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AN ELEMENTARY PROOF OF THE GENERALIZED ITÔ FORMULA

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ABSTRACT. For one-dimensional semimartingales, the Itô formula can be extended from C^2-functions to C^1-functions with a locally absolutely continuous derivative. We propose a new, different proof of this result, which is simple, straightforward and quite elementary, avoiding in particular the extensive theory of local times for semimartingales.

1. Introduction

Let X be a real-valued continuous semimartingale and $f \in C^2(\mathbb{R})$. Then $f(X)$ is again a semimartingale and the Itô formula provides its explicit decomposition into a local martingale and a continuous process of locally bounded variation. The assumption that f is twice continuously differentiable can be relaxed, in particular, the Itô formula holds without any change for functions $f \in C^1(\mathbb{R})$ with a locally absolutely continuous derivative, although the second derivative f'' is, in general, defined only almost everywhere on $\mathbb{R}$ and is merely locally integrable. This result finds traditionally its application in the proof of the Feller test for non-explosion without unnecessary continuity hypotheses on the drift and diffusion coefficients; recently it has been applied e.g. in the study of the stochastic Camassa-Holm equation (see Remark 1.3 below for a more detailed discussion). Surprisingly, it seems difficult to find this version of the Itô formula explicitly stated: we know only about the proof in the third printing of the second edition of Protter's book [6] and as an exercise it appears in the textbook [3]. In both cases, it is derived as a consequence of the Meyer-Itô formula for δ-convex functions f, hence it depends on the rather heavy machinery of local times for semimartingales. In this paper we propose a direct proof that presupposes only a basic knowledge of stochastic analysis. (In Remark 1.2 below we provide a comparison of our approach with the standard one.)

It is worth mentioning that the generalized Itô formula is established in [6] in a more general setting of càdlàg (i.e., right-continuous with left-limits) semimartingales. We consider the general result as well and discuss the minor (and easy) changes that must be done in our proof.

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Let us fix some notation. By $\mathscr{B}$ we denote the Borel σ-algebra on $\mathbb{R}$ and by λ the Lebesgue measure on $\mathscr{B}$. Let $I \subseteq \mathbb{R}$ be an open interval, we set

$$AC^1(I) = \{f \in C^1(I);\ f' \text{ absolutely continuous on } \bar{I}\},$$
$$AC^1_{\mathrm{loc}}(\mathbb{R}) = \{f \in C^1(\mathbb{R});\ f \in AC^1(I) \text{ for any bounded open interval } I \subseteq \mathbb{R}\}.$$

Recall that if $f \in AC^1_{\mathrm{loc}}(\mathbb{R})$ then the second derivative $f''(s)$ exists at almost every point $s \in \mathbb{R}$, $f'' \in L^1_{\mathrm{loc}}(\mathbb{R})$ and f' is an absolutely continuous antiderivative of f''.

If $f\colon I \longrightarrow \mathbb{R}$ is a continuous nondecreasing function, we shall occasionally denote by μ_f the Borel measure on I whose distribution function is f. By $\mathbb{R} \otimes \mathbb{R}^n$ we denote the space of all $1 \times n$ matrices over $\mathbb{R}$.

We aim at proving the following generalized Itô formula.

Proposition 1.1. *Let X be a real-valued continuous semimartingale defined on a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t), \boldsymbol{P})$ with a normal filtration. Let $f \in AC^1_{\mathrm{loc}}(\mathbb{R})$ and let $g\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a Borel function satisfying $g = f''$ λ-almost everywhere on $\mathbb{R}$. Then*

$$f(X_t) - f(X_0) = \int_0^t f'(X_s)\,\mathrm{d}X_s + \frac{1}{2}\int_0^t g(X_s)\,\mathrm{d}\langle X\rangle_s \tag{1.1}$$

for any $t \geq 0$ $\boldsymbol{P}$-almost surely.

In the course of the proof, we check that the second term on the right-hand side of (1.1) is well defined; this fact deserves being stated as a separate corollary.

