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A Class of Signed Rank Estimators in Regression Models with Random Covariates

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Abstract

This note proves the asymptotic uniform linearity of a weighted empirical process of residual signed ranks and a class of linear residual signed rank statistics with bounded scores in nonlinear parametric regression models when covariates are random and independent of the errors. This result is used to derive limiting distributions of a class of signed rank estimators of the underlying regression parameters in these models. The latter result is applied to the errors in variables linear regression model to show that these estimators are robust against large measurement error in the sense that the asymptotic relative efficiency of a class of signed rank estimators against the bias corrected least square estimator tends to infinity as the measurement error variance tends to infinity (in some cases monotonically), when covariates and regression and measurement errors have Gaussian distributions.

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1 Introduction

Hodges and Lehmann (1963) proposed classes of estimators of the one and two sample location parameters based on signed ranks and ordinary ranks of the residuals, respectively. Some members of these classes of estimators are asymptotically more efficient than the classical estimators based on the sample means at several error distributions, while at the same time they are robust against outliers. Jurečková (1969, 1971) developed analogs of the estimators based on ranks for the regression parameter vector in linear regression models with non-random covariates. A class of rank estimators for a class of nonlinear parametric regression models, where the covariates are random and independent of the regression errors, were developed in Koul (1996). In this note we derive analogous results for a class of signed rank

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estimators defined as minimizers of the Euclidean norm of the given class of linear residual signed rank statistics.

A crucial result needed for deriving the asymptotic distributions of these estimators is the so called asymptotic uniform linearity (AUL) of the defining linear signed rank statistics of the residuals. In this note we derive such a result for a weighted empirical process of residual signed ranks and a class of linear residual signed rank statistics in a class of nonlinear parametric regression models, where random covariates are independent of the regression errors. In the case of linear regression model with non-random covariates, the AUL of a class of linear residual signed rank statistics was established in Koul (1969, 2002) for bounded scores and by van Eeden (1972) for general square integrable scores.

These results are in turn used to derive the asymptotic distributions of a class of signed rank estimators of the regression parameter vector in a linear errors in variables regression model. We show that the asymptotic relative efficiency of this class of estimators, relative to the bias corrected least squares estimator, when covariates and errors are Gaussian, tends to infinity as the measurement error variance tends to infinity, monotonically in some cases. This is a desirable property to have from a practical point of view. One may say that these estimators are robust against large measurement error variance.

When there is no measurement error in covariates, several properties of signed rank estimators or their variants have been studied in the literature. Hettmansperger and McKean (1998) study their robustness against gross errors in linear regression models. Abebe et al. (2012) proves consistency of some variants of these estimators in some nonlinear regression models while Abebe and Bindele (2016) use them in variable selection in linear regression. Bindele (2014) derives the asymptotic distributions of a class of signed rank estimators in a class of nonlinear regression models when covariates are observable, independent of the regression errors that are dependent r.v.'s and satisfy some mixing conditions. Bindele (2015) study these estimator for nonlinear regression with responses missing at random while Bindele and Nguelifack (2019) investigate some generalized signed-rank estimator in regression models with non-ignorable missing responses. An application in big data set up of some extensions of these estimators is discussed in Bindele et al. (2022). Jureckova et al. (2016) derives asymptotic distributions of a class of R estimators, the estimators based on ordinary ranks, in errors in variables linear regression models where responses also have measurement error while Koul (2022) discusses an R estimator of the regression parameter vector in the errors in variables linear regression model.

Some notation For any two stochastic processes $\mathcal{W}_{nj}(u,t)$, $j=1,2, 0 \le u \le 1, t \in \mathbb{R}^q$ and any sequence $0 < a_n \to \infty$, by $W_{n1}(u,t) = W_{n2}(u,t) + u_p(a_n^{-1})$ is meant that

$$\sup_{0 \le u \le 1, \|t\| \le b} a_n \big| W_{n1}(u,t) - W_{n2}(u,t) \big| = o_p(1), \quad \forall\, 0 < b < \infty.$$

For any positive integer m, $\mathcal{N}_m(\nu, C)$ denotes the m-dimensional normal distribution with the mean vector ν and the covariance matrix C, $\mathcal{N} \equiv \mathcal{N}_1$.

2 Nonlinear Parametric Regression Model

This section introduces the nonlinear parametric regression model of interest. It also contains the proof of the AUL of a weighted empirical process of residual signed ranks and of a class of linear residual signed rank statistics for such a model.

To proceed further, let p, q be known positive integers, $\Theta \subseteq \mathbb{R}^q$ and γ be a known function from $\Theta \times \mathbb{R}^p$ to $\mathbb{R}$ such that $\gamma(\vartheta, z)$ is measurable in $z \in \mathbb{R}^p$, for every $\vartheta \in \Theta$. Let (Y, Z) be a random vector, where Y is 1-dimensional response variable and Z is p-dimensional random vector of covariates. In the nonlinear parametric regression model of interest here, these entities obey the following relation for some $\theta \in \Theta$:

$$Y = \gamma(\theta, Z) + e, \quad \text{where the random error } e \text{isindependentof} Z. \tag{2.1}$$

Let Y_i, Z_i, e_i, $1 \le i \le n$ be iid copies of Y, Z, e of the model Eq. 2.1.

About γ, we assume the following: There exists a vector $\dot\gamma$ of functions from $\Theta \times \mathbb{R}^p$ to $\mathbb{R}^q$ such that for every $\vartheta \in \Theta$, $\dot\gamma(\vartheta, z)$ is measurable in z and the following conditions hold:

$$\max_{1 \le i \le n, \|t\| \le b} n^{1/2} \big| \gamma(\theta + n^{-1/2}t, Z_i) - \gamma(\theta, Z_i) - n^{-1/2} t' \dot\gamma(\theta, Z_i) \big| = o_p(1), \quad \forall\, 0 < b < \infty, \tag{2.2}$$

$$E\big\|\dot\gamma(\theta, Z)\big\|^2 < \infty. \tag{2.3}$$

$$E\big\|\dot\gamma(\theta + n^{-1/2}t, Z) - \dot\gamma(\theta, Z)\big\|^2 = o(1), \qquad \forall\, t \in \mathbb{R}^q. \tag{2.4}$$

$$n^{1/2} E\big\|\dot\gamma(\theta + n^{-1/2}t, Z) - \dot\gamma(\theta, Z)\big\| = O(1), \quad \forall\, t \in \mathbb{R}^q. \tag{2.5}$$

Throughout this paper, for any vector x, x' denotes its transpose.

As an example, consider the model where $\gamma(\theta, z) = \theta_1 + \theta_2 e^{\theta_3 z}$, $\theta_1 \in \mathbb{R}, \theta_2 > 0, \theta_3 > 0$, $z > 0$. This type of model is used when relating the concentration of a substance Y to elapsed time Z. In this case all of the above conditions are satisfied with $q = 3, p = 1$ and $\dot{\gamma}(\theta, z) = (1, e^{\theta_3 z}, \theta_2 z e^{\theta_3 z})'$ provided $E\big(Z^2 e^{2(\theta_3+\delta)Z}\big) < \infty$, for some $\delta > 0$. Other examples of interest can be found in Seber and Wild (1989).

For the sake of brevity, let 0_q denote the vector of q 0's and

$$\begin{aligned}
&e(t) := Y - \gamma(\theta + n^{-1/2}t, Z), \quad \gamma_i(t) := \gamma(\theta + n^{-1/2}t, Z_i), \quad d_i(t) := \gamma_i(t) - \gamma_i(0_q),\\
&e_{it} := e_i - d_i(t), \quad \dot{\gamma}_i(t) := \dot{\gamma}(\theta + n^{-1/2}t, Z_i), \quad \gamma_i := \gamma_i(0_q), \quad \dot{\gamma}_i := \dot{\gamma}_i(0_q), \quad 1 \le i \le n,\\
&\Sigma_\theta := E\big(\dot{\gamma}(\theta, Z)\dot{\gamma}(\theta, Z)'\big), \qquad D_\gamma(t) := n^{-1/2}\sum_i \big[\dot{\gamma}_i(t) - \dot{\gamma}_i\big], \quad t \in \mathbb{R}^q.
\end{aligned}$$

We also assume the following: For every $\alpha > 0$, $\exists\, \delta > 0$ and $N_\alpha < \infty$ such that $\forall\, \|s\| \le b$,

$$P\Big(\sup_{\|t-s\|<\delta} n^{-1/2}\sum_i \big\|\dot{\gamma}_i(t) - \dot{\gamma}_i(s)\big\| \le \alpha\Big) \ge 1 - \alpha, \quad \forall\, n > N_\alpha. \tag{2.6}$$

In this note the index of the summation varies from 1 to n, unless specified otherwise. Note that $e_i \equiv Y_i - \gamma_i$, $1 \le i \le n$ are iid copies of $e = Y - \gamma(\theta, Z)$ and for each $t \in \mathbb{R}^q$ and $n \ge 1$, $e(t)$, $e_{it}, 1 \le i \le n$ are iid r.v.'s.

