v_j)^T\,\mathbb{B}(v_j). \tag{3.6}$$

We observe that the following properties hold:

$$\nabla v_j \rightharpoonup \nabla v \text{ weakly in } L^q(\partial\Omega;\mathbb{R}^{d\times d}) \text{ as } j\to+\infty, \tag{3.7}$$

$$\mathbb{A}(v_j) \rightharpoonup \mathbb{A}(v) \text{ weakly in } L^q(\partial\Omega;\mathbb{R}^{d\times d}) \text{ as } j\to+\infty, \tag{3.8}$$

$$\text{the sequence } (\mathbb{B}(v_j))_{j\in\mathbb{N}} \text{ is bounded in } L^q(\partial\Omega;\mathbb{R}^{d\times d}). \tag{3.9}$$

Indeed, (3.9) follows from the Sobolev trace embedding $W^{1,p}(\Omega) \hookrightarrow L^{(d-1)p/(d-p)}(\partial\Omega)$, since $d > p \geq dq/(q+1)$ implies $(d-1)p/(d-p) \geq (d-1)q$, and since $\mathbb{B}(v_j)$ is polynomial of degree $d-1$ in the entries of ∇v_j (and $\varepsilon_j < 1$). Else if $p \geq d$ we have the embedding $W^{1,p}(\Omega) \hookrightarrow L^r(\partial\Omega)$ for every $r \in [1,+\infty)$. For the same reason, (3.7) and its direct consequence (3.8) hold true, since $(d-1)q \geq q$.

In particular, ∇v has a $L^q(\partial\Omega)$ trace on $\partial\Omega$.

We define

$$Q_j := \{x \in \partial\Omega : |\mathbb{A}(v_j(x))| + \varepsilon_j|\mathbb{B}(v_j(x))| < 2^{-4}\varepsilon_j^{-1/4}\} \tag{3.10}$$

and we notice that

$S(\partial\Omega \setminus Q_j)$

$$\leq \int_{\partial\Omega\setminus Q_j} 16\varepsilon_j^{1/4}(|\mathbb{A}(v_j)| + \varepsilon_j|\mathbb{B}(v_j)|)\,dS \leq 16\varepsilon_j^{1/4}\int_{\partial\Omega}(|\mathbb{A}(v_j)| + \varepsilon_j|\mathbb{B}(v_j)|)\,dS$$

so that (3.8) and (3.9) imply that $S(\partial\Omega \setminus Q_j) \to 0$ as $j \to +\infty$. By using (3.10), (3.6) and $\varepsilon_j < 1$, it is not difficult to see that on Q_j there hold

$$|2\varepsilon_j\,\mathbb{A}(v_j)\,\mathbf{n}\cdot\mathbf{n}+\varepsilon_j^2\,\mathbb{D}(v_j)\,\mathbf{n}\cdot\mathbf{n}| \leq 2\varepsilon_j\,|\mathbb{A}(v_j)|+\varepsilon_j^2\,|\mathbb{D}(v_j)| < \frac{1}{2}\varepsilon_j^{3/4} < \frac{1}{2}, \quad (3.11)$$

$$\varepsilon_j|\mathbb{D}(v_j)| \leq \sqrt{\varepsilon_j} + 2\varepsilon_j|\mathbb{B}(v_j)|. \quad (3.12)$$

Thanks to (3.11), starting from (3.5) and using Taylor series, on Q_j we find that

$$\begin{aligned}
&|\mathrm{cof}\,(\mathbf{I}+\varepsilon_j\nabla v_j)\,\mathbf{n}| - 1\\
&\quad= \sqrt{1+2\varepsilon_j\mathbb{A}(v_j)\,\mathbf{n}\cdot\mathbf{n}+\varepsilon_j^2\,\mathbb{D}(v_j)\,\mathbf{n}\cdot\mathbf{n}} - 1\\
&\quad= \varepsilon_j\mathbb{A}(v_j)\,\mathbf{n}\cdot\mathbf{n} + \frac{\varepsilon_j^2}{2}\mathbb{D}(v_j)\,\mathbf{n}\cdot\mathbf{n} + \sum_{k=2}^{+\infty}\alpha_k(2\varepsilon_j\mathbb{A}(v_j)\,\mathbf{n}\cdot\mathbf{n}+\varepsilon_j^2\,\mathbb{D}(v_j)\,\mathbf{n}\cdot\mathbf{n})^k,
\end{aligned} \quad (3.13)$$

where $\alpha_k := \dfrac{(-1)^{k-1}(2k)!}{4^k(k!)^2(2k-1)}$ (in particular we have $\sum_{k=0}^{+\infty}\alpha_{k+2}\,2^{-k} < +\infty$) and

$$\begin{aligned}
&\sum_{k=2}^{+\infty}\alpha_k\,(2\varepsilon_j\mathbb{A}(v_j)\,\mathbf{n}\cdot\mathbf{n}+\varepsilon_j^2\,\mathbb{D}(v_j)\,\mathbf{n}\cdot\mathbf{n})^k\\
&\quad= \varepsilon_j^2\,(2\mathbb{A}(v_j)\,\mathbf{n}\cdot\mathbf{n}+\varepsilon_j\,\mathbb{D}(v_j)\,\mathbf{n}\cdot\mathbf{n})^2\sum_{k=0}^{+\infty}\alpha_{k+2}(2\varepsilon_j\,\mathbb{A}(v_j)\,\mathbf{n}\cdot\mathbf{n}+\varepsilon_j^2\,\mathbb{D}(v_j)\,\mathbf{n}\cdot\mathbf{n})^k\\
&\quad\leq \varepsilon_j^2\,(2\mathbb{A}(v_j)\,\mathbf{n}\cdot\mathbf{n}+\varepsilon_j\,\mathbb{D}(v_j)\,\mathbf{n}\cdot\mathbf{n})^2\sum_{k=0}^{+\infty}\alpha_{k+2}\,2^{-k} \leq \varepsilon_j^{3/2}\sum_{k=0}^{+\infty}\alpha_{k+2}\,2^{-k}.
\end{aligned} \quad (3.14)$$

From (3.13), we have

$$\begin{aligned}
\mathbb{1}_{Q_j}\frac{|\mathrm{cof}\,(\mathbf{I}+\varepsilon_j\nabla v_j)\,\mathbf{n}|-1}{\varepsilon_j} &= \mathbb{1}_{Q_j}\,\mathbb{A}(v_j)\,\mathbf{n}\cdot\mathbf{n} + \mathbb{1}_{Q_j}\,\frac{\varepsilon_j}{2}\mathbb{D}(v_j)\,\mathbf{n}\cdot\mathbf{n}\\
&\quad+\mathbb{1}_{Q_j}\,\frac{1}{\varepsilon_j}\sum_{k=2}^{+\infty}\alpha_k(2\varepsilon_j\mathbb{A}(v_j)\,\mathbf{n}\cdot\mathbf{n}+\varepsilon_j^2\,\mathbb{D}(v_j)\,\mathbf{n}\cdot\mathbf{n})^k,
\end{aligned}$$

where we may notice that the two terms on the right hand side converge strongly to 0 in $L^q(\partial\Omega)$, thanks to (3.9), (3.12) and (3.14).

Since $S(\partial\Omega \setminus Q_j) \to 0$ and since (3.8) holds, we deduce by the equiintegrability of $(\mathbb{A}(v_j))_{j\in\mathbb{N}}$ that $\mathbb{1}_{Q_j}\mathbb{A}(v_j)\,\mathbf{n}\cdot\mathbf{n} \rightharpoonup \mathbb{A}(v)\,\mathbf{n}\cdot\mathbf{n}$ weakly in $L^q(\partial\Omega)$ and thus

$$\mathbb{1}_{Q_j}\frac{|\mathrm{cof}\,(\mathbf{I}+\varepsilon_j\nabla v_j)\,\mathbf{n}|-1}{\varepsilon_j} \rightharpoonup \mathbb{A}(v)\,\mathbf{n}\cdot\mathbf{n} \quad \text{weakly in } L^q(\partial\Omega) \text{ as } j\to+\infty \quad (3.15)$$

so that

$$\liminf_{j\to+\infty}\frac{1}{\varepsilon_j^q}\int_{Q_j}||\mathrm{cof}\,(\mathbf{I}+\varepsilon_j\nabla v_j)\,\mathbf{n}|-1|^q\,dS\geq\int_{\partial\Omega}|\mathbb{A}(v)\,\mathbf{n}\cdot\mathbf{n}|^q\,dS.$$

In order to conclude, we are left to prove that

$$\lim_{j\to+\infty}\frac{1}{\varepsilon_j^q}\int_{\partial\Omega\setminus Q_j}||\mathrm{cof}\,(\mathbf{I}+\varepsilon_j\nabla v_j)\,\mathbf{n}|-1|^q\,dS=0. \tag{3.16}$$

But (3.5) yields

$$\begin{aligned}\frac{1}{\varepsilon_j}\,||\mathrm{cof}\,(\mathbf{I}+\varepsilon_j\nabla v_j)\,\mathbf{n}|-1| &= \frac{1}{\varepsilon_j}\,||\mathbf{n}+\varepsilon_j\mathbb{A}(v_j)\mathbf{n}+\varepsilon_j^2\,\mathbb{B}(v_j)\,\mathbf{n}|-|\mathbf{n}||\\ &\leq\frac{1}{\varepsilon_j}\,|\varepsilon_j\,\mathbb{A}(v_j)\,\mathbf{n}+\varepsilon_j^2\,\mathbb{B}(v_j)\,\mathbf{n}|\leq|\mathbb{A}(v_j)|+\varepsilon_j|\mathbb{B}(v_j)|,\end{aligned}$$

and since $S(\partial\Omega\setminus Q_j)\to 0$, (3.16) follows from (3.8) and (3.9).