Corollary 1.2. *Let X be a real-valued continuous semimartingale defined on a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t), \boldsymbol{P})$ with a normal filtration. Then*

$$\boldsymbol{P}\left\{\int_0^t |h(X_s)|\,\mathrm{d}\langle X\rangle_s < \infty \quad \text{for any } t \geq 0\right\} = 1 \tag{1.2}$$

whenever $h \in L^1_{\mathrm{loc}}(\mathbb{R})$.

In particular, if W is a one-dimensional Wiener process, then

$$\boldsymbol{P}\left\{\int_0^t h(W_s)\,\mathrm{d}s < \infty \quad \text{for any } t \geq 0\right\} = 1$$

for all nonnegative locally integrable Borel functions $h\colon \mathbb{R} \longrightarrow \mathbb{R}_+$, thus, as a byproduct, we get one implication in the Engelbert-Schmidt 0-1 law (see, e.g., [3: Proposition 3.6.27]).

Finally, let us turn to the extension of Proposition 1.1 to càdlàg semimartingales.

Proposition 1.3. *Let X be a real-valued càdlàg semimartingale defined on a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t), \boldsymbol{P})$ with a normal filtration. Let $f \in AC^1_{\mathrm{loc}}(\mathbb{R})$ and let $g\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a Borel function satisfying $g = f''$ λ-almost everywhere on $\mathbb{R}$. Then*

$$f(X_t) - f(X_0) = \int_0^t f'(X_{s-})\,\mathrm{d}X_s + \frac{1}{2}\int_0^t g(X_{s-})\,\mathrm{d}[X]^c_s + \sum_{s\in(0,t]}\bigl[f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s\bigr] \tag{1.3}$$

for any $t \geq 0$ $\boldsymbol{P}$-almost surely.

Recall that by $[X]^c$ the continuous part of the quadratic variation $[X]$ of the semimartingale X is denoted and $\Delta X_s = X_s - X_{s-}$. Again, as a consequence of Proposition 1.3, we get that for any real-valued càdlàg semimartingale X and $h \in AC^1_{\mathrm{loc}}(\mathbb{R})$, the sum

$$\sum_{s\in(0,t]} \big[h(X_s) - h(X_{s-}) - h'(X_{s-})\Delta X_s\big]$$

converges absolutely.

Remark 1.1.

(i) Clearly, f' is an antiderivative of g.

(ii) Let $X = X_0 + A + M$ be the canonical decomposition of the semimartingale X into a continuous process A of a locally bounded variation and a continuous local martingale M, $A_0 = M_0 = 0$. Then $\langle X\rangle = \langle M\rangle$ and, by definition,

$$\int_0^{\cdot} f'(X_s)\,\mathrm{d}X_s = \int_0^{\cdot} f'(X_s)\,\mathrm{d}A_s + \int_0^{\cdot} f'(X_s)\,\mathrm{d}M_s.$$

Both integrals on the right-hand side are well defined. Indeed, let $\omega \in \Omega$ be such that the trajectories $X(\cdot,\omega)$, $A(\cdot,\omega)$ and $\langle M\rangle(\cdot,\omega)$ are continuous and $A(\cdot,\omega)$ has bounded variation on $[0,t]$. Denote by $\widetilde{A}(u,\omega)$ the variation of $A(\cdot,\omega)$ on the interval $[0,u]$, $u \geq 0$. The set $L = \{X(s,\omega);\ 0 \leq s \leq t\}$ is compact, f' is continuous, in particular locally bounded, so $f' \circ X(\cdot,\omega)$ is bounded on $[0,t]$ and

$$\int_0^t |f'(X_s(\omega))|\,\mathrm{d}\tilde{A}_s(\omega) + \int_0^t |f'(X_s(\omega))|^2\,\mathrm{d}\langle M\rangle_s(\omega) < \infty.$$

This is, of course, well known, however, we shall use this argument so often that we decided to state it explicitly. (See e.g. [3: § 3.2] or [6: Chapter II] for the very basic facts about stochastic integrals we use in this paper.)

(iii) Proceeding in a completely analogous manner we can check that (1.2) is satisfied whenever h is, in addition, locally bounded. Hence Corollary 1.2 is non-trivial only for functions h which are locally integrable but not locally bounded.

