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YOUNG DIFFERENTIAL EQUATIONS DRIVEN BY BESOV–ORLICZ PATHS

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Abstract. In the article, we consider nonlinear differential equations driven by paths from the exponential Besov–Orlicz space $B^{\alpha}_{\Phi_\beta,q}$ for

$$\alpha \in (1/2,1), \quad \Phi_\beta(x) \sim \mathrm{e}^{x^\beta} - 1 \quad \text{with} \quad \beta \in (0,\infty), \quad \text{and} \quad q \in (0,\infty].$$

By appealing to the recently obtained sewing lemma for such paths, we construct a Young-type integral and show that such equations admit a unique solution that is again of exponential Besov–Orlicz regularity. The results cover equations driven by paths of a large number of stochastic processes that exhibit long-range dependence, e.g. fractional Brownian motion with Hurst parameter $H \in (1/2,1)$ or, more generally, any Hermite process.

1. Introduction.

In the article, we consider differential equations

$$\mathrm{d}Y_t = f(Y_t)\,\mathrm{d}X_t, \quad Y_0 = y, \tag{1}$$

on the interval $[0,T]$ where $y \in \mathbb{R}^m$ is a given initial condition, $f : \mathbb{R}^m \to \mathbb{R}^{m\times n}$ is a sufficiently smooth function, and $X : [0,T] \to \mathbb{R}^n$ is a function from the exponential Besov–Orlicz space $B^{\alpha}_{\Phi_\beta,q}$ with

$$\alpha \in (1/2,1), \quad \Phi_\beta(x) \sim \mathrm{e}^{x^\beta} - 1 \quad \text{for} \quad \beta \in (0,\infty), \quad \text{and} \quad q \in (0,\infty].$$

(See Section 2 for the precise definition of exponential Besov–Orlicz spaces.)

The need to analyze such equations arises from stochastic modeling. There, equations of the form (1) are used to describe the response of a dynamical system to random error. When the past events in the error significantly influence its future dynamics (such as trend in asset prices) while still exhibiting consistent behavior over different time scales (such as fluctuations of the asset prices when viewed over minutes, hours, or days), it is common to use long-range dependent self-similar stochastic process as models for the source of randomness [7, 23].

Typical examples of such processes are the so-called Hermite processes; see, e.g., [5, 26, 27]. The simplest Hermite process is the well-known fractional Brownian

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motion with Hurst parameter $H \in (1/2, 1)$; see, e.g., [1, 4, 16], that can be used in situations where Gaussianity is desirable. If, for example, one needs to account for a higher occurrence of extreme events (such as market crashes or spikes in network traffic) or to capture asymmetries of the data (such as negative skewness of asset returns due to frequent small gains and occasional large losses), a higher-order Hermite process, and in particular the Rosenblatt process; see, e.g., [22, 24, 25], might be considered as a useful alternative.

It is well-known that Hermite paths are Hölder continuous of an order greater than $1/2$, a fact that seems to be connected to long-range dependence, so that the integral on the right-hand side of (1) with X being a path of a Hermite process can be understood as the classical Young integral [29] (or its extension from, e.g., [30] or [28]) and the pathwise Hölder continuous solution can be obtained by classical arguments as in, e.g., [8, 12]. On the other hand, such regularity is far from optimal.

It fact, the smallest, currently known, function space to which Hermite paths belong is the exponential Besov–Orlicz space $B^H_{\Phi_{2/n},\infty}$ where n is the order of the Hermite process ($n = 1$ for the fractional Brownian motion and $n = 2$ for the Rosenblatt process) and $H \in (1/2, 1)$ is its self-similarity parameter; see [3, Corollary 4.2]. Therefore, in view of the Sobolev embedding

$$B^H_{\Phi_{2/n},\infty} \hookrightarrow B^H_{p,\infty} \hookrightarrow C^{H-\frac{1}{p}},$$

that is valid for any finite $p > 1/H$ and in which $B^H_{p,\infty}$ and $C^{H-\frac{1}{p}}$ denote the classical Besov and Hölder spaces, respectively (see Section 2 for the definitions), the result of [11, Theorem 4.2] can be used to obtain the $B^H_{p,\infty}$-regularity of the solution. It is clear, however, that this result does not use all the information on the regularity of the Hermite path (it does not, for example, reflect the Hermite order n) and it is natural to ask whether the solution also retains the exponential Besov–Orlicz regularity of the Hermite process.

It follows from the results in the present article that the answer to this question is affirmative. In particular, we use the recently obtained exponential Besov–Orlicz sewing lemma from [2, Theorem 4.11] to define a (Young-type) integral $\int f \, \mathrm{d}g$ for $f \in B^{\alpha_0}_{\Phi_{\beta_0},q_0}$ and $g \in B^{\alpha_1}_{\Phi_{\beta_1},q_1}$, where $\alpha_0, \alpha_1 \in (0, 1)$ satisfy $\alpha_0 + \alpha_1 > 1$, as the limit of the classical Riemann sums in the exponential Besov–Orlicz topology (see Theorem 3.1 for the exact statement) and show that the integral retains the precise regularity of the integrator g. Then we prove that equation (1) with $f \in C^{1,1}_b$ (i.e. bounded differentiable function whose Fréchet derivative is bounded and Lipschitz continuous) and $X \in B^{\alpha}_{\Phi_\beta,q}$ admits a solution $Y \in B^{\alpha}_{\Phi_\beta,q}$ (see Theorem 4.2), show that the solution map is continuous (see Theorem 4.3), and use these results to show that the solution to equation (1) driven by a path of the n^{th}-order Hermite process with Hurst parameter $H \in (1/2, 1)$ is again of $B^H_{\Phi_{2/n},\infty}$-regularity (see Corollary 5.1).

It should be noted that our results as well as methods are intimately connected to the rough path theory. In fact, the results contained in this article can be viewed as the level-1 rough exponential Besov–Orlicz case (i.e. corresponding to $\alpha \in (1/2, 1)$) and are therefore complementary to the level-2 rough exponential Besov–Orlicz case (i.e. corresponding to $\alpha \in (1/3, 1/2]$) considered in [2]. Similar techniques in different settings are also used in the series of articles [13, 14, 15], where rough paths of Sobolev regularity are treated, and in the articles [9, 10, 11, 20], where drivers of Besov regularity are considered.

The paper is organized as follows. In Section 2, we collect some notation used throughout the article, recall the function spaces that are needed for our analysis and their properties, and we give a sewing lemma that is used for the construction of the Young integral. In Section 3, we define the Young integral and in Section 4, the Young differential equations of the form (1) are treated. Finally, applications to stochastic differential equations driven by Hermite processes are given in Section 5.

2. **Preliminaries.**

2.1. **Notation and conventions.** Here we collect basic notation and conventions.

2.1.1. *Convention on constants.* The following convention on constants is used to simplify the exposition. We write $A \lesssim B$ if there is a finite positive constant C whose specific value is not important such that $A \leq CB$. If we wish to stress that C depends on some parameter θ, we write $A \lesssim_\theta B$. If A, B, and C depend on an additional parameter $T > 0$, i.e. we have that $A(T) \leq C(\theta, T)B(T)$ holds for all $T > 0$, we may wish to stress that $C(\theta, \cdot)$ is non-decreasing. In this case, we write $A(T) \lesssim_\theta^T B(T)$, or simply $A(T) \lesssim^T B(T)$, if the dependence of $C(\theta, T)$ on parameter θ is not important. Moreover, the value of the constant C can change from line to line without any additional comment.

