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RESEARCH

Rough Differential Equations Driven by Besov–Orlicz Paths

Petr Čoupek[1] · **František Hendrych**[1] · **Jakub Slavík**[2]

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Abstract
In the article, the rough path theory is extended to cover paths from the exponential Besov–Orlicz space

$$B^{\alpha}_{\Phi_\beta,q} \quad \text{for} \quad \alpha \in (1/3, 1/2], \quad \Phi_\beta(x) \sim \mathrm{e}^{x^\beta} - 1 \quad \text{with} \quad \beta \in (0,\infty), \quad \text{and} \quad q \in (0,\infty],$$

and the extension is used to treat nonlinear differential equations driven by such paths. The exponential Besov–Orlicz-type spaces, rough paths, and controlled rough paths are defined and analyzed, a sewing lemma for such paths is given, and the existence and uniqueness of the solution to differential equations driven by these paths is proved. The results cover equations driven by paths of continuous local martingales with Lipschitz continuous quadratic variation (e.g. the Wiener process) or by paths of fractionally filtered Hermite processes in the n^{th} Wiener chaos with Hurst parameter $H \in (1/3, 1/2]$ (e.g. the fractional Brownian motion).

Keywords Besov–Orlicz space · Rough path · Rough differential equation

Mathematics Subject Classification Primary 60L20 · Secondary 60H10 · 60G17 · 60G22

1 Introduction

In the last two decades, rough path theory and its extensions have had enormous impact in the field of differential equations (DEs) driven by singular functions. Such singularity was originally described in terms of Hölder continuity (or, almost equivalently, in terms of finite p-variation) but in recent years, several extensions to paths of Sobolev or Besov regularity have been given (see the series of papers [19–21] and [11, 12, 26], respectively).

✉ Petr Čoupek
coupek@karlin.mff.cuni.cz

František Hendrych
hendrychfrantisek@karlin.mff.cuni.cz

Jakub Slavík
slavik@utia.cas.cz

[1] Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Prague 8, Czech Republic

[2] Institute of Information Theory and Automation, Czech Academy of Sciences, Pod Vodárenskou věží 4, 180 00 Prague 8, Czech Republic

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From the perspective of stochastic differential equations (SDEs), i.e. DEs driven by paths generated from a stochastic processes, such extensions are very useful. Indeed, if we consider these equations, one would expect that the solution retains the noise regularity because, roughly speaking, the solution should behave like the noise on small scales. Consider, for example, the Wiener process $W = (W_t, t \in [0,1])$ as the noise source. By a straightforward application of the Kolmogorov continuity theorem, it is immediately seen that its paths lie, almost surely, in the Hölder space $C^{\frac{1}{2}-\varepsilon}([0,1])$ for any $\varepsilon \in (0, 1/2)$. One can therefore fix $\varepsilon \in (0, 1/6)$ and use the rough path machinery for Hölder continuous functions (e.g. [10, Theorem 8.4]) to enhance the Wiener path $W(\omega)$ to a Wiener rough path $\mathbf{W}(\omega) = (W(\omega), \mathbb{W}(\omega))$, where $\mathbb{W}(\omega)$ is the corresponding path of the, say, Stratonovich integral

$$\mathbb{W}_{s,t}(\omega) = \int_s^t (W_r - W_s) \circ \mathrm{d}W_r(\omega) = \frac{1}{2}(W_t(\omega) - W_s(\omega))^2,$$

to obtain the global solution to the rough differential equation (RDE)

$$\mathrm{d}Y_t(\omega) = f(Y_t(\omega))\,\mathrm{d}\mathbf{W}_t(\omega), \quad Y_0(\omega) = y, \tag{1}$$

for $y \in \mathbb{R}$ and $f \in C_b^3(\mathbb{R})$ that will again be of $C^{1/2-\varepsilon}$-regularity. The solution obtained in this manner then agrees with the solution to the corresponding Stratonovich SDE. However, somehow one feels that more information about the solution can be obtained if we could employ some extra information about the regularity of the Wiener path. It is known, for example, that Wiener paths belong, almost surely, to the Besov space $B^{1/2}_{p,\infty}([0,1])$ for all $p \in [1,\infty)$ but not to the space $B^{1/2}_{p,q}([0,1])$ for any $q < \infty$ (see [3]). Therefore, by choosing $p \in [1,\infty)$ and by appealing to the Besov extension of rough paths in [12] (namely to Theorem 5.6 therein), one in fact obtains that the solution to the RDE (1) is actually of $B^{1/2}_{p,\infty}$-regularity. On the other hand, it is also known that Wiener paths belong, almost surely, to the modular Hölder space $C^{|r\log r|^{1/2}}([0,1])$, see [18], but this space and the Besov space $B^{1/2}_{p,\infty}([0,1])$ are not included in one another. It was then soon realized that one can quantify the asymptotic growth of the L^p-modulus of continuity inside the $B^{1/2}_{p,\infty}$-norm and show that Wiener paths belong, almost surely, to the exponential Besov–Orlicz space $B^{1/2}_{\Phi_2,\infty}([0,1])$, where $\Phi_2(x) = \mathrm{e}^{x^2} - 1$, see [4], that lies in the intersection of the two spaces.

In fact, the exponential Besov–Orlicz spaces

$$B^{\alpha}_{\Phi_\beta,\infty}([0,1]) \quad \text{for} \quad \alpha \in (0,1) \quad \text{and} \quad \Phi_\beta \sim \mathrm{e}^{x^\beta} - 1 \quad \text{with} \quad \beta \in (0,\infty)$$

appear to form a very natural scale of function spaces for a multitude of stochastic processes. For example, such path regularity is obtained for continuous local martingales with Lipschitz continuous quadratic variation (whose prototypical example is the Wiener process) or for the fractionally filtered Hermite processes in the n^{th} Wiener chaos with the Hurst parameter $H \in (0,1)$ [1] (with examples such as the fractional Brownian motion [9] or the Rosenblatt process [31]) and while the former processes are known to have paths in $B^{1/2}_{\Phi_2,\infty}([0,1])$, see [23, Theorem 4.1], the latter have paths in $B^{H}_{\Phi_{2/n},\infty}([0,1])$, see [7, Corollary 4.2]. Other results on Besov–Orlicz regularity of stochastic processes are also given in [24, 35].

In the present article, we therefore aim to solve differential equations of the form

$$\mathrm{d}Y_t = f(Y_t)\,\mathrm{d}X_t, \quad Y_0 = y,$$

on the interval $[0,T]$ for $T \in (0,\infty)$, $y \in \mathbb{R}^m$, $f \in C_b^{2,1}(\mathbb{R}^m, \mathbb{R}^{m\times n})$, and a path X in the exponential Besov–Orlicz space $B^{\alpha}_{\Phi_\beta,q}([0,T];\mathbb{R}^n)$ for $\alpha \in (1/3, 1/2]$, $\beta \in (0,\infty)$,

$q \in (0, \infty]$, and $m, n \in \mathbb{N}$. In order to do so, we employ the rough path machinery; that is, initially, we extend the definition of the classical exponential Besov–Orlicz spaces to the exponential Besov–Orlicz-type spaces of multivariate maps that are suitable for rough path analysis and give several of their properties. This is done in Section 3. We then proceed, in Section 4, with the definition of an exponential Besov–Orlicz rough path $\mathbf{X}$ and show that the paths of the stochastic processes mentioned above can be indeed lifted to such rough paths. Subsequently, controlled rough paths are defined and their properties such as their stability under compositions with C_b^2-functions are given. We then prove a sewing lemma that is subsequently used to define a rough integral for paths of the considered Besov–Orlicz regularity. Finally, in Section 5, we consider the (rough) DE

$$\mathrm{d}Y_t = f(Y_t)\,\mathrm{d}\mathbf{X}_t, \quad Y_0 = y,$$

and we give the main result of the article in Theorem 5.2 where we show that the equation admits a unique solution of $B^{\alpha}_{\Phi_\beta, q}$-regularity. As a consequence, not only do we obtain $B^{1/2}_{\Phi_2,\infty}$-regularity of the solution to equation (1), which improves the known results on DEs driven by Wiener paths, but we also obtain Besov–Orlicz regularity of Stratonovich-type DEs driven by paths of other, possibly non-Gaussian and non-Markovian, stochastic processes.

2 Preliminaries: Hölder and Besov Spaces

Let us begin by listing the basic notation and function spaces used throughout the article and by recalling the definitions of Hölder-type and Besov-type spaces used in the theory of rough paths.

2.1 Basic Notation and Function Spaces

We use the following convention throughout the article: We write $A \lesssim B$ if there exists a finite positive constant C such that $A \leq CB$. If C depends on some parameter θ, we write either $A \lesssim_\theta B$ or $A \leq C(\theta)B$. If A, B, and C also depend on an additional parameter $T > 0$, i.e. the inequality $A(T) \leq C(\theta, T)B(T)$ holds for all $T > 0$, we may wish to stress that $C(\theta, \cdot)$ is nondecreasing (in particular, the value of $C(\theta, T)$ will not tend to infinity as $T \to 0+$), we write $A(T) \lesssim^T_\theta B(T)$. The value of the constant itself can change from one line to another without any additional comment.

In the list of function spaces below, let $n \in \mathbb{N}$, let $(\mathbb{V}_1, |\cdot|_{\mathbb{V}_1})$ and $(\mathbb{V}_2, |\cdot|_{\mathbb{V}_2})$ be finite-dimensional normed vector spaces, let (E, d) be a nonempty metric space and let $V \subseteq \mathbb{V}_1$. We note that the definitions of the spaces of E-valued functions below formally depend on the particular choice of $e_0 \in E$. However, with a different choice of e_0, only the respective norms differ while the function spaces themselves remain the same. For simplicity, we therefore assume that $e_0 \in E$ is fixed. When E is a Banach space, we naturally choose $e_0 = 0$.

- For $m, k \in \mathbb{N}$, we denote the space of $(m \times k)$-matrices by $\mathbb{R}^{m\times k}$. With $b \in \mathbb{R}^m$ and $\tilde{b} \in \mathbb{R}^k$, we note that the Hilbert–Schmidt norm $|\cdot|_{\mathbb{R}^{m\times k}}$ is compatible, i.e. $|b\tilde{b}^\top|_{\mathbb{R}^{m\times k}} \leq |b|_{\mathbb{R}^m}|\tilde{b}|_{\mathbb{R}^k}$, and symmetric, i.e. $|b\tilde{b}^\top|_{\mathbb{R}^{m\times k}} = |\tilde{b}b^\top|_{\mathbb{R}^{k\times m}}$.
- We denote the space of all bounded linear operators from $\mathbb{V}_1$ to $\mathbb{V}_2$ by $\mathcal{L}(\mathbb{V}_1; \mathbb{V}_2)$ and the space of n-ary linear operator from the n-product space $\mathbb{V}_1 \times \mathbb{V}_1 \times \cdots \times \mathbb{V}_1$ to $\mathbb{V}_2$ by $\mathcal{L}^{(n)}(\mathbb{V}_1 \times \mathbb{V}_1 \times \cdots \times \mathbb{V}_1; \mathbb{V}_2)$.

- We denote the space of all continuous functions from V to E by $C(V; E)$. Similarly, the space of all bounded continuous functions from V to E is denoted by $C_b(V; E)$. We equip $C_b(V; E)$ with the supremum norm $\|f\|_{C_b(V;E)} = \sup_{v\in V} d(f_v, e_0)$.
- If V is open, $f \in C(V; \mathbb{V}_2)$, and f is Fréchet-differentiable in V, we denote the Fréchet derivative of f by $Df : V \to \mathcal{L}(\mathbb{V}_1; \mathbb{V}_2)$. Similarly, the symbol $D^n f : V \to \mathcal{L}^{(n)}(\mathbb{V}_1 \times \mathbb{V}_1 \times \cdots \times \mathbb{V}_1; \mathbb{V}_2)$ stands for the n-th Fréchet derivative if it exists.
- For $n \in \mathbb{N}$, we denote

$$C_b^n(V; \mathbb{V}_2) = \left\{ f : V \to \mathbb{V}_2 \;\middle|\; D^k f \in C_b(V; \mathcal{L}^{(k)}(\mathbb{V}_1 \times \mathbb{V}_1 \times \cdots \times \mathbb{V}_1; \mathbb{V}_2)), k = 0, \ldots, n \right\},$$

  where, for $k = 0$, we identify the respective space of 0-ary operators with $\mathbb{V}_2$. We equip the space $C_b^n(V; \mathbb{V}_2)$ with the norm

$$\|f\|_{C_b^n(V;\mathbb{V}_2)} = \sum_{k=0}^{n} \|D^k f\|_{C_b(V;\mathcal{L}^{(k)}(\mathbb{V}_1\times\mathbb{V}_1\times\cdots\times\mathbb{V}_1;\mathbb{V}_2))}.$$

- For $\alpha \in (0, 1]$, the space of α-Hölder (resp. Lipschitz for $\alpha = 1$) continuous functions $C^{0,\alpha}(V; E)$ is defined by

$$C^{0,\alpha}(V; E) = \{f \in C(V; E) \mid [f]_{C^{0,\alpha}(V;E)} < \infty\},$$

  where

$$[f]_{C^{0,\alpha}(V;E)} = \sup_{\substack{v,\tilde{v}\in V \\ v\neq\tilde{v}}} \frac{d(f_v, f_{\tilde{v}})}{|v - \tilde{v}|^{\alpha}_{\mathbb{V}_1}}.$$

  Note that if $\alpha \in (0, 1)$, we simply write $C^{\alpha}(V; E) := C^{0,\alpha}(V; E)$.
- For $n \in \mathbb{N}$ and $\alpha \in (0, 1]$, we denote

$$C_b^{n,\alpha}(V; E) := \{f \in C_b^n(V; E) \mid D^n f \in C^{0,\alpha}(V; E)\}$$

  and equip it with the norm

$$\|f\|_{C_b^{n,\alpha}(V;E)} := \|f\|_{C_b^n(V;E)} + [D^n f]_{C^{0,\alpha}(V;\mathcal{L}^{(n)}(\mathbb{V}_1\times\mathbb{V}_1\times\cdots\times\mathbb{V}_1;\mathbb{V}_2))}.$$

- For $\mathcal{O} \subseteq \mathbb{R}^n$, $\mathfrak{L}(\mathcal{O})$ denotes the σ-algebra of Lebesgue measurable subsets of $\mathcal{O}$ and $\mathfrak{B}(E)$ denotes the σ-algebra of Borel measurable subsets of E. We write $L^0(\mathcal{O}; E)$ for the set of equivalence classes of measurable functions $f : (\mathcal{O}, \mathfrak{L}(\mathcal{O})) \to (E, \mathfrak{B}(E))$ with respect to equality almost everywhere.
- For $p \in (0, \infty]$ and $f \in L^p([0, T]; E)$, we define the L^p-modulus of continuity by

$$\omega_p(f, \tau) = \sup_{h\in[0,\tau]} \|d(f_\cdot, f_{\cdot+h})\|_{L^p([0,T-h])}, \quad \tau \in [0, T].$$

- Let $\alpha \in (0, 1)$ and $p, q \in (0, \infty]$. The Besov space $B^{\alpha}_{p,q}([0, T]; E)$ is defined by

$$B^{\alpha}_{p,q}([0, T]; E) = \left\{ f \in L^p([0, T]; E) \;\middle|\; [f]_{B^{\alpha}_{p,q}([0,T];E)} < \infty \right\},$$

  where

$$[f]_{B^{\alpha}_{p,q}([0,T];E)} = \left\| \frac{\omega_p(f, \tau)}{\tau^{\alpha}} \right\|_{L^q([0,T],\frac{\mathrm{d}\tau}{\tau})}.$$

  We remark that the form of $[\,\cdot\,]_{B^{\alpha}_{p,q}([0,T];E)}$ above corresponds to characterization of standard Besov space by differences, see e.g. [30, Section 5.2.2].

2.2 Hölder-type and Besov-type Spaces

We now recall the definitions of Hölder-type and Besov-type spaces used in the theory of rough paths. The Besov-type spaces were first defined in [12] to which we refer the reader for a more detailed exposition. Let us fix $d \in \{2, 3\}$, $T \in (0, \infty)$, and a normed vector space $(\mathbb{V}, |\cdot|_{\mathbb{V}})$ for the rest of this section. We denote

$$\triangle^d[0, T] = \{(u_1, \dots, u_d) \in [0, T]^d \mid u_1 \le \cdots \le u_d\}.$$

For $f : [0, T] \to \mathbb{V}$ and $\Xi : \triangle^2[0, T] \to \mathbb{V}$, we define $\delta f : \triangle^2[0, T] \to \mathbb{V}$ and $\delta\Xi : \triangle^3[0, T] \to \mathbb{V}$ by

$$\begin{aligned} \delta f_{s,t} &= f_t - f_s, && (s, t) \in \triangle^2[0, T], \\ \delta\Xi_{s,u,t} &= \Xi_{s,t} - \Xi_{s,u} - \Xi_{u,t}, && (s, u, t) \in \triangle^3[0, T]. \end{aligned}$$

We note that $\delta(\delta f) \equiv 0$. Next, we construct the space of measurable functions on the simplex $\triangle^d[0, T]$. Let $\Xi, \tilde{\Xi} : (\triangle^d[0, T], \mathfrak{L}(\triangle^d[0, T])) \to (\mathbb{V}, \mathfrak{B}(\mathbb{V}))$. For $d = 2$, we define

$$\Xi \sim_2 \tilde{\Xi} \text{ if } \Xi_{r,r+h} = \tilde{\Xi}_{r,r+h} \text{ for all } h \in [0, T] \text{ and almost all } r \in [0, T - h],$$

and, similarly, for $d = 3$, let

$$\Xi \sim_3 \tilde{\Xi} \text{ if } \Xi_{r,r+\theta h,r+h} = \tilde{\Xi}_{r,r+\theta h,r+h} \text{ for all } h \in [0, T], \text{ all } \theta \in [0, 1], \text{ and almost all } r \in [0, T - h].$$

It is straightforward to show that $\sim_d$ is an equivalence. We then define $L^{0;d}([0, T]; \mathbb{V})$ as the space of all equivalence classes of measurable functions $(\triangle^d[0, T], \mathfrak{L}(\triangle^d[0, T])) \to (\mathbb{V}, \mathfrak{B}(\mathbb{V}))$ with respect to equivalence $\sim_d$.

We can now recall the definition of the Hölder and Besov-type spaces. For $\alpha \in (0, \infty)$, we define the α-Hölder-type space by

$$C^{\alpha;2}([0, T]; \mathbb{V}) = \left\{\Xi : \triangle^2[0, T] \to \mathbb{V} \;\middle|\; \|\Xi\|_{C^{\alpha;2}([0,T];\mathbb{V})} < \infty\right\},$$

where

$$\|\Xi\|_{C^{\alpha;2}([0,T];\mathbb{V})} = \sup_{\substack{0<h\le T \\ 0\le r\le T-h}} \frac{|\Xi_{r,r+h}|_{\mathbb{V}}}{|h|^\alpha}.$$

Note that if $\alpha \in (0, 1)$, then $[f]_{C^\alpha([0,T];\mathbb{V})} = \|\delta f\|_{C^{\alpha;2}([0,T];\mathbb{V})}$.

For $p \in (0, \infty]$ and $\Xi \in L^{0;d}([0, T]; \mathbb{V})$, the L^p-modulus of continuity of Ξ is defined by

$$\omega_p^d(\Xi, \tau) = \begin{cases} \sup\limits_{h\in[0,\tau]} \|\Xi_{\cdot,\cdot+h}\|_{L^p([0,T-h];\mathbb{V})}, & d = 2, \\ \sup\limits_{\theta\in[0,1]} \sup\limits_{h\in[0,\tau]} \|\Xi_{\cdot,\cdot+\theta h,\cdot+h}\|_{L^p([0,T-h];\mathbb{V})}, & d = 3, \end{cases} \quad \tau \in [0, T],$$

and for $\alpha \in (0, \infty)$ and $p, q \in (0, \infty]$, we define the Besov-type space $B_{p,q}^{\alpha;d}([0, T]; \mathbb{V})$ by

$$B_{p,q}^{\alpha;d}([0, T]; \mathbb{V}) = \left\{\Xi \in L^{0;d}([0, T]; \mathbb{V}) \;\middle|\; \|\Xi\|_{B_{p,q}^{\alpha;d}([0,T];\mathbb{V})} < \infty\right\},$$

where

$$\|\Xi\|_{B_{p,q}^{\alpha;d}([0,T];\mathbb{V})} = \left\| \frac{\omega_p^d(\Xi, \tau)}{\tau^\alpha} \right\|_{L^q([0,T],\frac{d\tau}{\tau})}.$$

Note that for $\alpha \in (0, 1)$, it holds that $[f]_{B_{p,q}^\alpha([0,T];\mathbb{V})} = \|\delta f\|_{B_{p,q}^{\alpha;2}([0,T];\mathbb{V})}$.