Eventually, if we also have strong $W^{1,p}(\Omega;\mathbb{R}^{d\times d})$ convergence of ∇v_j to ∇v, we obtain strong convergence in (3.7) and (3.8), thus in (3.15), so that by taking (3.16) into account we deduce (3.3). ◻

Corollary 3.3 *Let $p\geq dq/(q+1)$.. Let $v\in W^{2,p}(\Omega;\mathbb{R}^d)$. Let $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ be a vanishing sequence and let $(v_j)_{j\in\mathbb{N}}\subset W^{2,p}(\Omega;\mathbb{R}^d)$ be a sequence such that ∇v_j weakly converge to ∇v in $W^{1,p}(\Omega;\mathbb{R}^{d\times d})$ as $j\to+\infty$. Then*

$$\lim_{j\to+\infty}\frac{1}{\varepsilon_j^q}\left|\int_{\partial\Omega}|\mathrm{cof}\,(\boldsymbol{I}+\varepsilon_j\nabla v_j)\,\boldsymbol{n}|dS-|\partial\Omega|\right|^q=\left|\int_{\partial\Omega}\mathbb{A}(v)\,\boldsymbol{n}\cdot\boldsymbol{n}\,dS\right|^q \tag{3.17}$$

and

$$\liminf_{j\to+\infty}\frac{1}{\varepsilon_j^q}\int_{\partial\Omega}|(\mathrm{cof}\,(\boldsymbol{I}+\varepsilon_j\nabla v_j)-\boldsymbol{I})\,\boldsymbol{n}|^q\,dS=\int_{\partial\Omega}|\mathbb{A}(v)\,\boldsymbol{n}|^q\,dS. \tag{3.18}$$

If ∇v_j strongly converge to ∇v in $W^{1,p}(\Omega;\mathbb{R}^{d\times d})$ as $j\to+\infty$, we also have

$$\lim_{j\to+\infty}\frac{1}{\varepsilon_j^q}\int_{\partial\Omega}|(\mathrm{cof}\,(\boldsymbol{I}+\varepsilon_j\nabla v_j)-\boldsymbol{I})\,\boldsymbol{n}|^q\,dS=\int_{\partial\Omega}|\mathbb{A}(v)\,\boldsymbol{n}|^q\,dS. \tag{3.19}$$

Proof By following the proof of Lemma 3.2, with Q_j still defined by (3.10), we see that that (3.17) follows from (3.15) and (3.16). On the other hand, (3.4) implies that

$$\frac{1}{\varepsilon_j}(\mathrm{cof}\,(\mathbf{I}+\varepsilon_j\nabla v_j)-\mathbf{I})\,\mathbf{n}=\mathbb{A}(v_j)\,\mathbf{n}+\varepsilon_j\mathbb{B}(v_j)\,\mathbf{n}$$

so that (3.18) follows from (3.8) and (3.9). If ∇v_j strongly converge to ∇v in $W^{1,p}(\Omega;\mathbb{R}^{d\times d})$ as $j\to+\infty$, then there is strong convergence in (3.8) so that (3.19) follows. □

We next prove existence of minimizers $\mathcal{G}_\varepsilon$, for fixed $\varepsilon>0$.

Lemma 3.4 *Let $p\geq dq/(q+1)$. If $d=2$ and $q=1$, assume in addition that $p>1$. Suppose that $\mathcal{L}$ is a bounded linear functional on $W^{2,p}(\Omega;\mathbb{R}^d)$, that W satisfies (2.1) and (2.2), and that (2.4) hold. Then, the functional $\mathcal{G}_\varepsilon$ attains a minimum on $W^{2,p}_\Gamma(\Omega;\mathbb{R}^d)$ for every $\varepsilon>0$.*

Proof We assume wlog that $\varepsilon=1$. For every $v\in W^{2,p}_\Gamma(\Omega;\mathbb{R}^d)$, by (2.4), (2.5) and (2.6) there holds

$$\mathcal{G}_1(v)\geq C_0\int_\Omega|\nabla^2 v|^p-\mathcal{L}(v)\geq C_0\int_\Omega|\nabla^2 v|^p-KC_{\mathcal{L}}\|\nabla^2 v\|_{L^p(\Omega;\mathbb{R}^{d\times d\times d})}.$$

Similarly to the proof of Lemma 3.1, since $p>1$ this estimate implies boundedness from below of functional $\mathcal{G}_1$ on $W^{2,p}_\Gamma(\Omega;\mathbb{R}^d)$, and since $\mathcal{G}_1(0)=0$, it implies that any minimizing sequence $(v_n)_{n\in\mathbb{N}}\subset W^{2,p}_\Gamma(\Omega;\mathbb{R}^d)$ is bounded in $W^{2,p}(\Omega;\mathbb{R}^d)$, thus weakly converging up to subsequences to some $v\in W^{2,p}_\Gamma(\Omega;\mathbb{R}^d)$. This yields up to subsequences pointwise a.e. convergence of ∇v_n to ∇v, so that since W is continuous and nonnegative, Fatou Lemma implies lower semicontinuity of the term involving W in functional $\mathcal{G}_1$ along the sequence $(v_n)_{n\in\mathbb{N}}$. On the other hand, $\mathcal{L}$ is obviously continuous along such sequence, while

$$\liminf_{n\to+\infty}\int_\Omega H(\nabla^2 v_n)\geq\int_\Omega H(\nabla^2 v)$$

is a consequence of the convexity of H from (2.4). Finally,

since $p>1$, the Sobolev trace embedding $W^{1,p}(\Omega;\mathbb{R}^d)\hookrightarrow L^1(\partial\Omega;\mathbb{R}^d)$ is compact, thus we have pointwise $S-a.e.$ convergence of ∇v_n to ∇v on $\partial\Omega$ (up to a subsequences), and by Fatou Lemma we get lower semicontinuity of the interfacial term of functional $\mathcal{G}_1$.

□

By means of the next two lemmas, we prove the Γ-convergence

Lemma 3.5 *(Γ-**limsup**) Let $p\geq dq/(q+1)$. Suppose that $\mathcal{L}$ is a bounded linear functional on $W^{2,p}(\Omega;\mathbb{R}^d)$, that W satisfies (2.1) and (2.2), and that (2.4) holds. Let $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ be a vanishing sequence. Let $v\in W^{2,p}_\Gamma(\Omega;\mathbb{R}^d)$. There exists a sequence $(v_j)_{j\in\mathbb{N}}\subset C^\infty(\overline{\Omega};\mathbb{R}^d)\cap W^{2,p}_\Gamma(\Omega;\mathbb{R}^d)$ such that*

$$v_j\to v\quad\textit{strongly in } W^{2,p}(\Omega;\mathbb{R}^d)\textit{ as } j\to+\infty$$

and

$$\lim_{j\to+\infty}\mathcal{G}_{\varepsilon_j}(v_j)=\mathcal{G}_*(v).$$

Proof By the density of $C^\infty(\overline{\Omega};\mathbb{R}^d)\cap W^{2,p}_\Gamma(\Omega;\mathbb{R}^d)$ in $W^{2,p}_\Gamma(\Omega;\mathbb{R}^d)$, with respect to the $W^{2,p}(\Omega,\mathbb{R}^d)$ convergence, there exists a sequence $(\tilde v_j)_{j\in\mathbb{N}}\subset C^\infty(\overline{\Omega};\mathbb{R}^d)\cap W^{2,p}_\Gamma(\Omega;\mathbb{R}^d)$ such that $\tilde v_j\to v$ strongly in $W^{2,p}(\Omega;\mathbb{R}^d)$ as $j\to+\infty$. We suppose wlog that the sequence $(\|\nabla\tilde v_j\|_{C^0(\overline{\Omega};\mathbb{R}^{d\times d})})_{j\in\mathbb{N}}$ is nondecreasing. We notice that it is converging if $p>d$, thanks to the Sobolev embedding $W^{1,p}(\Omega;\mathbb{R}^d)\hookrightarrow C^0(\overline{\Omega};\mathbb{R}^d)$ holding in such a case. In any case, if it is converging we let $v_j=\tilde v_j$ for every $j\in\mathbb{N}$. Else if it is diverging, we define a new sequence $(v_j)_{j\in\mathbb{N}}\subset C^\infty(\overline{\Omega};\mathbb{R}^d)\cap W^{2,p}_\Gamma(\Omega;\mathbb{R}^d)$ as follows:

$$\{\tilde v_1,\tilde v_1,\ \ldots,\tilde v_1,\ \tilde v_2,\tilde v_2,\ \ldots,\tilde v_2,\ \tilde v_3,\tilde v_3,\ \ldots,\tilde v_3,\ \tilde v_4,\tilde v_4\ldots\},$$

where each of the $\tilde v_k$'s is repeated t_k times, where t_1 is defined as the smallest positive integer such that

$$\varepsilon_i\|\nabla\tilde v_2\|_{C^0(\overline{\Omega};\mathbb{R}^{d\times d})}<\frac{1}{2}\quad\text{for every integer } i\geq 1+t_1,$$

whose existence is ensured by the fact that the sequence $(\varepsilon_j)_{j\in\mathbb{N}}$ is vanishing, and where t_k is then recursively defined as the smallest positive integer such that

$$\varepsilon_i\|\nabla\tilde v_{k+1}\|_{C^0(\overline{\Omega};\mathbb{R}^{d\times d})}<\frac{1}{k}\quad\text{for every integer } i\geq 1+t_1+\ldots+t_k.$$

It is clear that we have $v_j\to v$ strongly in $W^{2,p}(\Omega;\mathbb{R}^d)$ as $j\to+\infty$. Moreover, by construction we also have

$$\lim_{j\to+\infty}\varepsilon_j\|\nabla v_j\|_{C^0(\overline{\Omega};\mathbb{R}^{d\times d})}=0. \tag{3.20}$$

We notice that the assumptions on the strain energy density W imply that

$$\left|W(\mathbf{I}+\mathbf{F})-\frac{1}{2}\mathbf{F}^T\,D^2W(\mathbf{I})\,\mathbf{F}\right|\leq\omega(|\mathbf{F}|)\,|\mathbf{F}|^2$$

for every $\mathbf{F}\in\tilde{\mathcal{U}}\subset\subset\mathcal{U}$, where $\omega:[0,+\infty)\to[0,+\infty)$ denotes the modulus of uniform continuity of D^2W on $\tilde{\mathcal{U}}$, which is a continuous nondecreasing function that vanishes at 0. But (3.20) implies that, for every large enough j, we have $\varepsilon_j\nabla v_j(x)\in\tilde{\mathcal{U}}$ for every $x\in\overline{\Omega}$. Therefore, if j is large enough, we get

$$\left|\frac{1}{\varepsilon_j^2}W(\mathbf{I}+\varepsilon_j\nabla v_j(x))-\frac{1}{2}\mathbb{E}(v_j(x))\,D^2W(\mathbf{I})\,\mathbb{E}(v_j(x))\right|\leq\omega(\varepsilon_j|\nabla v_j(x)|)\,|\nabla v_j(x)|^2 \tag{3.21}$$

for every $x\in\overline{\Omega}$, having used the fact that by frame indifference $D^2W(\mathbf{I})$ acts as a quadratic form only on the symmetric part of ∇v_j.