2.1.2. *Standard spaces.* For $m, n \in \mathbb{N}$, we write $\mathbb{R}^{m\times n}$ for the space of $(m \times n)$ real matrices and endow it with the Hilbert-Schmidt norm that is denoted by $\|\cdot\|_{\mathbb{R}^{m\times n}}$. For a finite dimensional normed linear space $(V, \|\cdot\|_V)$, its subset $U \subseteq V$, and a finite dimensional Banach space $(E, \|\cdot\|_E)$, we denote the space of continuous functions from U to E by $C(U, E)$. The space of bounded continuous functions from U to E is denoted by $C_b(U, E)$ and it is endowed with the supremum norm

$$\|f\|_\infty := \sup_{u\in U} \|f_u\|_E, \quad f \in C_b(U, E).$$

For $\alpha \in (0, 1]$, the space of α-Hölder (resp. Lipschitz for $\alpha = 1$) continuous functions from U to E is denoted by $C^\alpha(U, E)$ (resp. $C^{0,1}(U, E)$ for $\alpha = 1$) and it is defined as the space of all functions $f \in C(U, E)$ for which the quantity

$$[f]_{C^\alpha(U,E)} := \sup_{\substack{u,v\in V,\\ u\neq v}} \frac{\|f_u - f_v\|_E}{\|u - v\|_V^\alpha}$$

is finite. If U is open and $f : U \to E$ is a Fréchet differentiable function, we denote by Df its Fréchet derivative. The space of all bounded continuous functions from U to E with a bounded and continuous Fréchet derivative is denoted by $C_b^1(U, E)$ and it is endowed with the norm

$$\|f\|_{C_b^1(U,E)} := \|f\|_\infty + \|Df\|_\infty, \quad f \in C_b^1(U, E).$$

Finally, we denote by $C_b^{1,1}(U, E)$ the space of all functions $f \in C_b^1(U, E)$ such that $Df \in C^{0,1}(U, E)$ and equip it with the norm

$$\|f\|_{C_b^{1,1}(U,E)} := \|f\|_{C_b^1(U;E)} + [Df]_{C^{0,1}(U,E)}.$$

Note that if $E = \mathbb{R}$, we simply omit it from the notation described above.

2.1.3. *Further notation.* Some further notation, typically used within the rough path theory, shall also be adopted to simplify the exposition. In particular, for $d \in \mathbb{N}$, we set

$$\triangle^d[0,T] := \{(u_1, u_2, \dots, u_d) \in [0,T]^d \mid u_1 \le u_2 \le \dots \le u_d\},$$

for a function $\Xi : [0,T] \to \mathbb{R}$, we set

$$\delta\Xi_{s,t} := \Xi_t - \Xi_s, \quad (s,t) \in \triangle^2[0,T],$$

and, similarly, for a function $\Xi : \triangle^2[0,T] \to \mathbb{R}$, we set

$$\delta\Xi_{s,u,t} := \Xi_{s,t} - \Xi_{s,u} - \Xi_{u,t}, \quad (s,u,t) \in \triangle^3[0,T].$$

2.2. **Function spaces.** For the analysis given in the present article, we will need certain function spaces. For brevity, we formulate all the definitions and results below for scalar functions but the higher yet finite dimensional case can be treated in the same manner.

Let $d \in \{1,2,3\}$ and $T \in (0,\infty)$. Let us begin by recalling the construction of the space of measurable functions on simplex $\triangle^d[0,T]$. Let Ξ, $\widetilde{\Xi}$ be two measurable maps from $(\triangle^d[0,T], \mathfrak{L}(\triangle^d[0,T]))$ to $(\mathbb{R}, \mathfrak{B}(\mathbb{R}))$. Here we denote by $\mathfrak{L}(\triangle^d[0,T])$ and $\mathfrak{B}(\mathbb{R})$, the σ-algebras of Lebesgue and Borel measurable subsets of $\triangle^d[0,T]$ and $\mathbb{R}$, respectively. We also define

$$\Xi \sim_1 \widetilde{\Xi} \quad \text{if} \quad \Xi_r = \widetilde{\Xi}_r, \text{ a.e. } r \in [0,T],$$

$$\Xi \sim_2 \widetilde{\Xi} \quad \text{if} \quad \Xi_{r,r+h} = \widetilde{\Xi}_{r,r+h}, \ \forall h \in [0,T], \text{ a.e. } r \in [0,T-h],$$

$$\Xi \sim_3 \tilde{\Xi} \quad \text{if} \quad \Xi_{r,r+\theta h,r+h} = \tilde{\Xi}_{r,r+\theta h,r+h}, \ \forall h \in [0,T], \forall \theta \in [0,1], \text{ a.e. } r \in [0,T-h].$$

Relation $\sim_d$ is an equivalence and we define $L^{0,d}([0,T])$ as the space of equivalence classes of measurable functions from $(\triangle^d[0,T], \mathfrak{L}(\triangle^d[0,T]))$ to $(\mathbb{R}, \mathfrak{B}(\mathbb{R}))$.

We now define the class of exponential Orlicz spaces that will be needed throughout the article. For a thorough exposition of the topic, we refer to, e.g., [19, 21]. Let $T \in (0,\infty)$ and let $\beta \in (0,\infty)$. Define the function $\Phi_\beta : [0,\infty) \to [0,\infty)$ by

$$\Phi_\beta(x) := \begin{cases} \mathrm{e}^{x^\beta} - 1, & \text{if } \beta \in [1,\infty), \\ E_\beta(x) - E_\beta(0), & \text{if } \beta \in (0,1), \end{cases}$$

where E_β is the extension of the convex part of $x \mapsto \mathrm{e}^{x^\beta}$ on (x_β, ∞) by its tangent line at $x_\beta \in (0,\infty)$. The *exponential Orlicz space* $L^{\Phi_\beta}([0,T])$ is then defined as the linear space

$$L^{\Phi_\beta}([0,T]) := \left\{ f \in L^{0,1}([0,T]) \;\middle|\; \exists \lambda \in (0,\infty) : \int_0^T \Phi_\beta\left(\frac{|f_r|}{\lambda}\right) \mathrm{d}r < \infty \right\}.$$

Endowed with the *Luxemburg (gauge) norm*

$$\|f\|_{L^{\Phi_\beta}([0,T])} := \inf\left\{ \lambda \in (0,\infty) \;\middle|\; \int_0^T \Phi_\beta\left(\frac{|f_r|}{\lambda}\right) \mathrm{d}r \le 1 \right\},$$

the space $L^{\Phi_\beta}([0,T])$ is a Banach space. Among all the properties of the exponential Orlicz spaces, let us mention the following Hölder-type inequality that we will need throughout the article.

Proposition 2.1 (cf. [2, Lemma 2.1 and Lemma 2.3]). *Let $\theta \in (0,\infty)$ and let $\theta_1, \theta_2 \in (1,\infty)$ satisfy $\theta_1\theta_2 = \theta_1 + \theta_2$. Let also $u, v \in L^{0,1}([0,T])$. Then there is the following inequality:*

$$\|uv\|_{L^{\Phi_\theta}([0,T])} \lesssim_\theta \|u\|_{L^{\Phi_{\theta\theta_1}}([0,T])} \|v\|_{L^{\Phi_{\theta\theta_2}}([0,T])}.$$

We are now ready to define the *spaces of exponential Besov–Orlicz type* that are suitable for our purposes. Let again $d \in \{1,2,3\}$ and $T \in (0,\infty)$, and let $\alpha \in (0,\infty)$, $\beta \in (0,\infty)$, and $q \in (0,\infty]$. Set

$$\omega^d_{\Phi_\beta}(\Xi,\tau) := \begin{cases} \sup_{h\in[0,\tau]} \|\delta\Xi_{r,r+h}\|_{L^{\Phi_\beta}([0,T-h],\,\mathrm{d}r)} & \text{if } d=1,\\ \sup_{h\in[0,\tau]} \|\Xi_{r,r+h}\|_{L^{\Phi_\beta}([0,T-h],\,\mathrm{d}r)} & \text{if } d=2,\\ \sup_{\theta\in[0,1]} \sup_{h\in[0,\tau]} \|\Xi_{r,r+\theta h,r+h}\|_{L^{\Phi_\beta}([0,T-h],\,\mathrm{d}r)} & \text{if } d=3, \end{cases}$$

for $\Xi \in L^{0,d}([0,T])$ and $\tau \in [0,T]$, and define

$$[\Xi]_{B^{\alpha,d}_{\Phi_\beta,q}([0,T])} := \|\tau^{-\alpha}\omega^d_{\Phi_\beta}(\Xi,\tau)\|_{L^q([0,T];\frac{\mathrm{d}\tau}{\tau})}$$

and

$$\|\Xi\|_{B^{\alpha,d}_{\Phi_\beta,q}([0,T])} := \begin{cases} \|\Xi\|_{L^{\Phi_\beta}([0,T])} + [\Xi]_{B^{\alpha,1}_{\Phi_\beta,q}([0,T])} & \text{if } d=1,\\ [\Xi]_{B^{\alpha,d}_{\Phi_\beta,q}([0,T])} & \text{if } d\in\{2,3\}, \end{cases}$$

for $\Xi \in L^{0,d}([0,T])$. *The exponential Besov–Orlicz-type space* $B^{\alpha,d}_{\Phi_\beta,q}([0,T])$ is defined as the linear space

$$B^{\alpha,d}_{\Phi_\beta,q}([0,T]) := \left\{ \Xi \in L^{0,d}([0,T]) \;\middle|\; \|\Xi\|_{B^{\alpha,d}_{\Phi_\beta,q}([0,T])} < \infty \right\}.$$