(iv) Let $N \in \mathscr{B}$, $\lambda(N) = 0$. Using Proposition 1.1 with the choice $f = 0$, $g = \mathbf{1}_N$, we arrive at

$$\int_0^t \mathbf{1}_N(X_s)\,\mathrm{d}\langle X\rangle_s = 0 \quad \boldsymbol{P}\text{-almost surely,} \tag{1.4}$$

in particular,

$$\boldsymbol{P}\big\{\omega \in \Omega;\ (1_N \circ X)(\cdot,\omega) = 0\ \mu_{\langle X\rangle(\omega)}\text{-almost everywhere on } [0,t]\big\} = 1.$$

(v) Let $\tilde{g}\colon \mathbb{R} \longrightarrow \mathbb{R}$ be another Borel function satisfying $\tilde{g} = f''$ λ-almost everywhere on $\mathbb{R}$. Set $M = \{\tilde{g} \neq g\}$, then $\lambda(M) = 0$ and by (1.4) we know that $(\tilde{g}\mathbf{1}_M) \circ X = 0$ $\mu_{\langle X\rangle}$-almost everywhere on $[0,t]$ $\boldsymbol{P}$-almost surely, whence

$$\int_0^t |\tilde{g}(X_s)|\,\mathrm{d}\langle X\rangle_s \leq \int_0^t |(\tilde{g}\mathbf{1}_{\mathbb{R}\setminus M})(X_s)|\,\mathrm{d}\langle X\rangle_s + \int_0^t |(\tilde{g}\mathbf{1}_M)(X_s)|\,\mathrm{d}\langle X\rangle_s \leq \int_0^t |g(X_s)|\,\mathrm{d}\langle X\rangle_s < \infty$$

$\boldsymbol{P}$-almost surely; similarly we obtain

$$\int_0^t \tilde{g}(X_s)\,\mathrm{d}\langle X\rangle_s = \int_0^t g(X_s)\,\mathrm{d}\langle X\rangle_s \quad \boldsymbol{P}\text{-almost surely.}$$

Therefore, in Proposition 1.1 we may replace g with $\tilde{g}$. In other words, Proposition 1.1 does not depend on a particular choice of a Borel function g as far as g satisfies $g = f''$ λ-almost everywhere on $\mathbb{R}$.

(vi) The assumption $f \in AC^1_{\mathrm{loc}}(\mathbb{R})$ is satisfied if f belongs to the Sobolev space $W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$ or, more generally, if the function $f \in C^1(\mathbb{R})$ has a locally Lipschitz continuous derivative.

(vii) Suppose that the assumptions of Proposition 1.1 are satisfied and, moreover, X is an Itô process. That is, there exist an n-dimensional $(\mathscr{F}_t)$-Wiener process W and $(\mathscr{F}_t)$-progressively measurable processes a and σ such that $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, $\sigma \in L^2_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{R}\otimes\mathbb{R}^n)$ $\boldsymbol{P}$-almost surely and

$$X = X_0 + \int_0^{\cdot} a(s)\,\mathrm{d}s + \int_0^{\cdot} \sigma(s)\,\mathrm{d}W_s \quad \boldsymbol{P}\text{-almost surely.}$$

Then

$$f(X_t) = f(X_0) = \int_0^t \Big\{ f'(X_s)a(s) + \frac{1}{2} g(X_s)\|\sigma(s)\|^2 \Big\}\,\mathrm{d}s + \int_0^t f'(X_s)\sigma(s)\,\mathrm{d}W_s \tag{1.5}$$

$\boldsymbol{P}$-almost surely.

(viii) Another generalized Itô formula for Itô processes was proposed by N. V. Krylov, see [4: § II.10]. It is a very useful result which holds for $\mathbb{R}^d$-valued processes as well. However, for $d = 1$ it is weaker than Proposition 1.1 in the form (1.5) as, roughly speaking, one has to assume also that $f'' \in L^2_{\mathrm{loc}}(\mathbb{R})$ and, $\boldsymbol{P}$-almost surely, a and $\|\sigma\|$ are bounded and $\|\sigma\|^2 > 0$ on $[0, t]$.