For the later use we note the following facts. By Eqs. 2.2 and 2.3,

$$\sup_{1\le i\le n, \|t\|\le b} \big|d_i(t)\big| \le \sup_{1\le i\le n, \|t\|\le b} \big|d_i(t) - n^{-1/2}t'\dot{\gamma}_i\big| + b\, n^{-1/2}\max_{1\le i\le n}\big\|\dot{\gamma}_i\big\| = o_p(1), \tag{2.7}$$

$$\sup_{\|t\|\le b}\sum_i \big|d_i(t)\big| \le \sum_i \big|d_i(t) - n^{-1/2}t'\dot{\gamma}_i\big| + b\, n^{-1/2}\sum_i \big\|\dot{\gamma}_i\big\| = O_p(n^{1/2}). \tag{2.8}$$

By Eqs. 2.3, 2.4 and 2.5, for every fixed $t \in \mathbb{R}^q$,

$$\max_{1\le i\le n} n^{-1/2}\|\dot{\gamma}_i(t)\| \le \max_{1\le i\le n} n^{-1/2}\|\dot{\gamma}_i(t) - \dot{\gamma}_i\| + \max_{1\le i\le n} n^{-1/2}\|\dot{\gamma}_i\| = o_p(1), \quad \|D_\gamma(t)\| = O_p(1).$$

Hence, by the compactness of the ball $\{t \in \mathbb{R}^q; \|t\| \le b\}$ and Eq. 2.6,

$$\sup_{1\le i\le n, \|t\|\le b} n^{-1/2}\|\dot{\gamma}_i(t)\| = o_p(1), \qquad \sup_{\|t\|\le b} \big\|D_\gamma(t)\| = O_p(1), \quad \forall\, 0 < b < \infty. \tag{2.9}$$

Next, define, for $y \in \mathbb{R}$, $t \in \mathbb{R}^q$,

$$S_\gamma(y,t) := n^{-1}\sum_i \dot{\gamma}_i(t) I\big(Y_i \le y + \gamma_i(t)\big) = n^{-1}\sum_i \dot{\gamma}_i(t) I\big(e_i \le y + d_i(t)\big).$$

Let K denote the distribution function (d.f.) of e and assume the following:

$$K \text{ has uniformly continuous density } \kappa \text{ on } \mathbb{R}, \kappa > 0 \text{ a.e.} \tag{2.10}$$

Upon taking $h_{ni}(\theta + n^{-1/2}t) \equiv \gamma_i(t)$, $\dot{\boldsymbol{h}}_i(\theta + n^{-1/2}t) \equiv \dot{\gamma}_i(t)$ and $X_i \equiv Y_i$ in (1.8) of Lemma 1.1 of Koul (1996), we obtain the following lemma:
Lemma 2.1 *Suppose the model Eq. 2.1 and the assumptions Eqs. 2.2–2.6 and 2.10 hold. Then, for every* $0 < b < \infty$,

$$\sup_{y\in\mathbb{R}, \|t\|\le b} \big\|n^{1/2}\big(S_\gamma(y,t) - S_\gamma(y,0)\big) - \Sigma_\theta t\,\kappa(y) - D_\gamma(t)K(y)\big\| = o_p(1). \tag{2.11}$$

AUL of weighted empirical process of residual signed ranks From now on assume additionally that the d.f. K is symmetric about the origin so that $K(0) = 1/2$ and $\gamma(\theta, Z)$ is the conditional median of Y, given Z. Let $\mathrm{sgn}(y) := I(y > 0) - I(y < 0)$. Define

$$R_{it}^+ := \sum_j I\big(|e_j - d_j(t)| \le |e_i - d_i(t)|\big), \qquad s_i(t) := \mathrm{sgn}\big(e_i - d_i(t)\big),$$
$$Z_\gamma^+(u,t) := n^{-1}\sum_i \dot{\gamma}_i(t) I\big(R_{it}^+ \le nu\big)s_i(t), \quad J_{nt}(x) := n^{-1}\sum_i I\big(|e_i - d_i(t)| \le x\big),$$
$$J(x) := P(|e| \le x) = 2K(x) - 1, \quad x \ge 0; \qquad \omega(u) := \kappa\big(K^{-1}(u)\big) - \kappa(0), \quad 0 \le u \le 1,$$
$$S_\gamma^+(u,t) := n^{-1}\sum_i \dot{\gamma}_i(t) I\big(|e_i - d_i(t)| \le J^{-1}(u)\big)s_i(t), \qquad t \in \mathbb{R}^q.$$

For any function $L(u,t)$ of u, t, $L(u)$ will stand for $L(u, 0_q)$. The following lemma gives the AUL of the weighted empirical process of residual signed ranks $Z_\gamma^+(u,t)$.

Lemma 2.2 *Suppose the nonlinear regression model Eq. 2.1 and the assumptions Eqs. 2.2–2.6, 2.10 hold and K is symmetric about* 0. *Then,*

$$\sup_{0\le u\le 1,\|t\|\le b}\left|n^{1/2}\big(Z_\gamma^+(u,t)-S_\gamma^+(u)\big)-2\Sigma_\theta t\,\omega(u)-D_\gamma(t)\right|=o_p(1),\quad \forall\, 0<b<\infty. \tag{2.12}$$

Proof Fix a $0<b<\infty$. Recall that for any d.f. G,

$$G(G^{-1}(u))\ge u,\quad \forall\, 0\le u\le 1,\quad \text{with equality holding if } G \text{ is continuous.} \tag{2.13}$$
$$G^{-1}(G(x))\le x,\quad \forall\, x\in\mathbb{R},\quad \text{with equality holding if } G \text{ is strictly increasing.}$$

Let

$$\mathcal{V}_\gamma^+(u,t):=n^{-1}\sum_i \dot\gamma_i(t)I\big(|e_i-d_i(t)|\le J_{nt}^{-1}(u)\big)s_i(t).$$

The continuity of K, see Eq. 2.10, and the independence of e and Z imply that for every $t\in\mathbb{R}^q$, the distribution of $e(t)$ is continuous. Hence, the set of ranks $R_{it}^+, 1\le i\le n$ is a permutation of integers $1,\cdots,n$, for every $t\in\mathbb{R}^q$, with probability 1. This fact and Eq. 2.13 imply that, almost surely, for all $1\le i\le n$, $0\le u\le 1$, $t\in\mathbb{R}^q$,

$$\Big\{|e_i-d_i(t)|\ge J_{nt}^{-1}(u)\Big\}\implies\Big\{R_{it}^+\ge nu\Big\}\implies\Big\{|e_i-d_i(t)|\ge J_{nt}^{-1}(u)\Big\}. \tag{2.14}$$

Rewrite

$$Z_\gamma^+(u,t)=n^{-1}\sum_i\dot\gamma_i(t)\Big\{1-I\big(R_{it}^+\ge nt\big)+I\big(R_{it}^+=nu\big)\Big\}s_i(t),$$
$$\mathcal{V}_\gamma^+(u,t)=n^{-1}\sum_i\dot\gamma_i(t)\Big\{1-I\big(|e_i-d_i(t)|\ge J_{nt}^{-1}(u)\big)+I\big(|e_i-d_i(t)|=J_{nt}^{-1}(u)\big)\Big\}s_i(t).$$

By Eq. 2.14,

$$n^{1/2}\big(Z_\gamma^+(u,t)-\mathcal{V}_\gamma^+(u,t)\big)=\sum_i n^{-1/2}\dot\gamma_i(t)\Big\{I\big(R_{it}^+=nu\big)-I\big(|e_i-d_i(t)|=J_{nt}^{-1}(u)\big)\Big\}.$$