2.3 Exponential Orlicz Spaces

Let us first recall some basic facts on exponential Orlicz spaces that will be needed for our analysis. For a thorough exposition, we refer the reader to the excellent monographs [25] and [27]. Let $\beta \in (0, \infty)$ and set $x_\beta = (\frac{1-\beta}{\beta})^{1/\beta}$ for $\beta < 1$ and $x_\beta = 0$ for $\beta \geq 1$. Define $\Psi_\beta, E_\beta, \Phi_\beta : [0, \infty) \to [0, \infty)$ by

$$\begin{aligned}\Psi_\beta(x) &= \exp(x^\beta) - 1,\\ E_\beta(x) &= \begin{cases}\Psi_\beta(x), & x \in [x_\beta, \infty),\\ \Psi_\beta(x_\beta) + \Psi'_\beta(x_\beta)(x - x_\beta), & x \in [0, x_\beta),\end{cases}\\ \Phi_\beta(x) &= E_\beta(x) - E_\beta(0),\end{aligned}$$

for $x \geq 0$. It is easily seen that Φ_β is a convex function such that $\Phi_\beta(0) = 0$ and $\lim_{x\to\infty} \Phi_\beta(x) = \infty$. Consequently, Φ_β is an example of Young function. Let $D \subseteq \mathbb{R}$ be a bounded (nondegenerate) interval. Recall that the *(real) exponential Orlicz space* $L^{\Phi_\beta}(D; \mathbb{R}) = L^{\Phi_\beta}(D) = L^{\Phi_\beta}$ is defined as the linear space

$$L^{\Phi_\beta}(D; \mathbb{R}) = \left\{ f \in L^0(D; \mathbb{R}) \,\middle|\, \exists \lambda \in (0, \infty) : \int_D \Phi_\beta\left(\frac{|f_r|}{\lambda}\right) \mathrm{d}r < \infty \right\}$$

and *the subspace of its finite elements* $L^{\Phi_\beta}_{\mathrm{fin}}(D; \mathbb{R}) = L^{\Phi_\beta}_{\mathrm{fin}}(D) = L^{\Phi_\beta}_{\mathrm{fin}}$ is defined as

$$L^{\Phi_\beta}_{\mathrm{fin}}(D; \mathbb{R}) = \left\{ f \in L^0(D; \mathbb{R}) \,\middle|\, \forall \lambda \in (0, \infty) : \int_D \Phi_\beta\left(\frac{|f_r|}{\lambda}\right) \mathrm{d}r < \infty \right\}.$$

Endowed with the *Luxemburg* (or *gauge*) *norm*

$$\|f\|_{L^{\Phi_\beta}} = \inf\left\{ \lambda \in (0, \infty) \,\middle|\, \int_D \Phi_\beta\left(\frac{|f_r|}{\lambda}\right) \mathrm{d}r \leq 1 \right\}, \quad f \in L^{\Phi_\beta},$$

the space L^{Φ_β} is a Banach space with $L^{\Phi_\beta}_{\mathrm{fin}}$ being its closed subspace. We emphasize that $L^{\Phi_\beta}_{\mathrm{fin}}(D; \mathbb{R}) \subsetneq L^{\Phi_\beta}(D; \mathbb{R})$ by, e.g., [25, Remark 4.12.4].

Roughly speaking, exponential Orlicz spaces measure the asymptotic growth of L^p-norms as p increases. More precisely, there is the following equivalence (see [4, Theorem (3.4)]):

$$\|\cdot\|_{L^{\Phi_\beta}([0,1])} \lesssim_\beta \sup_{p\in[1,\infty)} p^{-\frac{1}{\beta}} \|\cdot\|_{L^p([0,1])} \lesssim_\beta \|\cdot\|_{L^{\Phi_\beta}([0,1])}. \tag{2}$$

It is possible to construct exponential Orlicz spaces of metric space-valued functions. Let (E, d) be a nonempty metric space. Let again $\beta \in (0, \infty)$ and let $D \subseteq \mathbb{R}$ be a bounded nondegenerate interval. The *exponential Orlicz space of E-valued functions* $L^{\Phi_\beta}(D; E)$ is the space

$$L^{\Phi_\beta}(D; E) = \left\{ f \in L^0(D; E) \,\middle|\, \exists e_0 \in E : d(f., e_0) \in L^{\Phi_\beta}(D; \mathbb{R}) \right\}$$

and the *subspace of its finite elements* $L^{\Phi_\beta}_{\mathrm{fin}}(D; E)$ is

$$L^{\Phi_\beta}_{\mathrm{fin}}(D; E) = \left\{ f \in L^0(D; E) \,\middle|\, \exists e_0 \in E : d(f., e_0) \in L^{\Phi_\beta}_{\mathrm{fin}}(D; \mathbb{R}) \right\}.$$

The *Luxemburg norm* of $f \in L^{\Phi_\beta}(D; E)$ is defined by

$$\|f\|_{L^{\Phi_\beta}(D;E)} = \|d(f., e_0)\|_{L^{\Phi_\beta}(D;\mathbb{R})}$$

for $e_0 \in E$ fixed. There are the following observations for the space $L^{\Phi_\beta}(D; E)$ (and similarly for the space $L^{\Phi_\beta}_{\text{fin}}(D; E)$):

- If $f \in L^{\Phi_\beta}(D; E)$, then, for any $e \in E$, it holds that $d(f., e) \in L^{\Phi_\beta}(D; \mathbb{R})$. In particular, as discussed in Section 2.1, the choice of $e_0 \in E$ does not change the function space.
- Fix $T \in (0, \infty)$, $h \in [0, T]$, and $e \in E$. Then

$$f \in L^{\Phi_\beta}([0, T]; E) \quad \Longrightarrow \quad d(f_{.+h}, e), d(f., f_{.+h}) \in L^{\Phi_\beta}([0, T-h]; \mathbb{R}).$$

- If E is complete and separable, then $(L^{\Phi_\beta}(D; E), \|\cdot\|_{L^{\Phi_\beta}(D;E)})$ is also complete.
- If E is a linear space, then $L^{\Phi_\beta}(D; E)$ is also a linear space.

In the rest of the article, we will only work with exponential (Besov–)Orlicz(-type) spaces and we will omit the epithet "exponential" for simplicity. Below, we collect several properties of Orlicz spaces and Orlicz norms that will be frequently used throughout the whole text. To this end, let $T \in (0, \infty)$ be fixed. We start by a, fairly obvious, result that we often implicitly rely upon.

Lemma 2.1 *Let $\beta \in (0, \infty)$ and $p \in (0, \infty)$. Then, for all $f \in L^{\Phi_\beta}([0, T])$, it holds*

$$\| |f|^p \|_{L^{\Phi_\beta}([0,T])} = \|f\|^p_{L^{\Phi_{p\beta}}([0,T])}.$$

The following claim is an extension of the equivalence (2).

Lemma 2.2 *Let $\beta \in (0, \infty)$. Then, for $f \in L^{\Phi_\beta}([0, T])$, it holds*

$$\frac{1}{(1 \vee T)} \|f\|_{L^{\Phi_\beta}([0,T])} \lesssim_\beta \sup_{p \in [1,\infty)} p^{-\frac{1}{\beta}} \|f\|_{L^p([0,T])} \lesssim_\beta (1 \vee T) \|f\|_{L^{\Phi_\beta}([0,T])}.$$

Proof Define $f^T : [0, 1] \to \mathbb{R}$ by $f^T_r = f_{Tr}$ for $r \in [0, 1]$ if $T \geq 1$ and by $f^T_r = f_{r \wedge T} \mathbf{1}_{[0,T]}(r)$ for $r \in [0, 1]$ if $T < 1$. Then in all the four cases ($\beta \geq 1$ and $T \geq 1$, $\beta < 1$ and $T \geq 1$, $\beta \geq 1$ and $T < 1$, and $\beta < 1$ and $T < 1$), we obtain $f^T \in L^{\Phi_\beta}([0, 1])$ and, moreover, that

$$\|f^T\|_{L^{\Phi_\beta}([0,1])} \leq \|f\|_{L^{\Phi_\beta}([0,T])} \leq (1 \vee T) \|f^T\|_{L^{\Phi_\beta}([0,1])}$$

holds. (When $T < 1$, the inequality between the norms is trivial and when $T \geq 1$, one obtains the estimate by using the fact that for every $c \geq 1$, the inequality $c(\mathrm{e}^x - 1) \leq \mathrm{e}^{cx} - 1$ holds for every $x \geq 0$.) As we also have, for every $p \in [1, \infty)$, that

$$\|f\|_{L^p([0,T])} = (1 \vee T^{\frac{1}{p}}) \|f^T\|_{L^p([0,1])},$$

we obtain the claim by appealing to equivalence (2). □

There is also a Hölder-type inequality for the Luxemburg norm.

Lemma 2.3 *Let $\beta_1, \beta_2 \in (1, \infty)$ be such that $\beta_1 \beta_2 = \beta_1 + \beta_2$. Then the inequality*

$$\|fg\|_{L^{\Phi_1}([0,T])} \lesssim \|f\|_{L^{\Phi_{\beta_1}}([0,T])} \|g\|_{L^{\Phi_{\beta_2}}([0,T])}$$

holds for all $f, g \in L^0([0, T])$. Moreover, if $\beta \in (0, \infty)$, then

$$\|fg\|_{L^{\Phi_\beta}([0,T])} \lesssim_\beta \| |f|^\beta \|^{\frac{1}{\beta}}_{L^{\Phi_{\beta_1}}([0,T])} \| |g|^\beta \|^{\frac{1}{\beta}}_{L^{\Phi_{\beta_2}}([0,T])}. \tag{3}$$

Proof The first claim is a special case of Hölder's inequality for Orlicz spaces, see e.g. [27, Theorem 7, p. 64]). The rest follows from the first claim and Lemma 2.1 by the estimate

$$\|fg\|_{L^{\Phi_\beta}([0,T])} = \||fg|^\beta\|^{\frac{1}{\beta}}_{L^{\Phi_1}([0,T])} \lesssim \||f|^\beta\|^{\frac{1}{\beta}}_{L^{\Phi_{\beta_1}}([0,T])} \||g|^\beta\|^{\frac{1}{\beta}}_{L^{\Phi_{\beta_2}}([0,T])}.$$

□

We finish this section by an embedding result from [27, Theorem 3, p. 155] that can also be immediately obtained (albeit with a possibly worse constant) by repeated application of Lemma 2.2.

Proposition 2.4 *Let $\beta_1, \beta_2 \in (0,\infty)$ be such that $\beta_1 \le \beta_2$. If $f \in L^{\Phi_{\beta_2}}([0,T])$, then $f \in L^{\Phi_{\beta_1}}([0,T])$ and it holds*

$$\|f\|_{L^{\Phi_{\beta_1}}([0,T])} \lesssim_{\beta_1,\beta_2} (1 \vee T)\, \|f\|_{L^{\Phi_{\beta_2}}([0,T])}.$$

3 Exponential Besov–Orlicz-type Spaces

3.1 Univariate Besov–Orlicz Spaces

Let us now review basic results on the Besov–Orlicz spaces of functions with values in metric spaces that will be needed for our analysis. Let us fix $T \in (0,\infty)$, $\alpha \in (0,1)$, $\beta \in (0,\infty)$, $q \in (0,\infty]$, and a nonempty complete separable metric space (E,d) for the rest of this section.

Definition 3.1 For $f \in L^{\Phi_\beta}([0,T])$ and $\tau \in [0,T]$, set

$$\omega_{\Phi_\beta}(f,\tau) = \sup_{h\in[0,\tau]} \|d(f_\cdot, f_{\cdot+h})\|_{L^{\Phi_\beta}([0,T-h])}.$$

The *exponential Besov–Orlicz space* is defined as

$$B^\alpha_{\Phi_\beta,q}([0,T];E) = \left\{ f \in L^{\Phi_\beta}([0,T];E) \;\middle|\; [f]_{B^\alpha_{\Phi_\beta,q}([0,T];E)} < \infty \right\},$$

where

$$[f]_{B^\alpha_{\Phi_\beta,q}([0,T];E)} = \left\| \frac{\omega_{\Phi_\beta}(f,\tau)}{\tau^\alpha} \right\|_{L^q([0,T],\frac{\mathrm{d}\tau}{\tau})}.$$

There are the following observations for the space $B^\alpha_{\Phi_\beta,q}([0,T];E)$.

- The map $[\,\cdot\,]_{B^\alpha_{\Phi_\beta,q}([0,T];E)}$ is a (quazi)seminorm on $B^\alpha_{\Phi_\beta,q}([0,T];E)$ and a seminorm if $q \ge 1$.
- If E is a linear space, then $B^\alpha_{\Phi_\beta,q}([0,T];E)$ is also a linear space. In this case, we define

$$d_{B^\alpha_{\Phi_\beta,q}}(f,g) = \|f-g\|_{L^{\Phi_\beta}([0,T];E)} + [f-g]^{q\wedge 1}_{B^\alpha_{\Phi_\beta,q}([0,T];E)} \tag{4}$$

for $f,g \in B^\alpha_{\Phi_\beta,q}([0,T];E)$. Then it follows that $d_{B^\alpha_{\Phi_\beta,q}}$ is a metric on $B^\alpha_{\Phi_\beta,q}([0,T];E)$ and as the metric space (E,d) is complete and separable, the metric space

$$(B^\alpha_{\Phi_\beta,q}([0,T];E), d_{B^\alpha_{\Phi_\beta,q}([0,T];E)})$$

is also complete.

Next, we summarize the embeddings between Besov–Orlicz spaces and Besov and Hölder spaces.

Proposition 3.2 *Let $f \in B^{\alpha}_{\Phi_\beta,q}([0,T];E)$. Then for any $p \in [1,\infty)$ and $p' \in (\frac{1}{\alpha},\infty)$, it holds*

$$[f]_{B^{\alpha}_{p,q}([0,T];E)} \lesssim_{\beta,p} (1 \vee T)[f]_{B^{\alpha}_{\Phi_\beta,q}([0,T];E)},$$
$$[f]_{C^{\alpha-1/p'}([0,T];E)} \lesssim_{\alpha,\beta,p',q} (1 \vee T)[f]_{B^{\alpha}_{\Phi_\beta,q}([0,T];E)}.$$

In particular, the inclusion $B^{\alpha}_{\Phi_\beta,q}([0,T];E) \subseteq L^{\infty}([0,T];E) \subseteq L^{\Phi_\beta}_{\mathrm{fin}}([0,T];E)$ holds.

Proof The first embedding $B^{\alpha}_{\Phi_\beta,q}([0,T];E) \hookrightarrow B^{\alpha}_{p,q}([0,T];E)$ is an immediate consequence of Lemma 2.2. The second embedding then follows from the first one and the classical Sobolev embedding from, e.g., [29, Theorem 3.3.1]. Thus, since the elements of $B^{\alpha}_{\Phi_\beta,q}([0,T];E)$ are bounded, the remaining claim follows from a straightforward estimate.□

The Besov–Orlicz (quazi)seminorm allows a discrete characterization which will be useful below.

Proposition 3.3 *For $f \in B^{\alpha}_{\Phi_\beta,q}([0,T];E)$, it holds*

$$[f]_{B^{\alpha}_{\Phi_\beta,q}([0,T];E)} \lesssim_{\alpha,q} \left\| \left(T2^{-n}\right)^{-\alpha} \|d(f_\cdot, f_{\cdot+T2^{-n}})\|_{L^{\Phi_\beta}([0,T-T2^{-n}])} \right\|_{\ell^q(n=1,2,\dots)} \lesssim_{\alpha,q} [f]_{B^{\alpha}_{\Phi_\beta,q}([0,T];E)}.$$

Proof The proofs of both inequalities are similar to the proofs of the corresponding inequalities in [12, Lemma 2.2]. We therefore omit the details and only discuss the nonobvious differences.

The proof of the first inequality uses the continuity of the map $H : [0,T] \to [0,\infty)$ defined by $h \mapsto \|d(f_\cdot, f_{\cdot+h})\|_{L^{\Phi_\beta}([0,T-h])}$. While this continuity is almost trivial in the standard L^p-case, it requires a more careful approach in the case of exponential Orlicz spaces. The continuity of H at 0 follows immediately from $f \in B^{\alpha}_{\Phi_\beta,q}([0,T];E)$ by contradiction. To prove its continuity in $(0,T]$, let $0 \le h_1 < h_2 \le T$ be arbitrary. Then it holds that

$$H(h_2) \le \|d(f_\cdot, f_{\cdot+h_1})\|_{L^{\Phi_\beta}([0,T-h_2])} + \|d(f_{\cdot+h_1}, f_{\cdot+h_2})\|_{L^{\Phi_\beta}([0,T-h_2])} \le H(h_1) + H(h_2-h_1).$$

Moreover, by splitting the interval $[0,T-h_1]$ into $[0,T-h_2]$ and $[T-h_2,T-h_1]$, we deduce

$$\begin{aligned} H(h_1) &\le \|d(f_\cdot, f_{\cdot+h_1})\|_{L^{\Phi_\beta}([0,T-h_2])} + \|d(f_\cdot, f_{\cdot+h_1})\|_{L^{\Phi_\beta}([T-h_2,T-h_1])} \\ &\le H(h_2) + \|d(f_{\cdot+h_1}, f_{\cdot+h_2})\|_{L^{\Phi_\beta}([0,T-h_2])} + \|d(f_\cdot, f_{\cdot+h_1})\|_{L^{\Phi_\beta}([T-h_2,T-h_1])} \\ &\le H(h_2) + H(h_2-h_1) + \|d(f_\cdot, e_0)\|_{L^{\Phi_\beta}([T-h_2,T-h_1])} + \|d(f_\cdot, e_0)\|_{L^{\Phi_\beta}([T-(h_2-h_1),T])}. \end{aligned}$$

By combining the two estimates above, we obtain

$$|H(h_2) - H(h_1)| \le H(h_2-h_1) + \|d(f_\cdot, e_0)\|_{L^{\Phi_\beta}([T-h_2,T-h_1])} + \|d(f_\cdot, e_0)\|_{L^{\Phi_\beta}([T-(h_2-h_1),T])}.$$

The term $H(h_2-h_1)$ converges to 0 as $(h_2-h_1) \to 0$ by the continuity of H at 0. Let us show the convergence to 0 of the second term; the convergence of the third term follows similarly. Since $f \in L^{\Phi_\beta}_{\mathrm{fin}}([0,T];E)$ by Lemma 3.2, the convergence

$$\lim_{h_2 \to h_1+} \int_{T-h_2}^{T-h_1} \Phi_\beta\left(\frac{d(f_r, e_0)}{\lambda}\right) \mathrm{d}r = 0 \tag{5}$$

holds for all $\lambda \in (0,\infty)$ by the continuity of the Lebesgue integral. If $\|d(f_\cdot, e_0)\|_{L^{\Phi_\beta}([T-h_2,T-h_1])}$ does not converge to 0, we may find $\varepsilon \in (0,\infty)$ and a sequence $\{h^n\}_{n=1}^\infty$ satisfying $h^n \searrow h_1$ such that

$$\inf\left\{\lambda \in (0,\infty) \,\middle|\, \int_{T-h^n}^{T-h_1} \Phi_\beta\left(\frac{d(f_r,e_0)}{\lambda}\right) \mathrm{d}r \le 1\right\} > \varepsilon,$$

for all $n \in \mathbb{N}$. In particular, for all $n \in \mathbb{N}$, it holds

$$\int_{T-h^n}^{T-h_1} \Phi_\beta\left(\frac{d(f_r,e_0)}{\frac{\varepsilon}{2}}\right) \mathrm{d}r > 1,$$

which is a contradiction with (5). □

The following proposition discusses the embeddings of Besov–Orlicz spaces.

Proposition 3.4 *Let $\alpha_1, \alpha_2 \in (0,1)$, $\beta_1, \beta_2 \in (0,\infty)$, and $q_1, q_2 \in (0,\infty]$ satisfy $\alpha_1 \le \alpha_2$, $\beta_1 \le \beta_2$, and $q_1 \ge q_2$. Then for all $f \in L^0([0,T];E)$, the following inequalities hold:*

$$\begin{aligned}
[f]_{B^{\alpha_1}_{\Phi_\beta,q}([0,T];E)} &\le T^{\alpha_2-\alpha_1}[f]_{B^{\alpha_2}_{\Phi_\beta,q}([0,T];E)},\\
[f]_{B^{\alpha}_{\Phi_{\beta_1},q}([0,T];E)} &\lesssim_{\beta_1,\beta_2} (1 \vee T)\,[f]_{B^{\alpha}_{\Phi_{\beta_2},q}([0,T];E)},\\
[f]_{B^{\alpha}_{\Phi_\beta,q_1}([0,T];E)} &\lesssim_{\alpha,q_1,q_2} [f]_{B^{\alpha}_{\Phi_\beta,q_2}([0,T];E)}.
\end{aligned}$$

Proof The first inequality follows from the definition of the Besov–Orlicz (quazi)seminorm directly. The second inequality can then be deduced from Proposition 2.4 and the third inequality can be established by appealing to Proposition 3.3 and the monotonicity of ℓ^q-spaces. □

The smoothness parameter α plays a more prominent role than the fine parameter q. In particular, a Besov–Orlicz space with smaller α contains Besov–Orlicz spaces of any larger q. More precisely, there is the following embedding result:

Proposition 3.5 *If $\varepsilon \in (0,\alpha)$, $q_1, q_2 \in (0,\infty]$, and $f \in B^{\alpha}_{\Phi_\beta,q_2}([0,T];E)$, then*

$$[f]_{B^{\alpha-\varepsilon}_{\Phi_\beta,q_1}([0,T];E)} \lesssim^T_{\alpha,\varepsilon,q_1} [f]_{B^{\alpha}_{\Phi_\beta,\infty}([0,T];E)} \lesssim_{\alpha,q_2} [f]_{B^{\alpha}_{\Phi_\beta,q_2}([0,T];E)}.$$

Proof We have

$$\begin{aligned}
[f]_{B^{\alpha-\varepsilon}_{\Phi_\beta,q_1}([0,T];E)} &\lesssim_{\alpha,\varepsilon,q_1} \left(\sum_{n=1}^\infty \left(\left(T2^{-n}\right)^{-(\alpha-\varepsilon)} \|d(f_\cdot, f_{\cdot+T2^{-n}})\|_{L^{\Phi_\beta}([0,T-T2^{-n}])}\right)^{q_1}\right)^{\frac{1}{q_1}}\\
&\le T^\varepsilon \left(\sup_{n\in\mathbb{N}} \left(\left(T2^{-n}\right)^{-\alpha} \|d(f_\cdot, f_{\cdot+T2^{-n}})\|_{L^{\Phi_\beta}([0,T-T2^{-n}])}\right)^{q_1} \sum_{n=1}^\infty 2^{-n\varepsilon q_1}\right)^{\frac{1}{q_1}}\\
&\lesssim^T_{\alpha,\varepsilon,q_1} [f]_{B^{\alpha}_{\Phi_\beta,\infty}([0,T];E)}\\
&\lesssim_{\alpha,q_2} [f]_{B^{\alpha}_{\Phi_\beta,q_2}([0,T];E)}
\end{aligned}$$

where we used Proposition 3.3 in the first and the third inequality and the last inequality follows by the third assertion of Proposition 3.4.