Remark 1.2. Let us compare briefly our proof of Proposition 1.1 with the standard one (see [6: Theorem IV.71] or [3: Problem 3.7.3 and a hint on p. 236]). If $f \in AC^1_{\mathrm{loc}}(\mathbb{R})$, then f is δ-convex and the Radon measure $f''\lambda$ is its second derivative in the sense of distributions. Let g be as in Proposition 1.1 and X a continuous (real-valued) semimartingale, let us denote by $L(X) = (L^a_s(X),\, a \in \mathbb{R}, s \geq 0)$ its local time. By the Meyer-Itô formula

$$f(X_t) - f(X_0) = \int_0^t f'(X_s)\,\mathrm{d}X_s + \frac{1}{2}\int_{-\infty}^{\infty} L^a_t(X) g(a)\,\mathrm{d}a \tag{1.6}$$

$\boldsymbol{P}$-almost surely. Properties of the local time $L(X)$ imply

$$\int_{-\infty}^{\infty} g(a) L^a_t(X)\,\mathrm{d}a = \int_0^t g(X_s)\,\mathrm{d}\langle X\rangle_s \tag{1.7}$$

by [6: Corollary 1 to Theorem IV.70] or [3: Theorem 3.7.1(iv)]. Applying (1.7) we see that (1.6) implies (1.1). (Note that (1.7) is proved in [6] only for bounded functions g; in [3], the equality (1.7) is stated for nonnegative functions g but no argument why the integrals are finite is provided. However, it is easy to fill the gaps once we take into account that $L^{\bullet}_t(X)$ has a compact support $\boldsymbol{P}$-almost surely).

The proof of the Meyer-Itô formula, however, is much less elementary than the direct proof of Proposition 1.1 we propose in this paper.

Remark 1.3.

(a) In our view, a basic application of Proposition 1.1 is in Feller's theory of one-dimensional diffusions when an approach via stochastic differential equations is adopted (see e.g. [3: § 5.5C] for a brief introduction to the topic). There one needs to apply the Itô formula to (Carathéodory) solutions of ordinary differential equations $Lu = 0$, $Lu = \pm 1$ and $Lu = u$ (with suitable initial or boundary conditions) where L is the Kolmogorov operator associated with a stochastic differential equation

$$\mathrm{d}X = b(X)\,\mathrm{d}X + \sigma(X)\,\mathrm{d}W$$

and $b, \sigma\colon \mathbb{R} \longrightarrow \mathbb{R}$ are Borel functions such that

$$\frac{1 + |b|}{\sigma^2} \in L^1_{\mathrm{loc}}(\mathbb{R}).$$

These solutions are in $AC^1_{\mathrm{loc}}(\mathbb{R})$ but they belong to $C^2(\mathbb{R})$ only under an additional assumption that $b, \sigma \in C(\mathbb{R})$ and $\sigma^2 > 0$ on $\mathbb{R}$. This can be seen easily if explicit solutions are available as in the case of the scale function p solving $Lp = 0$, since

$$p\colon z \longmapsto \int_a^z \exp\Big(-\int_a^y \frac{2b(r)}{\sigma^2(r)}\,\mathrm{d}r\Big)\,\mathrm{d}y$$

for some $a \in \mathbb{R}$.

(b) In the paper [1], the generalized Itô formula is applied many times to functions from the space $W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$, either to functions of the type $x \mapsto x(|x| + 1)^\alpha$ with $\alpha \in (0, 1)$ (see e.g. [1: Proposition 3.2]), or to various cut-offs of unbounded smooth functions, see e.g. [1: formula (4.1)] for a typical choice.

(c) Lyapunov function like $V_p = |\cdot|^p$ with $p < 2$ are used in nonexplosion and stability criteria for stochastic differential equations, see e.g. [2: § V.5] or [5: § 4.1]. If $p \in (1, 2)$ then $V_p \in AC^1_{\mathrm{loc}}(\mathbb{R}) \setminus C^2(\mathbb{R})$.

We shall provide a detailed proof of Proposition 1.1 and Corollary 1.2 in Section 2 and a sketch of the proof of Proposition 1.3 in Section 3.