We notice that, since $p \geq dq/(q+1) \geq d/2$, if $d > p$ we have $p_* := dp/(d-p) \geq 2$, therefore we have the embedding $W^{1,p}(\Omega;\mathbb{R}^d) \hookrightarrow L^2(\Omega)$, still holding if $p \geq d$. Therefore, $\mathbb{E}(v_j) \to \mathbb{E}(v)$ strongly in $L^2(\Omega;\mathbb{R}^{d\times d})$. Hence,

$$\begin{aligned}\int_\Omega \left| \frac{1}{\varepsilon_j^2} W(\mathbf{I}+\varepsilon_j \nabla v_j) - \frac{1}{2}\mathbb{E}(v)\, D^2W(\mathbf{I})\, \mathbb{E}(v) \right| \\ \leq \int_\Omega \left| \frac{1}{\varepsilon_j^2} W(\mathbf{I}+\varepsilon_j \nabla v_j) - \frac{1}{2}\mathbb{E}(v_j)\, D^2W(\mathbf{I})\, \mathbb{E}(v_j) \right| \\ + \int_\Omega \left| \frac{1}{2}\mathbb{E}(v_j)\, D^2W(\mathbf{I})\, \mathbb{E}(v_j) - \frac{1}{2}\mathbb{E}(v)\, D^2W(\mathbf{I})\, \mathbb{E}(v) \right|\end{aligned}$$

and the second term in the right hand side goes to 0 as $j \to +\infty$. On the other hand, the first term in the right hand side gets estimated by means of (3.21) as

$$\int_\Omega \left| \frac{1}{\varepsilon_j^2} W(\mathbf{I}+\varepsilon_j \nabla v_j) - \frac{1}{2}\mathbb{E}(v_j)\, D^2W(\mathbf{I})\, \mathbb{E}(v_j) \right| \leq \omega(\varepsilon_j \|\nabla v_j\|_{C^0(\overline{\Omega};\mathbb{R}^{d\times d})}) \int_\Omega |\nabla v_j|^2$$

and so it vanishes as well as $j \to +\infty$, thanks to (3.20) and to the boundedness of $(\nabla v_j)_{j\in\mathbb{N}}$ in $L^2(\Omega;\mathbb{R}^{d,\times d})$, since $\omega(t) \to 0$ as $t \to 0$. We conclude that

$$\lim_{j\to+\infty} \frac{1}{\varepsilon_j^2} \int_\Omega W(\mathbf{I}+\varepsilon_j \nabla v_j) = \frac{1}{2}\int_\Omega \mathbb{E}(v)\, D^2W(\mathbf{I})\, \mathbb{E}(v).$$

With respect to the strong $W^{2,p}$ convergence, the interfacial term is continuous by Lemma 3.2, while the load term and the second gradient term are obviously continuous. The proof is concluded. □

Lemma 3.6 *(Γ-**liminf**) Let $p \geq dq/(q+1)$. Suppose that $\mathcal{L}$ is a bounded linear functional on $W^{2,p}(\Omega;\mathbb{R}^d)$, that W satisfies (2.1) and (2.2), and that (2.4) holds. Let $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ be a vanishing sequence. Let $v \in W^{2,p}_\Gamma(\Omega;\mathbb{R}^d)$. Let $(v_j)_{j\in\mathbb{N}} \subset W^{2,p}_\Gamma(\Omega;\mathbb{R}^d)$ be a sequence that weakly converges to v in $W^{2,p}(\Omega;\mathbb{R}^d)$. Then,*

$$\liminf_{j\to+\infty} \mathcal{G}_{\varepsilon_j}(v_j) \geq \mathcal{G}_*(v).$$

Proof Similarly to the proof of Lemma 3.5, we have the embedding of $W^{1,p}(\Omega)$ in $L^2(\Omega)$. Therefore, $\nabla v_j \to \nabla v$ weakly in $L^2(\Omega;\mathbb{R}^{d\times d})$. Let

$$H_j := \{x \in \Omega : \sqrt{\varepsilon_j}|\nabla v_j(x)| < 1\},$$

so that

$$|\Omega \setminus H_j| \leq \int_{\Omega\setminus H_j} \sqrt{\varepsilon_j}|\nabla v_j| \leq \sqrt{\varepsilon_j}\int_\Omega |\nabla v_j|,$$

thus $|\Omega \setminus H_j| \to 0$ as $j \to +\infty$. For every $x \in H_j$ we have $\varepsilon_j |\nabla v_j(x)| < \sqrt{\varepsilon_j}$ and thus (3.21) holds for every large enough j, yielding

$$\begin{aligned}\liminf_{j\to+\infty} \frac{1}{\varepsilon_j^2}\int_\Omega W(\mathbf{I}+\varepsilon_j \nabla v_j) &\geq \liminf_{j\to+\infty} \frac{1}{\varepsilon_j^2}\int_{H_j} W(\mathbf{I}+\varepsilon_j \nabla v_j)\\ &\geq \liminf_{j\to+\infty}\left(\frac{1}{2}\int_{H_j} \mathbb{E}(v_j)\, D^2W(\mathbf{I})\,\mathbb{E}(v_j) - \int_{H_j} \omega(\varepsilon_j|\nabla v_j|)\,|\nabla v_j|^2\right)\\ &\geq \liminf_{j\to+\infty}\left(\frac{1}{2}\int_{H_j} \mathbb{E}(v_j)\, D^2W(\mathbf{I})\,\mathbb{E}(v_j) - \int_{H_j} \omega(\sqrt{\varepsilon_j})|\nabla v_j|^2\right).\end{aligned}$$

But $\omega(\sqrt{\varepsilon_j}) \to 0$ as $j \to +\infty$ and $(\nabla v_j)_{j\in\mathbb{N}}$ is bounded in $L^2(\Omega;\mathbb{R}^{d\times d})$, therefore

$$\begin{aligned}&\liminf_{j\to+\infty} \frac{1}{\varepsilon_j^2}\int_\Omega W(\mathbf{I}+\varepsilon_j \nabla v_j)\\ &\quad\geq \liminf_{j\to+\infty} \frac{1}{2}\int_{H_j} \mathbb{E}(v_j)\, D^2W(\mathbf{I})\,\mathbb{E}(v_j) \geq \frac{1}{2}\int_\Omega \mathbb{E}(v)\, D^2W(\mathbf{I})\,\mathbb{E}(v),\end{aligned}$$

where the last inequality is due to the weak convergence of ∇v_j to ∇v in $L^2(\Omega;\mathbb{R}^{d\times d})$ and to the fact that $|\Omega \setminus H_j| \to 0$, yielding the weak $L^2(\Omega;\mathbb{R}^{d\times d})$ convergence of $\mathbb{1}_{H_j}\mathbb{E}(v_j)$ to $\mathbb{E}(v)$, and to the weak $L^2(\Omega;\mathbb{R}^{d\times d})$ lower semicontinuity of the map $\mathbf{G} \mapsto \int_\Omega \mathbf{G}^T\, D^2W(\mathbf{I})\,\mathbf{G}$.

With respect to the weak $W^{2,p}$ convergence, the interfacial term is lower semicontinuous by Lemma 3.2, the load term is continuous, and the second gradient term is lower semicontinuous, thanks to the assumptions (2.4). The proof is concluded. $\square$

Proof of Theorem 2.1 Since $W \geq 0$, by Young inequality along with (2.5), (2.4) and (2.6), calculations similar to (3.1) show that $\mathcal{G}_{\varepsilon_j}$ is bounded from below on $W^{2,p}_\Gamma(\Omega;\mathbb{R}^d)$, uniformly with respect to $j \in \mathbb{N}$. Moreover, by applying the same estimate to v_j, since $\mathcal{G}_{\varepsilon_j}(0) = 0$, we obtain

$$C_0 \int_\Omega |\nabla^2 v_j|^p \leq \mathcal{G}_{\varepsilon_j}(0) + 1 = 1$$

for every large enough j, thus showing that the sequence $(v_j)_{j\in\mathbb{N}}$ is uniformly bounded in $W^{2,p}(\Omega;\mathbb{R}^d)$. Having shown the Γ-convergence by means of Lemma 3.5 and Lemma 3.6, the proof concludes. Indeed, let $v \in W^{2,p}_\Gamma(\Omega;\mathbb{R}^d)$ be a weak $W^{2,p}(\Omega;\mathbb{R}^d)$ limit point of the sequence $(v_j)_{j\in\mathbb{N}}$. Let $\hat{v} \in W^{2,p}_\Gamma(\Omega;\mathbb{R}^d)$ and let $(\hat{v}_j)_{j\in\mathbb{N}} \subset W^{2,p}_\Gamma(\Omega)$ be a sequence such that $\mathcal{G}_{\varepsilon_j}(\hat{v}_j) \to \mathcal{G}_*(\hat{v})$ as $j \to +\infty$, whose existence is ensured by Lemma 3.5. By minimality of v_j and Lemma 3.6, we get

$$\mathcal{G}_*(v) \leq \liminf_{j\to+\infty} \mathcal{G}_{\varepsilon_j}(v_j) \leq \limsup_{j\to+\infty} \mathcal{G}_{\varepsilon_j}(\hat{v}_j) = \mathcal{G}_*(\hat{v}).$$