For $e \in E$, the subspace of $B^{\alpha}_{\Phi_\beta,q}([0,T];E)$ of all functions originating from e is denoted by

$$B^{\alpha}_{\Phi_\beta,q}([0,T];E,e) = \{f \in B^{\alpha}_{\Phi_\beta,q}([0,T];E) \mid f_0 = e\}. \tag{6}$$

Note that the space $B^{\alpha}_{\Phi_\beta,q}([0,T];E,e)$ is well-defined since $B^{\alpha}_{\Phi_\beta,q}([0,T];E) \subseteq C([0,T];E)$ by Proposition 3.2. In the rest of this section, we discuss Besov–Orlicz spaces of functions with values in a separable Banach space $\mathbb{V}$. In particular, we establish equivalence of certain norms and recall that the seminorm $[\cdot]_{B^{\alpha}_{\Phi_\beta,q}([0,T];\mathbb{V})}$ is a norm on affine subspaces of $B^{\alpha}_{\Phi_\beta,q}([0,T];\mathbb{V})$ of the form (6) for $q \geq 1$. This will be important for controlled rough paths in subsequent sections.

Proposition 3.6 *Let $(\mathbb{V}, |\cdot|_{\mathbb{V}})$ be a separable Banach space.*

i) The mappings

$$\begin{aligned}
N_{\Phi_\beta} &: B^{\alpha}_{\Phi_\beta,q}([0,T];\mathbb{V}) \to \mathbb{R} : f \mapsto (\|f\|_{L^{\Phi_\beta}([0,T];\mathbb{V})} + [f]_{B^{\alpha}_{\Phi_\beta,q}([0,T];\mathbb{V})}),\\
N_0 &: B^{\alpha}_{\Phi_\beta,q}([0,T];\mathbb{V}) \to \mathbb{R} : f \mapsto (|f_0|_{\mathbb{V}} + [f]_{B^{\alpha}_{\Phi_\beta,q}([0,T];\mathbb{V})}),\\
N_\infty &: B^{\alpha}_{\Phi_\beta,q}([0,T];\mathbb{V}) \to \mathbb{R} : f \mapsto (\|f\|_{L^{\infty}([0,T];\mathbb{V})} + [f]_{B^{\alpha}_{\Phi_\beta,q}([0,T];\mathbb{V})})
\end{aligned}$$

are equivalent quazinorms on $B^{\alpha}_{\Phi_\beta,q}([0,T];\mathbb{V})$, resp. equivalent norms if $q \geq 1$, and, for $f \in B^{\alpha}_{\Phi_\beta,q}([0,T];\mathbb{V})$, it holds

$$\begin{aligned}
T N_0(f) &\lesssim^{T}_{\alpha,\beta,q} N_{\Phi_\beta}(f) \lesssim^{T}_{\alpha,\beta,q} N_0(f),\\
T N_\infty(f) &\lesssim^{T}_{\alpha,\beta,q} N_{\Phi_\beta}(f) \lesssim^{T}_{\alpha,\beta,q} N_\infty(f).
\end{aligned}$$

Moreover, for all $p \in (\frac{1}{\alpha}, \infty)$, it holds

$$\|f\|_{L^{\infty}([0,T];\mathbb{V})} \lesssim_{\alpha,\beta,q,p} |f_0|_{\mathbb{V}} + T^{\alpha-\frac{1}{p}}(1 \vee T)[f]_{B^{\alpha}_{\Phi_\beta,q}([0,T];\mathbb{V})}.$$

ii) Let $v \in \mathbb{V}$ and set

$$\rho(f,g) = [f-g]^{q\wedge 1}_{B^{\alpha}_{\Phi_\beta,q}([0,T];\mathbb{V})}$$

for $f, g \in B^{\alpha}_{\Phi_\beta,q}([0,T];\mathbb{V},v)$. Then ρ is a metric on $B^{\alpha}_{\Phi_\beta,q}([0,T];\mathbb{V},v)$ and, $[\cdot]_{B^{\alpha}_{\Phi_\beta,q}([0,T];\mathbb{V})}$ is a quazinorm, resp. norm if $q \geq 1$, on $B^{\alpha}_{\Phi_\beta,q}([0,T];\mathbb{V},v)$. Moreover, $B^{\alpha}_{\Phi_\beta,q}([0,T];\mathbb{V},v)$ is a closed (affine) subspace of $B^{\alpha}_{\Phi_\beta,q}([0,T];\mathbb{V})$.

We note that the constants in Proposition 3.6 depend on T in a nondecreasing manner.

Proof The proof of the equivalence of the norms mirrors the standard proofs of the equivalence of the norms on spaces of Hölder-continuous functions and it is therefore omitted. The remaining estimate in i) can be established by

$$\|f\|_{L^{\infty}([0,T];\mathbb{V})} \leq |f_0|_{\mathbb{V}} + \sup_{r\in[0,T]} |f_r - f_0|_{\mathbb{V}} \leq |f_0|_{\mathbb{V}} + T^{\alpha-\frac{1}{p}}[f]_{C^{\alpha-1/p}([0,T];\mathbb{V})}$$

and the embedding into Hölder-continuous functions from Proposition 3.2. The properties of ρ and $f \mapsto [f]_{B^{\alpha}_{\Phi_\beta,q}([0,T];\mathbb{V})}$ follow from the definitions and the restriction $f_0 = v$. □

3.2 Multivariate Besov–Orlicz-type Spaces

In this section, Besov–Orlicz-type spaces of multivariate maps suitable for rough path analysis are defined and some of their properties are collected. In the whole section, let us fix a nonempty separable Banach space $(\mathbb{V}, |\cdot|_{\mathbb{V}})$, $d \in \{2, 3\}$, $T \in (0, \infty)$, $\alpha \in (0, \infty)$, $\beta \in (0, \infty)$, and $q \in (0, \infty]$.

Definition 3.7 For $\Xi \in L^{0;d}([0, T]; \mathbb{V})$ and $\tau \in [0, T]$, set

$$\omega^d_{\Phi_\beta}(\Xi, \tau) = \begin{cases} \sup\limits_{h\in[0,\tau]} \| |\Xi_{r,r+h}|_{\mathbb{V}} \|_{L^{\Phi_\beta}([0,T-h])}, & d = 2, \\ \sup\limits_{\theta\in[0,1]} \sup\limits_{h\in[0,\tau]} \| |\Xi_{r,r+\theta h,r+h}|_{\mathbb{V}} \|_{L^{\Phi_\beta}([0,T-h])}, & d = 3. \end{cases} \tag{7}$$

We define the *(multivariate) exponential Besov–Orlicz-type space* by

$$B^{\alpha;d}_{\Phi_\beta,q}([0, T]; \mathbb{V}) = \left\{ \Xi \in L^{0;d}([0, T]; \mathbb{V}) \,\middle|\, \|\Xi\|_{B^{\alpha;d}_{\Phi_\beta,q}([0,T];\mathbb{V})} < \infty \right\},$$

where

$$\|\Xi\|_{B^{\alpha;d}_{\Phi_\beta,q}([0,T];\mathbb{V})} = \left\| \frac{\omega^d_{\Phi_\beta}(\Xi, \tau)}{\tau^\alpha} \right\|_{L^q([0,T],\frac{\mathrm{d}\tau}{\tau})}.$$

There are the following observations for the space $B^{\alpha;d}_{\Phi_\beta,q}([0, T]; \mathbb{V})$.

- The map $\|\cdot\|_{B^{\alpha;d}_{\Phi_\beta,q}([0,T];\mathbb{V})}$ is a quazinorm on $B^{\alpha;d}_{\Phi_\beta,q}([0, T]; \mathbb{V})$ and a norm if $q \geq 1$.
- If we define

$$d_{B^{\alpha;d}_{\Phi_\beta,q}([0,T];\mathbb{V})}(\Xi, \tilde{\Xi}) = \|\Xi - \tilde{\Xi}\|^{q\wedge 1}_{B^{\alpha;d}_{\Phi_\beta,q}([0,T];\mathbb{V})} \tag{8}$$

  for $\Xi, \tilde{\Xi} \in B^{\alpha;d}_{\Phi_\beta,q}([0, T]; \mathbb{V})$, then the map $d_{B^{\alpha;d}_{\Phi_\beta,q}([0,T];\mathbb{V})}$ is a metric on $B^{\alpha;d}_{\Phi_\beta,q}([0, T]; \mathbb{V})$.

  As $(\mathbb{V}, |\cdot|_{\mathbb{V}})$ is a separable Banach space, the space $B^{\alpha;d}_{\Phi_\beta,q}([0, T]; \mathbb{V})$ is also a Banach space.

First, we summarize the relation of the Besov–Orlicz-type spaces to the Besov-type and Hölder-type spaces defined in Section 2.

Proposition 3.8 *Let $\Xi \in B^{\alpha;2}_{\Phi_\beta,q}([0, T]; \mathbb{V})$, then for all $p \in [1, \infty)$ it holds*

$$\|\Xi\|_{B^{\alpha;2}_{p,q}([0,T];\mathbb{V})} \lesssim_{\beta,p} (1 \vee T)\|\Xi\|_{B^{\alpha;2}_{\Phi_\beta,q}([0,T];\mathbb{V})}.$$

If $\mathbb{V} = \mathbb{R}^m$ and if, additionally, Ξ satisfies

$$|\delta\Xi_{s,u,t}|_{\mathbb{R}^m} \leq M \left(((u - s) \wedge (t - u))^\theta ((u - s) \vee (t - u))^{1-\theta} \right)^{\alpha-\frac{1}{p}} \tag{9}$$

for some $M \in (0, \infty)$, $\theta \in (0, \frac{1}{2}]$, $p \in (\frac{1}{\alpha}, \infty)$, and all $(s, u, t) \in \triangle^3[0, T]$, then Ξ has a continuous version and it holds

$$\|\Xi\|_{C^{\alpha-1/p;2}([0,T];\mathbb{R}^m)} \lesssim_{\theta,\alpha,\beta,p} (1 \vee T)\|\Xi\|_{B^{\alpha;2}_{\Phi_\beta,q}([0,T];\mathbb{R}^m)} + M.$$

Proof The first claim follows from the definitions of the respective norms and Lemma 2.2 immediately. The second claim can be established by [12, Proposition 2.7] and the first claim. □

Next, we give an interpolation result whose proof is similar to that of [12, Lemma 2.7] and is therefore omitted.

Lemma 3.9 *Let $\gamma \in (\alpha, \infty)$, $\beta' \in (\beta, \infty)$, and $\delta \in (0, \infty)$ satisfy $\delta(1 - \frac{\beta}{\beta'}) + \gamma \frac{\beta}{\beta'} - \alpha > 0$. Then*

$$\|\Xi\|_{B^{\alpha;2}_{\Phi_{\beta'},q}([0,T];\mathbb{V})} \lesssim_{\alpha,\gamma,\beta,\beta',q,\delta} T^{\delta(1-\frac{\beta}{\beta'})+\gamma\frac{\beta}{\beta'}-\alpha} \|\Xi\|^{1-\frac{\beta}{\beta'}}_{C^{\delta;2}([0,T];\mathbb{V})} \|\Xi\|^{\frac{\beta}{\beta'}}_{B^{\gamma;2}_{\Phi_\beta,q}([0,T];\mathbb{V})}$$

holds for any $\Xi \in L^{0;2}([0,T];\mathbb{V})$.

In the rest of the section, we discuss embeddings of the Besov–Orlicz-type spaces defined above. The monotonicity of Besov–Orlicz-type spaces with respect to the smoothness parameter α and the integrability parameter β can be established analogously as in Proposition 3.4 and therefore, we skip the proof.

Proposition 3.10 *If $\alpha_1, \alpha_2, \beta_1, \beta_2 \in (0, \infty)$ satisfy $\alpha_1 \le \alpha_2$ and $\beta_1 \le \beta_2$, then*

$$\|\Xi\|_{B^{\alpha_1;d}_{\Phi_\beta,q}([0,T];\mathbb{V})} \le T^{\alpha_2-\alpha_1} \|\Xi\|_{B^{\alpha_2;d}_{\Phi_\beta,q}([0,T];\mathbb{V})},$$
$$\|\Xi\|_{B^{\alpha;d}_{\Phi_{\beta_1},q}([0,T];\mathbb{V})} \lesssim_{\beta_1,\beta_2} (1 \vee T) \|\Xi\|_{B^{\alpha;d}_{\Phi_{\beta_2},q}([0,T];\mathbb{V})}$$

hold for $\Xi \in L^{0;d}([0,T];\mathbb{V})$.

Compared to the standard Besov–Orlicz, resp. Besov, spaces the definition of the Besov–Orlicz-type, resp. Besov-type, space does not enforce the boundedness of $\omega^d_{\Phi_\beta}(\Xi,\cdot)$ on $[0,T]$. Thus, the monotonicity of the Besov–Orlicz-type spaces with respect to the fine parameter q does not hold without additional assumptions. We demonstrate this in the following example.

Example 3.11 Let $\theta \in (1,\infty)$ and consider $\Xi^\theta : \triangle^2[0,T] \to \mathbb{R}$ such that

$$\Xi^\theta_{r,r+h} = \frac{(T-h)^{-\theta} \mathbf{1}_{[\frac{T}{2},T]}(h)}{\|1\|_{L^{\Phi_\beta}([0,T-h])}}, \quad h \in [0,T),\ r \in [0,T-h].$$

Then $\Xi^\theta \in B^{\alpha;2}_{\Phi_\beta,q}([0,T];\mathbb{R})$ if and only if $q < \frac{1}{\theta-1}$. Hence, choosing $0 < q_2 < \frac{1}{\theta-1} < q_1$, we observe $\|\Xi^\theta\|_{B^{\alpha;2}_{\Phi_\beta,q_1}([0,T];\mathbb{R})} = \infty$ while $\|\Xi^\theta\|_{B^{\alpha;2}_{\Phi_\beta,q_2}([0,T];\mathbb{R})} < \infty$.

Proposition 3.12 *Let $q_1, q_2 \in (0,\infty]$ be such that $q_2 \le q_1$ and let $\Xi \in B^{\alpha;2}_{\Phi_\beta,q_2}([0,T];\mathbb{R}^m)$. Then the following holds:*

(1) Let $\omega^2_{\Phi_\beta}(\Xi,\cdot)$ be bounded on $[0,T]$. Then $\Xi \in B^{\alpha;2}_{\Phi_\beta,q_1}([0,T];\mathbb{R}^m)$.

(2) If (9) holds for some $M \in (0,\infty)$, $\theta \in (0,\frac{1}{2}]$, $p \in (\frac{1}{\alpha},\infty)$, and all $(s,u,t) \in \triangle^3[0,T]$, then $\omega^2_{\Phi_\beta}(\Xi,\cdot)$ is bounded on $[0,T]$ and there is the following estimate:

$$\|\Xi\|_{B^{\alpha;2}_{\Phi_\beta,q_1}([0,T];\mathbb{R}^m)} \lesssim^T_{\alpha,\beta,p,q_1,q_2} \|\Xi\|_{B^{\alpha;2}_{\Phi_\beta,q_2}([0,T];\mathbb{R}^m)} + MT^{-\frac{1}{p}}.$$

With a slight abuse of terminology, we will often refer to the result of Proposition 3.12 as "the inclusion $B^{\alpha;2}_{\Phi_\beta,q_2} \subseteq B^{\alpha;2}_{\Phi_\beta,q_1}$" and we will not mention the additional assumptions explicitly.

Proof To prove the first claim, we first assume $q_2 < \infty$ and $q_1 = \infty$. Since $\Xi \in B^{\alpha;2}_{\Phi_\beta,q_2}([0,T];\mathbb{R}^m)$, the function $\tau \mapsto \omega^2_{\Phi_\beta}(\Xi,\tau)\tau^{-\alpha}$ is bounded near 0 and thus, by the assumption on boundedness of $\omega^2_{\Phi_\beta}(\Xi,\cdot)$, we obtain $\Xi \in B^{\alpha;2}_{\Phi_\beta,\infty}([0,T];\mathbb{R}^m)$. For $q_1 < \infty$, the inclusion $B^{\alpha;2}_{\Phi_\beta,q_2}([0,T];\mathbb{R}^m) \subseteq B^{\alpha;2}_{\Phi_\beta,q_1}([0,T];\mathbb{R}^m)$ then follows by the straightforward estimate

$$\|\Xi\|_{B^{\alpha;2}_{\Phi_\beta,q_1}([0,T];\mathbb{R}^m)} \le \|\Xi\|_{B^{\alpha;2}_{\Phi_\beta,\infty}([0,T];\mathbb{R}^m)}^{1-q_2/q_1} \|\Xi\|_{B^{\alpha;2}_{\Phi_\beta,q_2}([0,T];\mathbb{R}^m)}^{q_2/q_1} < \infty.$$

Regarding the second claim, Proposition 3.8 implies $\Xi \in C^{\alpha-1/p;2}([0,T];\mathbb{R}^m)$ and hence, by Lemma 2.2, it holds that

$$\omega^2_{\Phi_\beta}(\Xi,T) \lesssim^T_\beta \sup_{0\le h\le T}\sup_{p\ge 1} p^{-\frac{1}{\beta}} \left(\int_0^{T-h} |\Xi_{r,r+h}|^p \,\mathrm{d}r\right)^{\frac{1}{p}} \le \|\Xi\|_{C^{\alpha-1/p;2}([0,T];\mathbb{R}^m)} T^\alpha < \infty.$$

Let us show the explicit bound for $q_1 < \infty$; the case $q_1 = \infty$ can be established similarly. First, from (9), we deduce that

$$\|\Xi_{\cdot,\cdot+h}\|_{L^{\Phi_\beta}([0,T-h])} \le \|\Xi_{\cdot,\cdot+\frac{h}{2}}\|_{L^{\Phi_\beta}([0,T-h])} + \|\Xi_{\cdot+\frac{h}{2},\cdot+h}\|_{L^{\Phi_\beta}([0,T-h])} + MT^{\alpha-\frac{1}{p}}\|1\|_{L^{\Phi_\beta}([0,T-h])}$$

holds for every $h \in [0,T]$. Since $\|1\|_{L^{\Phi_\beta}([0,T-h])} \le C_\beta(1\vee T)$ for some $C_\beta \in (0,\infty)$ by a direct calculation, the above implies

$$\omega^2_{\Phi_\beta}(\Xi,T) \le 2\omega^2_{\Phi_\beta}\left(\Xi,\frac{T}{2}\right) + M_{\beta,T}T^{\alpha-\frac{1}{p}}, \tag{10}$$

where we write $M_{\beta,T} = MC_\beta(1\vee T)$ for brevity.