2. Proof of Proposition 1.1

P r o o f. Clearly, we may fix a $t > 0$.

Step 1. Let $V\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a locally bounded Borel function, we define

$$\mathfrak{e}V\colon \mathbb{R} \longrightarrow \mathbb{R},\ s \longmapsto \int_0^s V\,\mathrm{d}\lambda, \qquad \mathfrak{e}_2 V = \mathfrak{e}(\mathfrak{e}V).$$

(Henceforward, we use the convention $\int_0^s \cdots\,\mathrm{d}\lambda = -\int_s^0 \cdots\,\mathrm{d}\lambda$ for $s < 0$.) Obviously, $\mathfrak{e}V$ is locally absolutely continuous and $\mathfrak{e}_2 V \in AC^1_{\mathrm{loc}}(\mathbb{R})$. Let us denote by $\mathscr{I}$ the set of all locally bounded

Borel function $V\colon \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$\mathfrak{e}_2 V(X_t) - \mathfrak{e}_2 V(X_0) = \int_0^t \mathfrak{e}V(X_s)\,\mathrm{d}X_s + \frac{1}{2}\int_0^t V(X_s)\,\mathrm{d}\langle X\rangle_s$$

$\boldsymbol{P}$-almost surely. Plainly, $\mathscr{I}$ is a vector space and $C(\mathbb{R}) \subseteq \mathscr{I}$ by the classical Itô formula.

Step 2. We claim: if $V_n \in \mathscr{I}$, $n \geq 1$,

$$\sup_{n\geq 1}\sup_{s\in L}|V_n(s)| < \infty \quad \text{for any compact set } L \subseteq \mathbb{R}, \tag{2.1}$$

and $V\colon \mathbb{R} \longrightarrow \mathbb{R}$ is a function such that $V = \lim_{n\to\infty} V_n$ pointwise on $\mathbb{R}$ then $V \in \mathscr{I}$. By the dominated convergence theorem, which may be used owing to the assumption (2.1), we obtain

$$\lim_{n\to\infty} \mathfrak{e}V_n(u) = \mathfrak{e}V(u) \quad \text{and} \quad \lim_{n\to\infty} \mathfrak{e}_2 V_n(u) = \mathfrak{e}_2 V(u) \quad \text{for all } u \in \mathbb{R}, \tag{2.2}$$

moreover, a straightforward computation shows that

$$\sup_{n\geq 1}\sup_{s\in L}|\mathfrak{e}V_n(s)| < \infty \quad \text{and} \quad \sup_{n\geq 1}\sup_{s\in L}|\mathfrak{e}_2 V_n(s)| < \infty \quad \text{for any compact set } L \subseteq \mathbb{R}. \tag{2.3}$$

By (2.2)

$$\lim_{n\to\infty}\Big[\mathfrak{e}_2 V_n(X_t) - \mathfrak{e}_2 V_n(X_0)\Big] = \mathfrak{e}_2 V(X_t) - \mathfrak{e}_2 V(X_0) \quad \boldsymbol{P}\text{-almost surely},$$

and the dominated convergence theorem implies

$$\lim_{n\to\infty}\int_0^t \mathfrak{e}V_n(X_s)\,\mathrm{d}A_s = \int_0^t \mathfrak{e}V(X_s)\,\mathrm{d}A_s \quad \boldsymbol{P}\text{-almost surely},$$

where we used (2.3) with the choice $L = \{X(u,\omega);\ 0 \leq u \leq t\}$ to get an integrable majorant. Similarly,

$$\lim_{n\to\infty}\int_0^t \big|\mathfrak{e}V_n(X_s) - \mathfrak{e}V(X_s)\big|^2\,\mathrm{d}\langle M\rangle_s = 0 \quad \boldsymbol{P}\text{-almost surely},$$

whence

$$\lim_{n\to\infty}\int_0^t \mathfrak{e}V_n(X_s)\,\mathrm{d}M_s = \int_0^t \mathfrak{e}V(X_s)\,\mathrm{d}M_s \quad \text{in } \boldsymbol{P}\text{-probability}.$$

Finally, the same reasoning yields

$$\lim_{n\to\infty}\int_0^t V_n(X_s)\,\mathrm{d}\langle X\rangle_s = \int_0^t V(X_s)\,\mathrm{d}\langle X\rangle_s \quad \boldsymbol{P}\text{-almost surely}$$

and our claim follows.