The arbitrariness of $\hat{v} \in W^{2,p}_\Gamma(\Omega;\mathbb{R}^d)$ ends the proof. □

4 The Traction Problem: Proof of Theorem 2.2

We start by proving existence of minimizers for functional $\overline{\mathcal{G}}$ over $W^{2,p}(\Omega;\mathbb{R}^d)$. This is done after a preliminary lemma

Lemma 4.1 *Let $p>1$. Let $(u_j)_{j\in\mathbb{N}} \subset W^{2,p}(\Omega;\mathbb{R}^d)$ be a sequence such that $\mathbb{E}(u_j) \in L^2(\Omega;\mathbb{R}^{d\times d})$ for every $j\in\mathbb{N}$. Suppose that*

$$\sup_{j\in\mathbb{N}} \|\mathbb{E}(u_j)\|_{L^2(\Omega;\mathbb{R}^{d\times d})} + \sup_{j\in\mathbb{N}} \|\nabla^2 u_j\|_{L^p(\Omega;\mathbb{R}^{d\times d\times d})} < +\infty. \tag{4.1}$$

Then there exist $u \in W^{2,p}(\Omega;\mathbb{R}^d)$ with $\mathbb{E}(u) \in L^2(\Omega;\mathbb{R}^{d\times d})$, a sequence $(a_j)_{j\in\mathbb{N}} \subset \mathbb{R}^d$ and a sequence of skew-symmetric matrices $(\mathbf{W}_j)_{j\in\mathbb{N}} \subset \mathbb{R}^{d\times d}$ such that as $j\to+\infty$, along a suitable not relabeled subsequence,

$$\begin{aligned} &u_j - \mathbf{W}_j x - a_j \rightharpoonup u \quad \text{weakly in } W^{2,p}(\Omega;\mathbb{R}^d)\\ &\mathbb{E}(u_j) \rightharpoonup \mathbb{E}(u) \quad \text{weakly in } L^2(\Omega;\mathbb{R}^{d\times d}). \end{aligned} \tag{4.2}$$

Proof Assume first that $1<p\le 2$. By Poincaré inequality and by second Korn inequality, for every $j\in\mathbb{N}$ there exist $a_j\in\mathbb{R}^d$ and a skew-symmetric matrix $\mathbf{W}_j \in \mathbb{R}^{d\times d}$ such that

$$\|u_j - \mathbf{W}_j x - a_j\|_{L^p(\Omega;\mathbb{R}^d)} \le \mathfrak{q}\,\|\nabla u_j - \mathbf{W}_j\|_{L^p(\Omega;\mathbb{R}^{d\times d})} \le \mathfrak{q}\mathfrak{K}\|\mathbb{E}(u_j)\|_{L^p(\Omega;\mathbb{R}^{d\times d})}, \tag{4.3}$$

where $\mathfrak{q}$, $\mathfrak{K}$ are positive constants, only depending on Ω and p. We conclude from (4.1) that there exists $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ such that, by passing to a not relabeled subsequence, $u_j - \mathbf{W}_j x - a_j \rightharpoonup u$ weakly in $W^{1,p}(\Omega;\mathbb{R}^d)$ as $j\to+\infty$. But then (4.1) implies that $\mathbb{E}(u)\in L^2(\Omega;\mathbb{R}^{d\times d})$ and that $\mathbb{E}(u_j)\to\mathbb{E}(u)$ weakly in $L^2(\Omega;\mathbb{R}^{d\times d})$. The boundedness in $L^p(\Omega;\mathbb{R}^{d\times d\times d})$ of the sequence $(\nabla^2 u_j)_{j\in\mathbb{N}}$, still given by (4.1), allows to conclude.

Else if $p\ge 2$, Poincaré inequality yields for every $j\in\mathbb{N}$ the existence of $\mathbf{U}_j \in \mathbb{R}^{d\times d}$ such that

$$\begin{aligned} \|\mathbb{E}(u_j) - \mathbf{U}_j\|_{L^2(\Omega;\mathbb{R}^{d\times d})} &\le |\Omega|^{\frac{p-2}{2p}} \|\mathbb{E}(u_j) - \mathbf{U}_j\|_{L^p(\Omega;\mathbb{R}^{d\times d})}\\ &\le \mathfrak{q}|\Omega|^{\frac{p-2}{2p}} \|\nabla\mathbb{E}(u_j)\|_{L^p(\Omega;\mathbb{R}^{d\times d})}\\ &\le \mathfrak{q}|\Omega|^{\frac{p-2}{2p}} \|\nabla^2 u_j\|_{L^p(\Omega;\mathbb{R}^{d\times d})}. \end{aligned} \tag{4.4}$$

But (4.1) and (4.4) imply that the sequence $(\mathbf{U}_j)_{j\in\mathbb{N}}$ is uniformly bounded in $\mathbb{R}^{d\times d}$. Therefore still by (4.4), we get that $(\mathbb{E}(u_j))_{j\in\mathbb{N}}$ is uniformly bounded also in $L^p(\Omega;\mathbb{R}^{d\times d})$, so that we still have, by Korn and Poincaré inequalities, the validity of (4.3), where the right hand side is uniformly bounded with respect to j. We thus conclude as done in the case $1 < p \le 2$. □

Lemma 4.2 *Let $p \ge 2d/(d+2)$ and $p \ge dq/(d-1+q)$. If $d=2$ and $q=1$, let $p>1$. Suppose that W satisfies (2.1) and (2.2) and that H satisfies (2.4). Let $\mathcal{L}$ be a bounded linear functional over $W^{2,p\wedge 2}(\Omega;\mathbb{R}^d)$ that satisfies (2.7). Then, there exists a solution to each of the problems*

$$\min\{\overline{\mathcal{G}}(u) : u \in W^{2,p}(\Omega;\mathbb{R}^d)\} \quad \textit{and} \quad \min\{\mathcal{G}(u) : u \in W^{2,p}(\Omega;\mathbb{R}^d)\}.$$

Proof We consider the problem for functional $\overline{\mathcal{G}}$, the proof for the other one being analogous. As seen in the proof of Lemma 3.1, we have the Sobolev embedding $W^{1,p}(\Omega;\mathbb{R}^d) \hookrightarrow L^2(\Omega;\mathbb{R}^d)$ and the Sobolev trace embedding $W^{1,p}(\Omega;\mathbb{R}^d) \hookrightarrow L^q(\partial\Omega;\mathbb{R}^d)$, so that $\overline{\mathcal{G}}(v)$ is well defined and finite for every $v \in W^{2,p}(\Omega;\mathbb{R}^d)$. For every $v \in W^{2,p}(\Omega;\mathbb{R}^d)$ and every $\mathbf{R} \in \mathcal{S}^0_{\mathcal{L}}$, we define

$$\mathcal{H}(v,\mathbf{R}) := \frac{1}{2}\int_\Omega \mathbb{E}(v)\, D^2W(\mathbf{I})\, \mathbb{E}(v) + \int_\Omega H(\nabla^2 v) + \gamma \int_{\partial\Omega} |\mathbb{A}(v)\,\mathbf{n}\cdot\mathbf{n}|^q \, dS - \mathcal{L}(\mathbf{R}v).$$

Let $(u_j,\mathbf{R}_j)_{j\in\mathbb{N}} \subset W^{2,p}(\Omega;\mathbb{R}^d)\times\mathcal{S}^0_{\mathcal{L}}$ be a minimizing sequence for the minimization problem

$$\min\{\mathcal{H}(u,\mathbf{R}) : (u,\mathbf{R}) \in W^{2,p}(\Omega;\mathbb{R}^d)\times\mathcal{S}^0_{\mathcal{L}}\}.$$

Since $\mathcal{H}(0,\mathbf{I}) = 0$ we may assume wlog that $\mathcal{H}(u_j,\mathbf{R}_j) \le 1$ for every $j \in \mathbb{N}$. By (2.2) and (2.4), by Hölder inequality and by (2.8) with $r = 2\wedge p$, we get

$$\begin{aligned} C_0 \int_\Omega |\nabla^2 u_j|^p + C\int_\Omega |\mathbb{E}(u_j)|^2 &\le 1 + \mathcal{L}(\mathbf{R}_j u_j) \\ &\le 1 + Q\left(\|\mathbb{E}(u_j)\|_{L^2(\Omega;\mathbb{R}^{d\times d})} + \|\nabla^2 u_j\|_{L^p(\Omega;\mathbb{R}^{d\times d\times d})}\right), \end{aligned}$$

where Q is a suitable positive constant, only depending on p, Ω and $\mathcal{L}$. An application of Young inequality in the right hand side (similarly to the proof of Lemma 3.1) shows that the sequence $(u_j)_{j\in\mathbb{N}}$ satisfies (4.1). The same computation shows that the sequence $(\overline{\mathcal{G}}(u_j))_{j\in\mathbb{N}}$ is bounded from below. By Lemma 4.1, there exist $u \in W^{2,p}(\Omega;\mathbb{R}^d)$, a sequence $(a_j)_{j\in\mathbb{N}} \subset \mathbb{R}^d$ and a sequence of skew-symmetric matrices $(\mathbf{W}_j)_{j\in\mathbb{N}} \subset \mathbb{R}^{d\times d}$ such that (4.2) holds along a suitable not relabeled subsequence. Since $\mathcal{L}$ is a bounded linear functional over $W^{2,p}(\Omega;\mathbb{R}^d)$, and since (2.7) holds, we deduce

$$\lim_{j\to+\infty} \mathcal{L}(u_j) = \lim_{j\to+\infty} \mathcal{L}(u_j - \mathbf{W}_j x - a_j) = \mathcal{L}(u).$$