Next, we estimate $\|\Xi\|_{B^{\alpha;2}_{\Phi_\beta,q_1}([0,T];\mathbb{R}^m)}$ by a discrete sum similarly as in Proposition 3.3. In particular, we split the interval $[0,T]$ in the outer integral in the definition of $\|\Xi\|_{B^{\alpha;2}_{\Phi_\beta,q_1}([0,T];\mathbb{R}^m)}$ into intervals $[T2^{-n-1}, T2^{-n}]$, $n \in \mathbb{N}_0$, and use the estimates

$$\tau^{-\alpha q_1} \le (T2^{-n-1})^{-\alpha q_1} \quad \text{and} \quad \omega^2_{\Phi_\beta}(\Xi,\tau) \le \omega^2_{\Phi_\beta}(\Xi,T2^{-n})$$

that holds for $\tau \in [T2^{-n-1}, T2^{-n}]$ to obtain

$$\|\Xi\|_{B^{\alpha;2}_{\Phi_\beta,q_1}([0,T];\mathbb{R}^m)} \le (\log 2)^{\frac{1}{q_1}} \left(\frac{2}{T}\right)^\alpha \left(\sum_{n=0}^\infty \left(2^{\alpha n}\omega^2_{\Phi_\beta}(\Xi,T2^{-n})\right)^{q_1}\right)^{\frac{1}{q_1}}. \tag{11}$$

Applying (10) to the zeroth term of the sum in (11) and recalling the monotonicity of ℓ^q-spaces and the inequality $(a+b)^q \le 2^q(a^q+b^q)$ that holds for $a,b \in [0,\infty)$ and $q \in [0,\infty)$, one derives

$$\|\Xi\|_{B^{\alpha;2}_{\Phi_\beta,q_1}([0,T];\mathbb{R}^m)} \lesssim_{\alpha,q_1,q_2} T^{-\alpha}\left(\sum_{n=1}^\infty \left(2^{\alpha n}\omega^2_{\Phi_\beta}(\Xi,T2^{-n})\right)^{q_2}\right)^{\frac{1}{q_2}} + M_{\beta,T}T^{-\frac{1}{p}}.$$

The desired claim now follows from the inequality

$$\sum_{n=1}^\infty \left(2^{\alpha n}\omega^2_{\Phi_\beta}(\Xi,T2^{-n})\right)^{q_2} \le \frac{(2T)^{\alpha q_2}}{\log 2}\|\Xi\|^{q_2}_{B^{\alpha;2}_{\Phi_\beta,q_2}([0,T];\mathbb{R}^m)},$$

which can be proved similarly as (11). □

4 Exponential Besov–Orlicz rough Paths

In this section, we study the exponential Besov–Orlicz rough paths and the related results such as a sewing lemma, rough integral and its stability. Our treatment is not essentially different from the ones in, e.g., Friz and Hairer [10] and Friz and Seeger [12]. However, in contrast to the latter reference, we consider only compositions of controlled rough paths with C_b^3 functions, see the discussion above Theorem 4.11 below.

Let $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, $\beta \in (0, \infty)$, and $q \in (0, \infty]$ be fixed in the whole section.

4.1 The Space of Exponential Besov–Orlicz rough Paths

We begin by the definition of the Besov–Orlicz rough paths and provide some basic estimates.

Definition 4.1 Let $X \in B^{\alpha}_{\Phi_\beta, q}([0,T];\mathbb{R}^n)$ and $\mathbb{X} \in B^{2\alpha;2}_{\Phi_{\beta/2}, q/2}([0,T];\mathbb{R}^{n\times n})$ satisfy the so-called *Chen's relations*, i.e.

$$\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = (\delta X_{s,u})(\delta X_{u,t})^\top, \quad (s,u,t) \in \Delta^3[0,T].$$

Then the pair $\mathbf{X} = (X, \mathbb{X})$ is called an *exponential Besov–Orlicz rough path (over $\mathbb{R}^n$)*, or an *exponential Besov–Orlicz rough path lift of X*. We denote the space of all exponential Besov–Orlicz rough paths by $\mathscr{B}^{\alpha}_{\Phi_\beta, q} = \mathscr{B}^{\alpha}_{\Phi_\beta, q}([0,T];\mathbb{R}^n)$.

In what follows, we show that a Besov–Orlicz rough path can be identified with a function that takes values in a truncated tensor algebra of level 2 of Besov–Orlicz regularity. Recall that the truncated tensor algebras of level 1 and of level 2 are defined by

$$T_1^{(1)}(\mathbb{R}^n) = \{(1,b) \,|\, b \in \mathbb{R}^n\} \quad \text{and} \quad T_1^{(2)}(\mathbb{R}^n) = \{(1,b,c) \,|\, b \in \mathbb{R}^n, c \in \mathbb{R}^{n\times n}\},$$

respectively. We also define

$$\begin{aligned}
\delta_\lambda &: T_1^{(2)}(\mathbb{R}^n) \to T_1^{(2)}(\mathbb{R}^n) : (1,b,c) \mapsto (1, \lambda b, \lambda^2 c),\\
\tau_1 &: T_1^{(2)}(\mathbb{R}^n) \to T_1^{(1)}(\mathbb{R}^n) : (1,b,c) \mapsto (1,b),\\
\tau_2 &: T_1^{(2)}(\mathbb{R}^n) \to T_1^{(2)}(\mathbb{R}^n) : (1,b,c) \mapsto (1,b,c).
\end{aligned}$$

and recall that the tensor product $\otimes$ on $T_1^{(2)}(\mathbb{R}^n)$ is defined by

$$(1,b,c) \otimes (1,\tilde{b},\tilde{c}) = (1, b+\tilde{b}, c+\tilde{c}+b\tilde{b}^\top)$$

for $(1,b,c), (1,\tilde{b},\tilde{c}) \in T_1^{(2)}(\mathbb{R}^n)$. Clearly, $(1,0,0)$ is the unit element with respect to $\otimes$ and $(1,b,c)^{-1} = (1,-b,-c+bb^\top)$. Moreover, we define

$$N : T_1^{(2)}(\mathbb{R}^n) \to \mathbb{R} : (1,b,c) \mapsto \max\left\{|b|_{\mathbb{R}^n}, \sqrt{2|c|_{\mathbb{R}^{n\times n}}}\right\}$$

and

$$|||\cdot||| : T_1^{(2)}(\mathbb{R}^n) \to \mathbb{R} : \mathbf{x} \mapsto \frac{1}{2}(N(\mathbf{x}) + N(\mathbf{x}^{-1})).$$

It is possible to show that the map

$$d_{T_1^{(2)}(\mathbb{R}^n)} : T_1^{(2)}(\mathbb{R}^n) \times T_1^{(2)}(\mathbb{R}^n) \to \mathbb{R} : (\mathbf{x},\mathbf{y}) \mapsto |||\mathbf{x}^{-1} \otimes \mathbf{y}|||$$

is a metric on the space $T_1^{(2)}(\mathbb{R}^n)$ and that this space when endowed with metric $d_{T_1^{(2)}(\mathbb{R}^n)}$ is separable.

On the other hand, on the space $\mathscr{B}^{\alpha}_{\Phi_\beta,q}$, we define the map

$$|||\mathbf{X}|||_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}} = [X]_{B^{\alpha}_{\Phi_\beta,q}} + \|\mathbb{X}\|^{\frac{1}{2}}_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}}$$

for $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{B}^{\alpha}_{\Phi_\beta,q}$. With a slight abuse of terminology, we call $|||\cdot|||_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}}$ *the exponential Besov–Orlicz rough path norm* even though it is homogeneous only in the sense that

$$|||\delta_\lambda \mathbf{X}|||_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}} = |\lambda| |||\mathbf{X}|||_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}}$$

holds for any $\lambda \in \mathbb{R}$ and $\mathbf{X} \in \mathscr{B}^{\alpha}_{\Phi_\beta,q}$. For $\mathbf{X} = (X, \mathbb{X})$, $\tilde{\mathbf{X}} = (\tilde{X}, \tilde{\mathbb{X}}) \in \mathscr{B}^{\alpha}_{\Phi_\beta,q}$, we set

$$\rho_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}}(\mathbf{X}, \tilde{\mathbf{X}}) = d_{B^{\alpha}_{\Phi_\beta,q}}(X, \tilde{X}) + d_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}}(\mathbb{X}, \tilde{\mathbb{X}}),$$

where the metrics $d_{B^{\alpha}_{\Phi_\beta,q}}$ and $d_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}}$ are defined by (4) and (8), respectively. We call $\rho_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}}$ *the exponential Besov–Orlicz rough path metric*.

Finally, let us note that, for $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{B}^{\alpha}_{\Phi_\beta,q}$, the notation $\mathbf{X}^{(1)} := \delta X$ and $\mathbf{X}^{(2)} := \mathbb{X}$ will sometimes be used.

Lemma 4.2 *Let $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{B}^{\alpha}_{\Phi_\beta,q}$ and $p \in (\frac{1}{\alpha}, \infty)$. Then, for all $(s,u,t) \in \triangle^3[0,T]$, it holds*

$$|\delta\mathbb{X}_{s,u,t}|_{\mathbb{R}^{n\times n}} \lesssim^{T}_{\alpha,\beta,p,q} [X]^2_{B^{\alpha}_{\Phi_\beta,q}} \left(((u-s)\wedge(t-u))^{\frac{1}{2}}((u-s)\vee(t-u))^{\frac{1}{2}}\right)^{2(\alpha-\frac{1}{p})}$$

and

$$|\delta(\mathbb{X}-\tilde{\mathbb{X}})_{s,u,t}|_{\mathbb{R}^{n\times n}} \lesssim^{T}_{\alpha,\beta,p,q} \left([X]_{B^{\alpha}_{\Phi_\beta,q}}[X-\tilde{X}]_{B^{\alpha}_{\Phi_\beta,q}} + [X-\tilde{X}]_{B^{\alpha}_{\Phi_\beta,q}}[\tilde{X}]_{B^{\alpha}_{\Phi_\beta,q}}\right) \cdot \left(((u-s)\wedge(t-u))^{\frac{1}{2}}((u-s)\vee(t-u))^{\frac{1}{2}}\right)^{2(\alpha-\frac{1}{p})}.$$

Proof The first claim follows from Chen's relations, the compatibility of the $\mathbb{R}^{n\times n}$-norm and the embedding $B^{\alpha}_{\Phi_\beta,q} \hookrightarrow C^{\alpha-1/p}$ from Proposition 3.2 by

$$\begin{aligned}|\delta\mathbb{X}_{s,u,t}|_{\mathbb{R}^{n\times n}} &\le [X]^2_{C^{\alpha-1/p}}|u-s|^{\alpha-\frac{1}{p}}|t-u|^{\alpha-\frac{1}{p}} \\ &\lesssim^{T}_{\alpha,\beta,p,q} [X]^2_{B^{\alpha}_{\Phi_\beta,q}} \left(((u-s)\wedge(t-u)^{\frac{1}{2}}((u-s)\vee(t-u))^{\frac{1}{2}}\right)^{2(\alpha-\frac{1}{p})}.\end{aligned}$$

Similarly, we deduce

$$\delta(\mathbb{X}-\tilde{\mathbb{X}})_{s,u,t} = (\delta X_{s,u})(\delta(X-\tilde{X})_{u,t})^\top + (\delta(X-\tilde{X})_{s,u})(\delta\tilde{X}_{u,t})^\top$$

and establish the second claim analogously. □

Lemma 4.3 *Let $\mathbf{X}, \tilde{\mathbf{X}} \in \mathscr{B}^{\alpha}_{\Phi_\beta,q}$ and $p \in (\frac{1}{\alpha}, \infty)$. Then, for $k = 1, 2$, it holds*

$$\|\mathbf{X}^{(k)}\|^{\frac{1}{k}}_{C^{k(\alpha-1/p);2}} \lesssim^{T}_{\alpha,\beta,p,q} |||\tau_k\mathbf{X}|||_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}},$$

$$\|\mathbf{X}^{(k)} - \tilde{\mathbf{X}}^{(k)}\|_{C^{k(\alpha-1/p);2}} \lesssim^{T}_{\alpha,\beta,p,q} \sum_{j=1}^{k} \left(|||\tau_k\mathbf{X}|||_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}} \vee |||\tau_k\tilde{\mathbf{X}}|||_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}}\right)^{k-j} \|\mathbf{X}^{(j)} - \tilde{\mathbf{X}}^{(j)}\|_{B^{j\alpha;2}_{\Phi_{\beta/j},q/j}}.$$

Proof The result is an immediate consequence of similar bounds for Besov rough paths from [12, Proposition 5.1] thanks to the embeddings $B^{\alpha}_{\Phi_\beta,q} \hookrightarrow B^{\alpha}_{p,q}$ from Proposition 3.2 and $B^{j\alpha;2}_{\Phi_{\beta/j},q/j} \hookrightarrow B^{j\alpha;2}_{p/j,q/j}$ from Proposition 3.8. □

Lemma 4.4 *Let* $\mathbf{X}, \tilde{\mathbf{X}} \in \mathscr{B}^{\alpha}_{\Phi_\beta,q}$ *and* $p \in (\frac{1}{\alpha}, \infty)$. *Then it holds*

$$\|\mathbb{X}\|_{B^{\alpha;2}_{\Phi_\beta,q}} \lesssim^{T}_{\alpha,\beta,p,q} (T^{\alpha-\frac{1}{p}} \vee T^{\alpha-\frac{2}{p}}) |||\mathbf{X}|||^2_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}},$$

$$\|\mathbb{X}-\tilde{\mathbb{X}}\|_{B^{\alpha;2}_{\Phi_\beta,q}} \lesssim^{T}_{\alpha,\beta,p,q} (T^{\alpha-\frac{1}{p}} \vee T^{\alpha-\frac{3}{p}}) \sum_{j=1}^{2} \left(|||\mathbf{X}|||_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}} \vee |||\tilde{\mathbf{X}}|||_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}} \right)^{2-j} \|\mathbf{X}^{(j)} - \tilde{\mathbf{X}}^{(j)}\|_{B^{j\alpha;2}_{\Phi_{\beta/j},q/j}}.$$

Proof The proof is analogous to [12, Lemma 5.1] with the appropriate Besov–Orlicz variants of the interpolation result from Lemma 3.9 and the Hölder-type bound Lemma 4.3 instead of [12, Lemma 2.7] and [12, Proposition 5.1], respectively. We establish the first claim in detail to illustrate the origin of the negative powers of T.

From Lemma 3.9 (with $\delta = 2\alpha - \frac{2}{p}$, $\gamma = 2\alpha$, $\frac{\beta}{2}$ in place of β, and $\beta' = \beta$), the inclusion $B^{\alpha;2}_{\Phi_\beta,q_2} \subseteq B^{\alpha;2}_{\Phi_\beta,q_1}$ for $q_2 \le q_1$ from Proposition 3.12 (with $M = [X]^2_{B^{\alpha}_{\Phi_\beta,q}}$ from Lemma 4.2), and from the Hölder bound in Lemma 4.3, we obtain

$$\begin{aligned}\|\mathbb{X}\|_{B^{\alpha;2}_{\Phi_\beta,q}} &\lesssim_{\alpha,\beta,p,q} T^{\alpha-\frac{1}{p}} \|\mathbb{X}\|^{\frac{1}{2}}_{C^{2\alpha-2/p;2}} \|\mathbb{X}\|^{\frac{1}{2}}_{B^{2\alpha;2}_{\Phi_{\beta/2},q}} \\ &\lesssim^{T}_{\alpha,\beta,p,q} T^{\alpha-\frac{1}{p}} |||\mathbf{X}|||_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}} \left(\|\mathbb{X}\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} + [X]^2_{B^{\alpha}_{\Phi_\beta,q}} T^{-\frac{2}{p}} \right)^{\frac{1}{2}}.\end{aligned}$$

The claim then follows from the definition of the norm on $\mathscr{B}^{\alpha}_{\Phi_\beta,q}$. □

The result that connects the space of Besov–Orlicz rough paths to the space of $T^{(2)}_1(\mathbb{R}^n)$-valued functions of exponential Besov–Orlicz regularity is given now.

Proposition 4.5 *The following claims hold:*

i) If $(X, \mathbb{X}) \in \mathscr{B}^{\alpha}_{\Phi_\beta,q}([0,T];\mathbb{R}^n)$, *then the* $T^{(2)}_1(\mathbb{R}^n)$*-valued path* $\mathbf{x}_\cdot = (1, \delta X_{0,\cdot}, \mathbb{X}_{0,\cdot})$ *satisfies* $\mathbf{x} \in B^{\alpha}_{\Phi_\beta,q}([0,T]; T^{(2)}_1(\mathbb{R}^n), (1,0,0))$ *and*

$$|||\mathbf{X}|||_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}([0,T];\mathbb{R}^n)} \asymp_q [\mathbf{x}]_{B^{\alpha}_{\Phi_\beta,q}([0,T];T^{(2)}_1(\mathbb{R}^n))}. \tag{12}$$

ii) Conversely, if $\mathbf{x} \in B^{\alpha}_{\Phi_\beta,q}([0,T]; T^{(2)}_1(\mathbb{R}^n), (1,0,0))$, *define* $\mathbf{X} = (X, \mathbb{X})$ *by* $(1, \delta X_{s,t}, \mathbb{X}_{s,t}) = \mathbf{x}_s^{-1} \otimes \mathbf{x}_t$ *for all* $(s,t) \in \triangle^2[0,T]$. *Then* $\mathbf{X} \in \mathscr{B}^{\alpha}_{\Phi_\beta,q}([0,T];\mathbb{R}^n)$ *and* (12) *holds.*

Proof In the first claim, equivalence (12) can be established from the definitions above by elementary estimates and Lemma 2.1. From Lemma4.3, we deduce that $d_{T^{(2)}_1(\mathbb{R}^n)}((1, \delta X_{0,\cdot}, \mathbb{X}_{0,\cdot}), (1,0,0))$ is bounded on $[0,T]$ and thus $\mathbf{x} \in L^{\Phi_\beta}([0,T]; T^{(2)}_1(\mathbb{R}^n))$ by Proposition 3.6.

Regarding the second claim, note that $\mathbf{X}$ (and in particular X) is indeed well-defined since, if $\mathbf{x}_\cdot = (1, b_\cdot, c_\cdot)$ for some $b : [0,T] \to \mathbb{R}^n$ and $c : [0,T] \to \mathbb{R}^{n\times n}$, then

$$\mathbf{x}_s^{-1} \otimes \mathbf{x}_t = (1, \delta b_{s,t}, \delta c_{s,t} - b_s(\delta b_{s,t})^\top), \quad (s,t) \in \triangle^2[0,T],$$

hence $X = b$. The proof of the second claim follows similarly as the proof of the first one. □

4.2 Brownian Motion as an Exponential Besov–Orlicz Rough Path and Other Examples

Below, we show that any path $X \in B^{\alpha}_{\Phi_\beta,q}([0,T];\mathbb{R})$ can be lifted to a Besov–Orlicz rough path

$$\mathbf{X} = (X, \mathbb{X}) \in \mathscr{B}^{\alpha}_{\Phi_\beta,q}([0,T];\mathbb{R}).$$

Theorem 4.6 *Let $X \in B^{\alpha}_{\Phi_\beta,q}([0,T];\mathbb{R})$ and let $F \in B^{2\alpha;2}_{\Phi_{\beta/2},q/2}([0,T];\mathbb{R})$ be an additive function, i.e. let $\delta F_{s,u,t} = 0$ hold for all $(s,u,t) \in \triangle^3[0,T]$. Define $\mathbb{X} : \triangle^2[0,T] \to \mathbb{R}$ by*

$$\mathbb{X}_{s,t} = \frac{(\delta X_{s,t})^2}{2} + F_{s,t}, \quad (s,t) \in \triangle^2[0,T].$$

Then $\mathbb{X}$ belongs to the space $B^{2\alpha;2}_{\Phi_{\beta/2},q/2}([0,T];\mathbb{R})$ and satisfies Chen's relations.