Step 3. Let $U \subseteq \mathbb{R}$ be an arbitrary open set, then there exist $f_n \in C(\mathbb{R})$ such that $0 \leq f_n \nearrow \mathbf{1}_U$, thus $\mathbf{1}_U \in \mathscr{I}$ by Step 2. Set $\mathscr{A} = \{B \in \mathscr{B};\ \mathbf{1}_B \in \mathscr{I}\}$. To prove that $\mathscr{A} = \mathscr{B}$ it suffices to show that $\mathscr{A}$ is a Dynkin class as $\mathscr{A}$ contains the Euclidean topology which is a π-system. However, if $\Gamma, \Lambda \in \mathscr{A}$, $\Lambda \supseteq \Gamma$ then $\Lambda \setminus \Gamma \in \mathscr{A}$ due to the linear structure of $\mathscr{I}$, and if $\Gamma_n \in \mathscr{A}$, $\Gamma_n \nearrow \Gamma$, then $\mathbf{1}_\Gamma \in \mathscr{I}$ by Step 2. Therefore, all simple Borel functions on $\mathbb{R}$ are in $\mathscr{I}$ and invoking Step 2 twice we can check easily that all bounded Borel functions and then all locally bounded Borel functions are in $\mathscr{I}$.

Step 4. Let f and g satisfy the hypotheses of Proposition 1.1. Then the functions $g_n = g\mathbf{1}_{\{|g|\le n\}}$, $n \ge 1$, are bounded, so we know from Step 3 that

$$\mathfrak{e}_2 g_n(X_t) - \mathfrak{e}_2 g_n(X_0) = \int_0^t \mathfrak{e} g_n(X_s)\,\mathrm{d}X_s + \frac{1}{2}\int_0^t g_n(X_s)\,\mathrm{d}\langle X\rangle_s \tag{2.4}$$

$\boldsymbol{P}$-almost surely. Plainly, we may assume that $g \ge 0$, otherwise we would consider the nonnegative and nonpositive parts of g separately. Since $0 \le g_n \le g$, a simple calculation shows that

$$|\mathfrak{e} g_n| \le |\mathfrak{e} g| \quad \text{and} \quad |\mathfrak{e}_2 g_n| \le |\mathfrak{e}_2 g| \quad \text{on } \mathbb{R}. \tag{2.5}$$

As g is locally integrable, the functions $\mathfrak{e}g$ and $\mathfrak{e}_2 g$ are continuous and (2.5) implies that

$$\sup_{n\ge 1}\sup_{s\in L} |\mathfrak{e} g_n(s)| \le \sup_{s\in L} |\mathfrak{e} g| < \infty$$

and

$$\sup_{n\ge 1}\sup_{s\in L} |\mathfrak{e}_2 g_n(s)| \le \sup_{s\in L} |\mathfrak{e}_2 g| < \infty$$

for every compact set $L \subseteq \mathbb{R}$, moreover,

$$\lim_{n\to\infty} \mathfrak{e} g_n(u) = \mathfrak{e} g(u) \quad \text{and} \quad \lim_{n\to\infty} \mathfrak{e}_2 g_n(u) = \mathfrak{e}_2 g(u) \quad \text{for all } u \in \mathbb{R} \tag{2.6}$$

by the dominated convergence theorem. Therefore, an argument analogous to that employed in Step 2 shows that

$$\lim_{n\to\infty}\Big[\mathfrak{e}_2 g_n(X_t) - \mathfrak{e}_2 g_n(X_0)\Big] = \mathfrak{e}_2 g(X_t) - \mathfrak{e}_2 g(X_0) \quad \boldsymbol{P}\text{-almost surely}$$

and

$$\lim_{n\to\infty}\int_0^t \mathfrak{e} g_n(X_s)\,\mathrm{d}X_s = \int_0^t \mathfrak{e} g(X_s)\,\mathrm{d}X_s \quad \text{in } \boldsymbol{P}\text{-probability}.$$