The Sobolev trace embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$ shows that $\mathbb{A}(u_j)\,\mathbf{n}\cdot\mathbf{n} \rightharpoonup \mathbb{A}(u)\,\mathbf{n}\cdot\mathbf{n}$ weakly in $L^q(\partial\Omega)$, therefore

$$
\begin{aligned}
&\liminf_{j\to+\infty}\int_{\partial\Omega}|\mathbb{A}(u_j)\,\mathbf{n}\cdot\mathbf{n}|^q\,dS\\
&\quad=\liminf_{j\to+\infty}\int_{\partial\Omega}|\mathbb{A}(u_j-\mathbf{W}_jx-a_j)\,\mathbf{n}\cdot\mathbf{n}|^q\,dS\geq\int_{\partial\Omega}|\mathbb{A}(u)\,\mathbf{n}\cdot\mathbf{n}|^q\,dS.
\end{aligned}
$$

Along the sequence $(u_j)_{j\in\mathbb{N}}$, the first two terms of $\mathcal{H}$ are lower semicontinuous, thanks to (4.2). By possibly extracting a further not relabeled subsequence we have $\mathbf{R}_j\to\mathbf{R}\in\mathcal{S}^0_{\mathcal{L}}$ as $j\to+\infty$ and then $\mathcal{L}(\mathbf{R}_ju_j)\to\mathcal{L}(\mathbf{R}u)$. We conclude that

$$
\overline{\mathcal{G}}(u)\leq\mathcal{H}(u,\mathbf{R})\leq\liminf_{j\to+\infty}\mathcal{H}(u_j,\mathbf{R}_j)=\inf_{W^{2,p}(\Omega;\mathbb{R}^d)\times\mathcal{S}^0_{\mathcal{L}}}\mathcal{H}=\inf_{W^{2,p}(\Omega;\mathbb{R}^d)}\overline{\mathcal{G}},
$$

thus showing that u is a minimizer for $\overline{\mathcal{G}}$ over $W^{2,p}(\Omega;\mathbb{R}^d)$. □

Next we prove existence of minimizers over $W^{2,p}(\Omega;\mathbb{R}^d)$ for functional $\mathcal{G}_\varepsilon$, for fixed ε.

Lemma 4.3 *Let $p\geq dq/(q+1)$. If $d=2$ and $q=1$, assume in addition that $p>1$. Let $\mathcal{L}$ be a bounded linear functional on $W^{2,p\wedge 2}(\Omega;\mathbb{R}^d)$ that satisfies (2.7). Suppose that W satisfies (2.1) and (2.3), and that H satisfies (2.4). Then, the functional $\mathcal{G}_\varepsilon$ attains a minimum on $W^{2,p}(\Omega;\mathbb{R}^d)$, for every $\varepsilon>0$.*

Proof We fix wlog $\varepsilon=1$. For every $v\in W^{2,p}(\Omega;\mathbb{R}^d)$, the conditions on p imply that $v\in W^{1,2}(\Omega;\mathbb{R}^d)$ by Sobolev embedding, and thanks to (2.11) and (2.3) we get

$$
\begin{aligned}
\mathcal{G}_1(v)\;&\geq\int_\Omega W(\mathbf{I}+\nabla v)+\int_\Omega H(\nabla^2 v)-\mathcal{L}(v)\\
&\geq\frac{\bar{C}}{C_\Omega}\int_\Omega|\mathbf{I}+\nabla v-\mathbf{R}|^2+\int_\Omega H(\nabla^2 v)-\mathcal{L}(v),
\end{aligned}
$$

for some suitable $\mathbf{R}\in SO(d)$, depending on v, therefore there exists $c>0$ (only depending on $\bar{C}$, C_Ω and d), such that, also using (2.4),

$$
c+\mathcal{G}_1(v)\geq\int_\Omega|\nabla v|^2+\int_\Omega H(\nabla^2 v)-\mathcal{L}(v)\geq\int_\Omega|\nabla v|^2+C_0\int_\Omega|\nabla^2 v|^p-\mathcal{L}(v).
$$

Similarly to the proof of Lemma 4.2, the latter estimate can be combined with (2.8) with $r=2\wedge p$ and with Young inequality to obtain that $\mathcal{G}_1$ is bounded from below over $W^{2,p}(\Omega;\mathbb{R}^d)$ and that every minimizing sequence $(v_n)_{n\in\mathbb{N}}\subset W^{2,p}(\Omega;\mathbb{R}^d)$ of functional $\mathcal{G}_1$ is such that $(\nabla v_n)_{n\in\mathbb{N}}$ is bounded in $L^2(\Omega;\mathbb{R}^{d\times d})$ and such that $(\nabla^2 v_n)_{n\in\mathbb{N}}$ is bounded in $L^p(\Omega;\mathbb{R}^{d\times d\times d})$. By arguing as in the proof of Lemma 4.2, we may use Poincaré inequality and deduce that there exists $v\in W^{2,p}(\Omega;\mathbb{R}^d)$ with $\nabla v\in L^2(\Omega;\mathbb{R}^{d\times d})$ and a sequence $(a_n)_{n\in\mathbb{N}}\subset\mathbb{R}^d$ such that, as $n\to+\infty$ along a suitable subsequence, $v_n-a_n\rightharpoonup v$ weakly in $W^{2,p}(\Omega;\mathbb{R}^d)$. In particular, up to extraction of a further subsequence, $\nabla v_n\to\nabla v$ a.e. in Ω, thus by Fatou lemma and the continuity of W we obtain the lower semicontinuity of the integral involving W along the sequence $(v_n)_{n\in\mathbb{N}}$, while the lower semicontinuity of the term involving

H is ensured by (2.4). We have by (2.7) $\mathcal{L}(v_n) = \mathcal{L}(v_n - a_n) \to \mathcal{L}(v)$ as $n \to +\infty$ since $\mathcal{L}$ is a bounded linear functional over $W^{2,p}(\Omega; \mathbb{R}^d)$. Finally, the interfacial term of functional $\mathcal{G}_1$ is lower semicontinuous along the sequence $(v_n)_{n\in\mathbb{N}}$, by the same argument in the proof of Lemma 3.4. We conclude that v is a minimizer of $\mathcal{G}_1$ over $W^{2,p}(\Omega; \mathbb{R}^d)$. □

We need three lemmas in order to prove Theorem 2.2, proving, respectively, compactness, lower bound and upper bound.

Lemma 4.4 *Let $p \geq dq/(q+1)$. If $d = 2$ and $q = 1$, assume in addition that $p > 1$. Let $M > 0$. Let W satisfy (2.1) and (2.3). Let $\mathcal{L}$ be a bounded linear functional over $W^{2,2\wedge p}(\Omega; \mathbb{R}^d)$ that satisfies (2.7) and (2.9). Let $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ be a vanishing sequence. Let $(v_j)_{j\in\mathbb{N}} \subset W^{2,p}(\Omega; \mathbb{R}^d)$ be a sequence such that $\mathcal{G}_{\varepsilon_j}(v_j) \leq M$ for every $j \in \mathbb{N}$. Then there exist $\boldsymbol{R} \in \mathcal{S}^0_{\mathcal{L}}$, a sequence $(\boldsymbol{R}_j)_{j\in\mathbb{N}} \subset SO(d)$, and $u \in W^{2,p}(\Omega; \mathbb{R}^d)$ such that, letting*

$$u_j := \boldsymbol{R}_j^T v_j + \frac{\boldsymbol{R}_j^T x - x}{\varepsilon_j}, \tag{4.5}$$

in the limit as $j \to +\infty$ (possibly along a not relabeled subsequence) there hold

$$\boldsymbol{R}_j \to \boldsymbol{R} \quad \text{and} \quad \nabla u_j \rightharpoonup \nabla u \quad \text{weakly in } W^{1,p}(\Omega; \mathbb{R}^{d\times d}).$$

Proof A consequence of (2.11) and of (2.3) is that there exists a a sequence $(\mathbf{R}_j)_{j\in\mathbb{N}} \subset SO(d)$ and a constant $c > 0$ (only depending on W and Ω) such that

$$c\int_\Omega |\nabla u_j|^2 = \frac{c}{\varepsilon_j^2}\int_\Omega |\mathbf{I} - \mathbf{R}_j + \varepsilon_j \nabla v_j|^2 \leq \frac{1}{\varepsilon_j^2}\int_\Omega W(\mathbf{I} + \varepsilon_j \nabla v_j), \tag{4.6}$$

and then we deduce, since $\mathcal{G}_{\varepsilon_j}(v_j) \leq M$ and since (2.9) holds, that

$$\begin{aligned} c\int_\Omega |\nabla u_j|^2 + \int_\Omega H(\nabla^2 u_j) &\leq M + \mathcal{L}(v_j) \\ &= M + \frac{1}{\varepsilon_j}\mathcal{L}(\mathbf{R}_j x - x) + \mathcal{L}(\mathbf{R}_j u_j) \leq M + \mathcal{L}(\mathbf{R}_j u_j). \end{aligned} \tag{4.7}$$