Proof Since $\|(\delta X)^2\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} = [X]^2_{B^{\alpha}_{\Phi_\beta,q}}$ holds by the definitions of the respective quazi(semi)norms and Lemma 2.1, the desired bound can be established by the (quazi-)triangle inequality as

$$\|\mathbb{X}\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} \lesssim_q \|(\delta X)^2\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} + \|F\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} = [X]^2_{B^{\alpha}_{\Phi_\beta,q}} + \|F\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} < \infty.$$

Chen's relations can be verified directly. □

Let us fix a probability basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,1]}, \mathsf{P})$ rich enough to carry any of the processes below. Theorem 4.6 enables us to construct a rough path lift for any ($\mathbb{R}$-valued) continuous local martingale X with Lipschitz continuous quadratic variation. It is well-known that for such a process, it holds that $X(\omega) \in B^{1/2}_{\Phi_2,\infty}([0,1];\mathbb{R})$ for a.a. $\omega \in \Omega$; see [23, Theorem 4.1]. Set

$$\mathbb{X}^{\mathrm{It\hat{o}}}_{s,t} = \frac{(\delta X_{s,t})^2}{2} - \frac{\delta\langle X\rangle_{s,t}}{2} \quad \text{and} \quad \mathbb{X}^{\mathrm{Strat}}_{s,t} = \frac{(\delta X_{s,t})^2}{2}$$

for $(s,t) \in \triangle^2[0,1]$ where $\langle X\rangle$ stands for the (probabilistic) quadratic variation of X. The mapping $\mathbb{X}^{\mathrm{It\hat{o}}}_{s,t}$ coincides with the Itô integral $\int_s^t (\delta X_{s,r})\,\mathrm{d}X_r$ while the mapping $\mathbb{X}^{\mathrm{Strat}}_{s,t}$ coincides with the Stratonovich integral $\int_s^t (\delta X_{s,r}) \circ \mathrm{d}X_r$. By appealing to Lemma 4.6 with $F^{\mathrm{It\hat{o}}}_{s,t} = -\frac{1}{2}\delta\langle X\rangle_{s,t}$ and $F^{\mathrm{Strat}}_{s,t} = 0$, we obtain

$$\mathbf{X}^{\mathrm{It\hat{o}}}(\omega) = (X(\omega), \mathbb{X}^{\mathrm{It\hat{o}}}(\omega)) \in \mathscr{B}^{1/2}_{\Phi_2,\infty} \quad \text{for a.a. } \omega \in \Omega,$$
$$\mathbf{X}^{\mathrm{Strat}}(\omega) = (X(\omega), \mathbb{X}^{\mathrm{Strat}}(\omega)) \in \mathscr{B}^{1/2}_{\Phi_2,\infty} \quad \text{for a.a. } \omega \in \Omega.$$

A canonical example of these constructions is the standard scalar Brownian motion W for which we have that $W(\omega) \in B^{1/2}_{\Phi_2,\infty}([0,1];\mathbb{R})$ holds for a.a. $\omega \in \Omega$ (see [4, Theorem 5.8] for a stand-alone proof). As far as Gaussian processes with lower regularity are concerned, the scalar fractional Brownian motion W^H with Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2}]$ can be mentioned. By [34, Corollary 5.3] (see also [5]), we have that $W^H(\omega) \in B^{H}_{\Phi_2,\infty}([0,1];\mathbb{R})$ holds for a.a. $\omega \in \Omega$ and therefore, we can apply Theorem 4.6 with $\mathbb{W}^H_{s,t} = \frac{1}{2}(\delta W^H_{s,t})^2$ and $F^H_{s,t} = 0$ to obtain

$$\mathbf{W}^H(\omega) = (W^H(\omega), \mathbb{W}^H(\omega)) \in \mathscr{B}^{H}_{\Phi_2,\infty} \quad \text{for a.a. } \omega \in \Omega.$$

Finally, such regularity is also obtained for non-Gaussian processes. For example, if $n \in \mathbb{N}$ and $\beta_1, \beta_2 \in \mathbb{R}$ are real numbers such that

$$0 < \beta_1 + \frac{n}{2}(\beta_2 - 1) + 1 < 1, \quad 1 - \frac{1}{n} < \beta_2 < 1, \quad \text{and} \quad \frac{1}{3} < \kappa = \beta_1 + \frac{n}{2}(\beta_2 - 1) + 1 \leq \frac{1}{2},$$

then it holds for the fractionally filtered Hermite process $Z^{\beta_1,\beta_2,n}$ (see [7, p.318] and also [1, Theorem 3.27] for the definition) that $Z^{\beta_1,\beta_2,n}(\omega) \in B^{\kappa}_{\Phi_{2/n},\infty}([0,1];\mathbb{R})$ for a.a. $\omega \in \Omega$ (see [7, Corollary 4.2]). As such, these processes can be also lifted via Theorem 4.6 with $\mathbb{Z}^{\beta_1,\beta_2,n}_{s,t} = \frac{1}{2}(\delta Z^{\beta_1,\beta_2,n}_{s,t})^2$ and $F^{\beta_1,\beta_2,n}_{s,t} = 0$ to a rough path that satisfies

$$\mathbf{Z}^{\beta_1,\beta_2,n}(\omega) = (Z^{\beta_1,\beta_2,n}(\omega), \mathbb{Z}^{\beta_1,\beta_2,n}(\omega)) \in \mathscr{B}^{\kappa}_{\Phi_{2/n},\infty} \quad \text{for a.a. } \omega \in \Omega.$$

Remark 4.7 It is natural to ask whether the above results can be extended to cover multidimensional continuous local martingales or fractionally filtered Hermite processes. We believe that such an extension is indeed possible in both cases. For continuous local martingales, one can define the iterated integral as in [13, Chapter 14] and combine similar arguments from there with the arguments in [23]. Similarly, for fractionally filtered Hermite processes, the iterated integral can be defined as in [10, Chapter 10] and one can combine arguments from there with arguments from [7]. This would, in particular, require the extension of [7, Theorem 3.3] to multivariate maps. We intend to investigate the feasibility of such extensions in our future research.

4.3 Controlled Rough Paths

In this section, we define a Besov–Orlicz variant of the controlled rough path from [15], establish some basic estimates, and study the composition of controlled rough paths with sufficiently smooth functions.

For the rest of this section, let $n, m, k \in \mathbb{N}$ be fixed.

Definition 4.8 Given $X \in B^{\alpha}_{\Phi_\beta,q}([0,T];\mathbb{R}^n)$ and $Y \in B^{\alpha}_{\Phi_\beta,q}([0,T];\mathbb{R}^{m\times k})$, we say that Y is *controlled* by X if there is $Y' \in B^{\alpha}_{\Phi_\beta,q}([0,T];\mathcal{L}(\mathbb{R}^n;\mathbb{R}^{m\times k}))$ such that the remainder R^Y given by

$$\delta Y_{s,t} = Y'_s \delta X_{s,t} + R^Y_{s,t}, \quad (s,t) \in \triangle^2[0,T],$$

satisfies $R^Y \in B^{2\alpha;2}_{\Phi_{\beta/2},q/2}([0,T];\mathbb{R}^{m\times k})$. We denote the space of all controlled rough paths, i.e. the space of all such pairs (Y,Y'), by $\mathscr{D}^{\alpha;X}_{\Phi_\beta,q}([0,T];\mathbb{R}^{m\times k})$. For $(Y,Y') \in \mathscr{D}^{\alpha;X}_{\Phi_\beta,q}([0,T];\mathbb{R}^{m\times k})$, we define

$$[(Y,Y')]_{\mathscr{D}^{\alpha;X}_{\Phi_\beta,q}([0,T];\mathbb{R}^{m\times k})} = [Y']_{B^{\alpha}_{\Phi_\beta,q}([0,T];\mathcal{L}(\mathbb{R}^n;\mathbb{R}^{m\times k}))} + \|R^Y\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}([0,T];\mathbb{R}^{m\times k})}.$$

Additionally, for $\tilde{X} \in B^{\alpha}_{\Phi_\beta,q}([0,T];\mathbb{R}^n)$ and $(\tilde{Y},\tilde{Y}') \in \mathscr{D}^{\alpha;\tilde{X}}_{\Phi_\beta,q}([0,T];\mathbb{R}^{m\times k})$, we denote

$$d_{X,\tilde{X},\mathscr{D}^{\alpha}_{\Phi_\beta,q}([0,T])}((Y,Y'),(\tilde{Y},\tilde{Y}')) = d_{B^{\alpha}_{\Phi_\beta,q}([0,T];\mathcal{L}(\mathbb{R}^n;\mathbb{R}^{m\times k}))}(Y',\tilde{Y}') + d_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}([0,T];\mathbb{R}^{m\times k})}(R^Y,R^{\tilde{Y}}).$$

If there is no risk of ambiguity, we will often omit the domains and codomains and write, e.g., only $\mathscr{D}^{\alpha;X}_{\Phi_\beta,q}$ and $d_{X,\tilde{X},\mathscr{D}^{\alpha}_{\Phi_\beta,q}}$ instead of $\mathscr{D}^{\alpha;X}_{\Phi_\beta,q}([0,T];\mathbb{R}^{m\times k})$ and $d_{X,\tilde{X},\mathscr{D}^{\alpha}_{\Phi_\beta,q}([0,T])}$,

respectively. We follow a similar convention for other spaces as well. If $X = \tilde{X}$, we use the notation $d_{X,\mathscr{D}^\alpha_{\Phi_\beta,q}} = d_{X,\tilde{X},\mathscr{D}^\alpha_{\Phi_\beta,q}}$. Let us note that we follow the convention from [10] in the sense that the coefficients of $\mathscr{D}^{\alpha;X}_{\Phi_\beta,q}$ correspond to the regularity of the remainder R^Y. In the often referenced paper [12], the coefficients correspond to the regularity of the underlying path X instead.

For brevity, we will often omit the norm-related subscripts and write only, e.g., $|X_t|$, $|Y_t|$, $|Y'_t|$, and $|R^Y_{s,t}|$ instead of $|X_t|_{\mathbb{R}^n}$, $|Y_t|_{\mathbb{R}^{m\times k}}$, $|Y'_t|_{\mathcal{L}(\mathbb{R}^n;\mathbb{R}^{m\times k})}$, and $|R^Y_{s,t}|_{\mathbb{R}^{m\times k}}$, respectively.

Lemma 4.9 *Let $X \in B^\alpha_{\Phi_\beta,q}([0,T];\mathbb{R}^n)$, $(Y,Y') \in \mathscr{D}^{\alpha;X}_{\Phi_\beta,q}([0,T];\mathbb{R}^{m\times k})$, and $p \in (\frac{1}{\alpha},\infty)$. Then, for any $(s,u,t) \in \triangle^3([0,T])$, it holds that*

$$|\delta R^Y_{s,u,t}|_{\mathbb{R}^{m\times k}} \lesssim^T_{\alpha,\beta,p,q} [Y']_{B^\alpha_{\Phi_\beta,q}}[X]_{B^\alpha_{\Phi_\beta,q}}\Big(((u-s)\wedge(t-u)^{\frac{1}{2}}((u-s)\vee(t-u))^{\frac{1}{2}}\Big)^{2(\alpha-\frac{1}{p})}$$

and

$$\begin{aligned}|\delta(R^Y - R^{\tilde{Y}})_{s,u,t}|_{\mathbb{R}^{m\times k}} \lesssim^T_{\alpha,\beta,p,q} &\Big([Y'-\tilde{Y}']_{B^\alpha_{\Phi_\beta,q}}[X]_{B^\alpha_{\Phi_\beta,q}} + [\tilde{Y}']_{B^\alpha_{\Phi_\beta,q}}[X-\tilde{X}]_{B^\alpha_{\Phi_\beta,q}}\Big)\\ &\cdot\Big(((u-s)\wedge(t-u)^{\frac{1}{2}}((u-s)\vee(t-u))^{\frac{1}{2}}\Big)^{2(\alpha-\frac{1}{p})}.\end{aligned}$$

Proof From the definition of R^Y, we deduce

$$\delta R^Y_{s,u,t} = -Y'_s\delta X_{u,t} + Y'_u\delta X_{u,t} = (\delta Y'_{s,u})(\delta X_{u,t}), \quad (s,u,t)\in\triangle^3[0,T].$$

The first claim can be therefore established analogously to Lemma 4.2. The second claim follows similarly from

$$\delta(R^Y - R^{\tilde{Y}})_{s,u,t} = (\delta(Y'-\tilde{Y}')_{s,u})(\delta X_{u,t}) + (\delta\tilde{Y}'_{s,u})(\delta(X-\tilde{X})_{u,t}), \quad (s,u,t)\in\triangle^3[0,T].$$

□

Lemma 4.10 *Let $p \in (\frac{1}{\alpha},\infty)$, $X,\tilde{X} \in B^\alpha_{\Phi_\beta,q}([0,T];\mathbb{R}^n)$, $(Y,Y') \in \mathscr{D}^{\alpha;X}_{\Phi_\beta,q}([0,T];\mathbb{R}^{m\times k})$, and $(\tilde{Y},\tilde{Y}') \in \mathscr{D}^{\alpha;\tilde{X}}_{\Phi_\beta,q}([0,T];\mathbb{R}^{m\times k})$. Then*

$$\begin{aligned}\|R^Y\|_{C^{2\alpha-2/p;2}} &\lesssim^T_{\alpha,\beta,p,q} \|R^Y\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} + [Y']_{B^\alpha_{\Phi_\beta,q}}[X]_{B^\alpha_{\Phi_\beta,q}},\\ \|R^Y - R^{\tilde{Y}}\|_{C^{2\alpha-2/p;2}} &\lesssim^T_{\alpha,\beta,p,q} \|R^Y - R^{\tilde{Y}}\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}}\\ &\quad + [Y'-\tilde{Y}']_{B^\alpha_{\Phi_\beta,q}}[X]_{B^\alpha_{\Phi_\beta,q}} + [\tilde{Y}']_{B^\alpha_{\Phi_\beta,q}}[X-\tilde{X}]_{B^\alpha_{\Phi_\beta,q}},\end{aligned}$$

and

$$\begin{aligned}[Y]_{B^\alpha_{\Phi_\beta,q}} &\lesssim^T_{\alpha,\beta,p,q} |Y'_0|_{\mathcal{L}(\mathbb{R}^n;\mathbb{R}^{m\times k})}[X]_{B^\alpha_{\Phi_\beta,q}}\\ &\quad + (T^{\alpha-\frac{1}{p}}\vee T^{\alpha-\frac{3}{p}})\Big([Y']_{B^\alpha_{\Phi_\beta,q}}[X]_{B^\alpha_{\Phi_\beta,q}} + \|R^Y\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}}\Big),\\ [Y-\tilde{Y}]_{B^\alpha_{\Phi_\beta,q}} &\lesssim^T_{\alpha,\beta,p,q} |Y'_0-\tilde{Y}'_0|_{\mathcal{L}(\mathbb{R}^n;\mathbb{R}^{m\times k})}[X]_{B^\alpha_{\Phi_\beta,q}} + |\tilde{Y}'_0|_{\mathcal{L}(\mathbb{R}^n;\mathbb{R}^{m\times k})}[X-\tilde{X}]_{B^\alpha_{\Phi_\beta,q}}\\ &\quad + (T^{\alpha-\frac{1}{p}}\vee T^{\alpha-\frac{3}{p}})\Big([Y'-\tilde{Y}']_{B^\alpha_{\Phi_\beta,q}}[X]_{B^\alpha_{\Phi_\beta,q}} + [\tilde{Y}']_{B^\alpha_{\Phi_\beta,q}}[X-\tilde{X}]_{B^\alpha_{\Phi_\beta,q}}\Big)\\ &\quad + (T^{\alpha-\frac{1}{p}}\vee T^{\alpha-\frac{3}{p}})\|R^Y - R^{\tilde{Y}}\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}}.\end{aligned}$$

Proof The first two estimates are direct consequences of [12, Lemma 5.2] by the embeddings $B^{\alpha}_{\Phi_\beta,q} \hookrightarrow B^{\alpha}_{p,q}$ and $B^{\alpha;2}_{\Phi_\beta,q} \hookrightarrow B^{\alpha;2}_{p,q}$ from Proposition 3.2 and Proposition 3.8, respectively.

To establish the third estimate, let us first derive an auxiliary estimate of R^Y. By the interpolation inequality from Lemma 3.9 (with $\delta = 2\alpha - \frac{2}{p}$, $\frac{\beta}{2}$ in place of β, and $\beta' = \beta$), Young's inequality, the first estimate in this lemma and the inclusion $B^{2\alpha;2}_{\Phi_{\beta/2},q/2} \subseteq B^{2\alpha;2}_{\Phi_{\beta/2},q}$ from Proposition 3.12 (with $M = [Y']_{B^{\alpha}_{\Phi_\beta,q}}[X]_{B^{\alpha}_{\Phi_\beta,q}}$ from Lemma 4.9), we deduce

$$\begin{aligned}
\|R^Y\|_{B^{\alpha;2}_{\Phi_\beta,q}} &\lesssim_{\alpha,\beta,p,q} T^{\alpha-\frac{1}{p}} \left(\|R^Y\|_{C^{2\alpha-2/p;2}} + \|R^Y\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q}} \right) \\
&\lesssim^{T}_{\alpha,\beta,p,q} T^{\alpha-\frac{1}{p}} \left(\|R^Y\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} + [Y']_{B^{\alpha}_{\Phi_\beta,q}}[X]_{B^{\alpha}_{\Phi_\beta,q}} + [Y']_{B^{\alpha}_{\Phi_\beta,q}}[X]_{B^{\alpha}_{\Phi_\beta,q}} T^{-\frac{2}{p}} \right) \\
&\lesssim (T^{\alpha-\frac{1}{p}} \vee T^{\alpha-\frac{3}{p}}) \left(\|R^Y\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} + [Y']_{B^{\alpha}_{\Phi_\beta,q}}[X]_{B^{\alpha}_{\Phi_\beta,q}} \right).
\end{aligned}$$

Now, the estimate from Proposition 3.6 yields

$$\begin{aligned}
[Y]_{B^{\alpha}_{\Phi_\beta,q}} &\lesssim_q \|Y'\|_{L^\infty}[X]_{B^{\alpha}_{\Phi_\beta,q}} + \|R^Y\|_{B^{\alpha;2}_{\Phi_\beta,q}} \\
&\lesssim^{T}_{\alpha,\beta,p,q} \left(|Y'_0| + T^{\alpha-\frac{1}{p}}[Y']_{B^{\alpha}_{\Phi_\beta,q}} \right)[X]_{B^{\alpha}_{\Phi_\beta,q}} + \|R^Y\|_{B^{\alpha;2}_{\Phi_\beta,q}},
\end{aligned}$$

which leads to the desired bound by the auxiliary estimate above.

The remaining estimate follows from

$$\delta(Y - \tilde{Y})_{s,t} = (Y' - \tilde{Y}')_s \delta X_{s,t} + \tilde{Y}'_s \delta(X - \tilde{X})_{s,t} + R^Y_{s,t} - R^{\tilde{Y}}_{s,t}, \quad (s,t) \in \triangle^2[0,T],$$

in a similar manner. □

Next, we establish that a composition of a controlled rough path with a sufficiently regular function is a controlled rough path. A similar result was established for Sobolev-type spaces in [19, Lemma 3.7] with $f \in C^3_b$ and for Besov-type spaces in [12, Proposition 5.2] with $f \in C^{2,\nu}_b$ for some $\nu \in (0,1)$. Below, we provide a proof for Besov–Orlicz-type spaces with $f \in C^{2,1}_b$.

Theorem 4.11 *Let $X, \tilde{X} \in B^{\alpha}_{\Phi_\beta,q}([0,T];\mathbb{R}^n)$ and $(Y,Y'), (\tilde{Y},\tilde{Y}') \in \mathscr{D}^{\alpha;X}_{\Phi_\beta,q}([0,T];\mathbb{R}^{m\times k})$ be given.*

(1) Let $f \in C^2_b(\mathbb{R}^{m\times k},\mathbb{R}^{m\times n})$ and define

$$f(Y)'_\cdot = ((Df)(Y_\cdot))Y'_\cdot.$$

Then $(f(Y), f(Y)') \in \mathscr{D}^{\alpha;X}_{\Phi_\beta,q}([0,T];\mathbb{R}^{m\times n})$ and there are the bounds

$$[f(Y)]_{B^{\alpha}_{\Phi_\beta,q}} \le \|Df\|_{L^\infty}[Y]_{B^{\alpha}_{\Phi_\beta,q}}$$

and

$$\begin{aligned}
[(f(Y), f(Y)')]_{\mathscr{D}^{\alpha;X}_{\Phi_\beta,q}} &\lesssim^{T}_{\alpha,\beta,q} \|f\|_{C^2_b} \left(1 + [X]_{B^{\alpha}_{\Phi_\beta,q}}\right) \\
&\quad \cdot \left(\left(|Y'_0| + [(Y,Y')]_{\mathscr{D}^{\alpha;X}_{\Phi_\beta,q}} \right) \vee \left(|Y'_0| + [(Y,Y')]_{\mathscr{D}^{\alpha;X}_{\Phi_\beta,q}} \right)^2 \right).
\end{aligned}$$

(2) Let $f \in C_b^{2,1}(\mathbb{R}^{m\times k}, \mathbb{R}^{m\times n})$ and assume that the estimate

$$\left(|Y_0'| + [(Y, Y')]_{\mathscr{D}^{\alpha;X}_{\Phi_\beta,q}}\right) \vee \left(|\tilde{Y}_0'| + [(\tilde{Y}, \tilde{Y}')]_{\mathscr{D}^{\alpha;\tilde{X}}_{\Phi_\beta,q}}\right) \vee [X]_{B^\alpha_{\Phi_\beta,q}} \vee [\tilde{X}]_{B^\alpha_{\Phi_\beta,q}} \le M$$

holds for some $M \in (0,\infty)$. Then there are the bounds

$$\begin{aligned}[f(Y)' - f(\tilde{Y})']_{B^\alpha_{\Phi_\beta,q}} &\lesssim^{T,M}_{\alpha,\beta,q} \|f\|_{C_b^{2,1}} \\ &\cdot \left(|Y_0 - \tilde{Y}_0| + |Y_0' - \tilde{Y}_0'| + [X - \tilde{X}]_{B^\alpha_{\Phi_\beta,q}} + d_{X,\tilde{X},\mathscr{D}^\alpha_{\Phi_\beta,q}}((Y,Y'),(\tilde{Y},\tilde{Y}'))\right) \qquad (13)\end{aligned}$$

and

$$\begin{aligned}\|R^{f(Y)} - R^{f(\tilde{Y})}\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} &\lesssim^{T,M}_{\alpha,\beta,q} \|f\|_{C_b^{2,1}} \\ &\cdot \left(|Y_0 - \tilde{Y}_0| + |Y_0' - \tilde{Y}_0'| + [X - \tilde{X}]_{B^\alpha_{\Phi_\beta,q}} + d_{X,\tilde{X},\mathscr{D}^\alpha_{\Phi_\beta,q}}((Y,Y'),(\tilde{Y},\tilde{Y}'))\right).\end{aligned}$$

Proof The proof runs similarly as the proof of [12, Proposition 5.2] with minor differences such as the use of Lemma 4.10 in place of [12, Lemma 5.2] with a fixed choice of $p \in (\frac{3}{\alpha}, \infty)$ where required. We provide a more detailed proof of (13).