It can be shown that $\|\cdot\|_{B^{\alpha,d}_{\Phi_\beta,q}([0,T])}$ is a quasinorm (norm if $q \geq 1$) on $B^{\alpha,d}_{\Phi_\beta,q}([0,T])$. Moreover, it can also be shown that when endowed with the metric

$$d_{B^{\alpha,d}_{\Phi_\beta,q}([0,T])}(\Xi,\widetilde{\Xi}) := \begin{cases} \|\Xi-\widetilde{\Xi}\|_{L^{\Phi_\beta}([0,T])} + [\Xi-\widetilde{\Xi}]^{q\wedge 1}_{B^{\alpha,1}_{\Phi_\beta,q}([0,T])}, & \text{if } d=1,\\ [\Xi-\widetilde{\Xi}]^{q\wedge 1}_{B^{\alpha,d}_{\Phi_\beta,q}([0,T])}, & \text{if } d\in\{2,3\}, \end{cases}$$

the space $B^{\alpha,d}_{\Phi_\beta,q}([0,T])$ becomes a complete metric space.

It is clear that for $\alpha \in (0,1)$, the space $B^{\alpha,1}_{\Phi_\beta,q}([0,T])$ coincides with the usual exponential Besov–Orlicz space $B^{\alpha}_{\Phi_\beta,q}([0,T])$. As such, the space is monotone in all the parameters in the usual manner, cf., e.g., [2, Proposition 3.4], and, additionally, it can be embedded into the space of continuous functions:

Proposition 2.2 (cf. [2, Proposition 3.2]). *Let $\alpha \in (0,1)$, $\beta \in (0,\infty)$, $q \in (0,\infty]$, and $p \in (\frac{1}{\alpha},\infty)$ and let $f \in B^{\alpha,1}_{\Phi_\beta,q}([0,T])$. Then there is the following estimate:*

$$[f]_{C^{\alpha-\frac{1}{p}}([0,T])} \lesssim_{\alpha,\beta,q,p} (1\vee T)[f]_{B^{\alpha,1}_{\Phi_\beta,q}([0,T])}.$$

On the other hand, the space $B^{\alpha,2}_{\Phi_\beta,q}([0,T])$ differs from the usual exponential Besov–Orlicz space $B^{\alpha}_{\Phi_\beta,q}(\triangle^2[0,T])$. To see this, we will need the Hölder-type space

$$C^{\alpha,2}([0,T]) := \{\Xi : \triangle^2[0,T] \to \mathbb{R} \,\big|\, \|\Xi\|_{C^{\alpha,2}([0,T])} < \infty\},$$

where $\alpha \in (0,\infty)$, $T \in (0,\infty)$, and

$$\|\Xi\|_{C^{\alpha,2}([0,T])} := \sup_{\substack{h\in(0,T]\\ r\in[0,T-h]}} \frac{|\Xi_{r,r+h}|}{|h|^\alpha},$$

from, e.g., [8]. A Besov–Orlicz–Hölder interpolation result is given now.

Proposition 2.3 (cf. [2, Lemma 3.9]). *Let $\alpha_1,\alpha_2 \in (0,\infty)$, $\beta_1,\beta_2 \in (0,\infty)$, $\delta \in (0,\infty)$, and $q \in (0,\infty]$ satisfy*

$$\alpha_1 \le \alpha_2, \quad \beta_1 \le \beta_2, \quad \text{and} \quad \delta\left(1-\frac{\beta_1}{\beta_2}\right)+\alpha_2\frac{\beta_1}{\beta_2}-\alpha_1 > 0.$$

Then the inequality

$$\|\Xi\|_{B^{\alpha_1,2}_{\Phi_{\beta_2},q}([0,T])} \lesssim_{\substack{\alpha_i,\beta_i,q,\delta\\ (i\in\{1,2\})}} T^{\delta\left(1-\frac{\beta_1}{\beta_2}\right)+\alpha_2\frac{\beta_1}{\beta_2}-\alpha_1}\|\Xi\|^{1-\frac{\beta_1}{\beta_2}}_{C^{\delta,2}([0,T])}\|\Xi\|^{\frac{\beta_1}{\beta_2}}_{B^{\alpha_2,2}_{\Phi_{\beta_1},q}([0,T])}$$

holds for any $\Xi \in L^{0,2}([0,T])$.

Monotonicity in the regularity parameter α and the intergrability parameter β still holds, cf. [2, Proposition 3.10]. However, since the definition of the space $B^{\alpha,2}_{\Phi_\beta,q}([0,T])$ does not enforce the boundedness of $\omega^2_{\Phi_\beta}(\Xi,\cdot)$ on $[0,T]$, the spaces are not monotone in the fine parameter q without additional assumptions:

Proposition 2.4 (cf. [2, Proposition 3.12]). *Let $\alpha,\beta \in (0,\infty)$ and let $q_1,q_2 \in (0,\infty]$ satisfy $q_1 \ge q_2$. Let $\Xi \in B^{\alpha,2}_{\Phi_\beta,q_2}([0,T])$ be such that $\omega^2_{\Phi_\beta}(\Xi,\cdot)$ is bounded on $[0,T]$. Then $\Xi \in B^{\alpha,2}_{\Phi_\beta,q_1}([0,T])$. If, in addition, there are $\rho \in (0,\frac12]$, $M \in (0,\infty)$, and $p \in (\frac{1}{\alpha},\infty)$ such that*

$$|\delta\Xi_{s,u,t}| \le M\left(((u-s)\wedge(t-u))^\rho((u-s)\vee(t-u))^{1-\rho}\right)^{\alpha-\frac1p}$$

holds for all $(s,u,t) \in \triangle^3[0,T]$, then there is the estimate

$$\|\Xi\|_{B^{\alpha,2}_{\Phi_\beta,q_1}([0,T])} \lesssim^T_{\alpha,\beta,p,q_1,q_2} \|\Xi\|_{B^{\alpha,2}_{\Phi_\beta,q_2}([0,T])} + MT^{-\frac1p}.$$

There is also a version of the Sobolev embedding of the exponential Besov–Orlicz-type spaces into the Hölder-type spaces that requires an additional assumption.

Proposition 2.5 (cf. [2, Proposition 3.8]). *Let $\alpha,\beta \in (0,\infty)$, and $q \in (0,\infty]$ and let $\Xi \in B^{\alpha,2}_{\Phi_\beta,q}([0,T])$ be such that there are $\rho \in (0,\frac12]$, $M \in (0,\infty)$, and $p \in (\frac{1}{\alpha},\infty)$ with which the inequality*

$$|\delta\Xi_{s,u,t}| \le M\left(((u-s)\wedge(t-u))^\rho((u-s)\vee(t-u))^{1-\rho}\right)^{\alpha-\frac1p}$$

holds for all $(s,u,t) \in \triangle^3[0,T]$. Then Ξ has a continuous version and it holds that

$$\|\Xi\|_{C^{\alpha-\frac1p,2}([0,T])} \lesssim_{\rho,\alpha,\beta,p} (1\vee T)\|\Xi\|_{B^{\alpha,2}_{\Phi_\beta,q}([0,T])} + M.$$

2.3. **Sewing lemma.** In what follows, we briefly recall the exponential Besov–Orlicz sewing lemma from [2, Theorem 4.11] that will be used for the construction of the integral $\int_0^\bullet f\,\mathrm{d}g$ for functions f and g that are of exponential Besov–Orlicz regularity in Section 3. Again, the result is formulated for scalar functions, but the higher yet finite dimensional case can be reduced to this setting.