By including (2.8) with $r = 2 \wedge p$ and Hölder inequality, we obtain

$$c\int_\Omega |\nabla u_j|^2 + \int_\Omega H(\nabla^2 u_j) \leq M + Q\|\nabla u_j\|_{L^2(\Omega;\mathbb{R}^{d\times d})} + Q\|\nabla^2 u_j\|_{L^p(\Omega;\mathbb{R}^{d\times d\times d})}.$$

where $Q > 0$ is a suitable constant, only depending on Ω, p and on $C_{\mathcal{L}}$ from (2.5), and then by Young inequality we get

$$\begin{aligned} &c\int_\Omega |\nabla u_j|^2 + \int_\Omega H(\nabla^2 u_j) \\ &\quad \leq M + \frac{Q^2}{2\delta^2} + \frac{\delta^2}{2}\int_\Omega |\nabla u_j|^2 + \frac{p-1}{p}\left(\frac{Q}{\delta}\right)^{\frac{p}{p-1}} + \frac{\delta^p}{p}\int_\Omega |\nabla^2 u_j|^p \end{aligned}$$

for every $\delta > 0$. Choosing small enough δ, we see that the sequence $(\nabla u_j)_{j\in\mathbb{N}}$ is bounded in $L^2(\Omega;\mathbb{R}^{d\times d})$ and that the sequence $(\nabla^2 u_j)_{j\in\mathbb{N}}$ is bounded in $L^p(\Omega;\mathbb{R}^{d\times d\times d})$. By Poincaré inequality, there exists a sequence $(a_j)_{j\in\mathbb{N}} \subset \mathbb{R}^d$ and a positive constant $\mathfrak{c}$ (only depending on Ω) such that $\|u_j - a_j\|_{L^2(\Omega;\mathbb{R}^d)} \le \mathfrak{c}\|\nabla u_j\|_{L^2(\Omega;\mathbb{R}^{d\times d})}$ for every $j \in \mathbb{N}$. Therefore, by passing to a suitable not relabeled subsequence, we have the existence of $u \in W^{1,2}(\Omega;\mathbb{R}^d)$ such that $u_j - a_j \rightharpoonup u$ weakly in $L^2(\Omega;\mathbb{R}^d)$ and $\nabla u_j \to \nabla u$ weakly in $L^2(\Omega;\mathbb{R}^{d\times d})$ as $j \to +\infty$; the uniform bound on the sequence $(\nabla^2 u_j)_{j\in\mathbb{N}}$ in $L^p(\Omega;\mathbb{R}^{d\times d\times d})$ and Poincaré inequality again allow to conclude that $u \in W^{2,p}(\Omega;\mathbb{R}^d)$ and that $\nabla u_j \rightharpoonup \nabla u$ weakly in $W^{1,p}(\Omega;\mathbb{R}^{d\times d})$.

On the other hand, by (4.7) and (2.9) we have

$$0 \le -\frac{1}{\varepsilon_j}\mathcal{L}(\mathbf{R}_j x - x) \le M + \mathcal{L}(\mathbf{R}_j u_j).$$

But then (2.5) and the already established uniform bounds of the sequence $(\nabla u_j)_{j\in\mathbb{N}}$ in $L^2(\Omega;\mathbb{R}^{d\times d})$ and of the sequence $(\nabla^2 u_j)_{j\in\mathbb{N}}$ in $L^p(\Omega;\mathbb{R}^{d\times d\times d})$ yield

$$\lim_{j\to+\infty} \mathcal{L}(\mathbf{R}_j x - x) = 0.$$

This shows that any limit point of the sequence $(\mathbf{R}_j)_{j\in\mathbb{N}}$ belongs to $\mathcal{S}^0_{\mathcal{L}}$. □

Lemma 4.5 *Let $p \ge dq/(q+1)$. Let W satisfy (2.1) and (2.2). Let $\mathcal{L}$ be a bounded linear functional over $W^{2,2\wedge p}(\Omega;\mathbb{R}^d)$ that satisfies (2.7) and (2.9). Let $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ be a vanishing sequence. Let $u \in W^{2,p}(\Omega;\mathbb{R}^d)$. Let $(v_j)_{j\in\mathbb{N}} \subset W^{2,p}(\Omega;\mathbb{R}^d)$ and $(\boldsymbol{R}_j)_{j\in\mathbb{N}} \subset SO(d)$ be sequences such that*

$$\nabla u_j \rightharpoonup \nabla u \quad \textit{weakly in } W^{1,p}(\Omega;\mathbb{R}^{d\times d}) \quad \textit{as } j \to +\infty,$$

where $u_j := \boldsymbol{R}_j^T v_j + \frac{1}{\varepsilon_j}(\boldsymbol{R}_j^T x - x)$, and $\boldsymbol{R}_j \to \boldsymbol{R} \in \mathcal{S}^0_{\mathcal{L}}$ as $j \to +\infty$. Then,

$$\overline{\mathcal{G}}(u) \le \liminf_{j\to+\infty} \mathcal{G}_{\varepsilon_j}(v_j).$$

Proof Let $T_j := \{x \in \Omega : \sqrt{\varepsilon_j}|\nabla u_j(x)| < 1\}$, so that

$$|\Omega \setminus T_j| \le \int_{\Omega\setminus T_j} \sqrt{\varepsilon_j}|\nabla u_j| \le \sqrt{\varepsilon_j}\int_\Omega |\nabla u_j|,$$

thus $|\Omega \setminus T_j| \to 0$ as $j \to +\infty$. By repeating the argument in the proof of Lemma 3.6, for every $x \in T_j$ we have $\varepsilon_j|\nabla u_j(x)| < \sqrt{\varepsilon_j}$ and thus (3.21) holds for every large enough j, yielding

$$\begin{aligned}
&\liminf_{j\to+\infty}\frac{1}{\varepsilon_j^2}\int_\Omega W(\mathbf{I}+\varepsilon_j\nabla v_j)\\
&\quad=\liminf_{j\to+\infty}\frac{1}{\varepsilon_j^2}\int_\Omega W(\mathbf{I}+\varepsilon_j\nabla u_j)\geq\liminf_{j\to+\infty}\frac{1}{\varepsilon_j^2}\int_{T_j} W(\mathbf{I}+\varepsilon_j\nabla u_j)\\
&\quad\geq\liminf_{j\to+\infty}\left(\frac{1}{2}\int_{T_j}\mathbb{E}(u_j)\,D^2W(\mathbf{I})\,\mathbb{E}(u_j)-\int_{T_j}\omega(\varepsilon_j|\nabla u_j|)\,|\nabla u_j|^2\right)\\
&\quad\geq\liminf_{j\to+\infty}\left(\frac{1}{2}\int_\Omega(\mathbb{1}_{T_j}\mathbb{E}(u_j))\,D^2W(\mathbf{I})\,(\mathbb{1}_{T_j}\mathbb{E}(u_j))-\int_{T_j}\omega(\sqrt{\varepsilon_j})|\nabla u_j|^2\right),
\end{aligned}$$

where the first equality is due to the frame indifference of W. But $\omega(\sqrt{\varepsilon_j})\to 0$ as $j\to+\infty$ and $(\nabla u_j)_{j\in\mathbb{N}}$ is bounded in $L^2(\Omega;\mathbb{R}^{d\times d})$ because of the embedding of $W^{1,p}(\Omega)$ in $L^2(\Omega)$ yielding $\nabla u_j\to\nabla u$ in $L^2(\Omega;\mathbb{R}^{d\times d})$. Moreover, for every $\eta\in L^2(\Omega;\mathbb{R}^{d\times d})$ we have

$$\left|\int_\Omega \eta:\mathbb{1}_{\Omega\setminus T_j}\mathbb{E}(u_j)\right|\leq\|\nabla u_j\|_{L^2(\Omega;\mathbb{R}^{d\times d})}\left(\int_{\Omega\setminus T_j}|\eta|^2\right)^{\frac{1}{2}}$$

where the right hand side goes to zero as $j\to+\infty$ since $|\Omega\setminus T_j|\to 0$, so that $\mathbb{1}_{\Omega\setminus T_j}\mathbb{E}(u_j)\rightharpoonup 0$ weakly in $L^2(\Omega;\mathbb{R}^{d\times d})$, and by writing $\mathbb{1}_{T_j}\mathbb{E}(u_j)=\mathbb{E}(u_j)-\mathbb{1}_{\Omega\setminus T_j}\mathbb{E}(u_j)$ we see that $\mathbb{1}_{T_j}\mathbb{E}(u_j)\rightharpoonup\mathbb{E}(u)$ weakly in $L^2(\Omega;\mathbb{R}^{d\times d})$. We conclude that

$$\liminf_{j\to+\infty}\frac{1}{\varepsilon_j^2}\int_\Omega W(\mathbf{I}+\varepsilon_j\nabla v_j)\geq\frac{1}{2}\int_\Omega\mathbb{E}(u)\,D^2W(\mathbf{I})\,\mathbb{E}(u),\tag{4.8}$$

thanks to the weak $L^2(\Omega;\mathbb{R}^{d\times d})$ semicontinuity of the map $\mathbf{F}\mapsto\int_\Omega\mathbf{F}^T\,D^2W(\mathbf{I})\,\mathbf{F}$.