Hence, by Eq. 2.9,

$$\sup_{0\le u\le 1,\|t\|\le b}n^{1/2}\big|Z_\gamma^+(u,t)-\mathcal{V}_\gamma^+(u,t)\big|\le 2n^{-1/2}\sup_{1\le i\le n,\|t\|\le b}\big\|\dot\gamma_i(t)\big\|=o_p(1).$$

Note that $J^{-1}(u) = K^{-1}\big((u+1)/2\big) \geq 0$, for all $0 \leq u \leq 1$. Now, rewrite

$$\begin{aligned} S_\gamma^+(u,t) &:= n^{-1}\sum_i \dot\gamma_i(t)\Big[I\big(0 < e_i - d_i(t) \leq J^{-1}(u)\big) - I\big(-J^{-1}(u) \leq e_i - d_i(t) < 0\big)\Big] \\ &= S_\gamma(J^{-1}(u),t) + S_\gamma(-J^{-1}(u),t) - 2S_\gamma(0,t) \\ &\qquad + n^{-1}\sum_i \dot\gamma_i(t)\Big[I\big(e_i - d_i(t) = 0\big) - I\big(e_i - d_i(t) = J^{-1}(u)\big)\Big]. \end{aligned}$$

Let $\widehat{S}_\gamma(x,t) := S_\gamma(x,t) + S_\gamma(-x,t) - 2S_\gamma(0,t)$, $x \geq 0$. Then, by Eq. 2.9,

$$\sup_{0\leq u\leq 1, \|t\|\leq b} n^{1/2}\big|S_\gamma^+(u,t) - \widehat{S}_\gamma(J^{-1}(u),t)\big| \leq 2n^{-1/2} \max_{1\leq i\leq n, \|t\|\leq b} \big\|\dot\gamma_i(t)\big\| = o_p(1). \tag{2.15}$$

By Eqs. 2.11, 2.15, and the symmetry of K about 0, $J^{-1}(0) = 0$, $D_\gamma(0) \equiv 0$,

$$\begin{aligned} n^{1/2}S_\gamma^+(u,t) &= n^{1/2}\widehat{S}_\gamma\big(J^{-1}(u),t\big) + u_p(1) \qquad (2.16) \\ &= n^{1/2}\Big[S_\gamma\big(J^{-1}(u)\big) + S_\gamma\big(-J^{-1}(u)\big) - 2S_\gamma(0)\Big] \\ &\qquad + 2\Sigma_\theta\, t\Big[\kappa\big(J^{-1}(u)\big) - \kappa\big(J^{-1}(0)\big)\Big] + D_\gamma(t) + u_p(1) \\ &= n^{1/2}\widehat{S}_\gamma\big(J^{-1}(u)\big) + 2\Sigma_\theta\, t\,\omega(u)\big) + D_\gamma(t) + u_p(1) \\ &= n^{1/2}S_\gamma^+(u) + 2\Sigma_\theta\, t\,\omega(u)\big) + D_\gamma(t) + u_p(1). \end{aligned}$$

By the symmetry of K and the independence of Z and e, $E\big(S_\gamma^+(u)\big) \equiv 0$ and

$$nE\big(S_\gamma^+(u)S_\gamma^+(v)\big) = \Sigma_\theta \min(u,v), \quad 0 \leq u, v \leq 1.$$

Let $\dot\gamma_{ij}$ denote the jth coordinated of $\dot\gamma_i$, $1 \leq i \leq n$ and $S_{\gamma j}^+(u)$ denote the corresponding coordinate of $S_\gamma^+(u)$, $1 \leq j \leq q$, so that $S_\gamma^+(u) \equiv \big(S_{\gamma 1}^+(u), \cdots, S_{\gamma q}^+(u)\big)'$ and

$$S_{\gamma j}^+(u) = n^{-1}\sum_i \dot\gamma_{ij} I\big(|e_i| \leq J^{-1}(u)\big)\mathrm{sgn}(e_i), \quad 0 \leq u \leq 1.$$

For each $1 \leq j \leq q$, apply Theorem 1.1 of Koul and Ossiander (1994) with $\eta_{ni} \equiv J\big(|e_i|\big)$ and $\gamma_{ni} \equiv \dot\gamma_{ij}\mathrm{sgn}(e_i)$. By the symmetry of K about 0 and the independence between Z_i and e_i, we see that this γ_{ni} is independent of this

η_{ni}, for each $1 \le i \le n$. Also, Eq. 2.10 ensures the satisfaction of the other assumptions of the above theorem in this case. Hence, by Theorem 1.1 of Koul and Ossiander (1994), $n^{1/2} S_\gamma^+(u), 0 \le u \le 1$ converges weakly in the Skorokhod space $D^q[0,1]$ and uniform metric to $\boldsymbol{\mathcal{B}} := \big(\mathcal{B}_1, \cdots, \mathcal{B}_q\big)'$, where for each $1 \le j \le q$, $\mathcal{B}_j$ is a continuous mean zero Gaussian process with $\mathrm{Cov}\big(\boldsymbol{\mathcal{B}}(u), \boldsymbol{\mathcal{B}}(v)\big) = \Sigma_\theta \min(u,v)$, $0 \le u, v \le 1$. Moreover, for every $\epsilon > 0$, there exists a $\delta > 0$ and $N_\epsilon < \infty$ such that

$$P\Big(\sup_{|u-v|<\delta} n^{1/2}\big\|S_\gamma^+(u) - S_\gamma^+(v)\big\| > \epsilon\Big) < \epsilon, \quad \forall\, n > N_\epsilon. \tag{2.17}$$

We also need the following result:

$$\sup_{x\ge 0,\, \|t\|\le b} \big|J_{nt}(x) - J(x)\big| = o_p(1). \tag{2.18}$$

Proof of Eq. 2.18. Let $\bar J_{nt}(x) := n^{-1}\sum_i \Big[K\big(x + d_i(t)\big) - K\big(-x + d_i(t)\big)\Big]$. Write $J_{nt}(x) - J(x) = J_{nt}(x) - \bar J_{nt}(x) + \bar J_{nt}(x) - J(x)$, $x \ge 0$, so that

$$\sup_{x\ge 0,\, \|t\|\le b} \big|J_{nt}(x) - J(x)\big| \le \sup_{x\ge 0,\, \|t\|\le b} \big|J_{nt}(x) - \bar J_{nt}(x)\big| + \sup_{x\ge 0,\, \|t\|\le b} \big|\bar J_{nt}(x) - J(x)\big|. \tag{2.19}$$

Let $\delta_n := \sup_{1\le i\le n, \|t\|\le b}\big|d_i(t)\big|$. By Eq. 2.7, $n^{1/2}\delta_n = O_p(1)$. Hence, by Eq. 2.10

$$\begin{aligned}\mathcal{K}_n &:= \sup_{1\le i\le n, \|t\|\le b, x\in\mathbb{R}} n^{1/2}\big|K\big(x + d_i(t)\big) - K(x) - d_i(t)\kappa(x)\big| \\ &= \sup_{1\le i\le n, \|t\|\le b, x\in\mathbb{R}} n^{1/2}\Big|\int_0^{d_i(t)} \big[\kappa(x+z) - \kappa(x)\big]dz\Big| \le n^{1/2}\delta_n \sup_{|y-z|\le\delta_n}\big|\kappa(y) - \kappa(z)\big| = o_p(1).\end{aligned}$$

By the symmetry of K, $\kappa(x) \equiv \kappa(-x)$, so that

$$\begin{aligned}\bar J_{nt}(x) - J(x) = n^{-1}\sum_i \Big[&K\big(x + d_i(t)\big) - K(x) - d_i(t)\kappa(x) \\ &- K\big(-x + d_i(t)\big) + K(-x) + d_i(t)\kappa(-x)\Big],\end{aligned} \tag{2.20}$$

$$\sup_{x\ge 0, \|t\|\le b} n^{1/2}\big|\bar J_{nt}(x) - J(x)\big| \le 2\mathcal{K}_n = o_p(1).$$

Next, to deal with the first term in the upper bound of Eq. 2.19, let

$$K_{nt}(y) := n^{-1}\sum_i I\big(e_i \le y + d_i(t)\big), \qquad \bar{K}_{nt}(y) := n^{-1}\sum_i K\big(y + d_i(t)\big),$$
$$\Delta_n(y,t) := K_{nt}(y) - \bar{K}_{nt}(y), \quad K_n(y) := K_{n0}(y), \quad \mathcal{D}_n(y) := K_n(y) - K(y), \quad y \in \mathbb{R}.$$