Set $\tilde{Z} = f(\tilde{Y})$ and $\tilde{Z}' = Df(\tilde{Y})\tilde{Y}'$, then the (quazi-)triangle inequality implies

$$\begin{aligned}[f(Y)' - f(\tilde{Y}')]_{B^\alpha_{\Phi_\beta,q}} &= [Df(Y)Y' - Df(\tilde{Y})\tilde{Y}']_{B^\alpha_{\Phi_\beta,q}} \\ &\lesssim_q [Df(Y)(Y' - \tilde{Y}')]_{B^\alpha_{\Phi_\beta,q}} + [(Df(Y) - Df(\tilde{Y}))\tilde{Y}']_{B^\alpha_{\Phi_\beta,q}}.\end{aligned}$$

Regarding the first term, since Df is Lipschitz, we observe

$$\begin{aligned}|\delta(Df(Y)(Y' - \tilde{Y}'))_{s,t}| &\le |Df(Y_t)\delta(Y' - \tilde{Y}')_{s,t}| + |\delta(Df(Y))_{s,t}(Y_s' - \tilde{Y}_s')| \\ &\le \|Df\|_{L^\infty}|\delta(Y' - \tilde{Y}')_{s,t}| + \|D^2 f\|_{L^\infty}|\delta Y_{s,t}||Y_s' - \tilde{Y}_s'|\end{aligned}$$

for all $(s,t) \in \triangle^2[0,T]$. From Proposition 3.6, it follows

$$\begin{aligned}[Df(Y)(Y' - \tilde{Y}')]_{B^\alpha_{\Phi_\beta,q}} &\lesssim_q \|Df\|_{L^\infty}[Y' - \tilde{Y}']_{B^\alpha_{\Phi_\beta,q}} + \|D^2 f\|_{L^\infty}[Y]_{B^\alpha_{\Phi_\beta,q}}\|Y' - \tilde{Y}'\|_{L^\infty} \\ &\lesssim^{T,M}_q \|f\|_{C_b^2}\left(|Y_0' - \tilde{Y}_0'| + [Y' - \tilde{Y}']_{B^\alpha_{\Phi_\beta,q}}\right).\end{aligned}$$

Similarly, we rewrite the second term as

$$\delta((Df(Y) - Df(\tilde{Y}))\tilde{Y}')_{s,t} = (Df(Y_t) - Df(\tilde{Y}_t))(\delta\tilde{Y}'_{s,t}) + \delta(Df(Y) - Df(\tilde{Y}))_{s,t}\tilde{Y}_s'$$

for all $(s,t) \in \triangle^2[0,T]$. By the Lipschitz continuity of Df, we have

$$|\delta((Df(Y) - Df(\tilde{Y}))\tilde{Y}')_{s,t}| \le \|D^2 f\|_{L^\infty}|Y_t - \tilde{Y}_t||\delta\tilde{Y}'_{s,t}| + |\delta(Df(Y) - Df(\tilde{Y}))_{s,t}|\|\tilde{Y}'\|_{L^\infty}$$

for all $(s,t) \in \triangle^2[0,T]$. Then, Proposition 3.6 and the choice of M yield

$$\begin{aligned}&[(Df(Y) - Df(\tilde{Y}))\tilde{Y}']_{B^\alpha_{\Phi_\beta,q}} \\ &\lesssim_q \|D^2 f\|_{L^\infty}\|Y - \tilde{Y}\|_{L^\infty}[\tilde{Y}']_{B^\alpha_{\Phi_\beta,q}} + [Df(Y) - Df(\tilde{Y})]_{B^\alpha_{\Phi_\beta,q}}\|\tilde{Y}'\|_{L^\infty} \\ &\lesssim^{T,M}_q \|D^2 f\|_{L^\infty}\left(|Y_0 - \tilde{Y}_0| + [Y - \tilde{Y}]_{B^\alpha_{\Phi_\beta,q}}\right) + [Df(Y) - Df(\tilde{Y})]_{B^\alpha_{\Phi_\beta,q}}.\end{aligned}$$

Since it holds that

$$\delta(Df(Y) - Df(\tilde{Y}))_{s,t} = \int_0^1 D^2 f(\theta Y_t + (1-\theta)\tilde{Y}_t)\delta(Y-\tilde{Y})_{s,t}\,\mathrm{d}\theta - \int_0^1 (D^2 f(\theta Y_t + (1-\theta)\tilde{Y}_t) - D^2 f(\theta Y_s + (1-\theta)\tilde{Y}_s))(Y_s - \tilde{Y}_s)\,\mathrm{d}\theta,$$

we may use Lipschitz continuity of $D^2 f$ to deduce

$$|\delta(Df(Y) - Df(\tilde{Y}))_{s,t}| \le \|D^2 f\|_{L^\infty} |\delta(Y-\tilde{Y})_{s,t}| + \|D^2 f\|_{C^{0,1}} \left(|\delta Y_{s,t}| + |\delta \tilde{Y}_{s,t}|\right) \|Y - \tilde{Y}\|_{L^\infty}$$

for all $(s,t) \in \triangle^2[0,T]$. Thus, we obtain, by Proposition 3.6, the estimate

$$\begin{aligned}[Df(Y) - Df(\tilde{Y})]_{B^\alpha_{\Phi_\beta,q}} &\lesssim_q \|D^2 f\|_{L^\infty} [Y-\tilde{Y}]_{B^\alpha_{\Phi_\beta,q}} + \|D^2 f\|_{C^{0,1}} \left([Y]_{B^\alpha_{\Phi_\beta,q}} + [\tilde{Y}]_{B^\alpha_{\Phi_\beta,q}}\right) \|Y-\tilde{Y}\|_{L^\infty} \\ &\lesssim_q^{T,M} \|f\|_{C_b^{2,1}} \left([Y-\tilde{Y}]_{B^\alpha_{\Phi_\beta,q}} + |Y_0 - \tilde{Y}_0| + [Y-\tilde{Y}]_{B^\alpha_{\Phi_\beta,q}}\right),\end{aligned}$$

which finishes the proof of (13) by appealing to Lemma 4.10.

4.4 Sewing Lemma

By a partition of $[0,1]$, we understand a finite sequence $\pi = \{t_i\}_{i=0}^N$ such that $t_0 = 0$, $t_N = 1$ and $t_i < t_{i+1}$ for all $i = 0, \dots, N-1$. The norm of the partition $\pi = \{t_i\}_{i=0}^N$ is defined by $|\pi| = \max\{t_{i+1} - t_i \mid 0 \le i \le N-1\}$. A sequence of partitions $\{\pi^n\}_{n=0}^\infty$ is said to have *vanishing norms* if $|\pi^n| \to 0$ as $n \to \infty$.

For a mapping $\Xi \in L^{0;2}([0,T];\mathbb{R}^m)$ and π, a partition of $[0,1]$, we define the partial sum $I^\pi \Xi : \triangle^2[0,T] \to \mathbb{R}^m$ by

$$(I^\pi \Xi)_{s,t} = \sum_{i=0}^{N-1} \Xi_{s+t_i(t-s),s+t_{i+1}(t-s)}.$$

Theorem 4.12 (Sewing Lemma). *Assume that $\Xi \in L^{0;2}([0,T];\mathbb{R}^m)$. Let $\alpha \in (0,1)$, $\gamma \in (1,\infty)$, $\beta_1, \beta_2 \in (0,\infty)$, and $q_1, q_2 \in (0,\infty]$ be such that*

$$\Xi \in B^{\alpha;2}_{\Phi_{\beta_1},q_1}([0,T];\mathbb{R}^m) \quad \text{and} \quad \delta\Xi \in B^{\gamma;3}_{\Phi_{\beta_2},q_2}([0,T];\mathbb{R}^m).$$

Then there exists $\mathscr{I} = \mathscr{I}(\Xi) \in L^0([0,T];\mathbb{R}^m)$ such that the mapping $\Xi \mapsto \mathscr{I}(\Xi)$ is linear, the convergence

$$\lim_{N\to\infty} \|\delta\mathscr{I} - I^{\pi_N}\Xi\|_{B^{\gamma;2}_{\Phi_{\beta_2},r}} = 0 \tag{14}$$

holds for an arbitrary sequence $\{\pi_N\}_{N=1}^\infty$ of partitions of $[0,1]$ with vanishing norms and any r satisfying $r \ge q_2$ and $r > \frac{1}{\gamma}$, and such that there is the estimate

$$\|\delta\mathscr{I} - \Xi\|_{B^{\gamma;2}_{\Phi_{\beta_2},q_2}} \lesssim_{\gamma,q_2} \|\delta\Xi\|_{B^{\gamma;3}_{\Phi_{\beta_2},q_2}}. \tag{15}$$

Moreover, if there exists $\rho \in (0, \frac{1}{2}]$ and $M \in (0,\infty)$ such that

$$|\delta\Xi_{s,u,t}| \le M\left(((u-s)\wedge(t-u))^\rho ((u-s)\vee(t-u))^{1-\rho}\right)^{\gamma-\frac{1}{p}} \tag{16}$$

holds for any $(s,u,t) \in \triangle^3[0,T]$ *and some* $p \in (\frac{1}{\gamma}, \infty)$*, then* $\mathscr{I}$ *is continuous on* $[0,T]$*, the bound*

$$[\mathscr{I}]_{B^{\alpha}_{\Phi_{\beta_1\wedge\beta_2},q_1\vee q_2}} \lesssim_{\alpha,\gamma,\beta_1,\beta_2,p,q_1,q_2,T} \|\Xi\|_{B^{\alpha;2}_{\Phi_{\beta_1},q_1}} + \|\delta\Xi\|_{B^{\gamma;3}_{\Phi_{\beta_2},q_2}} + M \tag{17}$$

holds and $\mathscr{I}$ *is uniquely determined by prescribing* $\mathscr{I}_0 = \Theta$ *for some* $\Theta \in \mathbb{R}^m$*. If, in addition,* $\beta_2 < \beta_1$*, then, for any* $p \in (\frac{1}{\alpha} \vee \frac{1}{\gamma-\alpha}, \infty)$*, we have*

$$\|\delta\mathscr{I} - \Xi\|_{C^{\gamma-1/p;2}} \lesssim^T_{\gamma,p,q_2,\rho} \|\delta\Xi\|_{B^{\gamma;3}_{\Phi_{\beta_2},q_2}} + M \tag{18}$$

and

$$[\mathscr{I}]_{B^{\alpha}_{\Phi_{\beta_1},q_1\vee q_2}} \lesssim^T_{\alpha,\gamma,\beta_1,\beta_2,p,q_1,q_2,\rho} \|\Xi\|_{B^{\alpha;2}_{\Phi_{\beta_1},q_1}} + T^{\gamma-\frac{1}{p}-\alpha+\frac{\beta_2}{p\beta_1}}\left(\|\delta\Xi\|_{B^{\gamma;3}_{\Phi_{\beta_2},q_2}} + M\right) + T^{-\frac{1}{p}}M. \tag{19}$$

Proof The existence of $\mathscr{I} \in L^0([0,T];\mathbb{R}^m)$ satisfying convergence (14) and bound (15) is proved similarly as in [12, Theorem 3.1] since we can replace the Besov(-type) norms by Besov–Orlicz(-type) norms in a straightforward manner. The linearity of the map $\Xi \mapsto I\Xi$ can be established from the linearity of the partial sums $I^{\pi_n}(a\Xi + b\tilde{\Xi}) = aI^{\pi_n}\Xi + bI^{\pi_n}\tilde{\Xi}$ holding for any $a, b \in \mathbb{R}$ and $\tilde{\Xi}$ of the same regularity as Ξ by (14).

Assume that (16) holds. Bound (17) follows from $[\mathscr{I}]_{B^{\alpha}_{\Phi_{\beta_1\wedge\beta_2},q_1\vee q_2}} = \|\delta\mathscr{I}\|_{B^{\alpha;2}_{\Phi_{\beta_1\wedge\beta_2},q_1\vee q_2}}$ by the triangle inequality, Proposition 3.10, (15), and Proposition 3.12 with (16). Since $\delta\mathscr{I} \in B^{\alpha;2}_{\Phi_{\beta_1\wedge\beta_2},q_1\vee q_2}$ and $\delta(\delta\mathscr{I}) = 0$, $\delta\mathscr{I}$ has a continuous and bounded modification by Proposition 3.8 and, similarly, so does Ξ by (16). The continuity of $\delta\mathscr{I}$ also implies the continuity of $\mathscr{I}$.

Let us now establish that the choice of $\mathscr{I}_0$ identifies the map $\mathscr{I}$ uniquely. Let $\tilde{\mathscr{I}} \in C([0,T],\mathbb{R}^m)$ be such that (15) holds with $\tilde{\mathscr{I}}$ in place of $\mathscr{I}$ and let $\tilde{\mathscr{I}}_0 = \Theta$. Since

$$|\delta(\mathscr{I} - \tilde{\mathscr{I}})_{r,r+h}| \le |(\delta\mathscr{I} - \Xi)_{r,r+h}| + |(\delta\tilde{\mathscr{I}} - \Xi)_{r,r+h}|$$

holds for all $(r, r+h) \in \triangle^2[0,T]$, we may directly estimate

$$[\mathscr{I} - \tilde{\mathscr{I}}]_{B^{\gamma}_{\Phi_{\beta_2},q_2}} = \|\delta(\mathscr{I} - \tilde{\mathscr{I}})\|_{B^{\gamma;2}_{\Phi_{\beta_2},q_2}} \lesssim_q \|\mathscr{I} - \Xi\|_{B^{\gamma;2}_{\Phi_{\beta_2},q_2}} + \|\tilde{\mathscr{I}} - \Xi\|_{B^{\gamma;2}_{\Phi_{\beta_2},q_2}} < \infty.$$

From the embeddings in Proposition 3.2 and the standard embeddings of Besov spaces from, e.g., [29, Sections 3.2.4 and 3.3.1], we obtain

$$B^{\gamma}_{\Phi_{\beta_2},q_2} \hookrightarrow B^{\gamma}_{p,q_2} \hookrightarrow B^{\gamma}_{p,\infty} \hookrightarrow B^{\gamma-\frac{1}{p}}_{\infty,\infty} = C^{\gamma-\frac{1}{p}}$$

for any $p \in [1,\infty)$. Hence, for p sufficiently large so that $\gamma - \frac{1}{p} > 1$, it follows that $\mathscr{I} - \tilde{\mathscr{I}}$ is constant on $[0,T]$. By $\mathscr{I}_0 - \tilde{\mathscr{I}}_0 = 0$, it follows that $\mathscr{I} = \tilde{\mathscr{I}}$.

Bounds (18) and (19) are proved similarly to [12, Theorem 3.3 part (a)]. Choose any $p \in (\frac{1}{\alpha} \vee \frac{1}{\gamma-\alpha}, \infty)$. Proposition 3.8 and bound (15) implies

$$\|\delta\mathscr{I} - \Xi\|_{C^{\gamma-1/p}} \lesssim^T_{\gamma,\beta_2,p,\rho} \|\delta\mathscr{I} - \Xi\|_{B^{\gamma;2}_{\Phi_{\beta_2},q_2}} + M \lesssim_{\gamma,q_2} \|\delta\Xi\|_{B^{\gamma;3}_{\Phi_{\beta_2},q_2}} + M.$$

Finally, Lemma 3.9 (with $\delta = \gamma - \frac{1}{p}$, $\beta' = \beta_1$, $\beta = \beta_2$, and $q = q_2$) and the above inequality yield

$$\begin{aligned}\|\delta\mathscr{I} - \Xi\|_{B^{\alpha;2}_{\Phi_{\beta_1},q_2}} &\lesssim_{\alpha,\gamma,\beta_1,\beta_2,p,q_2} T^{\gamma-\alpha-\frac{1}{p}+\frac{\beta_2}{p\beta_1}} \|\delta\mathscr{I} - \Xi\|^{1-\frac{\beta_2}{\beta_1}}_{C^{\gamma-1/p;2}} \|\delta\mathscr{I} - \Xi\|^{\frac{\beta_2}{\beta_1}}_{B^{\gamma;2}_{\Phi_{\beta_2},q_2}} \\ &\lesssim^{T}_{\gamma,\beta_2,p,q_2,\rho} T^{\gamma-\alpha-\frac{1}{p}+\frac{\beta_2}{p\beta_1}} \left(\|\delta\Xi\|_{B^{\gamma;3}_{\Phi_{\beta_2},q_2}} + M\right)^{1-\frac{\beta_2}{\beta_1}} \left(\|\delta\Xi\|_{B^{\gamma;3}_{\Phi_{\beta_2},q_2}} + M\right)^{\frac{\beta_2}{\beta_1}} \\ &= T^{\gamma-\alpha-\frac{1}{p}+\frac{\beta_2}{p\beta_1}} \left(\|\delta\Xi\|_{B^{\gamma;3}_{\Phi_{\beta_2},q_2}} + M\right).\end{aligned}$$

Hence, as we have $\delta(\delta\mathscr{I} - \Xi) = -\delta\Xi$, Proposition 3.8 gives

$$\begin{aligned}[\mathscr{I}]_{B^{\alpha}_{\Phi_{\beta_1},q_1\vee q_2}} &\le \|\Xi\|_{B^{\alpha;2}_{\Phi_{\beta_1},q_1\vee q_2}} + \|\delta\mathscr{I} - \Xi\|_{B^{\alpha;2}_{\Phi_{\beta_1},q_1\vee q_2}} \\ &\lesssim^{T}_{\alpha,\gamma,q_1,q_2} \|\Xi\|_{B^{\alpha;2}_{\Phi_{\beta_1},q_1}} + T^{-\frac{1}{p}} M + \|\delta\mathscr{I} - \Xi\|_{B^{\alpha;2}_{\Phi_{\beta_1},q_2}} + T^{-\frac{1}{p}} M \\ &\lesssim^{T}_{\alpha,\gamma,\beta_1,\beta_2,p,q_2,\rho} \|\Xi\|_{B^{\alpha;2}_{\Phi_{\beta_1},q_1}} + T^{\gamma-\alpha-\frac{1}{p}+\frac{\beta_2}{p\beta_1}} \left(\|\delta\Xi\|_{B^{\gamma;3}_{\Phi_{\beta_2},q_2}} + M\right) + T^{-\frac{1}{p}} M.\end{aligned}$$

□

4.5 Rough Integral

We are now ready to construct the rough integral using the sewing lemma from the previous section. Recall that $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, $\beta \in (0, \infty)$, $q \in (0, \infty]$, let $k = n$, and let $\mathbf{X} \in \mathscr{B}^{\alpha}_{\Phi_\beta,q}([0,T];\mathbb{R}^n)$, $(Y, Y') \in \mathscr{D}^{\alpha;X}_{\Phi_\beta,q}([0,T];\mathbb{R}^{m\times n})$ be fixed. We define the approximation of the rough integral $\Xi = \Xi_{\mathbf{X}}(Y, Y') : \triangle^2[0,T] \to \mathbb{R}^m$ by

$$\Xi_{s,t} = Y_s(X_t - X_s) + Y'_s \mathbb{X}_{s,t}, \quad (s,t) \in \triangle^2[0,T]. \tag{20}$$

Notice that $\Xi \in L^{0;d}([0,T];\mathbb{R}^m)$. Indeed, we can suppose that Ξ is continuous because it has a continuous modification due to Proposition 3.2, Proposition 3.8, and Lemma 4.2. We begin with a simple auxiliary bound.

Lemma 4.13 *For $(s,u,t) \in \triangle^3[0,T]$ and $\theta \in (0,\infty)$, it holds*

$$|u-s|^{2\theta}|t-u|^{\theta} + |u-s|^{\theta}|t-u|^{2\theta} \le 2\left(((u-s)\wedge(t-u))^{\frac{1}{3}}((u-s)\vee(t-u))^{\frac{2}{3}}\right)^{3\theta}.$$

Proof If $u - s \le t - u$, then

$$\begin{aligned}|u-s|^{2\theta}|t-u|^{\theta} + |u-s|^{\theta}|t-u|^{2\theta} &= |t-u|^{2\theta}|u-s|^{\theta}\left(\frac{|u-s|^{\theta}}{|t-u|^{\theta}} + 1\right) \\ &\le 2|t-u|^{2\theta}|u-s|^{\theta} \\ &= 2\left(((u-s)\wedge(t-u))^{\frac{1}{3}}((u-s)\vee(t-u))^{\frac{2}{3}}\right)^{3\theta}.\end{aligned}$$

The case $u - s > t - u$ follows similarly. □

Next, we summarize the basic properties of the rough approximation Ξ.