By a *partition* of the interval $[0,1]$, we understand a finite sequence $\pi = \{t_i\}_{i=0}^N$ such that $t_0 = 0$, $t_N = 1$, and such that $t_i < t_{i+1}$ holds for all $i \in \{0,1,\dots,N-1\}$. We denote the set of all such partitions by $\mathcal{P}([0,1])$. The *mesh size* of partition

$\pi = \{t_i\}_{i=0}^N \in \mathcal{P}([0,1])$ is defined by $|\pi| := \max_{i=0,1,\dots,N-1} |t_{i+1} - t_i|$. A sequence of partitions $\{\pi_n\}_{n=1}^\infty \subseteq \mathcal{P}([0,1])$ is said to have *vanishing mesh size* if $|\pi_n| \xrightarrow[n\to\infty]{} 0$.

For $\Xi \in L^{0,2}([0,T];\mathbb{R})$ and $\pi = \{t_i\}_{i=0}^N \in \mathcal{P}([0,1])$, we define the partial sum $I^\pi \Xi : \triangle^2[0,T] \to \mathbb{R}$ by

$$(I^\pi \Xi)_{s,t} := \sum_{i=0}^{N-1} \Xi_{s+t_i(t-s),s+t_{i+1}(t-s)}.$$

Theorem 2.6 (cf. [2, Theorem 4.11])**.** *Let $\alpha \in (1,\infty)$, $\beta_1, \beta_2 \in (0,\infty)$, $\gamma \in (0,1)$, and $q_1, q_2 \in (0,\infty]$. Assume that $\Xi \in L^{0,2}([0,T])$ is such that*

$$\Xi \in B^{\gamma,2}_{\Phi_{\beta_1},q_1}([0,T]) \quad \text{and} \quad \delta\Xi \in B^{\alpha,3}_{\Phi_{\beta_2},q_2}([0,T]). \tag{2}$$

Then there exists $\mathscr{I} = \mathscr{I}(\Xi) \in L^{0,1}([0,T])$ such that the mapping $\Xi \mapsto \mathscr{I}(\Xi)$ is linear, the convergence

$$\lim_{N\to\infty} \|\delta\mathscr{I} - I^{\pi_N}\Xi\|_{B^{\alpha,2}_{\Phi_{\beta_2},r}([0,T])} = 0$$

holds for any sequence of partitions $\{\pi_N\}_{N=1}^\infty \subseteq \mathcal{P}([0,1])$ with vanishing mesh size and any $r \in (0,\infty)$ satisfying $r > \frac{1}{\alpha}$ and $r \geq q_2$, and such that there is the estimate

$$\|\delta\mathscr{I} - \Xi\|_{B^{\alpha,2}_{\Phi_{\beta_2},q_2}([0,T])} \lesssim_{\alpha,q_2} \|\delta\Xi\|_{B^{\alpha,3}_{\Phi_{\beta_2},q_2}([0,T])}. \tag{3}$$

If, moreover, there is $\rho \in (0,\frac{1}{2}]$, $M \in (0,\infty)$, and $p \in (\frac{1}{\alpha},\infty)$ such that

$$|\delta\Xi_{s,u,t}| \leq M\left(((u-s)\wedge(t-u))^\rho((u-s)\vee(t-u))^{1-\rho}\right)^{\alpha-\frac{1}{p}} \tag{4}$$

holds for any $(s,u,t) \in \triangle^3[0,T]$, then $\mathscr{I} \in B^{\gamma,1}_{\Phi_{\beta_1\wedge\beta_2},q_1\vee q_2}([0,T])$, which, in particular, means that $\mathscr{I}$ is continuous on the interval $[0,T]$ and it is uniquely determined by prescribing $\mathscr{I}_0 = \Theta$ for some $\Theta \in \mathbb{R}$. Additionally, there is the estimate

$$\|\delta\mathscr{I} - \Xi\|_{C^{\alpha-\frac{1}{p},2}([0,T])} \lesssim^T_{\alpha,\beta_2,q_2,p,\rho} \|\delta\Xi\|_{B^{\alpha,3}_{\Phi_{\beta_2},q_2}([0,T])} + M. \tag{5}$$

3. **Young integral.** Let us now turn our attention to the construction of the integral $\int_0^\bullet f\,\mathrm{d}g$ where f and g are functions from exponential Besov–Orlicz spaces such that the sum of their regularity coefficients is greater than one. The integral is constructed as the limit of the usual Riemann-type sums by means of the sewing lemma in Theorem 2.6. Additionally, we provide an estimate on its (quasi)seminorm that will be useful for the fixed-point argument in Section 4. Note again that while the result is stated for scalar functions, the higher yet finite dimensional case can be treated in the same manner.

Theorem 3.1. *Let $\alpha_0, \alpha_1 \in (0,1)$, $\beta_0, \beta_1 \in (0,\infty)$, and let $q_0, q_1 \in (0,\infty]$. Define*

$$\alpha := \alpha_0 + \alpha_1, \quad \frac{1}{\beta} := \frac{1}{\beta_0} + \frac{1}{\beta_1}, \quad \text{and} \quad \frac{1}{q} := \frac{1}{q_0} + \frac{1}{q_1}.$$

Furthermore, let $f \in B^{\alpha_0,1}_{\Phi_{\beta_0},q_0}([0,T])$ and $g \in B^{\alpha_1,1}_{\Phi_{\beta_1},q_1}([0,T])$. If $\alpha > 1$, then there is a unique function $\mathscr{I} \equiv \mathscr{I}(f,g) \in C([0,T])$ that satisfies $\mathscr{I}_0 = 0$, for which the assignment $(f,g) \mapsto \mathscr{I}(f,g)$ is bilinear, and such that

$$\lim_{N\to\infty} \|\delta\mathscr{I} - I^{\pi_N} f\delta g\|_{B^{\alpha,2}_{\Phi_\beta,r}([0,T])} = 0 \tag{6}$$

holds for any sequence of partitions $\{\pi_N\}_{N=1}^{\infty} \subseteq \mathcal{P}([0,1])$ *with vanishing mesh size and any* $r \in (0,\infty)$ *satisfying* $r > \frac{1}{\alpha}$ *and* $r \geq q$. *Moreover, there is* $\varepsilon \in (0, \alpha_0)$ *such that*

$$[\mathscr{I}]_{B^{\alpha_1,1}_{\Phi_{\beta_1},q_1}([0,T])} \lesssim^{T}_{\substack{\alpha_i,\beta_i,q_i,\varepsilon \\ (i\in\{1,2\})}} \left(|f_0| + T^{\alpha_0-\varepsilon}[f]_{B^{\alpha_0,1}_{\Phi_{\beta_0},q_0}([0,T])} \right) [g]_{B^{\alpha_1,1}_{\Phi_{\beta_1},q_1}([0,T])}. \tag{7}$$

Proof. Let $\Xi : \triangle^2[0,T] \to \mathbb{R}$ be the map defined by

$$\Xi_{s,t} := f_s \delta g_{s,t}, \quad (s,t) \in \triangle^2[0,T],$$

and denote, for brevity,

$$[f]_0 := [f]_{B^{\alpha_0,1}_{\Phi_{\beta_0},q_0}([0,T])}, \quad \text{and} \quad [g]_1 := [g]_{B^{\alpha_1,1}_{\Phi_{\beta_1},q_1}([0,T])}.$$

It follows from the definition of the Besov–Orlicz quasinorm and from the identity

$$\|\delta u\|_{B^{a,2}_{\Phi_b,v}([0,T])} = [u]_{B^{a,1}_{\Phi_b,v}([0,T])}, \tag{8}$$

that holds for any $u \in L^{0,1}([0,T])$, $a \in (0,1)$, $b \in (0,\infty)$, and $v \in (0,\infty]$, that

$$\|\Xi\|_{B^{\alpha_1,2}_{\Phi_{\beta_1},q_1}([0,T])} \leq \|f\|_\infty \|\delta g\|_{B^{\alpha_1,2}_{\Phi_{\beta_1},q_1}([0,T])} = \|f\|_\infty [g]_1. \tag{9}$$