By the Glivenko-Cantelli Lemma,

$$\sup_{y\in\mathbb{R}} \big|\mathcal{D}_n(y)\big| \to 0, \quad \text{a.s.} \tag{2.21}$$

Rewrite, $J_{nt}(x) - \bar{J}_{nt}(x) = \Delta_n(x,t) + \Delta_n(-x,t) - 2\Delta_n(0,t)$, $n \ge 1, x \ge 0, t \in \mathbb{R}^q$. With δ_n as above, $-\delta_n \le d_i(t) \le \delta_n$, for all $1 \le i \le n, \|t\| \le b$. Hence, by the monotonicity of the indicator and d.f.'s, for all $x \ge 0$,

$$K_n(\pm x - \delta_n) \le K_{nt}(\pm x) \le K_n(\pm x + \delta_n), \quad K(\pm x - \delta_n) \le \bar{K}_{nt}(\pm x) \le K(\pm x + \delta_n).$$

These bounds in turn imply that for all $x \ge 0, \|t\| \le b$,

$$\big|\Delta_n(\pm x,t)\big| \le \big|\mathcal{D}_n(\pm x + \delta_n)\big| + \big|\mathcal{D}_n(\pm x - \delta_n)\big| + \big[K\big(\pm x + \delta_n\big) - K\big(\pm x - \delta_n\big)\big].$$

Hence, by Eqs. 2.7 and 2.21, $\sup_{x\ge 0,\|t\le b} \big|\Delta_n(\pm x,t)\big| \le 2\sup_{y\in\mathbb{R}}\big|\mathcal{D}_n(y)\big| + 2\|\kappa\|_\infty \delta_n = o_p(1)$ and $\sup_{x\ge 0,\|t\|\le b}\big|J_{nt}(x) - \bar{J}_{nt}(x)\big| = o_p(1)$. Combine this fact with Eqs. 2.20 and 2.19 to conclude Eq. 2.18.

To proceed further we need the following fact implied by Eq. 2.18.

$$\begin{aligned}\sup_{0\le u\le 1}\big|JJ_{nt}^{-1}(u) - u\big| &= \sup_{0\le u\le 1}\big|JJ_{nt}^{-1}(u) - J_{nt}J_{nt}^{-1}(u) + J_{nt}J_{nt}^{-1}(u) - u\big| \\ &\le \sup_{x\ge 0}\big|J_{nt}(x) - J(x)\big| + n^{-1} = o_p(1).\end{aligned} \tag{2.22}$$

Next, consider the process $\mathcal{V}_\gamma^+$. By Eq. 2.10, $J = 2K - 1$ is continuous and strictly increasing, $J^{-1}J(x) \equiv x$ and $\mathcal{V}_\gamma^+(u,t) \equiv S_\gamma^+\big(JJ_{nt}^{-1}(u), u\big)$. Therefore, by Eqs. 2.16, 2.17, 2.22 and the uniform continuity of ω on $[0,1]$,

$$\begin{aligned}n^{1/2}\mathcal{V}_\gamma^+(u,t) &= n^{1/2}S_\gamma^+\big(JJ_{nt}^{-1}(u)\big) + 2\Sigma_\theta t\,\omega\big(JJ_{nt}^{-1}(u)\big) + D_\gamma(t) + u_p(1) \\ &= n^{1/2}S_\gamma^+(u) + 2\Sigma_\theta t\,\omega(u) + D_\gamma(t) + u_p(1).\end{aligned}$$

This completes the proof of Eq. 2.12 and of Lemma 2.2 also.

AUL of linear residual signed rank statistics We shall use the AUL result Eq. 2.12 to derive the AUL of a class of linear residual signed rank statistics. Let $\Omega := \{\psi : \psi$ a real valued nondecreasing right continuous function on $[0,1]$ having left limits such that $\psi(1) = 1\}$ and $\Omega_s := \{\psi : \psi \in \Omega$ such that $\psi(u) \equiv -\psi(1-u),\ 0 \le u \le 1\}$. For a $\psi \in \Omega_s$, the function $\varphi(u) := \psi\big((u+1)/2\big)$ is a d.f. on $[0,1]$. The linear residual signed rank statistic corresponding to a score function ψ is defined to be

$$\mathcal{Z}_\gamma^+(\psi, t) := n^{-1} \sum_i \dot{\gamma}_i(t) \varphi\Big(\frac{R_{it}^+}{n}\Big) s_i(t), \quad t \in \mathbb{R}^q.$$

To proceed further, we recall the following fact. Let $\mathcal{U}_1, \mathcal{U}_2$ be two right continuous functions of locally bounded variations on $[0,\infty)$. Then, for any $0 \le v < \infty$,

$$\int_{u\in(0,v]} \mathcal{U}_2(u) d\mathcal{U}_1(u) = \mathcal{U}_1(v)\mathcal{U}_2(v) - \mathcal{U}_1(0)\mathcal{U}_2(0) - \int_{u\in(0,v]} \mathcal{U}_1(u_-) d\mathcal{U}_2(u).$$

Apply this formula with $\mathcal{U}_1(u) = Z_\gamma^+(u,t)$, $\mathcal{U}_2(u) = \varphi(u)$, $v = 1$ and use the fact that $Z_\gamma^+(0,t) \equiv 0$ to obtain that that uniformly in $\psi \in \Omega$ and $\|t\| \le b$,

$$\mathcal{Z}_\gamma^+(\psi, t) = \int_0^1 \varphi(u) Z_\gamma^+(du, t) = Z_\gamma^+(1,t) - \int_0^1 Z_\gamma^+(u_-, t) d\varphi(u).$$

By Eq. 2.9,

$$\sup_{\|t\|\le b, \psi\in\Omega} \Big\| \int_0^1 n^{1/2} \big[Z_\gamma^+(u_-, t) - Z_\gamma^+(u,t)\big] d\varphi(u) \Big\| \le \sup_{\|t\|\le b, 1\le i\le n} n^{-1/2} \|\dot{\gamma}_i(t)\| = o_p(1).$$

Therefore,

$$\begin{aligned} \mathcal{Z}_\gamma^+(\psi, t) &= Z_\gamma^+(1,t) - \int_0^1 Z_\gamma^+(u,t) d\varphi(u) + u_p\big(n^{-1/2}\big) \qquad (2.23) \\ &= S_\gamma^+(1) + 2\Sigma_\theta t\, \omega(u) + D_\gamma(t) - \int \Big[S_\gamma^+(u) + 2\Sigma_\theta t\, \omega(u) + D_\gamma(t)\Big] d\varphi(u) + u_p\big(n^{-1/2}\big) \\ &= \Big[S_\gamma^+(1) - \int_0^1 S_\gamma^+(u) d\varphi(u)\Big] + 2\Sigma_\theta\, t \Big\{\omega(1) - \int_0^1 \omega(u) d\varphi(u)\Big\} + u_p\big(n^{-1/2}\big). \end{aligned}$$

For $\psi \in \Omega_s$, $\varphi\big(2K(x) - 1\big) = \psi\big(K(x)\big)$ and

$$\omega(1) - \int_0^1 \omega(u) d\varphi(u) = -\kappa(0) - \int_0^1 \big[\kappa\big(J^{-1}(u)\big) - \kappa(0)\big] d\varphi(u) = -\int_0^1 \kappa\big(J^{-1}(u)\big) d\varphi(u)$$

$$= -\int_0^1 \kappa\big(K^{-1}(\frac{u+1}{2})\big)d\varphi(u) = -\int_0^\infty \kappa(x)d\varphi\big(2K(x)-1\big) = -\int_0^\infty \kappa(x)d\psi\big(K(x)\big).$$

Let $\mathcal{I}(\psi) := \int_{-\infty}^{\infty} \kappa(x)d\psi\big(K(x)\big)$. By changing $x = -y$ and using the symmetry of κ, K about 0 and the symmetry of ψ about $1/2$, $\int_0^\infty \kappa(x)d\psi\big(K(x)\big) = \int_{-\infty}^0 \kappa(y)d\psi(K(y))$, for all $\psi \in \Omega_s$. Hence, $2\int_0^\infty \kappa(x)d\varphi$ $\big(K(x)\big)$
$= \int_{-\infty}^{\infty} \kappa(x)d\psi\big(K(x)\big) = \mathcal{I}(\psi)$. Moreover,

$$\widehat{S}_\gamma^+(\psi) := S^+(1) - \int_0^1 S_\gamma^+(u)d\varphi(u) = \int_0^1 \varphi(u)dS_\gamma^+(u) = n^{-1}\sum_i \dot{\gamma}_i\,\psi\big(K(|e_i|)\big)\mathrm{sgn}(e_i).$$