Assume further that the estimate

$$\int_0^t |g(X_s)|\,\mathrm{d}\langle X\rangle_s < \infty \quad \boldsymbol{P}\text{-almost surely} \tag{2.7}$$

has been already established. Then we may use g as an integrable majorant to get

$$\lim_{n\to\infty}\int_0^t g_n(X_s)\,\mathrm{d}\langle X\rangle_s = \int_0^t g(X_s)\,\mathrm{d}\langle X\rangle_s \quad \boldsymbol{P}\text{-almost surely},$$

hence passing to the limit $n \to \infty$ in (2.4) we arrive at

$$\mathfrak{e}_2 g(X_t) - \mathfrak{e}_2 g(X_0) = \int_0^t \mathfrak{e} g(X_s)\,\mathrm{d}X_s + \frac{1}{2}\int_0^t g(X_s)\,\mathrm{d}\langle X\rangle_s$$

$\boldsymbol{P}$-almost surely. However,

$$\mathfrak{e}_2 g\colon s \longmapsto f(s) - f(0) - f'(0)s,$$

therefore

$$f(X_s) - f'(0)X_s - f(X_0) + f'(0)X_0 = \int_0^t \big\{f'(X_s) - f'(0)\big\}\,\mathrm{d}X_s + \frac{1}{2}\int_0^t g(X_s)\,\mathrm{d}\langle X\rangle_s$$

$\boldsymbol{P}$-almost surely and (1.1) follows.

It remains to check (2.7). Recall that we denoted by $X = X_0 + A + M$ the canonical decomposition of X and by $\tilde{A}$ the variation of the process A.

By the monotone convergence theorem,

$$\lim_{n\to\infty} \int_0^t g_n(X_s)\,\mathrm{d}\langle X\rangle_s = \int_0^t g(X_s)\,\mathrm{d}\langle X\rangle_s \quad \boldsymbol{P}\text{-almost surely,}$$

therefore, it suffices to prove that the sequence

$$\left(\int_0^t g_n(X_s)\,\mathrm{d}\langle X\rangle_s\right)_{n=1}^{\infty} = \left(\mathfrak{e}_2 g_n(X_t) - \mathfrak{e}_2 g_n(X_0) - \int_0^t \mathfrak{e} g_n(X_s)\,\mathrm{d}X_s\right)_{n=1}^{\infty}$$

$\boldsymbol{P}$-almost surely has a bounded subsequence. By (2.5)

$$\left|\mathfrak{e}_2 g_n(X_t) - \mathfrak{e}_2 g_n(X_0)\right| \le |\mathfrak{e}_2 g(X_t)| + |\mathfrak{e}_2 g(X_0)|$$

and the sequence $(\mathfrak{e}_2 g_n(X_t) - \mathfrak{e}_2 g_n(X_0))_{n=1}^{\infty}$ is bounded $\boldsymbol{P}$-almost surely. Further, we get

$$\begin{aligned}\left|\int_0^t \mathfrak{e} g_n(X_s)\,\mathrm{d}A_s\right| &\le \int_0^t |\mathfrak{e} g_n(X_s)|\,\mathrm{d}\tilde{A}_s \\ &\le \int_0^t |\mathfrak{e} g(X_s)|\,\mathrm{d}\tilde{A}_s \quad \boldsymbol{P}\text{-almost surely.}\end{aligned} \tag{2.8}$$

Finally, owing to (2.5), (2.6) and the dominated convergence theorem, we obtain

$$\lim_{n\to\infty} \int_0^t \left|\mathfrak{e} g_n(X_s) - \mathfrak{e} g(X_s)\right|^2 \mathrm{d}\langle M\rangle_s = 0 \quad \boldsymbol{P}\text{-almost surely,}$$

consequently

$$\lim_{n\to\infty} \int_0^t \mathfrak{e} g_n(X_s)\,\mathrm{d}M_s = \int_0^t \mathfrak{e} g(X_s)\,\mathrm{d}M_s \quad \text{in } \boldsymbol{P}\text{-probability,}$$

and there exists a subsequence (g_{n_k}) such that

$$\lim_{k\to\infty} \int_0^t \mathfrak{e} g_{n_k}(X_s)\,\mathrm{d}M_s = \int_0^t \mathfrak{e} g(X_s)\,\mathrm{d}M_s \quad \boldsymbol{P}\text{-almost surely.} \tag{2.9}$$