Moreover, it also follows from the identity

$$\delta\Xi_{s,u,t} = -\delta f_{s,u}\delta g_{u,t}, \quad (s,u,t) \in \triangle^3[0,T],$$

the Hölder-type inequality for Orlicz spaces in Proposition 2.1, the classical Hölder inequality, and equality (8) that

$$\begin{aligned}
&\|\delta\Xi\|_{B^{\alpha,3}_{\Phi_\beta,q}([0,T])} \\
&\quad = \left\| \tau^{-\alpha} \sup_{\theta\in[0,1]} \sup_{h\in[0,\tau]} \|\delta f_{\cdot,\cdot+\theta h}\delta g_{\cdot+\theta h,\cdot+h}\|_{L^{\Phi_\beta}([0,T-h])} \right\|_{L^q([0,T];\frac{\mathrm{d}\tau}{\tau})} \\
&\quad \lesssim_\beta \left\| \tau^{-\alpha} \sup_{\theta\in[0,1]} \sup_{h\in[0,\tau]} \|\delta f_{\cdot,\cdot+\theta h}\|_{L^{\Phi_{\beta_0}}([0,T-h])} \|\delta g_{\cdot+\theta h,\cdot+h}\|_{L^{\Phi_{\beta_1}}([0,T-h])} \right\|_{L^q([0,T];\frac{\mathrm{d}\tau}{\tau})} \\
&\quad \leq \left\| \tau^{-\alpha} \sup_{h\in[0,\tau]} \|\delta f_{\cdot,\cdot+h}\|_{L^{\Phi_{\beta_0}}([0,T-h])} \sup_{h\in[0,\tau]} \|\delta g_{\cdot,\cdot+h}\|_{L^{\Phi_{\beta_1}}([0,T-h])} \right\|_{L^q([0,T];\frac{\mathrm{d}\tau}{\tau})} \\
&\quad \leq \|\delta f\|_{B^{\alpha_0;2}_{\Phi_{\beta_0},q_0}([0,T])} \|\delta g\|_{B^{\alpha_1;2}_{\Phi_{\beta_1},q_1}([0,T])} \\
&\quad = [f]_0[g]_1.
\end{aligned} \tag{10}$$

Both expressions on the right-hand sides of estimates (9) and (10) are finite by the assumption that $f \in B^{\alpha_0,1}_{\Phi_{\beta_0},q_0}([0,T]) \subseteq C([0,T])$ and $g \in B^{\alpha_1,1}_{\Phi_{\beta_1},q_1}([0,T])$. Thus, assumption (2) of Theorem 2.6 is verified and we immediately obtain the existence of $\mathscr{I}$ that satisfies the local approximation property (6). Moreover, let $p_0, p_1 \in (0,\infty)$ be such that $\alpha_0 - \frac{2}{p_0} > 0$ and $\alpha_0 - \frac{1}{p_0} = \alpha_1 - \frac{1}{p_1}$. Then by appealing to Proposition 2.2, we obtain the estimate

$$\begin{aligned}
|\delta\Xi_{s,u,t}| &= |\delta f_{s,u}||\delta g_{u,t}| \\
&\leq (u-s)^{\alpha_0-\frac{1}{p_0}} [f]_{C^{\alpha_0-\frac{1}{p_0}}([0,T])} (t-u)^{\alpha_1-\frac{1}{p_1}} [g]_{C^{\alpha_1-\frac{1}{p_1}}([0,T])} \\
&\lesssim^{T} (u-s)^{\alpha_0-\frac{1}{p_0}} [f]_0 (t-u)^{\alpha_1-\frac{1}{p_1}} [g]_1 \\
&= [f]_0[g]_1 \left((u-s) \wedge (t-u) \right)^{\frac{1}{2}} \left((u-s) \vee (t-u) \right)^{\frac{1}{2}} \Big)^{\alpha-\frac{1}{p}}
\end{aligned} \tag{11}$$

where in the last line we set $\frac{1}{p} := \frac{1}{p_0} + \frac{1}{p_1}$. Thus, condition (4) of Theorem 2.6 is also verified with, in particular,

$$M := C_T[f]_0[g]_1, \tag{12}$$

where C_T is the constant realizing estimate (11), and continuity and uniqueness of $\mathscr{I}$ follows. Let us prove estimate (7) now. We have

$$[\mathscr{I}]_{B^{\alpha_1,1}_{\Phi_{\beta_1},q_1}([0,T])} = \|\delta\mathscr{I}\|_{B^{\alpha_1,2}_{\Phi_{\beta_1},q_1}([0,T])} \le \|\Xi\|_{B^{\alpha_1,2}_{\Phi_{\beta_1},q_1}([0,T])} + \|\delta\mathscr{I} - \Xi\|_{B^{\alpha_1,2}_{\Phi_{\beta_1},q_1}([0,T])}.$$

For the first term, we obtain

$$\begin{aligned}\|\Xi\|_{B^{\alpha_1,2}_{\Phi_{\beta_1},q_1}([0,T])} &\le \|f\|_\infty [g]_1 \\ &\le \left(|f_0| + T^{\alpha_0-\frac{1}{p}}[f]_{C^{\alpha_0-\frac{1}{p}}([0,T])}\right)[g]_1 \\ &\lesssim^T \left(|f_0| + T^{\alpha_0-\frac{1}{p}}[f]_0\right)[g]_1 \end{aligned} \tag{13}$$

from (9) by using Proposition 2.2. For the second term, we obtain the estimate

$$\begin{aligned}&\|\delta\mathscr{I} - \Xi\|_{B^{\alpha_1,2}_{\Phi_{\beta_1},q_1}([0,T])} \\ &\lesssim T^{\alpha_0-\frac{1}{p}\left(1-\frac{\beta}{\beta_1}\right)} \|\delta\mathscr{I} - \Xi\|^{1-\frac{\beta}{\beta_1}}_{C^{\alpha-\frac{1}{p},2}([0,T])} \|\delta\mathscr{I} - \Xi\|^{\frac{\beta}{\beta_1}}_{B^{\alpha,2}_{\Phi_\beta,q_1}([0,T])} \\ &\lesssim^T T^{\alpha_0-\frac{1}{p}\left(1-\frac{\beta}{\beta_1}\right)} \|\delta\mathscr{I} - \Xi\|^{1-\frac{\beta}{\beta_1}}_{C^{\alpha-\frac{1}{p},2}([0,T])} \left(\|\delta\mathscr{I} - \Xi\|_{B^{\alpha,2}_{\Phi_\beta,q}([0,T])} + MT^{-\frac{1}{p}}\right)^{\frac{\beta}{\beta_1}}\end{aligned}$$

where we used Proposition 2.3 and Proposition 2.4 (for which we note that if (4) holds for Ξ it also holds for $\delta\mathscr{I} - \Xi$ because of the identity $\delta(\delta\mathscr{I}) = 0$). From the above inequality we obtain the estimate

$$\|\delta\mathscr{I} - \Xi\|_{B^{\alpha_1,2}_{\Phi_{\beta_1},q_1}([0,T])} \lesssim^T T^{\alpha_0-\frac{1}{p}} \left(\|\delta\Xi\|_{B^{\alpha,3}_{\Phi_\beta,q}([0,T])} + M\right) \lesssim^T T^{\alpha_0-\frac{1}{p}}[f]_0[g]_1 \tag{14}$$

by using estimates (3) and (5), obtained from the sewing lemma, and, subsequently, estimate (10) and the particular form of constant M in (12). Putting together estimates (13) and (14) yields inequality (7) with $\varepsilon := \frac{1}{p}$. □

Definition 3.2. Let $\alpha_0, \alpha_1 \in (0,1)$, $\beta_0, \beta_1 \in (0,\infty)$, and let $q_0, q_1 \in (0,\infty]$. Let $f \in B^{\alpha_0,1}_{\Phi_{\beta_0},q_0}([0,T])$ and $g \in B^{\alpha_1,1}_{\Phi_{\beta_1},q_1}([0,T])$ and assume that $\alpha_0+\alpha_1 > 1$. The unique continuous function $\mathscr{I} = \mathscr{I}(f,g)$ from Theorem 3.1 is called the *Young integral of f with respect to g* and we will write $\int_0^t f_s \,\mathrm{d}g_s := \mathscr{I}(f,g)_t$, $t \in [0,T]$.