Because the r.v. e is symmetrically distributed around 0 and independent of Z, $E\big(\widehat{S}_\gamma^+(\psi)\big) \equiv 0$ and $nE\big(\widehat{S}_\gamma^+(\psi)\widehat{S}_\gamma^+(\psi)'\big) = \Sigma_\theta\tau_\psi^2$, where $\tau_\psi^2 := E\big(\psi^2(K(|e|)\big) = 2\int_{1/2}^1 \psi^2(u)du = \int_0^1 \psi^2(v)dv$. The last equality is obtained by changing the variable $u = 1-v$ and using the assumption $\psi(v) \equiv -\psi(1-v)$ so that $\int_{1/2}^1 \psi^2(u)du = \int_0^{1/2}\psi^2(v)dv$. By the classical CLT, under the assumption Eq. 2.3,

$$n^{1/2}\widehat{S}_\gamma^+(\psi) \to_D \mathcal{N}_q\big(0, \Sigma_\theta\tau_\psi^2\big). \tag{2.24}$$

The above derivations together then yield the following theorem describing the AUL of a class of linear residual signed rank statistics $\mathcal{Z}_\gamma^+(\psi,t)$, uniformly in $\psi \in \Omega_s$ and $\|t\| \le b$:

Theorem 2.1 *For the nonlinear regression model Eq. 2.1 and under the assumptions Eqs. 2.2–2.6, 2.10 and symmetry of the error d.f. K about 0, the following holds:*

$$\sup_{\psi\in\Omega_s,\|t\|\le b}\big\|n^{1/2}(\mathcal{Z}_\gamma^+(\psi,t) - \widehat{S}_\gamma^+(\psi)\big) + \Sigma_\theta t\mathcal{I}(\psi)\big\| = o_p(1), \quad \forall\, 0<b<\infty. \tag{2.25}$$

Consequently, by Eq. 2.24, for every $\psi \in \Omega_s, t\in\mathbb{R}^q$, $\mathcal{Z}_\gamma^+(\psi,t) \to_D \mathcal{N}_q\big(-$ $\Sigma_\theta t\mathcal{I}(\psi), \Sigma_\theta\tau_\psi^2\big)$.
This theorem does not require the existence of any moment of the regression error e.

3 Signed Rank Estimators

In this section we shall define a class of signed rank estimators and describe their asymptotic distributions. Recall, $\varphi(u) \equiv \psi\big((u+1)/2\big)$, $\psi \in \Omega_s$. Define,

$$\begin{aligned} r^+_{i\vartheta} &:= \sum_j I\Big(|Y_j - \gamma(\vartheta, Z_j)| \le |Y_i - \gamma(\vartheta, Z_i)|\Big), \quad 1 \le i \le n, \\ T_\psi(\vartheta) &:= n^{-1/2} \sum_i \dot\gamma(\vartheta, Z_i)\varphi\Big(\frac{r^+_{i\vartheta}}{n}\Big)\mathrm{sgn}\big(Y_i - \gamma(\vartheta, Z_i)\big), \quad \vartheta \in \mathbb{R}^q, \\ M_\psi(\vartheta) &:= \big\|T_\psi(\vartheta)\big\|^2, \qquad \widehat\theta_\psi := \mathrm{arginf}_{\vartheta\in\Theta} M_\psi(\vartheta). \end{aligned}$$

Because of the assumed independence of Z and e and the symmetry of e about 0, the r.v.'s Z_i, $|e_i|$ and $\mathrm{sgn}(e_i)$ are mutually independent for every $i = 1, \cdots, n$, and

$$E\big(T_\psi(\theta)\big) = n^{-1/2} \sum_i E\Big(\dot\gamma(\theta, Z_i)\varphi\big(\frac{r^+_{i\theta}}{n}\big)\Big)E\Big(\mathrm{sgn}\big(e_i\big)\Big) = 0. \tag{3.1}$$

Hence the corresponding signed rank estimator $\widehat\theta_\psi$ will not have any asymptotic bias.

We are interested in deriving the asymptotic distribution of $\widetilde\vartheta_\psi := n^{1/2}\big(\widehat\theta_\psi - \theta\big)$. But $\widetilde\vartheta_\psi \equiv \mathrm{arginf}_{t\in\mathbb{R}^q} M_\psi\big(\theta + n^{-1/2}t\big)$. If we let $R^+_{it} := r^+_{i,\theta+n^{-1/2}t}$, then $M_\psi\big(\theta + n^{-1/2}t\big) \equiv \mathcal{Z}^+_\gamma(\psi, t)$. Let

$$\begin{aligned} Q_\psi(t) &:= \big\|\widehat S^+_\gamma(\psi) - \Sigma_\theta t\mathcal{I}(\psi)\big\|^2 = M(\theta) - 2\mathcal{I}(\psi)t'\Sigma_\theta \widehat S^+_\gamma(\psi) + \mathcal{I}^2(\psi)t'\Sigma_\theta\Sigma_\theta t, \\ \widetilde t_\psi &:= \mathrm{arginf}_{t\in\mathbb{R}^q} Q_\psi(t) \equiv \big(\Sigma_\theta\mathcal{I}(\psi)\big)^{-1}\widehat S^+_\gamma(\psi). \end{aligned}$$

By Eq. 2.25, for every $0 < b < \infty$,

$$\sup_{\psi\in\Omega_s, \|t\|\le b} \big|M_\psi\big(\theta + n^{-1/2}t\big) - Q_\psi(t)\big| = o_p(1). \tag{3.2}$$

In other words, the sequence of objective functions $M_\psi\big(\theta+n^{-1/2}t\big), n \ge 1$ of t is asymptotically uniformly quadratic in t over all bounded sets and $\psi \in \Omega_s$. The sequence of the corresponding minimizers $\widetilde\vartheta_\psi$ will be close to the sequence of minimizers $\widetilde t_\psi$ of $Q_\psi(t)$ if we can verify that $\|\widetilde\vartheta_\psi\| = O_p(1)$ and $\|\widetilde t_\psi\| = O_p(1)$. But this claim about $\widetilde t_\psi$ follows from Eq. 2.24, which yields that $\widetilde t_\psi \to_D \mathcal{N}_q\big(0, \Sigma_\theta^{-1}\tau^2_\psi/\mathcal{I}^2(\psi)\big)$. To establish the former condition, we need the following additional assumption:

$$\begin{aligned}&\text{For every } \epsilon > 0, 0 < \alpha < \infty, \text{ there exist } N_\epsilon \text{ and } b_{\epsilon,\alpha} \text{ such that} \\ &P\Big(\inf_{\|t\|>b_{\epsilon,\alpha}} M_\psi\big(\theta + n^{-1/2}t\big) \geq \alpha\Big) \geq 1-\epsilon, \qquad \forall\, n > N_\epsilon.\end{aligned} \tag{3.3}$$

By arguing as in the proof of Theorem 5.4.1 of Chapter 5.4 in Koul (2002), we can verify that under Eqs. 2.24, 3.2 and 3.3, $\|n^{1/2}(\widehat{\theta}_\psi - \theta)\| = O_p(1)$, for every $\psi \in \Omega_s$.