By the Sobolev embedding $W^{1,p}(\Omega;\mathbb{R}^{d\times d})\hookrightarrow L^2(\Omega;\mathbb{R}^{d\times d})$, holding since $p>d/2$, we get $\nabla u_j\rightharpoonup\nabla u$ weakly in $L^2(\Omega;\mathbb{R}^{d\times d})$. By Poincaré inequality, we deduce the existence of a sequence $(a_j)_{j\in\mathbb{N}}\subset\mathbb{R}^d$ and of $\bar{u}\in L^2(\Omega;\mathbb{R}^d)$ such that $u_j-a_j\rightharpoonup\bar{u}$ weakly in $L^2(\Omega;\mathbb{R}^{d\times d})$ and such that $\nabla\bar{u}=\nabla u$ on Ω. Since $\nabla^2u_j\rightharpoonup\nabla^2u$ weakly in $L^p(\Omega;\mathbb{R}^{d\times d\times d})$, we deduce that $u_j-a_j\rightharpoonup\bar{u}$ weakly in $W^{2,2\wedge p}(\Omega;\mathbb{R}^d)$. Therefore, since (2.7) holds and since $\mathcal{L}$ is a bounded linear functional over $W^{2,2\wedge p}(\Omega;\mathbb{R}^d)$, we get $\mathcal{L}(u_j)=\mathcal{L}(u_j-a_j)\to\mathcal{L}(\bar{u})=\mathcal{L}(u)$ as $j\to+\infty$, and since $\mathbf{R}_j\to\mathbf{R}$, we obtain

$$\lim_{j\to+\infty}\mathcal{L}(\mathbf{R}_ju_j)=\lim_{j\to+\infty}\mathcal{L}(\mathbf{R}_j(u_j-a_j))=\mathcal{L}(\mathbf{R}\bar{u})=\mathcal{L}(\mathbf{R}u).$$

By taking (2.9) into account, we have

$$-\mathcal{L}(v_j)=-\frac{1}{\varepsilon_j}\mathcal{L}(\mathbf{R}_jx-x)-\mathcal{L}(\mathbf{R}_ju_j)\geq-\mathcal{L}(\mathbf{R}_ju_j)$$

and therefore

$$\liminf_{j\to+\infty} -\mathcal{L}(v_j) \ge -\mathcal{L}(\mathbf{R}u) \ge -\max_{\mathbf{R}\in\mathcal{S}^0_{\mathcal{L}}} \mathcal{L}(\mathbf{R}u). \tag{4.9}$$

Since $x+\varepsilon_j v_j = \mathbf{R}_j(x+\varepsilon_j u_j)$, and since $\operatorname{cof}(\mathbf{RF}) = \mathbf{R}\operatorname{cof}\mathbf{F}$ for every $\mathbf{R}\in SO(d)$ and every $\mathbf{F}\in\mathbb{R}^{d\times d}$, we have

$$\int_{\partial\Omega} ||\operatorname{cof}(\mathbf{I}+\varepsilon_j\nabla v_j)\,\mathbf{n}|-1|^q\,dS = \int_{\partial\Omega} ||\operatorname{cof}(\mathbf{I}+\varepsilon_j\nabla u_j)\,\mathbf{n}|-1|^q\,dS,$$

therefore Lemma 3.2 implies

$$\liminf_{j\to+\infty}\frac{1}{\varepsilon_j^q}\int_{\partial\Omega} ||\operatorname{cof}(\mathbf{I}+\varepsilon_j\nabla v_j)\,\mathbf{n}|-1|^q\,dS \ge \int_{\partial\Omega}|\mathbb{A}(u)\,\mathbf{n}\cdot\mathbf{n}|^q. \tag{4.10}$$

The weak lower semicontinuity of the L^p norm for the second gradient term, along with (4.8), (4.9) and (4.10), entails the desired result. □

Lemma 4.6 *Let $p \ge dq/(q+1)$. Let W satisfy (2.1) and (2.2). Let $\mathcal{L}$ be a bounded linear functional over $W^{2,2\wedge p}(\Omega;\mathbb{R}^d)$ that satisfies (2.7) and (2.9). Let $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ be a vanishing sequence. For every $u\in W^{2,p}(\Omega;\mathbb{R}^d)$, there exist a sequence $(u_j)_{j\in\mathbb{N}}\subset W^{2,p}(\Omega;\mathbb{R}^d)$ and $\boldsymbol{R}_u\in\mathcal{S}^0_{\mathcal{L}}$ such that*

$$u_j \to u \ \ \textit{strongly in } W^{2,p}(\Omega;\mathbb{R}^d) \tag{4.11}$$

and such that

$$\limsup_{j\to+\infty}\mathcal{G}_{\varepsilon_j}(v_j) \le \overline{\mathcal{G}}(u),$$

where $v_j := \boldsymbol{R}_u u_j + \dfrac{1}{\varepsilon_j}(\boldsymbol{R}_u x - x)$.

Proof Let $(u_j)_{j\in\mathbb{N}}\subset C^\infty(\overline{\Omega};\mathbb{R}^d)\cap W^{2,p}(\Omega;\mathbb{R}^d)$ be a sequence that strongly converges to u in $W^{2,p}(\Omega;\mathbb{R}^d)$. If $p<2$, we also have $\nabla u_j\to\nabla u$ strongly in $L^2(\Omega;\mathbb{R}^{d\times d})$ as $j\to+\infty$ by Sobolev embedding, since $p\ge dq/(q+1)$. By the same argument of the proof of Lemma 3.5, it is possible to assume that $\varepsilon_j\|\nabla u_j\|_{C^0(\overline{\Omega};\mathbb{R}^{d\times d})}\to 0$ as $j\to+\infty$; therefore, by repeating the arguments therein we get

$$\lim_{j\to+\infty}\frac{1}{\varepsilon_j^2}\int_\Omega W(\mathbf{I}+\varepsilon_j\nabla u_j) = \frac{1}{2}\int_\Omega \mathbb{E}(u)\,D^2W(\mathbf{I})\,\mathbb{E}(u).$$

We have $\int_\Omega|\nabla^2 u_j|^p \to \int_\Omega|\nabla^2 u|^p$ as $j\to+\infty$, and moreover by Lemma 3.2 we have

$$\lim_{j\to+\infty}\frac{1}{\varepsilon_j^q}\int_{\partial\Omega} ||\operatorname{cof}(\mathbf{I}+\varepsilon_j\nabla u_j)\,\mathbf{n}|-1|^q\,dS = \int_{\partial\Omega}|\mathbb{A}(u)\,\mathbf{n}\cdot\mathbf{n}|^q\,dS.$$

Let now $\mathbf{R}_u$ be a minimizer of the function

$$\mathbf{R} \mapsto \frac{1}{2}\int_\Omega \mathbb{E}(u)\, D^2W(\mathbf{I})\, \mathbb{E}(u) + \int_\Omega H(\nabla^2 u) + \gamma \int_{\partial\Omega} |\mathbb{A}(u)\, \mathbf{n}\cdot\mathbf{n}|^q \, dS - \mathcal{L}(\mathbf{R}u)$$

over $\mathcal{S}^0_{\mathcal{L}}$, and let $v_j := \mathbf{R}_u u_j + \frac{1}{\varepsilon_j}(\mathbf{R}_u x - x)$, $j \in \mathbb{N}$. We notice that by frame indifference all the terms in $\mathcal{G}_{\varepsilon_j}$, excluding the load term, are the same if evaluated at u_j or v_j. In particular, we get

$$\begin{aligned}
\limsup_{j\to+\infty} |\mathcal{G}_{\varepsilon_j}(v_j) - \overline{\mathcal{G}}(u)| &= \limsup_{j\to+\infty} |\frac{1}{\varepsilon_j}\mathcal{L}(\mathbf{R}_u x - x) + \mathcal{L}(\mathbf{R}_u u_j) - \mathcal{L}(\mathbf{R}_u u)| \\
&= \limsup_{j\to+\infty} |\mathcal{L}(\mathbf{R}_u u_j) - \mathcal{L}(\mathbf{R}_u u)| \\
&\le \limsup_{j\to+\infty} C_{\mathcal{L}} \Big((1+\mathfrak{c})\mathfrak{K} \,\|\nabla u_j - \nabla u\|_{L^{2\wedge p}(\Omega;\mathbb{R}^{d\times d})} + \|\nabla^2 u_j - \nabla^2 u\|_{L^{2\wedge p}(\Omega;\mathbb{R}^{d\times d\times d})}\Big) \\
&= 0,
\end{aligned}$$

having used $\mathbf{R}_u \in \mathcal{S}^0_{\mathcal{L}}$, (2.8) with $r = 2\wedge p$, and (4.11). The proof is concluded. □

Proof of Theorem 2.2 Let $(v_j)_{j\in\mathbb{N}} \subset W^{2,p}(\Omega, \mathbb{R}^d)$ be a sequence of minimizers of $\mathcal{G}_{\varepsilon_j}$ over $W^{2,p}(\Omega;\mathbb{R}^d)$. Since $\mathcal{G}_{\varepsilon_j}(0) = 0$, we may assume wlog that $\mathcal{G}_{\varepsilon_j}(v_j) \le 1$ for every $j \in \mathbb{N}$. By Lemma 4.4 and Lemma 4.5, there exist $u_* \in W^{2,p}(\Omega;\mathbb{R}^d)$, $\mathbf{R} \in \mathcal{S}^0_{\mathcal{L}}$ and a sequence $(\mathbf{R}_j)_{j\in\mathbb{N}} \subset SO(d)$ such that, by possibly passing to a not relabeled subsequence, there hold $\mathbf{R}_j \to \mathbf{R}$ and $\nabla u_j \to \nabla u_*$ weakly in $W^{1,p}(\Omega;\mathbb{R}^{d\times d})$ as $j \to +\infty$, where $u_j := \mathbf{R}_j^T v_j + \frac{1}{\varepsilon_j}(\mathbf{R}_j^T x - x)$, and

$$\overline{\mathcal{G}}(u_*) \le \liminf_{j\to+\infty} \mathcal{G}_{\varepsilon_j}(v_j). \tag{4.12}$$

Let now $\tilde{u} \in W^{2,p}(\Omega;\mathbb{R}^d)$. By Lemma 4.6, there exist $\mathbf{R}_{\tilde{u}} \in \mathcal{S}^0_{\mathcal{L}}$ and a sequence $(\tilde{u}_j)_{j\in\mathbb{N}} \subset W^{2,p}(\Omega;\mathbb{R}^d)$ such that, letting $\tilde{v}_j := \mathbf{R}_{\tilde{u}}\tilde{u}_j + \frac{1}{\varepsilon_j}(\mathbf{R}_{\tilde{u}} x - x)$, there holds

$$\limsup_{j\to+\infty} \mathcal{G}_{\varepsilon_j}(\tilde{v}_j) \le \overline{\mathcal{G}}(\tilde{u}). \tag{4.13}$$