4. **Young differential equations.** In this section, we consider differential equations driven by paths of exponential Besov–Orlicz regularity in the Young regime and prove the existence and uniqueness of solutions to such equations. We start with an auxiliary result on compositions of Besov–Orlicz and smooth functions.

Lemma 4.1. *Let $m, n \in \mathbb{N}$ and let $\alpha \in (0,1)$, $\beta \in (0,\infty)$, and $q \in (0,\infty]$.*

1. *Let $\Xi \in B^{\alpha,1}_{\Phi_\beta,q}([0,T],\mathbb{R}^m)$ and $F \in C^1_b(\mathbb{R}^m,\mathbb{R}^{m\times n})$. Then $F(\Xi)$ is an element of $B^{\alpha,1}_{\Phi_\beta,q}([0,T],\mathbb{R}^{m\times n})$ and we have the estimates*

$$\|F(\Xi)\|_{L^{\Phi_\beta}([0,T],\mathbb{R}^{m\times n})} \lesssim_\beta \|F(0)\|_{\mathbb{R}^{m\times n}} + \|DF\|_\infty \|\Xi\|_{L^{\Phi_\beta}([0,T],\mathbb{R}^m)} \tag{15}$$

and

$$[F(\Xi)]_{B^{\alpha,1}_{\Phi_\beta,q}([0,T],\mathbb{R}^{m\times n})} \le \|DF\|_\infty [\Xi]_{B^{\alpha,1}_{\Phi_\beta,q}([0,T],\mathbb{R}^m)}. \tag{16}$$

2. Let $\Xi, \Xi' \in B^{\alpha,1}_{\Phi_\beta,q}([0,T],\mathbb{R}^m)$ and $F \in C^{1,1}_b(\mathbb{R}^m,\mathbb{R}^{m\times n})$. Then we also have

$$\begin{aligned}[F(\Xi)-F(\Xi')]_{B^{\alpha,1}_{\Phi_\beta,q}([0,T],\mathbb{R}^{m\times n})}&\\ \lesssim^T_{\alpha,\beta,q} \|F\|_{C^{1,1}_b(\mathbb{R}^m,\mathbb{R}^{m\times n})}&\\ \cdot\left(1+[\Xi]_{B^{\alpha,1}_{\Phi_\beta,q}([0,T],\mathbb{R}^m)}+[\Xi']_{B^{\alpha,1}_{\Phi_\beta,q}([0,T],\mathbb{R}^m)}\right)&\\ \cdot\left(\|\Xi_0-\Xi_0'\|_{\mathbb{R}^m}+[\Xi-\Xi']_{B^{\alpha,1}_{\Phi_\beta,q}([0,T],\mathbb{R}^m)}\right).&\end{aligned} \tag{17}$$

Proof. Note first that we have

$$g(Z_t)-g(Z'_s)=\left(\int_0^1 (Dg)(\theta Z_t+(1-\theta)Z'_s)\,\mathrm{d}\theta\right)(Z_t-Z'_s) \tag{18}$$

for $g \in C^1(\mathbb{R}^m,\mathbb{R}^{m\times n})$, $Z, Z' \in B^{\alpha,1}_{\Phi_\beta,q}([0,T],\mathbb{R}^m)$, and $s,t \in [0,\tau]$ by the Fundamental Theorem of Calculus applied to the function

$$G:[0,1]\to\mathbb{R}^{m\times n},\quad \theta\mapsto g(\theta Z_t+(1-\theta)Z'_s)).$$

By adding and subtracting $F(0)$ inside the norm and using equality (18) (with $g=F$, $Z=\Xi$, $Z'=0$, and $s=t=r$), we obtain the estimate

$$\|F(\Xi_r)\|_{\mathbb{R}^{m\times n}}\le\|F(0)\|_{\mathbb{R}^{m\times n}}+\|DF\|_\infty\|\Xi_r\|_{\mathbb{R}^m}$$

for $r\in(0,T)$ from which inequality (15) follows by monotonicity of the exponential Orlicz norm. Similarly, by using the same equality (with $g=F$, $Z=Z'=\Xi$, $s=r$, and $t=r+h$), we also obtain

$$\|F(\Xi_{r+h})-F(\Xi_r)\|_{\mathbb{R}^{m\times n}}\le\|DF\|_\infty\|\Xi_{r+h}-\Xi_r\|_{\mathbb{R}^m}$$

for $h\in[0,T)$ and $r\in(0,T-h)$ from which we obtain inequality (16).

We prove the second claim of the lemma now. By using equality (18) (with $g=F$, $Z=\Xi$, $Z'=\Xi'$, and $s=t=r+h$), adding and subtracting the term $(DF)(\theta\Xi_{r+h}+(1-\theta)\Xi'_{r+h})(\Xi_r-\Xi'_r)$ inside the integrand, and using boundedness and Lipschitz continuity of Df, we obtain the estimate

$$\begin{aligned}\big\|(F(\Xi_{r+h})-F(\Xi'_{r+h}))-(F(\Xi_r)-F(\Xi'_r))\big\|_{\mathbb{R}^{m\times n}}&\\ \le\|DF\|_\infty\|(\Xi_{r+h}-\Xi'_{r+h})-(\Xi_r-\Xi'_r)\|_{\mathbb{R}^m}&\\ +\|DF\|_{C^{0,1}}\|\Xi-\Xi'\|_\infty\left(\|\Xi_{r+h}-\Xi_r\|_{\mathbb{R}^m}+\|\Xi'_{r+h}-\Xi'_r\|_{\mathbb{R}^m}\right)&\end{aligned}$$

for $h\in[0,T)$ and $r\in(0,T-h)$ from which we obtain

$$\begin{aligned}[F(\Xi)-F(\Xi')]_{B^{\alpha,1}_{\Phi_\beta,q}([0,T],\mathbb{R}^{m\times n})}&\lesssim\|DF\|_\infty[\Xi-\Xi']_{B^{\alpha,1}_{\Phi_\beta,q}([0,T],\mathbb{R}^m)}\\ &+\|DF\|_{C^{0,1}}\|\Xi-\Xi'\|_\infty\left([\Xi]_{B^{\alpha,1}_{\Phi_\beta,q}([0,T],\mathbb{R}^m)}+[\Xi']_{B^{\alpha,1}_{\Phi_\beta,q}([0,T],\mathbb{R}^m)}\right)\end{aligned}$$

by monotonicity of the exponential Orlicz norm and the (quasi)triangle inequality. By Proposition 2.2, we have, for every $\kappa\in(0,\alpha)$, the estimate

$$\begin{aligned}\|\Xi-\Xi'\|_\infty&\lesssim^T\|\Xi_0-\Xi_0'\|_{\mathbb{R}^m}+[\Xi-\Xi']_{C^{\alpha-\kappa}([0,T],\mathbb{R}^m)}\\ &\lesssim^T\|\Xi_0-\Xi_0'\|_{\mathbb{R}^m}+[\Xi-\Xi']_{B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^m)}.\end{aligned}$$

Inequality (17) is obtained by combining the last two estimates. □

Existence and uniqueness of the solution to differential equations driven by exponential Besov–Orlicz paths in the Young regime can be addressed now. To this end, let $m, n \in \mathbb{N}$, $f \in C_b^{1,1}(\mathbb{R}^m, \mathbb{R}^{m\times n})$, $X \in B^{\alpha,1}_{\Phi_\beta,q}([0,T],\mathbb{R}^n)$ for $\alpha \in (1/2,1)$, $\beta \in (0,\infty)$, and $q \in (0,\infty]$, and $y \in \mathbb{R}^m$ be fixed. Consider the differential equation

$$Y_t = y + \int_0^t f(Y_s)\,\mathrm{d}X_s, \quad t \in [0,T], \tag{YDE}$$

where the integral is understood as the Young integral defined in Definition 3.2. We have the following existence and uniqueness result.