Summarizing, we state the following theorem giving the asymptotic distribution of $\widetilde{\theta}_\psi$:

Theorem 3.1 *Suppose assumptions Eqs. 2.1, 2.2–2.6, 2.10 and 3.3 hold. Then, for every $\psi \in \Omega_s$, $\big\|n^{1/2}\big(\widehat{\theta}_\psi - \theta\big)\big\| = O_p(1)$ and*

$$n^{1/2}\big(\widehat{\theta}_\psi - \theta\big) = \mathcal{I}^{-1}(\psi)\Sigma_\theta^{-1} n^{1/2}\widehat{S}_\gamma^+(\psi) + o_p(1) \to_D \mathcal{N}_q\Big(0, \Sigma_\theta^{-1}\frac{\tau_\psi^2}{\mathcal{I}^2(\psi)}\Big). \tag{3.4}$$

Argue as in the proof of Lemma 5.5.4 on pages 183-186 of Koul (2002) to show that the following condition Eq. 3.5 implies Eq. 3.3.

$$\eta' T_\psi\big(\theta + n^{-1/2}s\eta\big) \text{ is monotonic in } s \in \mathbb{R}, \text{for every} \eta \in \mathbb{R}^q, \|\eta\| = 1, n \geq 1, \text{a.s.} \tag{3.5}$$

Least Absolute Deviation Estimator Let

$$\mathcal{T}(\vartheta) := n^{-1/2}\sum_i \dot{\gamma}(\vartheta, Z_i)\mathrm{sgn}\big(Y_i - \gamma(\vartheta, Z_i)\big), \quad \widehat{\theta}_{\ell ad} := \mathrm{arginf}_{\vartheta\in\Theta}\big\|\mathcal{T}(\vartheta)\big\|^2.$$

The entity $\widehat{\theta}_{\ell ad}$ is equivalent to the least absolute deviation (LAD) estimator. Using the arguments similar to those used in the proof of Theorem 3.1, one can verify that the following holds under the assumptions Eqs. 2.1, 2.2–2.6, 3.3 and the assumption that K has density κ that is continuous in a neighborhood of 0 and $\kappa(0) > 0$:

$$n^{1/2}\big(\widehat{\theta}_{\ell ad} - \theta\big) = -\frac{1}{2\kappa(0)}\Sigma_\theta^{-1} n^{1/2} S_\gamma^+(1,0) + o_p(1) \to_D \mathcal{N}_q\Big(0, \Sigma_\theta^{-1}/\big(4\kappa^2(0)\big)\Big).$$

Remark 3.1 **Linear Model.** Suppose $q = p$ and h is a p-dimensional vector of real valued measurable functions defined on $\mathbb{R}^p$. In Eq. 2.1, take $\gamma(\vartheta, Z) \equiv \vartheta' h(Z)$. Assume $E\big(\|h(Z)\|^2\big) < \infty$ and that $E\big(h(Z)h(Z)'\big)$ is positive definite. Then, $n^{-1/2} \max_{1\le i\le n} \|h(Z_i)\| = o_p(1)$, $\dot\gamma_i(t) \equiv \dot\gamma_i \equiv h(Z_i)$ and the assumptions Eqs. 2.2–2.6 are all trivially satisfied. Here,

$$R^+_{it} = \sum_j I\Big(|e_j - n^{-1/2}t'h(Z_j)| \le |e_i - n^{-1/2}t'h(Z_i)|\Big), \quad (3.6)$$

$$T_\psi\big(\theta + n^{-1/2}t\big) := n^{-1/2}\sum_i h(Z_i)\varphi\big(\frac{R^+_{it}}{n}\big)\mathrm{sgn}\big(e_i - n^{-1/2}t'h(Z_i)\big),$$

$$n^{1/2}\big(\widehat\theta_\psi - \theta\big) := \mathrm{arginf}_{t\in\mathbb{R}^q}\big\|T_\psi\big(\theta + n^{-1/2}t\big)\big\|^2.$$

From van Eeden (1972), we deduce that $\eta' T_\psi(\theta + n^{-1/2}\eta s)$ is monotonic in $s \in \mathbb{R}$, $\forall\,\|\eta\| = 1$, $\forall\,\psi \in \Omega_s$, so that Eq. 3.5 is satisfied here and the estimator $\widehat\theta_\psi$ satisfies Eq. 3.4 with $\Sigma_\theta \equiv \Sigma = E\big(h(Z)h(Z)'\big)$.

4 Errors in Variables Linear Regression

In this section we use the above results to derive the limiting distributions of a class of signed rank estimators of the regression parameter in an errors in variables (EIVs) linear regression model and discuss their asymptotic relative efficiencies compared to the bias corrected least square estimator.

Consider the EIVs linear regression model where the response Y, unobservable predictor X and its observable cohort Z obey the following relations for some $\theta \in \mathbb{R}^p$.

$$Y = \theta' X + \varepsilon, \quad Z = X + U, \quad X, \varepsilon, U \text{ mutually independent and } E(U) = 0.$$

The importance of this model in environmental, health and social sciences is well illustrated in the monographs of Carroll et al. (2006), Cheng and Van Ness (1999), Fuller (1987) and Yi (2017).

Here we shall additionally assume that $E\big(\|X\|^2 + \|U\|^2\big) < \infty$ and the distributions of X and U are known. In the absence of any additional information, for model identifiability, one often assumes that some characteristics of the distribution of U are known. Here, to illustrate the robustness of signed rank estimators against large measurement error, we assume the full knowledge of the distributions of X and U.

Then $h(z) := z - E(U|Z = z)$, $z \in \mathbb{R}^p$ is a known function of z, $E(\|h(Z)\|^2) < \infty$, and the above model becomes the parametric regression model

$$Y = \theta' h(Z) + e, \qquad e = \varepsilon - \theta' V, \qquad V := U - E(U|Z). \tag{4.1}$$

Let $Y_i, X_i, \varepsilon_i, U_i$, $1 \le i \le n$ be iid copies of Y, X, ε, U obeying the model Eq. 4.1.

Let F denote the d.f. of ε and assume the following:

$$F \text{ is symmetric around 0 and has uniformly continuous density } f, f > 0 \text{a.e.} \tag{4.2}$$

$$V \text{ is independent of } Z \text{ and , with } H \text{ denoting the d.f. of } V, \tag{4.3}$$
$$dH(-v) = -dH(v),\ \forall\, v \in \mathbb{R}^p, \text{ i.e., } V \text{ is symmetric around the origin.}$$

Note that we do not assume the finiteness of any moment of ε.

Let K, κ denote the d.f. and density of the e of Eq. 4.1, respectively. By the independence of ε and V and by Eqs. 4.2 and 4.3,

$$K(x) \equiv \int F(x + \theta' v) dH(v) \text{ is symmetric around 0 and } \kappa(x) \equiv \int f(x + \theta' v) dH(v). \tag{4.4}$$

The EIVs model Eq. 4.1 with the above assumptions is an example of the linear regression model of Remark 3.1. And by Eq. 3.1, the corresponding signed rank statistics T_ψ of Eq. 3.6 satisfy $E\big(T_\psi(\theta)\big) = 0$. Hence, unlike the R estimators studied in Koul (2022) or the least squares estimator, there is no need to do any bias correction for defining the signed rank estimators $\widehat{\theta}_\psi$ in the above EIV's model. Thus, Eq. 3.4 holds for this $\widehat{\theta}_\psi$ with $\Sigma_\theta \equiv \Sigma := E\big\{\big(Z - E(U|Z)\big)\big(Z - E(U|Z)\big)'\big\}$.

Example 4.1 We shall discuss an example of linear EIV regression model where the distributions of X, ε, U are such that the assumption Eq. 4.3 of the independence between Z and V is satisfied. We shall also show that in this case, if the regression errors are Gaussian then the asymptotic relative efficiency (ARE) of a class of signed rank estimators relative to the bias corrected least squares (BCLS) estimator tends to infinity as the measurement error variance tends to infinity, monotonically in some cases.

Suppose $p = 1$, $X \sim_D \mathcal{N}(\mu_X, \sigma_X^2)$, $U \sim_D \mathcal{N}(0, \sigma_U^2)$, with $\mu_X, \sigma_X^2 > 0$ and $\sigma_U^2 \ge 0$ known, and X and U are independent r.v.'s. Then $\mu_Z := E(Z) = \mu_X$,

$Z = X + U \sim_D \mathcal{N}(\mu_X, \sigma_X^2 + \sigma_U^2)$, $\mathrm{Cov}(Z, U) = \sigma_U^2$ so that

$$\binom{Z}{U} \sim_D \mathcal{N}_2\left(\binom{\mu_Z}{0}, \begin{pmatrix} \sigma_X^2 + \sigma_U^2 & \sigma_U^2 \\ \sigma_U^2 & \sigma_U^2 \end{pmatrix}\right).$$

Let $r^2 := \sigma_U^2/(\sigma_X^2 + \sigma_U^2)$. The conditional distribution of U, given $Z = z$, is $\mathcal{N}((z - \mu_Z)r^2, r^2\sigma_X^2)$. Hence, $E(U|Z = z) = (z - \mu_Z)r^2$ and $h(z) = z - E(U|Z = z) = z - r^2(z - \mu_Z)$. For any $z \in \mathbb{R}$, the conditional distribution of $V = U - E(U|Z = z) = U - (z - \mu_Z)r^2$, given $Z = z$, is $\mathcal{N}(0, r^2\sigma_X^2)$, which does not depend on z, so that V is independent of Z and symmetrically distributed around zero. Hence, $e = \varepsilon - \theta' V$ is also independent of Z. Its density is $\kappa(x) = \int f(x + \theta v) d\Phi\big(v/r\sigma_X\big)$, where Φ is the d.f. of a $\mathcal{N}(0, 1)$ r.v. Moreover, because $0 \le r^2 < 1$ and $\sigma_X^2 > 0$,

$$\Sigma = \sigma_h^2 := Eh^2(Z) = (1 - r^2)^2(\sigma_X^2 + \sigma_U^2) + \mu_Z^2 \ge (1 - r^2)^2(\sigma_X^2 + \sigma_U^2) > 0.$$

To summarize: Suppose F satisfied Eq. 4.2 and (X, U) satisfy the above normality assumption. Then, $0 < \sigma_h^2 := Eh^2(Z) < \infty$ and $n^{1/2}\big(\widehat{\theta}_\psi - \theta\big) \to_D \mathcal{N}\Big(0, \tau_\psi^2/\big(\sigma_h^2 \mathcal{I}^2(\psi)\big)\Big)$.