Lemma 4.14 *Let $\Xi : \triangle^2[0,T] \to \mathbb{R}^m$ be as in (20). Then, for $(s,u,t) \in \triangle^3[0,T]$, it holds*

$$\delta\Xi_{s,u,t} = -R^Y_{s,u}\delta X_{u,t} - \delta Y'_{s,u}\mathbb{X}_{u,t}$$

and, for $p \in (\frac{1}{\alpha}, \infty)$, we have

$$|\delta\Xi_{s,u,t}|_{\mathbb{R}^m} \lesssim^T_{\alpha,\beta,p,q} \left(\|R^Y\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} [X]_{B^\alpha_{\Phi_\beta,q}} + [Y']_{B^\alpha_{\Phi_\beta,q}} |||\mathbf{X}|||^2_{\mathscr{B}^\alpha_{\Phi_\beta,q}} \right) \cdot \left(((u-s)\wedge(t-u))^{\frac{1}{3}} ((u-s)\vee(t-u))^{\frac{2}{3}} \right)^{3(\alpha-\frac{1}{p})}.$$

Proof The identity follows directly from Chen's relations and the definition of R^Y. The bound can be then established in a straightforward manner from the $k(\alpha-1/p)$-Hölder continuity of R^Y, X, Y', and $\mathbb{X}$ for appropriate $k = 1, 2$ with the embeddings into the Besov–Orlicz and the Besov–Orlicz-type spaces from Lemma 4.10, Proposition 3.2, and Lemma 4.3 together with Lemma 4.13. □

Lemma 4.15 *Let $\tilde{\mathbf{X}} = (\tilde{X}, \tilde{\mathbb{X}}) \in \mathscr{B}^\alpha_{\Phi_\beta,q}([0,T];\mathbb{R}^n)$ and $(\tilde{Y}, \tilde{Y}') \in \mathscr{D}^{\alpha;\tilde{X}}_{\Phi_\beta,q}([0,T];\mathbb{R}^{m\times n})$ and let $\tilde{\Xi} = \tilde{\Xi}_{\tilde{\mathbf{X}}}(\tilde{Y}, \tilde{Y}')$ be defined by (20). Assuming that*

$$\left(|Y_0|_{\mathbb{R}^{m\times n}} + |Y'_0|_{\mathcal{L}(\mathbb{R}^n;\mathbb{R}^{m\times n})} + [(Y,Y')]_{\mathscr{D}^{\alpha;X}_{\Phi_\beta,q}} \right) \vee |||\mathbf{X}|||_{\mathscr{B}^\alpha_{\Phi_\beta,q}} \vee \left(|\tilde{Y}_0|_{\mathbb{R}^{m\times n}} + |\tilde{Y}'_0|_{\mathcal{L}(\mathbb{R}^n;\mathbb{R}^{m\times n})} + [(\tilde{Y},\tilde{Y}')]_{\mathscr{D}^{\alpha;\tilde{X}}_{\Phi_\beta,q}} \right) \vee |||\tilde{\mathbf{X}}|||_{\mathscr{B}^\alpha_{\Phi_\beta,q}} \le M$$

holds for some $M \in [0,\infty)$, then, for all $(s,u,t) \in \triangle^3[0,T]$, we have the estimate

$$|\delta(\Xi - \tilde{\Xi})_{s,u,t}| \lesssim^{T,M}_{\alpha,\beta,p,q} \left(((u-s)\wedge(t-u))^{\frac{1}{3}} ((u-s)\vee(t-u))^{\frac{2}{3}} \right)^{3(\alpha-\frac{1}{p})} \cdot \left(\|R^Y - R^{\tilde{Y}}\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} + [Y' - \tilde{Y}']_{B^\alpha_{\Phi_\beta,q}} + [X - \tilde{X}]_{B^\alpha_{\Phi_\beta,q}} + \|\mathbb{X} - \tilde{\mathbb{X}}\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} \right).$$

Proof Since it is readily seen that

$$\delta(\Xi - \tilde{\Xi})_{s,u,t} = (R^{\tilde{Y}} - R^Y)_{s,u}\delta X_{u,t} + R^{\tilde{Y}}_{s,u}\delta(\tilde{X} - X)_{u,t} + \delta(\tilde{Y}' - Y')_{s,u}\mathbb{X}_{u,t} + \delta\tilde{Y}'_{s,u}(\tilde{\mathbb{X}} - \mathbb{X})_{u,t}$$

holds for all $(s,u,t) \in \triangle^3[0,T]$, we may use the Hölder continuity similarly as in the proof of Lemma 4.14 to deduce

$$|\delta(\Xi - \tilde{\Xi})_{s,u,t}|_{\mathbb{R}^m} \lesssim^M \left(|u-s|^{2\alpha-\frac{2}{p}} |t-u|^{\alpha-\frac{1}{p}} + |u-s|^{\alpha-\frac{1}{p}} |t-u|^{2\alpha-\frac{2}{p}} \right) \cdot \left(\|R^Y - R^{\tilde{Y}}\|_{C^{2\alpha-2/p;2}} + [X - \tilde{X}]_{C^{\alpha-1/p}} + [Y' - \tilde{Y}']_{C^{\alpha-1/p}} + \|\mathbb{X} - \tilde{\mathbb{X}}\|_{C^{2\alpha-2/p;2}} \right).$$

The rest of the claim follows from the estimates in Lemma 4.10, Proposition 3.2 and Proposition 3.8 and the bound in Lemma 4.13. □

We are now ready to establish the existence of the rough integral that corresponds to the local approximation Ξ.

Theorem 4.16 *The mapping*

$$\mathcal{I} = \mathcal{I}_{\mathbf{X}}(Y,Y') : [0,T] \to \mathbb{R}^m, \quad t \mapsto \mathscr{J}(\Xi_{\mathbf{X}}(Y,Y'))_t,$$

where $\mathscr{I}(\Xi_{\mathbf{X}}(Y, Y'))$ *is the map from Theorem 4.12 and* $\Xi_{\mathbf{X}}(Y, Y')$ *is defined in (20), is well-defined. Moreover,* $\mathcal{I}_0 = 0$*, the mapping*

$$\mathscr{I} : \mathscr{D}^{\alpha;X}_{\Phi_\beta,q}([0,T];\mathbb{R}^{m\times n}) \to \mathscr{D}^{\alpha;X}_{\Phi_\beta,q}([0,T];\mathbb{R}^m), \quad (\bar{Y},\bar{Y}') \mapsto \left(\mathcal{I}_{\mathbf{X}}(\bar{Y},\bar{Y}'),\bar{Y}\right)$$

is linear, the limit

$$\lim_{N\to\infty} \|\delta\mathcal{I} - I^{\pi_N}\Xi\|_{B^{3\alpha;2}_{\Phi_{\beta/3},\bar{q}}} = 0 \tag{21}$$

holds for any $\bar{q} > \frac{1}{3\alpha}$, $\bar{q} \geq \frac{q}{3}$, *and any sequence* $\{\pi_N\}_{N=1}^\infty$ *of partitions of* $[0,1]$ *with vanishing norm, and the bounds*

$$\|\delta\mathcal{I} - \Xi\|_{B^{3\alpha;2}_{\Phi_{\beta/3},q/3}} \lesssim^T_{\alpha,\beta,q} \|\delta\Xi\|_{B^{3\alpha;3}_{\Phi_{\beta/3},q/3}} \lesssim^T_q \|R^Y\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} [X]_{B^\alpha_{\Phi_\beta,q}} + [Y']_{B^\alpha_{\Phi_\beta,q}} |||\mathbf{X}|||_{\mathscr{B}^\alpha_{\Phi_\beta,q}} \tag{22}$$

and

$$\begin{aligned}[\mathcal{I}]_{B^\alpha_{\Phi_\beta,q}} \lesssim_{\alpha,\beta,q,T} &\left(|Y_0|_{\mathbb{R}^{m\times n}} + [Y]_{B^\alpha_{\Phi_\beta,q}} + \|R^Y\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}}\right)[X]_{B^\alpha_{\Phi_\beta,q}} \\ &+ \left(|Y_0'|_{\mathcal{L}(\mathbb{R}^n;\mathbb{R}^{m\times n})} + [Y']_{B^\alpha_{\Phi_\beta,q}}\right)|||\mathbf{X}|||^2_{\mathscr{B}^\alpha_{\Phi_\beta,q}}\end{aligned} \tag{23}$$

hold. Additionally, for any $p \in (\frac{3}{\alpha}, \infty)$*, it holds that*

$$\begin{aligned}\|R^{\mathcal{I}}\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} \lesssim^T_{\alpha,\beta,p,q} &|Y_0'|_{\mathcal{L}(\mathbb{R}^n;\mathbb{R}^{m\times n})}\|\mathbb{X}\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} \\ &+ T^{\alpha-\frac{1}{p}}\left(|||\mathbf{X}|||_{\mathscr{B}^\alpha_{\Phi_\beta,q}} \vee |||\mathbf{X}|||^2_{\mathscr{B}^\alpha_{\Phi_\beta,q}}\right)[(Y,Y')]_{\mathscr{D}^{\alpha;X}_{\Phi_\beta,q}}.\end{aligned} \tag{24}$$

Proof Note first that it holds that

$$\|\Xi\|_{B^{\alpha;2}_{\Phi_\beta,q}} \lesssim^T_{\alpha,\beta,q} \|Y\|_{L^\infty}[X]_{B^\alpha_{\Phi_\beta,q}} + \|Y'\|_{L^\infty}|||\mathbf{X}|||^2_{\mathscr{B}^\alpha_{\Phi_\beta,q}}. \tag{25}$$

Recalling the definition of Ξ in (20), a straightforward application of the Hölder-type inequality in Orlicz spaces from Lemma 2.3 (with $\beta_1 = 3$, $\beta_2 = \frac{3}{2}$, and $\beta/3$ in place of β) and Lemma 2.1 leads to

$$\|\delta\Xi\|_{B^{3\alpha;3}_{\Phi_{\beta/3},q/3}} \lesssim^T_q \|R^Y\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}}[X]_{B^\alpha_{\Phi_\beta,q}} + [Y']_{B^\alpha_{\Phi_\beta,q}}\|\mathbb{X}\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}}. \tag{26}$$

Since (25), (26), and the bound in Lemma 4.14 verify the assumptions of the sewing lemma in Theorem 4.12, there exists $\mathcal{I} \in L^{\Phi_\beta}_{\text{fin}}([0,T];\mathbb{R}^m)$ that satisfies $\delta\mathcal{I} - \Xi \in B^{3\alpha;2}_{\Phi_{\beta/3},q/3}([0,T];\mathbb{R}^m)$ and (22) and, for which (21) holds for any sequence $\{\pi_N\}_{N=1}^\infty$ of partitions of $[0,1]$ with vanishing norms and for all $\bar{q} > \frac{1}{3\alpha}$ and $\bar{q} \geq \frac{q}{3}$. Estimate (23) follows from (19) in Theorem 4.12 by (25), (26), and by the bound in Lemma 4.14 with Proposition 3.6 and the choice, e.g., $p = 2/\alpha$.

It remains to establish (24). Let $p \in (\frac{3}{\alpha}, \infty)$. By setting $\mathcal{I}' = Y$, we observe that

$$\delta\mathcal{I} - \Xi = \delta\mathcal{I} - \mathcal{I}'\delta X - Y'\mathbb{X} = R^{\mathcal{I}} - Y'\mathbb{X}$$

holds and thus,

$$\|R^{\mathcal{I}}\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} \leq \|Y'\mathbb{X}\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} + \|\delta\mathcal{I} - \Xi\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}}$$

is obtained. Regarding the first term in the above inequality, by Proposition 3.6 and the definition of norms on $\mathscr{D}^{\alpha;X}_{\Phi_\beta,q}$ and $\mathscr{B}^{\alpha}_{\Phi_\beta,q}$, we deduce

$$\begin{aligned}\|Y'\mathbb{X}\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} &\leq \|Y'\|_{L^\infty}\|\mathbb{X}\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}}\\ &\lesssim^T_{\alpha,\beta,p,q} |Y'_0|_{\mathcal{L}(\mathbb{R}^n;\mathbb{R}^{m\times n})}\|\mathbb{X}\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} + T^{\alpha-\frac{1}{p}}[(Y,Y')]_{\mathscr{D}^{\alpha;X}_{\Phi_\beta,q}}|||\mathbf{X}|||^2_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}}.\end{aligned}$$

For the second term, let us denote $\mathcal{R}\Xi = \delta\mathcal{I} - \Xi$ for brevity. Since $\delta\mathcal{R}\Xi = \delta(\delta\mathcal{I} - \Xi) = -\delta\Xi$, the inclusion $B^{3\alpha;2}_{\Phi_{\beta/3},q/3} \subseteq C^{3(\alpha-1/p)}$ from Proposition 3.8 (with the bound from Lemma 4.14) and (26) yield

$$\begin{aligned}\|\mathcal{R}\Xi\|_{C^{3(\alpha-\frac{1}{p})}} &\lesssim^T_{\alpha,p,q} \|\delta\Xi\|_{B^{3\alpha;3}_{\Phi_{\beta/3},q/3}} + \|R^Y\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}}[X]_{B^{\alpha}_{\Phi_\beta,q}} + [Y']_{B^{\alpha}_{\Phi_\beta,q}}|||\mathbf{X}|||^2_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}}\\ &\leq \left(|||\mathbf{X}|||_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}} \vee |||\mathbf{X}|||^2_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}}\right)[(Y,Y')]_{\mathscr{D}^{\alpha;X}_{\Phi_\beta,q}}.\end{aligned} \tag{27}$$

By appealing to the interpolation inequality from Lemma 3.9 (with $3\delta = \alpha - \frac{3}{p}$, $\beta' = \frac{\beta}{2}$, and $\frac{\beta}{3}$ in place of β), the Young inequality, the inclusion $B^{3\alpha;2}_{\Phi_{\beta/3},q/3} \subseteq B^{3\alpha;2}_{\Phi_{\beta/3},q/2}$ from Proposition 3.12 (with M from Lemma 4.14), and to estimate (22), we obtain

$$\begin{aligned}\|\mathcal{R}\Xi\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} &\lesssim^T_{\alpha,\beta,p,q} T^{2\alpha-\frac{1}{p}}\left(\|\mathcal{R}\Xi\|_{C^{3\alpha-1/p;2}} + \|\mathcal{R}\Xi\|_{B^{3\alpha;2}_{\Phi_{\beta/3},q/2}}\right)\\ &\lesssim^T_{\alpha,q} T^{2\alpha-\frac{1}{p}}\left(\|\mathcal{R}\Xi\|_{C^{3\alpha-1/p;2}} + \|\mathcal{R}\Xi\|_{B^{3\alpha;2}_{\Phi_{\beta/3},q/3}}\right)\\ &\quad + T^{2\alpha-\frac{4}{p}}\left(\|R^Y\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}}[X]_{B^{\alpha}_{\Phi_\beta,q}} + [Y']_{B^{\alpha}_{\Phi_\beta,q}}\|\mathbb{X}\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}}\right)\\ &\lesssim_{\alpha,q} T^{2\alpha-\frac{1}{p}}\left(\|\mathcal{R}\Xi\|_{C^{3\alpha-1/p;2}} + \|\delta\Xi\|_{B^{3\alpha;2}_{\Phi_{\beta/3},q/3}}\right)\\ &\quad + T^{2\alpha-\frac{4}{p}}\left(\|R^Y\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}}[X]_{B^{\alpha}_{\Phi_\beta,q}} + [Y']_{B^{\alpha}_{\Phi_\beta,q}}\|\mathbb{X}\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}}\right).\end{aligned}$$

Hence, bounds (27) and (26) imply

$$\|\mathcal{R}\Xi\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} \lesssim^T_{\alpha,\beta,p,q} T^{2\alpha-\frac{4}{p}}\left(|||\mathbf{X}|||_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}} \vee |||\mathbf{X}|||^2_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}}\right)[(Y,Y')]_{\mathscr{D}^{\alpha;X}_{\Phi_\beta,q}}$$

and the proof of (24) is concluded by gathering the estimates above. □

Definition 4.17 For $\mathbf{X} \in \mathscr{B}^{\alpha}_{\Phi_\beta,q}([0,T];\mathbb{R}^n)$ and $(Y,Y') \in \mathscr{D}^{\alpha;X}_{\Phi_\beta,q}([0,T];\mathbb{R}^{m\times n})$, we define the *rough integral of Y with respect to the rough path* $\mathbf{X}$ by

$$\int_0^\cdot Y_s\,\mathrm{d}\mathbf{X}_s : [0,T] \to \mathbb{R}^m, \quad \int_0^\cdot Y_s\,\mathrm{d}\mathbf{X}_s = \mathcal{I}_{\mathbf{X}}(Y,Y').,$$

where $\mathcal{I}_{\mathbf{X}}(Y,Y')$ is defined in Theorem 4.16.

We conclude this section with a continuity estimate for the rough integral.

Theorem 4.18 *Let* $\mathbf{X}, \tilde{\mathbf{X}} \in \mathscr{B}^{\alpha}_{\Phi_\beta,q}([0,T];\mathbb{R}^n)$, $(Y,Y') \in \mathscr{D}^{\alpha;X}_{\Phi_\beta,q}([0,T];\mathbb{R}^{m\times n})$, *and* $(\tilde{Y},\tilde{Y}') \in \mathscr{D}^{\alpha;\tilde{X}}_{\Phi_\beta,q}([0,T];\mathbb{R}^{m\times n})$. *Assume that*

$$\left(|Y_0|_{\mathbb{R}^{m\times n}} + |Y_0'|_{\mathcal{L}(\mathbb{R}^n;\mathbb{R}^{m\times n})} + [(Y,Y')]_{\mathscr{D}^{\alpha;X}_{\Phi_\beta,q}}\right) \vee |||\mathbf{X}|||_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}} \vee \left(|\tilde{Y}_0|_{\mathbb{R}^{m\times n}} + |\tilde{Y}_0'|_{\mathcal{L}(\mathbb{R}^n;\mathbb{R}^{m\times n})} + [(\tilde{Y},\tilde{Y}')]_{\mathscr{D}^{\alpha;\tilde{X}}_{\Phi_\beta,q}}\right) \vee |||\tilde{\mathbf{X}}|||_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}} \le M$$

holds for some $M \in (0,\infty)$. *Let* $\tilde{\Xi} = \Xi_{\tilde{\mathbf{X}}}(\tilde{Y},\tilde{Y}')$ *and* $\tilde{\mathcal{I}} = \mathcal{I}_{\tilde{\mathbf{X}}}(\tilde{Y},\tilde{Y}')$, *where* Ξ *and* $\mathcal{I}$ *are defined in* (20) *and Theorem 4.16, respectively. Then, for any* $p \in (\frac{4}{\alpha},\infty)$, *it holds that*

$$\begin{aligned} d_{X,\tilde{X},\mathscr{D}^{\alpha}_{\Phi_\beta,q}}((\mathcal{I},Y),(\tilde{\mathcal{I}},\tilde{Y})) &\lesssim^{T,M}_{\alpha,\beta,p,q} |Y_0-\tilde{Y}_0|^{\frac{q}{2}\wedge 1}_{\mathbb{R}^{m\times n}} + |Y_0'-\tilde{Y}_0'|^{\frac{q}{2}\wedge 1}_{\mathcal{L}(\mathbb{R}^n;\mathbb{R}^{m\times n})} \\ &\quad + \rho_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}}(\mathbf{X},\tilde{\mathbf{X}})^{\frac{q}{2}\wedge 1} + (T^{\alpha-\frac{4}{p}})^{\frac{q}{2}\wedge 1}\left(\|R^Y - R^{\tilde{Y}}\|^{\frac{q}{2}\wedge 1}_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} + [Y'-\tilde{Y}']^{\frac{q}{2}\wedge 1}_{B^{\alpha}_{\Phi_\beta,q}}\right). \end{aligned}$$

Proof The proof is similar to the proof of [12, Theorem 5.5]. However, since the resulting bounds are different and since these bounds are important for our main result, we include the details.

We will often implicitly use the inequality $x \le (1\vee M)x^{q\wedge 1}$ that holds for all $q \in (0,\infty)$ and $x \in [0,M]$ and the following two estimates. From Lemma 4.4, we observe

$$\|\mathbb{X}\|_{B^{\alpha;2}_{\Phi_\beta,q}} \lesssim^{T}_{\alpha,\beta,p,q} (T^{\alpha-\frac{1}{p}} \vee T^{\alpha-\frac{2}{p}})|||\mathbf{X}|||^2_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}} \lesssim^{T,M} T^{\alpha-\frac{2}{p}}$$

and

$$\begin{aligned} \|\mathbb{X}-\tilde{\mathbb{X}}\|_{B^{\alpha;2}_{\Phi_\beta,q}} &\lesssim^{T}_{\alpha,\beta,p,q} \sum_{i=1}^{2}\left(|||\mathbf{X}|||_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}} \vee |||\tilde{\mathbf{X}}|||_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}}\right)^{2-i}\|\mathbf{X}^{(i)}-\tilde{\mathbf{X}}^{(i)}\|_{B^{i\alpha;2}_{\Phi_{\beta/i},q/i}} \\ &\lesssim^{T,M}_{q} \rho_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}}(\mathbf{X},\tilde{\mathbf{X}}). \end{aligned}$$

Also note that by Proposition 3.6, Lemma 4.10, and the choice of M, it holds

$$\begin{aligned} \|Y\|_{L^\infty} &\lesssim^{T}_{\alpha,\beta,p,q} |Y_0| + T^{\alpha-\frac{1}{p}}[Y]_{B^{\alpha}_{\Phi_\beta,q}} \\ &\lesssim^{T,M}_{\alpha,\beta,p,q} |Y_0| + T^{\alpha-\frac{1}{p}}\left(|Y_0'| + (T^{\alpha-\frac{1}{p}} \vee T^{\alpha-\frac{3}{p}})\left([Y']_{B^{\alpha}_{\Phi_\beta,q}} + \|R^Y\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}}\right)\right) \\ &\lesssim^{T,M} 1. \end{aligned}$$

Similarly, Proposition 3.6 implies $\|Y'\|_{L^\infty} \lesssim^{T,M}_{\alpha,\beta,p,q} 1$. Bounds of the same type obviously hold also for $\tilde{Y}$ and $\tilde{Y}'$. In the rest of the proof, we will use these estimates implicitly as well.