Theorem 4.2. *Problem* (YDE) *admits a unique solution* $Y \in B^{\alpha,1}_{\Phi_\beta,q}([0,T],\mathbb{R}^m)$.

Proof. To prove the claim, we aim to appeal to Banach's Fixed Point Theorem. We start by specifying the complete metric space in which we subsequently construct the contraction map. If $\tau \in (0, T\wedge 1]$, we define

$$B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^m;y) := \left\{ Y \in B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^m) \,\middle|\, Y_0 = y \right\}.$$

Because of the embedding $B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^m) \subseteq C([0,\tau],\mathbb{R}^m)$ (see Proposition 2.2), the space $B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^m;y)$ is well-defined. Moreover, the map $[\,\cdot\,]_{B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^m)}$ is a quasinorm (norm if $q \geq 1$) on $B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^m;y)$, the map

$$\rho(\Xi,\widetilde{\Xi}) := [\Xi - \widetilde{\Xi}]^{q\wedge 1}_{B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^m)}$$

is a metric on $B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^m;y)$, and the space $(B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^m;y),\rho)$ is a complete metric space (see [2, Proposition 3.6]). Consequently, if $K \in (0,\infty)$ and if we define

$$B_y(\tau,K) := \left\{ Y \in B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^m;y) \,\middle|\, [Y]_{B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^m)} \leq K \right\},$$

we obtain that $(B_y(\tau,K),\rho)$ is also a complete metric space.

It follows by Lemma 4.1 that $f(Y) \in B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^{m\times n})$ and, as we also have that $X \in B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^n)$, we see that the function $SY : [0,\tau] \to \mathbb{R}^m$ defined by

$$(SY)_t := y + \int_0^t f(Y_s)\,\mathrm{d}X_s, \quad t \in [0,\tau],$$

is a well-defined element of $B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^m)$ that satisfies

$$\begin{aligned}&[SY]_{B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^m)}\\ &\quad\lesssim^\tau \left(\|f(Y_0)\|_{\mathbb{R}^{m\times n}} + \tau^{\alpha-\varepsilon}[f(Y)]_{B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^{m\times n})} \right) [X]_{B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^n)}\end{aligned} \tag{19}$$

with some $\varepsilon \in (0,\alpha)$ by Theorem 3.1.

In what follows it is shown that we can choose constants τ and K in such a way that map S is a contraction on $(B_y(\tau,K),\rho)$.

We show first that τ and K can be chosen in such a way that $SY \in B_y(\tau,K)$. To this end, suppose that $Y \in B_y(\tau,K)$ for some $\tau \in (0,T\wedge 1]$ and $K \in (0,\infty)$. We have that $(SY)_0 = y$ and it follows from the above discussion that $SY \in B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^m)$. Moreover, we obtain from estimates (16) and (19) that

$$[SY]_{B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^m)} \lesssim^\tau \left(\|f\|_\infty + \tau^{\alpha-\varepsilon}\|Df\|_\infty K \right) [X]_{B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^n)}$$

holds for some $\alpha \in (0,\varepsilon)$ and it is clear that the right-hand side of the above estimate can be made smaller than K by choosing τ sufficiently small and K sufficiently large. With these choices of τ and K, we thus obtain that $SY \in B_y(\tau,K)$ as desired.

We show now that τ and K can be chosen in such a way that SY is a contraction on $(B_y(\tau,K),\rho)$. Suppose that $Y,Y' \in B_y(\tau,K)$ for some $\tau \in (0,T\wedge 1]$ and $K \in (0,\infty)$. By the first part of the proof, we immediately have that both SY and SY' are well-defined elements of $B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^m)$ and, moreover, it follows from (7) that their difference

$$(SY)_t - (SY')_t = \int_0^t (f(Y_r) - f(Y'_r))\,\mathrm{d}X_r, \quad t \in [0,\tau],$$

satisfies

$$\begin{aligned}&[SY - SY']_{B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^m)}\\ &\qquad \lesssim^{\tau} \tau^{\alpha-\varepsilon}[f(Y)-f(Y')]_{B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^{m\times n})}[X]_{B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^m)} \qquad (20)\end{aligned}$$

for some $\varepsilon \in (0,\alpha)$. By using Lemma 4.1, we obtain

$$[f(Y)-f(Y')]_{B^{\alpha,1}_{\Phi_\beta,q}([0,\tau];\mathbb{R}^{m\times n})} \lesssim^{\tau} (1+2K)\|f\|_{C^{1,1}_b}[Y-Y']_{B^{\alpha,1}_{\Phi_\beta,q}([0,\tau];\mathbb{R}^m)},$$

which, when inserted into inequality (20), yields

$$\begin{aligned}&[SY - SY']_{B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^m)}\\ &\qquad \lesssim^{\tau} \tau^{\alpha-\varepsilon}(1+2K)\|f\|_{C^{1,1}_b}[X]_{B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^m)}[Y-Y']_{B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^m)}.\end{aligned}$$

It is clear that the constant in front of the exponential Besov–Orlicz (quasi)seminorm of $Y-Y'$ on the right-hand side of the above inequality does not depend on Y or Y' and that it can be made smaller than 1 for any K by choosing τ sufficiently small.

Thus it is shown that there exist τ and K such that the map $S : B_y(\tau,K) \to B_y(\tau,K)$ is a contraction and by appealing to Banach's Fixed Point Theorem, we obtain the existence and uniqueness of the solution to (YDE) on interval $[0,\tau]$. It is clear that this procedure can be subsequently performed on intervals of the form $[i\tau,(i+1)\tau]$, $i \in \mathbb{N}_0$, until interval $[0,T]$ is covered and this proves the theorem. □

We conclude this section by discussing the regularity of the solution map.

Theorem 4.3. *The solution map*

$$\Psi : \mathbb{R}^m \times C^{1,1}_b(\mathbb{R}^m,\mathbb{R}^{m\times n}) \times B^{\alpha,1}_{\Phi_\beta,q}([0,T],\mathbb{R}^n) \to B^{\alpha,1}_{\Phi_\beta,q}([0,T],\mathbb{R}^m)$$

defined by $\Psi(y,f,X) := Y$, *where* Y *is the solution to problem* (YDE), *is locally Lipschitz continuous.*

Proof. Let $M \in (0,\infty)$ and let $y^1,y^2 \in \mathbb{R}^m$, $f^1,f^2 \in C^{1,1}_b(\mathbb{R}^m,\mathbb{R}^{m\times n})$, and $X^1,X^2 \in B^{\alpha,1}_{\Phi_\beta,q}([0,T],\mathbb{R}^n)$ be such that

$$[X^1]_{B^{\alpha,1}_{\Phi_\beta,q}([0,T],\mathbb{R}^n)} \vee [X^2]_{B^{\alpha,1}_{\Phi_\beta,q}([0,T],\mathbb{R}^n)} \vee \|f^1\|_{C^{1,1}_b} \vee \|f^2\|_{C^{1,1}_b} \le M.$$