Now suppose further that $F(y) \equiv \Phi(y/\sigma_\varepsilon)$, for some $\sigma_\varepsilon > 0$. Let $w^2 := \sigma_\varepsilon^2 + \theta^2 r^2 \sigma_X^2$. Then, $e = \varepsilon - \theta V \sim_D \mathcal{N}(0, w^2)$, i.e., $K(x) \equiv \Phi(x/w)$.

Take $\psi(u) \equiv 2u - 1$ and write $\widehat{\theta}_I$ for the corresponding $\widehat{\theta}_\psi$. Then, $\tau_\psi^2 = \int_0^1 (2u - 1)^2 du = 1/3$ and $\mathcal{I}(\psi) = 1/w\sqrt{\pi}$, $\tau_\psi^2/\mathcal{I}^2(\psi) = \pi w^2/3$.

Next, consider the BCLS estimator of θ defined as follows. For any set of bivariate r.v.'s $(\xi_i, \zeta_i), 1 \le i \le n$, let $\bar{\xi} := n^{-1}\sum_i \xi_i$ and $S_{\xi\zeta} := n^{-1}\sum_i (\xi_i - \bar{\xi})(\zeta_i - \bar{\zeta})$. Then, $\widehat{\theta}_{bcls} := S_{ZY}/\big(S_{ZZ} - \sigma_u^2\big)$, $\widehat{\theta}_{bcls} - \theta = \big(S_{ZY} - \theta S_{ZZ} + \theta\sigma_U^2\big)/\big(S_{ZZ} - \sigma_u^2\big)$.

Theorem 2.2.1 (page 108) of Fuller (1987) applied with $p = 1$ yields that $n^{1/2}\big(\widehat{\theta}_{bcls} - \theta\big) \to_D \mathcal{N}(0, \sigma_{bc\ell s}^2)$, where

$$\sigma_{bc\ell s}^2 := \frac{w^2(\sigma_X^2 + \sigma_U^2) + 2\theta^2\sigma_U^4}{\sigma_X^4}.$$

Therefore,

$$\mathrm{ARE}(\widehat{\theta}_I, \widehat{\theta}_{bc\ell s}) = \frac{3}{\pi}\frac{\sigma_h^2\sigma_{bc\ell s}^2}{w^2}.$$

Use $r^2 = \sigma_U^2/(\sigma_X^2+\sigma_U^2)$ to rewrite $\sigma_h^2 = (1-r^2)^2(\sigma_X^2+\sigma_U^2)+\mu_Z^2 = \Big(\frac{\sigma_X^4}{\sigma_X^2+\sigma_U^2}+\mu_Z^2\Big)$ and

$$\begin{aligned}\frac{\sigma_h^2\sigma_{bc\ell s}^2}{w^2} &= \Big\{\frac{\sigma_X^4}{(\sigma_X^2+\sigma_U^2)}+\mu_Z^2\Big\}\Big\{\frac{\sigma_X^2+\sigma_U^2}{\sigma_X^4}+2\theta^2\frac{\sigma_U^4}{\sigma_X^4 w^2}\Big\}\\ &= 1+2\theta^2\frac{\sigma_U^4}{(\sigma_X^2+\sigma_U^2)w^2}+\mu_Z^2\Big\{\frac{\sigma_X^2+\sigma_U^2}{\sigma_X^4}+2\theta^2\frac{\sigma_U^4}{\sigma_X^4 w^2}\Big\}.\end{aligned}$$

Use $w^2 = \sigma_\varepsilon^2 + \theta^2\{\sigma_U^2/(\sigma_X^2+\sigma_U^2)\}\sigma_X^2$ to rewrite

$$\frac{\sigma_U^4}{(\sigma_X^2+\sigma_U^2)w^2} = \frac{\sigma_U^2}{\sigma_\varepsilon^2(\frac{\sigma_X^2}{\sigma_U^2}+1)+\theta^2\sigma_X^2}, \qquad \frac{\sigma_U^4}{w^2} = \frac{\sigma_U^2(\sigma_X^2+\sigma_U^2)}{\frac{\sigma_\varepsilon^2\sigma_X^2}{\sigma_U^2}+\sigma_\varepsilon^2+\theta^2\sigma_X^2}.$$

Both of these expressions are increasing in σ_U^2. Both tend to 0 and ∞, as σ_U^2 tends to 0 and ∞, monotonically and respectively. Moreover, $(\sigma_X^2+\sigma_U^2)/\sigma_X^4 \to 1/\sigma_X^2, \infty$, as $\sigma_U^2 \to 0, \infty$, monotonically and respectively. Hence, using the fact $\mu_Z = \mu_X$,

$$\text{ARE}(\widehat{\theta}_I, \widehat{\theta}_{bc\ell s}) = \frac{3}{\pi}\frac{\sigma_h^2\sigma_{bc\ell s}^2}{w^2}\begin{cases}\downarrow \frac{3\mu_X^2}{\pi\sigma_X^2}, & \text{as } \sigma_U^2 \downarrow 0,\\ \uparrow \infty, & \text{as } \sigma_U^2 \uparrow \infty.\end{cases}$$

Similarly, using the fact that here $4\kappa^2(0) = 2/(\pi\omega^2)$, we obtain that

$$\text{ARE}(\widehat{\theta}_{\ell ad}, \widehat{\theta}_{bc\ell s}) = 4\kappa^2(0)\sigma_h^2\sigma_{bc\ell s}^2 = \frac{2}{\pi}\frac{\sigma_h^2\sigma_{bc\ell s}^2}{w^2}\begin{cases}\downarrow \frac{2\mu_X^2}{\pi\sigma_X^2}, & \text{as } \sigma_U^2 \downarrow 0,\\ \uparrow \infty, & \text{as } \sigma_U^2 \uparrow \infty.\end{cases}$$

In other words, the estimators $\widehat{\theta}_I$, $\widehat{\theta}_{\ell ad}$ are asymptotically far more efficient, compared to the BCLS estimator, against the increasing measurement error, at the above chosen Gaussian distributions of ε, X, U. One may say that for the large samples, these signed rank estimators are robust against the large measurement errors at the given Gaussian distributions, a highly desirable property to have from a practical point of view.