By combining (4.12) and (4.13), since $(v_j)_{j\in\mathbb{N}}$ is a sequence of minimizers, we deduce

$$\overline{\mathcal{G}}(u_*) \le \liminf_{j\to+\infty} \mathcal{G}_{\varepsilon_j}(v_j) \le \limsup_{j\to+\infty} \mathcal{G}_{\varepsilon_j}(\tilde{v}_j) \le \overline{\mathcal{G}}(\tilde{u}). \tag{4.14}$$

Then, the arbitrariness of $\tilde{u}$ shows that u_* minimizes $\overline{\mathcal{G}}$ over $W^{2,p}(\Omega;\mathbb{R}^d)$, and choosing $\tilde{u} = u_*$ in (4.14) we obtain (2.10). □

Proof of Corollary 2.3 The proof is the same as that of Theorem 2.1 and Theorem 2.2. The only difference is indeed in the interface terms. However, the limiting behavior of the interface terms of functional $\mathcal{F}_\varepsilon$ and $\mathcal{I}_\varepsilon$ is given by Corollary (3.3), which can be used in place of Lemma 3.2. This shows the validity of the result for the

Dirichlet problem. Concerning the pure traction problem, the proof is again the same for functional $\mathcal{G}_\varepsilon$, taking also advantage of the frame indifference of the interfacial term therein the allows to perform the argument in the proof of Lemma 4.6. □

In order to prove Theorem 2.4, we give a preliminary compactness result.

Lemma 4.7 *Let $p \geq dq/(q+1)$. If $q = 1$ and $d = 2$, let also $p > 1$. Let $M > 0$. Let W satisfy (2.1) and (2.3). Let $\mathcal{L}$ be a bounded linear functional over $W^{2,2\wedge p}(\Omega;\mathbb{R}^d)$ that satisfies (2.7) and (2.9). Let $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ be a vanishing sequence. Let $(v_j)_{j\in\mathbb{N}} \subset W^{2,p}(\Omega;\mathbb{R}^d)$ be a sequence such that $\mathcal{I}_{\varepsilon_j}(v_j) \leq M$ for every $j \in \mathbb{N}$. Then there exist $v \in W^{2,p}(\Omega;\mathbb{R}^d)$ such that there hold $\nabla v_j \rightharpoonup \nabla v$ weakly in $W^{1,p}(\Omega;\mathbb{R}^{d\times d})$ in the limit as $j \to +\infty$ (possibly along a not relabeled subsequence).*

Proof By (2.11) and (2.3), there exists a sequence $(\mathbf{R}_j)_{j\in\mathbb{N}} \subset SO(d)$ such that (4.6) holds, where u_j is defined by (4.5). Therefore,

$$
\begin{aligned}
c\int_\Omega |\nabla u_j|^2 + \int_\Omega H(\nabla^2 u_j) + \frac{\gamma}{\varepsilon_j^q}\int_{\partial\Omega} |(\mathrm{cof}\,(\mathbf{I}+\varepsilon_j\nabla v_j)-\mathbf{I})\,\mathbf{n}|^q\,dS \\
\leq \frac{1}{\varepsilon_j^2}\int_\Omega W(\mathbf{I}+\varepsilon_j\nabla v_j) + \frac{\gamma}{\varepsilon_j^p}\int_\Omega H(\varepsilon_j\nabla^2 v_j) + \frac{1}{\varepsilon_j^q}\int_{\partial\Omega} |(\mathrm{cof}\,(\mathbf{I}+\varepsilon_j\nabla v_j)-\mathbf{I})\,\mathbf{n}|^q\,dS \\
\leq \mathcal{I}_{\varepsilon_j}(v_j) + \mathcal{L}(v_j) \leq M + \frac{1}{\varepsilon_j}\mathcal{L}(\mathbf{R}_j x - x) + \mathcal{L}(\mathbf{R}_j u_j) \leq M + \mathcal{L}(\mathbf{R}_j u_j)
\end{aligned}
$$

having used (2.9). As seen in the proof of Lemma 4.4, this shows that there exists $u \in W^{2,p}(\Omega;\mathbb{R}^d)$ such that $\nabla u_j \rightharpoonup \nabla u$ weakly in $W^{1,p}(\Omega;\mathbb{R}^{d\times d})$ along a suitable not relabeled subsequence, and that

$$
\frac{1}{\varepsilon_j^q}\int_{\partial\Omega} |(\mathrm{cof}\,(\mathbf{I}+\varepsilon_j\nabla v_j)-\mathbf{I})\,\mathbf{n}|^q\,dS = \int_{\partial\Omega}\left|\frac{\mathbf{R}_j(\mathrm{cof}\,(\mathbf{I}+\varepsilon_j\nabla u_j)-\mathbf{I})\,\mathbf{n}}{\varepsilon_j} + \frac{(\mathbf{R}_j-\mathbf{I})\,\mathbf{n}}{\varepsilon_j}\right|^q dS
$$

is uniformly bounded w.r.t. j. However, the sequence $(\varepsilon_j^{-1}\mathbf{R}_j(\mathrm{cof}\,(\mathbf{I}+\varepsilon_j\nabla u_j) - \mathbf{I})\,\mathbf{n})_{j\in\mathbb{N}}$ is uniformly bounded in $L^q(\partial\Omega;\mathbb{R}^d)$, thanks to Corollary 3.15. Therefore, we deduce that

$$
\sup_{j\in\mathbb{N}}\int_{\partial\Omega}\left|\frac{(\mathbf{R}_j-\mathbf{I})}{\varepsilon_j}\,\mathbf{n}\right|^q dS < +\infty. \tag{4.15}
$$

Let $\mathbf{V}_j := \varepsilon_j^{-1}(\mathbf{R}_j - \mathbf{I})$ and $\mathbf{Z}_j := \mathbf{V}_j/|\mathbf{V}_j|$. We claim that the sequence $(\mathbf{V}_j)_{j\in\mathbb{N}} \subset \mathbb{R}^{d\times d}$ is bounded. Indeed, suppose not. Then there exists a suitable subsequence along which $|\mathbf{V}_j|$ diverge and $\mathbf{Z}_j \to \mathbf{Z}$ for some suitable $\mathbf{Z} \in \mathbb{R}^{d\times d}$ with $|\mathbf{Z}| = 1$. Thus

$$
|\mathbf{V}_j|^{-q}\int_{\partial\Omega}|\mathbf{V}_j\,\mathbf{n}|^q\,dS = \int_{\partial\Omega}|\mathbf{Z}_j\,\mathbf{n}|^q\,dS \to \int_{\partial\Omega}|\mathbf{Z}\,\mathbf{n}|^q\,dS
$$

as $j \to +\infty$. But $|\mathbf{V}_j| \to +\infty$ and (4.15) then imply $\int_{\partial\Omega} |\mathbf{Z}\,\mathbf{n}|^q \, dS = 0$, which is a contradiction since $|\mathbf{Z}| = 1$. The claim is proven and it implies the result, since $\nabla u_j \rightharpoonup \nabla u$ weakly in $W^{1,p}(\Omega; \mathbb{R}^{d\times d})$ and since u_j and v_j are related by (4.5). □

Proof of Theorem 2.4 We preliminarily notice that existence of minimizers over $W^{2,p}(\Omega; \mathbb{R}^d)$ of $\mathcal{I}_{\varepsilon_j}$, for every fixed j, and of $\mathcal{I}_*$ are obtained in the same way as done in Lemma 4.3 and Lemma 4.2, respectively.

We first check Gamma liminf inequality, that is we let $v \in W^{2,p}(\Omega; \mathbb{R}^d)$, we let $(v_j)_{j\in\mathbb{N}} \subset W^{2,p}(\Omega; \mathbb{R}^d)$ be a sequence such that $\nabla v_j \rightharpoonup \nabla v$ weakly in $W^{1,p}(\Omega; \mathbb{R}^{d\times d})$ as $j \to +\infty$,

and we check that

$$\mathcal{I}_*(v) \le \liminf_{j\to+\infty} \mathcal{I}_{\varepsilon_j}(v_j).$$

This is obtained in the very same way as in the proof of Lemma 4.5. Indeed, after defining $T_j := \{x \in \Omega : \sqrt{\varepsilon_j}|\nabla v_j(x)| < 1\}$ we follow the argument therein and obtain (4.8) with v in place of u. Similarly by using Poincaré inequality and (2.7), we get $\mathcal{L}(v_j) \to \mathcal{L}(v)$ as $j \to +\infty$. We also have lower semicontinuity of L^p norm of the second gradient, as well as lower semicontinuity of the interfacial term, thanks to Corollary 3.15.

As second step, we check Gamma limsup inequality, that is, we check that for every $v \in W^{2,p}(\Omega; \mathbb{R}^d)$ there exists a sequence $(v_j)_{j\in\mathbb{N}} \subset W^{2,p}(\Omega; \mathbb{R}^d)$ such that $v_j \to v$ strongly in $W^{2,p}(\Omega; \mathbb{R}^d)$ and such that

$$\limsup_{j\to+\infty} \mathcal{I}_{\varepsilon_j}(v_j) \le \mathcal{I}_*(v).$$

The argument is the same that was used for proving Lemma 4.6. Let $(v_j)_{j\in\mathbb{N}} \subset C^\infty(\overline{\Omega}; \mathbb{R}^d) \cap W^{2,p}(\Omega; \mathbb{R}^d)$ be a sequence that strongly converges to v in $W^{2,p}(\Omega; \mathbb{R}^d)$, constructed as in proof of Lemma 3.5. Thus, we get continuity along this sequence of all the terms in the energy but the load term, also using Corollary 3.15. Concerning the load term, we directly get $\mathcal{L}(v_j) \to \mathcal{L}(v)$ by means of (2.8).

Having proven compactness and Gamma convergence, the result follows. □

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