Convergent sequences are bounded, so combining (2.8) and (2.9), we complete the proof. □

3. Proof of Proposition 1.3

Proof. We shall use the notation introduced in the proof of Proposition 1.1. By the usual Itô formula,

$$\mathfrak{e}_2 V(X_t) - \mathfrak{e}_2 V(X_0) = \int_0^t \mathfrak{e} V(X_{s-})\,\mathrm{d}X_s + \frac{1}{2}\int_0^t V(X_{s-})\,\mathrm{d}[X]_s^c + \mathfrak{S}_t V$$

with

$$\mathfrak{S}_t V = \sum_{0<s\le t} \big[\mathfrak{e}_2 V(X_s) - \mathfrak{e}_2 V(X_{s-}) - \mathfrak{e} V(X_{s-})\Delta X_s\big]$$

$\boldsymbol{P}$-almost surely whenever $V \in C(\mathbb{R})$. We have to show that exactly the same approximation procedure as in the continuous case may be employed, i.e., that we can handle the additional term $\mathfrak{S}_t V$ using the same approximations. We may assume from the beginning of the proof that $g \ge 0$. Then, tracing the proof of Proposition 1.1, we can check easily that it suffices to consider only nonnegative nondecreasing approximating sequences. Hence to check that Step 2 of the proof remains valid, we have to prove that if $V_n \in \mathscr{I}$, $n \ge 1$, (2.1) is satisfied and $0 \le V_1 \le V_2 \le \cdots \nearrow V$ pointwise on $\mathbb{R}$, then

$$\lim_{n\to\infty} \mathfrak{S}_t V_n = \mathfrak{S}_t V \quad \boldsymbol{P}\text{-almost surely.} \tag{3.1}$$

Note that if $h \in AC^1_{\mathrm{loc}}(\mathbb{R})$ with $h'' \ge 0$ λ-almost everywhere on $\mathbb{R}$, then a straightforward calculation shows that

$$h(x) - h(y) - h'(y)(x-y) = \int_y^x \int_y^z h''(r)\,\mathrm{d}r\,\mathrm{d}z \ge 0 \quad \text{for all } x, y \in \mathbb{R}. \tag{3.2}$$

Therefore, if $h_1, h_2 \in AC^1_{\mathrm{loc}}(\mathbb{R})$ and $0 \le h_1'' \le h_2''$ λ-almost everywhere on $\mathbb{R}$, then

$$h_2(x) - h_2(y) - h_2'(y)(x-y) \ge h_1(x) - h_1(y) - h_1'(y)(x-y) \quad \text{for all } x, y \in \mathbb{R}$$

by (3.2). Consequently, $0 \le \mathfrak{S}_t V_1 \le \mathfrak{S}_t V_2 \le \cdots$ and (3.1) follows by the monotone convergence theorem. Once Step 2 is established, Step 3 and the first part of Step 4 require no essential changes. To complete the proof of (1.3) it remains to show that the sequence $(\mathfrak{S}_t g_n)_{n=1}^\infty$ $\boldsymbol{P}$-almost surely has a bounded subsequence, where $g_n = g\mathbf{1}_{\{g\le n\}}$, $n \ge 1$. However,

$$\left(\int_0^t g_n(X_{s-})\,\mathrm{d}[X]^c_s + \mathfrak{S}_t g_n\right)_{n=1}^\infty = \left(\mathfrak{e}_2 g_n(X_t) - \mathfrak{e}_2 g_n(X_0) - \int_0^t \mathfrak{e} g_n(X_{s-})\,\mathrm{d}X_s\right)_{n=1}^\infty,$$

the sequence on the left-hand side consists of sums of two nonnegative terms and $\boldsymbol{P}$-almost sure existence of a bounded subsequence of the sequence on the the right-hand side may be justified as in the proof of Proposition 1.1. □

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