Let $\dot{\Omega}_s := \{\psi \in \Omega_s, \psi$ is differentiable having continuous derivative $\dot{\psi}$ on $[0,1]\}$. Then, more generally, $\text{ARE}(\widehat{\theta}_\psi, \widehat{\theta}_{bc\ell s}) \to \infty$, as $\sigma_U^2 \to \infty$, for all

$\psi \in \dot{\Omega}_s$. To see this, rewrite

$$\mathrm{ARE}\big(\widehat{\theta}_\psi, \widehat{\theta}_{bc\ell s}\big) = \frac{\sigma^2_{bc\ell s}\sigma^2_h \mathcal{I}^2(\psi)}{\tau^2_\psi}.$$

Let ϕ denote the density of Φ and $\delta^2 := \sigma^2_\varepsilon + \theta^2\sigma^2_X$. Note that $w^2 = \sigma^2_\varepsilon + \theta^2\sigma^2_X\sigma^2_U/(\sigma^2_X + \sigma^2_U) \to \sigma^2_\varepsilon, \delta^2$, as $\sigma^2_U \to 0, \infty$, respectively. This fact together with the continuity of $\phi, \Phi, \dot{\psi}$ and the Dominated Convergence Theorem yield that

$$\begin{aligned}\mathcal{I}_w(\psi) := \mathcal{I}(\psi) = \frac{1}{w}\int_{-\infty}^{\infty}\phi\big(\frac{x}{w}\big)d\psi\big(\Phi\big(\frac{x}{w}\big)\big) = \frac{1}{w^2}\int_{-\infty}^{\infty}\phi^2\big(\frac{x}{w}\big)\dot{\psi}\big(\Phi\big(\frac{x}{w}\big)\big)dx \\ \to 0 < \mathcal{I}_{\sigma^2_X}(\psi) < \infty, \text{as } \sigma^2_U \to 0, \\ \to 0 < \mathcal{I}_{\delta^2}(\psi) < \infty, \text{as } \sigma^2_U \to \infty.\end{aligned}$$

Moreover,

$$\sigma^2_{bc\ell s}\sigma^2_h := \frac{w^2(\sigma^2_X + \sigma^2_U) + 2\theta^2\sigma^4_U}{\sigma^4_X}\Big[(1-r^2)^2(\sigma^2_X + \sigma^2_U) + \mu^2_Z\Big] \to \begin{cases}\mu^2_X, & \text{as } \sigma^2_U \to 0, \\ \infty, & \text{as } \sigma^2_U \to \infty.\end{cases}$$

Therefore, $\mathrm{ARE}\big(\widehat{\theta}_\psi, \widehat{\theta}_{bc\ell s}\big) \to \infty$, as $\sigma^2_U \to \infty$.

Of course an advantage of the BCLS estimator is that its definition needs only σ^2_U to be known and no other knowledge of the distributions of X and U.

It is interesting to note that the $\mathrm{ARE}(\widehat{\theta}_I, \widehat{\theta}_{\ell ad})$ is the same as in the case of no measurement error, because $\mathrm{ARE}\big(\widehat{\theta}_I, \widehat{\theta}_{\ell ad}\big) = \big(\pi w^2/2\sigma^2_h\big)/\big(\pi w^2/3\sigma^2_h\big) = 3/2$. More generally, for any $\psi_1, \psi_2 \in \dot{\Omega}_s$,

$$\mathrm{ARE}\big(\widehat{\theta}_{\psi_1}, \widehat{\theta}_{\psi_2}\big) = \frac{\tau^2_{\psi_2}\mathcal{I}(\psi_1)}{\tau^2_{\psi_1}\mathcal{I}(\psi_2)} \to \begin{cases}\tau^2_{\psi_2}\mathcal{I}_{\sigma^2_X}(\psi_1)/\tau^2_{\psi_1}\mathcal{I}_{\sigma^2_X}(\psi_2), \text{as } \sigma^2_U \to 0, \\ \tau^2_{\psi_2}\mathcal{I}_{\delta^2}(\psi_1)/\tau^2_{\psi_1}\mathcal{I}_{\delta^2}(\psi_2), \text{as } \sigma^2_U \to \infty.\end{cases}$$

We shall now describe the above ARE's in the case $p > 1$ briefly. Let X, U be independent random vectors with $X \sim_D \mathcal{N}_p(\mu_X, \Sigma_X)$, $U \sim_D \mathcal{N}_p(0, \Sigma_U)$, where Σ_X, Σ_U are known $p \times p$ positive definite matrices. Then, $Z = X +$

$U \sim_D \mathcal{N}_p(\mu_Z, \Sigma_X + \Sigma_U)$, $\mathrm{Cov}(Z, U) = \Sigma_U$ and because $\mu_X = \mu_Z$,

$$(Z, U) \sim_D \mathcal{N}_{2p}\left(\begin{pmatrix}\mu_Z \\ 0\end{pmatrix}, \begin{pmatrix}\Sigma_X + \Sigma_U & \Sigma_U \\ \Sigma_U & \Sigma_U\end{pmatrix}\right).$$

Hence, the conditional distribution of U, given $Z = z$, is

$$\mathcal{N}_p\Big(\Sigma_U(\Sigma_X + \Sigma_U)^{-1}(z - \mu_Z), \, \Sigma_U - \Sigma_U(\Sigma_X + \Sigma_U)^{-1}\Sigma_U\Big).$$

Let $R := \Sigma_U(\Sigma_X + \Sigma_U)^{-1}$. Then $E(U|Z = z) = R(z - \mu_Z)$ and the conditional distribution of $V = U - R(Z - \mu_Z)$, given Z, is $\mathcal{N}_p(0, \Sigma_U - R\Sigma_U)$, which again does not depend on Z and hence V is independent of Z. Note that $\Sigma_U - R\Sigma_U = \big[I - \Sigma_U(\Sigma_X + \Sigma_U)^{-1}\big]\Sigma_U = \Sigma_X(\Sigma_X + \Sigma_U)^{-1}\Sigma_U$.

Moreover, with $Z_c := Z - \mu_Z$, $E(Z_c Z_c') = \Sigma_X + \Sigma_U$, $h(Z) = Z - E(U|Z) = (I - R)Z_c + \mu_Z$ and $\Sigma = E(h(Z)h(Z)') = (I - R)(\Sigma_X + \Sigma_U)(I - R)' + \mu_Z\mu_Z'$. Since $\varepsilon \sim \mathcal{N}(0, \sigma_\varepsilon^2)$, then $\zeta = \varepsilon - \theta' V \sim \mathcal{N}\big(0, \omega^2\big)$, where now $\omega^2 := \sigma_\varepsilon^2 + \theta'\Sigma_X(\Sigma_X + \Sigma_U)^{-1}\Sigma_U\theta$. Hence, $\sqrt{n}(\widehat{\theta}_I - \theta) \to_D \mathcal{N}\big(0, (\pi\omega^2/3)\Sigma^{-1}\big)$.

Here the BCLS estimator is $\widehat{\theta}_{bc\ell s} = (\hat{\Sigma}_Z - \Sigma_U)^{-1}\hat{\Sigma}_{YZ}$,where $\hat{\Sigma}_Z := n^{-1}\sum_{i=1}^n (Z_i - \bar{Z})(Z_i - \bar{Z})', \hat{\Sigma}_{YZ} := n^{-1}\sum_{i=1}^n (Z_i - \bar{Z})(Y_i - \bar{Y})$. By Theorem 2.2.1 (page 108) of Fuller (1987),$\sqrt{n}(\widetilde{\theta}_{bc\ell s} - \theta) \to_D \mathcal{N}(0, \mathcal{D})$, where $\mathcal{D} = \Sigma_X^{-1}\alpha + \Sigma_X^{-1}[\Sigma_U\alpha + \Sigma_U\theta\theta'\Sigma_U]\Sigma_X^{-1}, \quad \alpha = \sigma_\varepsilon^2 + \theta'\Sigma_U\theta$.

Suppose $\hat{\theta}_j, j = 1, 2$ are the two estimators of a p-dimensional parameter θ such that $n^{1/2}(\hat{\theta}_j - \theta) \to_D \mathcal{N}_p(0, \Sigma_j), j = 1, 2$. Let $|\Sigma_j|$ denote the determinant of Σ_j, $j = 1, 2$. Suppose $|\Sigma_j| \neq 0, j = 1, 2$. Then $\mathrm{ARE}(\hat{\theta}_2, \hat{\theta}_1) = (|\Sigma_1|/|\Sigma_2|)^{1/p}$. See, e.g., Lehmann (1999). Using this definition, from the above facts we readily obtain the following:

$$\mathrm{ARE}(\widetilde{\theta}_I, \widetilde{\theta}_{bc\ell s}) = (3/(\omega^2\pi))^{1/p}(|\mathcal{D}|/|\Sigma^{-1}|)^{1/p} = (3/(\omega^2\pi))^{1/p}\Big\{|\mathcal{D}| \times |\Sigma|\Big\}^{1/p}.$$

Suppose Σ_X and Σ_U are diagonal matrices with all positive entries. One can verify that as the diagonal elements of Σ_U tend to infinity, $\alpha \to \infty$, $\omega^2 \to \sigma_\varepsilon^2 + \theta'\Sigma_X\theta > 0$ and $\mathrm{ARE}(\widetilde{\theta}_I, \widetilde{\theta}_{bc\ell s}) \to \infty$.

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Declarations

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