We aim to show that there is the estimate

$$[Y^1-Y^2]_{B^{\alpha,1}_{\Phi_\beta,q}([0,T],\mathbb{R}^m)} \lesssim \|y^1-y^2\|_{\mathbb{R}^m} + \|f^1-f^2\|_{C^{1,1}_b} + [X^1-X^2]_{B^{\alpha,1}_{\Phi_\beta}([0,T],\mathbb{R}^n)} \qquad (21)$$

for the processes Y^1 and Y^2 that satisfy

$$\mathrm{d}Y^i_t = f^i(Y^i_t)\,\mathrm{d}X^i_t, \quad Y^i_0 = y^i,$$

on $[0,T]$ for $i=1,2$. Such an estimate is sufficient because the map

$$|||\cdot|||_{B^{\alpha,1}_{\Phi_\beta,q}([0,T],\mathbb{R}^m)} : B^{\alpha,1}_{\Phi_\beta,q}([0,T],\mathbb{R}^m) \to [0,\infty), \quad f \mapsto \|f_0\|_{\mathbb{R}^m} + [f]_{B^{\alpha,1}_{\Phi_\beta,q}([0,T],\mathbb{R}^m)}$$

is a (quasi)norm on $B^{\alpha,1}_{\Phi_\beta,q}([0,T],\mathbb{R}^m)$ that is equivalent to $\|\cdot\|_{B^{\alpha,1}_{\Phi_\beta,q}([0,T],\mathbb{R}^m)}$ by, e.g., [2, Proposition 3.6]. Now, to prove (21) note that the estimate

$$\|(Y^1_{r+h} - Y^2_{r+h}) - (Y^1_r - Y^2_r)\|_{\mathbb{R}^m} \le \left\|I^1_{r,r+h}\right\|_{\mathbb{R}^m} + \left\|I^2_{r,r+h}\right\|_{\mathbb{R}^m} + \left\|I^3_{r,r+h}\right\|_{\mathbb{R}^m}$$

where

$$\begin{aligned} I^1_{r,r+h} &:= \int_r^{r+h} (f^1(Y^1_s) - f^2(Y^1_s))\,\mathrm{d}X^1_s, \\ I^2_{r,r+h} &:= \int_r^{r+h} f^2(Y^2_s)\,\mathrm{d}(X^1_s - X^2_s), \\ I^3_{r,r+h} &:= \int_r^{r+h} (f^2(Y^1_s) - f^2(Y^2_s))\,\mathrm{d}X^1_s, \end{aligned}$$

holds for every $\tau \in (0,T]$, $h \in (0,\tau)$, and $r \in (0,\tau-h)$. It follows that

$$[Y^1 - Y^2]_{B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^m)} \lesssim_q [I^1]_{B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^m)} + [I^2]_{B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^m)} + [I^3]_{B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^m)}$$

by monotonicity of the Orlicz norm and the (quasi) triangle inequality. We now estimate all the three (quasi)seminorms on the right-hand side of the above inequality. To this end, denote by $C_1(\tau)$ the constant from inequality (7) and by $C_2(\tau)$ the constant from inequality (17). Since both these constants, as well as the (quasi)seminorm $[X]_{B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^m)}$, are nondecreasing in τ, it follows from the proof of Theorem 4.2 that if $\tau \in (0, T\wedge 1]$ and $K \in (0,\infty)$ satisfy

$$\tau^{\alpha-\varepsilon} C(T) M^2 (1+2K) < 1 \quad \text{and} \quad K \ge \frac{C(T)M^2}{1 - C(T)M^2\tau^{\alpha-\varepsilon}},$$

where $C(T) := C_1(T) \vee (C_1(T)C_2(T))$, then we obtain the estimate

$$[Y^1]_{B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^m)} \vee [Y^2]_{B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^m)} \le K.$$

Such choice of τ and K is indeed possible; to see this, write $A := C(T)M^2$, $y := K$, and $x := \tau^{\alpha-\varepsilon}$ and notice that the system $y \ge \frac{A}{1-Ax}$ and $y < \frac{1}{2Ax} - \frac{1}{2}$ admits a non-empty solution region. Now, with this choice of τ and K, we can estimate

$$\begin{aligned} [I^1]_{B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^m)} &\lesssim \|f^1 - f^2\|_{C^{1,1}_b}, \\ [I^2]_{B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^m)} &\lesssim [X^1 - X^2]_{B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^m)} \end{aligned}$$

by using inequality (7) and Lemma 4.1. Similarly, by using these two results, combined with the Mean Value Inequality, we also obtain the estimate

$$\begin{aligned} [I^3]_{B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^m)} &\le C(T)M\|y^1 - y^2\|_{\mathbb{R}^m} \\ &\quad + C(T)\tau^{\alpha-\varepsilon}M^2(1+2K)\left(\|y^1 - y^2\|_{\mathbb{R}^m} + [Y^1 - Y^2]_{B^{\alpha,1}_{\Phi_\beta,q}([0,\tau],\mathbb{R}^m)}\right) \end{aligned}$$

and we see that the constant in front of the Besov–Orlicz (quasi)seminorm of the difference $Y^1 - Y^2$ is smaller than one so that estimate (21) on the interval $[0,\tau]$ can be obtained. Estimate (21) on the interval $[0,T]$ is obtained by iterating these arguments on the intervals of the form $[i\tau,(i+1)\tau]$, $i \in \mathbb{N}_0$, until $[0,T]$ is covered. □

5. **Application to SDEs driven by Hermite processes.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a two-sided Wiener process $W = (W_t)_{t\in\mathbb{R}}$ defined on it. Assume that $\mathcal{F}$ is generated by W. Consider the *Hermite process* $Z_n^H = (Z_n^H(t))_{t\in[0,1]}$ of order $n \in \mathbb{N}$ with Hurst parameter $H \in (1/2, 1)$ that is defined by

$$Z_n^H(t) := I_n(L_n^H(t)), \quad t \in [0,1],$$

where I_n denotes the multiple Wiener-Itô integral of order n (see, e.g., [17, 18]) and where $L_n^H : [0,1] \to L^2(\mathbb{R}^n)$ is defined by

$$L_n^H(t; x_1, x_2, \dots, x_n) := C_n^H \int_0^t \prod_{j=1}^n (u - x_j)_+^{-\left(\frac{1}{2}+\frac{1-H}{n}\right)} \mathrm{d}u, \quad x_1, x_2, \dots, x_n \in \mathbb{R},$$

with $(v)_+ := \max\{0, v\}$ and with C_n^H being the constant such that $\mathbb{E}(Z_n^H(1))^2 = 1$ holds. The concise treatment of Hermite processes can be found, e.g., in [26, Section 3.1] and the references therein, but let us recall that the Hermite process Z_n^H has a version with paths in the exponential Besov–Orlicz space $B^{H,1}_{\Phi_{2/n},\infty}([0,1])$, see [3, Corollary 4.2]. Let us therefore consider this version and denote it by Z_n^H again.

Corollary 5.1. *Let $f \in C_b^{1,1}(\mathbb{R})$ and $y \in \mathbb{R}$. For $\mathbb{P}$-almost every $\omega \in \Omega$, there exists a unique solution $Y(\omega)$ to the Young differential equation*

$$Y_t(\omega) = y + \int_0^t f(Y_s(\omega))\, \mathrm{d}Z_n^H(s;\omega), \quad t \in [0,T], \tag{22}$$

that belongs to $B^{H,1}_{\Phi_{2/n},\infty}([0,1])$. Moreover, process Y has a measurable version.

Proof. The existence and uniqueness of a solution to equation (22) for a fixed path of the Hermite process follows immediately by combining [3, Corollary 4.2] and Theorem 4.2. As Y belongs to $B^{H,1}_{\Phi_{2/n},\infty}([0,1])$ almost surely, we immediately obtain from the Besov–Orlicz–Hölder embedding in Proposition 2.2 that Y is almost surely continuous, therefore also continuous in probability, and the existence of its measurable version follows from, e.g., [6, Theorem 2.6]. □

6. **Concluding remarks.** In this article, we have addressed the existence and uniqueness of solutions to the differential equation (YDE), where the driving process belongs to an exponential Besov–Orlicz space with a regularity parameter within the Young regime. Specifically, we have established that this equation admits a unique solution that retains exponential Besov–Orlicz regularity, as demonstrated in Theorem 4.2. As a consequence, we have derived novel regularity results for the sample paths of solutions to stochastic differential equations driven by Hermite processes, as presented in Corollary 5.1, thereby extending the findings of [3]. Furthermore, we have proven the continuity of the solution map in Theorem 4.3, which suggests the potential for probabilistic applications, such as support theorems or large deviation principles, in the spirit of [12, Chapter 19]. We anticipate that such results could be established, provided that a support theorem and a large deviation principle in the exponential Besov–Orlicz topology for general Hermite processes are first developed. However, as these results do not appear to be available in the existing literature, we intend to pursue them in our future research.

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