By Proposition 3.6, Lemma 4.10, and by the inequality $\alpha - \frac{3}{p} \leq \alpha - \frac{1}{p}$, we estimate

$$\begin{aligned}
&d_{X,\tilde{X},\mathscr{D}^\alpha_{\Phi_\beta,q}}((\mathcal{I},Y),(\tilde{\mathcal{I}},\tilde{Y}')) \\
&\lesssim_q \|Y-\tilde{Y}\|_{L^{\Phi_\beta}} + [Y-\tilde{Y}]^{q\wedge 1}_{B^\alpha_{\Phi_\beta,q}} + \|R^{\mathcal{I}} - R^{\tilde{\mathcal{I}}}\|^{\frac{q}{2}\wedge 1}_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} \\
&\lesssim^{T,M}_{\alpha,\beta,p,q} |Y_0-\tilde{Y}_0| + [Y-\tilde{Y}]^{q\wedge 1}_{B^\alpha_{\Phi_\beta,q}} + \|R^{\mathcal{I}} - R^{\tilde{\mathcal{I}}}\|^{\frac{q}{2}\wedge 1}_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} \\
&\lesssim^{T,M}_{\alpha,\beta,p,q} |Y_0-\tilde{Y}_0| + |Y'_0-\tilde{Y}'_0|^{q\wedge 1} + \rho^{q\wedge 1}_{\mathscr{B}^\alpha_{\Phi_\beta,q}}(\mathbf{X},\tilde{\mathbf{X}}) \\
&\qquad + (T^{\alpha-\frac{3}{p}})^{q\wedge 1}[Y'-\tilde{Y}']^{q\wedge 1}_{B^\alpha_{\Phi_\beta,q}} + \|R^{\mathcal{I}} - R^{\tilde{\mathcal{I}}}\|^{\frac{q}{2}\wedge 1}_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}}.
\end{aligned}$$

Recalling the equalities $\mathcal{R}\Xi = \delta\mathcal{I} - \Xi = \delta\mathcal{I} - Y\delta X - Y'\mathbb{X}$ and $\mathcal{R}\tilde{\Xi} = \delta\tilde{\mathcal{I}} - \tilde{Y}\delta\tilde{X} - \tilde{Y}'\tilde{\mathbb{X}}$, we can estimate the last term by

$$\|R^{\mathcal{I}} - R^{\tilde{\mathcal{I}}}\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} \lesssim_q \|\mathcal{R}\Xi - \mathcal{R}\tilde{\Xi}\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} + \|Y'\mathbb{X} - \tilde{Y}'\tilde{\mathbb{X}}\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}}. \tag{28}$$

Before we proceed to the estimates of the first term on the right-hand side of (28), we establish several auxiliary bounds. Similarly as in the proof of Theorem 4.16, we use the bound from Lemma 4.15 with Lemma 2.3 and Lemma 2.1 to deduce

$$\|\delta(\Xi-\tilde{\Xi})\|_{B^{3\alpha;3}_{\Phi_{\beta/3},q/3}} \lesssim^{T,M}_q \rho_{\mathscr{B}^\alpha_{\Phi_\beta,q}}(\mathbf{X},\tilde{\mathbf{X}}) + [Y'-\tilde{Y}']_{B^\alpha_{\Phi_\beta,q}} + \|R^Y - R^{\tilde{Y}}\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}}. \tag{29}$$

By Proposition 3.6 and Lemma 4.10 and by observing that $\alpha - \frac{1}{p} \leq 2\alpha - \frac{4}{p}$ holds by the choice of p, we have

$$\begin{aligned}
\|Y-\tilde{Y}\|_{L^\infty} &\lesssim^T_{\alpha,\beta,p,q} |Y_0-\tilde{Y}_0| + T^{\alpha-\frac{1}{p}}[Y-\tilde{Y}]_{B^\alpha_{\Phi_\beta,q}} \\
&\lesssim^{T,M}_{\alpha,\beta,p,q} |Y_0-\tilde{Y}_0| + |Y'_0-\tilde{Y}'_0| + \rho_{\mathscr{B}^\alpha_{\Phi_\beta,q}}(\mathbf{X},\tilde{\mathbf{X}}) \\
&\qquad + T^{\alpha-\frac{1}{p}}\left([Y'-\tilde{Y}']_{B^\alpha_{\Phi_\beta,q}} + \|R^Y - R^{\tilde{Y}}\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}}\right)
\end{aligned}$$

and, by Proposition 3.6, we also have

$$\|Y'-\tilde{Y}'\|_{L^\infty} \lesssim^T_{\alpha,\beta,p,q} |Y'_0-\tilde{Y}'_0|_{\mathcal{L}(\mathbb{R}^n;\mathbb{R}^{m\times n})} + T^{\alpha-\frac{1}{p}}[Y'-\tilde{Y}']_{B^\alpha_{\Phi_\beta,q}}.$$

Noticing that $\delta\mathcal{R}(\Xi-\tilde{\Xi}) = -\delta(\Xi-\tilde{\Xi})$, the embedding into Hölder-type functions from Proposition 3.8 (with the bound from Lemma 4.15) and (29) yield

$$\begin{aligned}
\|\mathcal{R}(\Xi-\tilde{\Xi})\|_{C^{3(\alpha-\frac{1}{p});2}} &\lesssim^{T,M}_{\alpha,p,q} \|\delta(\Xi-\tilde{\Xi})\|_{B^{3\alpha;3}_{\Phi_{\beta/3},q/3}} + \rho_{\mathscr{B}^\alpha_{\Phi_\beta,q}}(\mathbf{X},\tilde{\mathbf{X}}) \\
&\qquad + [Y'-\tilde{Y}']_{B^\alpha_{\Phi_\beta,q}} + \|R^Y - R^{\tilde{Y}}\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} \\
&\lesssim^{T,M}_q \rho_{\mathscr{B}^\alpha_{\Phi_\beta,q}}(\mathbf{X},\tilde{\mathbf{X}}) + [Y'-\tilde{Y}']_{B^\alpha_{\Phi_\beta,q}} + \|R^Y - R^{\tilde{Y}}\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}}.
\end{aligned}$$

Finally, we use the interpolation inequality from Lemma 3.9 (with $\alpha = 2\alpha$, $\gamma = 3\alpha$, $\frac{\beta}{3}$ in place of β, $\beta' = \frac{\beta}{2}$, and $\frac{q}{2}$ in place of q), the Young inequality, the inclusion $B^{3\alpha;2}_{\Phi_{\beta/3},q/3} \subseteq B^{3\alpha;2}_{\Phi_{\beta/3},q/2}$

from Proposition 3.12 (with M from Lemma 4.14), estimate (15) from Theorem 4.12 (applied to $\Xi - \tilde{\Xi}$ in place of Ξ thanks to linearity) and the bound above with estimate (29) to get

$$\begin{aligned}
\|\mathcal{R}(\Xi - \tilde{\Xi})\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} &\lesssim_{\alpha,\beta,p,q} T^{\alpha-\frac{1}{p}} \left(\|\mathcal{R}(\Xi - \tilde{\Xi})\|_{C^{3(\alpha-\frac{1}{p});2}} + \|\mathcal{R}(\Xi - \tilde{\Xi})\|_{B^{3\alpha;3}_{\Phi_{\beta/3},q/2}} \right) \\
&\lesssim^{T}_{\alpha,p} T^{\alpha-\frac{1}{p}} \left(\|\mathcal{R}(\Xi - \tilde{\Xi})\|_{C^{3(\alpha-\frac{1}{p});2}} + \|\mathcal{R}(\Xi - \tilde{\Xi})\|_{B^{3\alpha;2}_{\Phi_{\beta/3},q/3}} \right) \\
&\quad + T^{\alpha-\frac{1}{p}} T^{-\frac{3}{p}} \left([Y' - \tilde{Y}']_{B^{\alpha}_{\Phi_\beta,q}} + \|R^Y - R^{\tilde{Y}}\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} \right) + \rho_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}}(\mathbf{X}, \tilde{\mathbf{X}}) \\
&\lesssim^{T,M}_{\alpha,\beta,p,q} \rho_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}}(\mathbf{X}, \tilde{\mathbf{X}}) + T^{\alpha-\frac{4}{p}} \left([Y' - \tilde{Y}']_{B^{\alpha}_{\Phi_\beta,q}} + \|R^Y - R^{\tilde{Y}}\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} \right).
\end{aligned}$$

It remains to estimate the second term on the right-hand side of (28). By the (quazi-)triangle inequality, the choice of M and the bounds above, we deduce

$$\begin{aligned}
\|Y'\mathbb{X} - \tilde{Y}'\tilde{\mathbb{X}}\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} &\lesssim_q \|Y'\|_{L^\infty} \|\mathbb{X} - \tilde{\mathbb{X}}\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} + \|Y' - \tilde{Y}'\|_{L^\infty} \|\tilde{\mathbb{X}}\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} \\
&\lesssim^{T,M}_{\alpha,\beta,p,q} \rho_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}}(\mathbf{X}, \tilde{\mathbf{X}}) + |Y'_0 - \tilde{Y}'_0| + T^{\alpha-\frac{4}{p}} [Y' - \tilde{Y}']_{B^{\alpha}_{\Phi_\beta,q}}.
\end{aligned}$$

The proof is concluded by collecting the bounds above and straightforward estimates using the inequality $x \leq (1 \vee M) x^{q \wedge 1}$ from the beginning of the proof. □

5 Rough Differential Equations with Besov–Orlicz Signals

In this section, we state and prove our main result – the existence and uniqueness of solutions to rough differential equations driven by exponential Besov–Orlicz signals. We fix $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, $\beta \in (0, \infty)$, and $q \in (0, \infty]$ for the whole section. Additionally, let $\mathbf{X} \in \mathscr{B}^{\alpha}_{\Phi_\beta,q}([0,T]; \mathbb{R}^n)$ and $f \in C^{2,1}_b(\mathbb{R}^m, \mathbb{R}^{m\times n})$. First, we define the notion of a solution to a rough differential equation.

Definition 5.1 We say that $(Y, Y') \in \mathscr{D}^{\alpha;X}_{\Phi_\beta,q}([0,T]; \mathbb{R}^m)$ is a *solution* to the rough differential equation (RDE)

$$\mathrm{d}Y_t = f(Y_t)\,\mathrm{d}\mathbf{X}_t, \quad Y_0 = y \in \mathbb{R}^m, \tag{30}$$

if

$$Y_t = y + \int_0^t f(Y_s)\,\mathrm{d}\mathbf{X}_s,$$

where the integral $\int_0^t f(Y_s)\,\mathrm{d}\mathbf{X}_s$ is the rough integral from Definition 4.17, holds for all $t \in [0,T]$.

We note that the rough integral $\int_0^\cdot f(Y_s)\,\mathrm{d}\mathbf{X}_s$ in the above definition is well-defined. Indeed, Theorem 4.11 guarantees that

$$(f(Y), f(Y)') \in \mathscr{D}^{\alpha;X}_{\Phi_\beta,q}([0,T]; \mathbb{R}^{m\times n}). \tag{31}$$

Therefore, Theorem 4.16 ensures that

$$y + \int_0^\cdot f(Y)\,\mathrm{d}\mathbf{X} \in B^{\alpha}_{\Phi_\beta,q}([0,T]; \mathbb{R}^m) \tag{32}$$

and that

$$R^{y+\int_0^\cdot f(Y)\,\mathrm{d}\mathbf{X}} \in B^{2\alpha;2}_{\Phi_{\beta/2},q/2}([0,T];\mathbb{R}^m). \tag{33}$$

We are now ready to state the main result.

Theorem 5.2 *For any $y \in \mathbb{R}^m$, there exists a unique solution $(Y, Y') \in \mathscr{D}^{\alpha;X}_{\Phi_\beta,q}([0,T];\mathbb{R}^m)$ to RDE (30).*

Before we proceed to the proof, we include an auxiliary completeness result used in the proof of Theorem 5.2.

Lemma 5.3 *The space*

$$\mathscr{Y}_T = \{(Y,Y') \in \mathscr{D}^{\alpha;X}_{\Phi_\beta,q}([0,T];\mathbb{R}^m) \mid Y_0 = y,\, Y'_0 = f(y)\}$$

equipped with the metric $d_{X,\mathscr{D}^\alpha_{\Phi_\beta,q}}$ from Definition 4.8 is complete.

Proof Let $\{(Y^{(n)}, Y'^{(n)})\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathscr{Y}_T$. By Definition 4.8, we need to find a subsequence converging to a point in $\mathscr{Y}_T$ in the metrics on the spaces $B^\alpha_{\Phi_\beta,q}$ and $B^{2\alpha;2}_{\Phi_{\beta/2},q/2}$ defined in (4) and (8), respectively. By the completeness of the affine subspaces from Proposition 3.6 and the bounds in Lemma 4.10, we may find $Y \in B^\alpha_{\Phi_\beta,q}([0,T];\mathbb{R}^m)$, $Y' \in B^\alpha_{\Phi_\beta,q}([0,T];\mathcal{L}(\mathbb{R}^n;\mathbb{R}^m))$, and $R^Y \in B^{2\alpha;2}_{\Phi_{\beta/2},q/2}([0,T];\mathbb{R}^m)$ such that $Y_0 = y$, $Y'_0 = f(y)$, and

$$\lim_{n\to\infty} [Y^{(n)} - Y]_{B^\alpha_{\Phi_\beta,q}} = 0, \quad \lim_{n\to\infty} [Y'^{(n)} - Y']_{B^\alpha_{\Phi_\beta,q}} = 0, \quad \lim_{n\to\infty} \|R^{Y^{(n)}} - R\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} = 0.$$

Moreover, the estimate from Propostion 3.6 yields

$$\|Y'^{(n)} - Y'\|_{L^\infty} \vee \|Y'^{(n)} - Y'\|_{L^{\Phi_\beta}} \lesssim_{\alpha,\beta,q,T} [Y'^{(n)} - Y']_{B^\alpha_{\Phi_\beta,q}}. \tag{34}$$

Choose $\varepsilon \in (0,\alpha)$ arbitrary. Then, using $R^{Y^{(n)}} = \delta Y^{(n)} - Y'^{(n)}\delta X$ and the embedding $B^{2\alpha;2}_{\Phi_{\beta/2},q} \hookrightarrow B^{\alpha-\varepsilon;2}_{\Phi_{\beta/2},q}$ from Proposition 3.10, we obtain

$$\begin{aligned}
&\|R - (\delta Y - Y'\delta X)\|_{B^{\alpha-\varepsilon;2}_{\Phi_{\beta/2},q/2}} \\
&\quad\lesssim_q \|R - R^{Y^{(n)}}\|_{B^{\alpha-\varepsilon;2}_{\Phi_{\beta/2},q/2}} + \|(\delta Y^{(n)} - Y'^{(n)}\delta X) - (\delta Y - Y'\delta X)\|_{B^{\alpha-\varepsilon;2}_{\Phi_{\beta/2},q/2}} \\
&\quad\lesssim^T_{\alpha,\beta,q,\varepsilon} \|R - R^{Y^{(n)}}\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} + \|\delta(Y^{(n)} - Y)\|_{B^{\alpha-\varepsilon;2}_{\Phi_\beta,q/2}} + \|Y'^{(n)} - Y'\|_{L^\infty}\|\delta X\|_{B^{\alpha-\varepsilon;2}_{\Phi_\beta,q/2}} \\
&\quad= \|R - R^{Y^{(n)}}\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} + [Y^{(n)} - Y]_{B^{\alpha-\varepsilon}_{\Phi_\beta,q/2}} + \|Y'^{(n)} - Y'\|_{L^\infty}[X]_{B^{\alpha-\varepsilon}_{\Phi_\beta,q/2}}.
\end{aligned}$$

Hence, the embedding $B^\alpha_{\Phi_\beta,q} \hookrightarrow B^{\alpha-\varepsilon}_{\Phi_\beta,q/2}$ from Proposition 3.5 and estimate (34) yield

$$\begin{aligned}
&\|R - (\delta Y - Y'\delta X)\|_{B^{\alpha-\varepsilon;2}_{\Phi_{\beta/2},q/2}} \\
&\qquad\lesssim_{\alpha,\beta,q,\varepsilon,T} \|R - R^{Y^{(n)}}\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} + [Y^{(n)} - Y]_{B^\alpha_{\Phi_\beta,q}} + [Y'^{(n)} - Y']_{B^\alpha_{\Phi_\beta,q}}[X]_{B^\alpha_{\Phi_\beta,q}}.
\end{aligned}$$

As $n \to \infty$, the right-hand side approaches zero. Consequently, $(Y,Y') \in \mathscr{D}^{\alpha;X}_{\Phi_\beta,q}([0,T];\mathbb{R}^m)$, implying that $(Y,Y') \in \mathscr{Y}_T$. Finally, estimate (34) yields

$$\lim_{n\to\infty} d_{X,\mathscr{D}^\alpha_{\Phi_\beta,q}}((Y^{(n)},Y'^{(n)}),(Y,Y')) = 0,$$

which concludes the proof. □

Proof of Theorem 5.2 We aim to use the Banach fixed-point argument to prove Theorem 5.2. To this end, let

$$\bar{Y}_t = y + f(y)X_t \quad \text{and} \quad \bar{Y}'_t \equiv f(y) \quad \text{for } t \in [0, T],$$

and note that $R^{\bar{Y}} \equiv 0$ on $\triangle^2[0, T]$. For $M \geq 0$, we denote the closed ball in $\mathscr{Y}_T$ of radius M centred in $(\bar{Y}, \bar{Y}')$ by

$$B_{\mathscr{Y}_T}((\bar{Y}, \bar{Y}'), M) = \{(Y, Y') \in \mathscr{Y}_T \mid d_{X,\mathscr{D}^\alpha_{\Phi_\beta,q}}((Y, Y'), (\bar{Y}, \bar{Y}')) \leq M\}.$$

By Lemma 5.3, $B_{\mathscr{Y}_T}((\bar{Y}, \bar{Y}'), M)$ is complete. Let $\mathscr{Z} : \mathscr{Y}_T \to \mathscr{Y}_T$ be defined by

$$\mathscr{Z}(Y, Y') = \left(y + \int_0^\cdot f(Y)\,\mathrm{d}\mathbf{X},\, f(Y)\right).$$

The map $\mathscr{Z}$ is well-defined thanks to regularities (31), (32), and (33), and the initial conditions $(\int_0^\cdot f(Y)\,\mathrm{d}\mathbf{X})_0 = 0$ and $f(Y_0) = f(y)$. We will find finite positive constants M_0 large enough and T_0 small enough, both independent of y, such that

i) $\mathscr{Z}(B_{\mathscr{Y}_{T_0}}((\bar{Y}, \bar{Y}'), M_0)) \subseteq B_{\mathscr{Y}_{T_0}}((\bar{Y}, \bar{Y}'), M_0)$, i.e.

$$\|f(Y) - \bar{Y}'\|_{L^{\Phi_\beta}} + [f(Y) - \bar{Y}']^{q\wedge 1}_{B^\alpha_{\Phi_\beta,q}} + \|R^{y+\int_0^\cdot f(Y)\,\mathrm{d}\mathbf{X}} - R^{\bar{Y}}\|^{\frac{q}{2}\wedge 1}_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} \leq M_0$$

for all $(Y, Y') \in B_{\mathscr{Y}_{T_0}}((\bar{Y}, \bar{Y}'), M_0)$; and

ii) $\mathscr{Z}$ is a contraction on $B_{\mathscr{Y}_{T_0}}((\bar{Y}, \bar{Y}'), M_0)$, i.e.

$$d_{X,\mathscr{D}^\alpha_{\Phi_\beta,q}}(\mathscr{Z}(Y, Y'), \mathscr{Z}(\tilde{Y}, \tilde{Y}')) \leq \tilde{C} d_{X,\mathscr{D}^\alpha_{\Phi_\beta,q}}((Y, Y'), (\tilde{Y}, \tilde{Y}'))$$

for some $\tilde{C} < 1$ and all $(Y, Y'), (\tilde{Y}, \tilde{Y}') \in B_{\mathscr{Y}_{T_0}}((\bar{Y}, \bar{Y}'), M_0)$. We remark that the smallness of the contraction factor $\tilde{C}$ comes from the continuity estimate in Theorem 4.18, see (41) below.

The Banach fixed-point theorem then yields the existence and uniqueness of a local solution, i.e. a solution on $[0, T_0]$, that can be extended to the full interval $[0, T]$ in a finite number of steps.

To establish claim 5 above, fix $M \in (0, \infty)$, $(Y, Y') \in B_{\mathscr{Y}_T}((\bar{Y}, \bar{Y}'), M)$, and $p \in (\frac{4}{\alpha}, \infty)$. Since

$$[(Y, Y')]_{\mathscr{D}^{\alpha;X}_{\Phi_\beta,q}} = [Y']_{B^\alpha_{\Phi_\beta,q}} + \|R^Y\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} = [Y' - f(y)]_{B^\alpha_{\Phi_\beta,q}} + \|R^Y - R^{\bar{Y}}\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}},$$

the estimate

$$\begin{aligned}[(Y, Y')]_{\mathscr{D}^{\alpha;X}_{\Phi_\beta,q}} &\leq d_{X,\mathscr{D}^\alpha_{\Phi_\beta,q}}((Y, Y'), (\bar{Y}, \bar{Y}'))^{1/(q\wedge 1)} + d_{X,\mathscr{D}^\alpha_{\Phi_\beta,q}}((Y, Y'), (\bar{Y}, \bar{Y}'))^{1/(\frac{q}{2}\wedge 1)} \\ &\leq M^{1/(q\wedge 1)} + M^{1/(\frac{q}{2}\wedge 1)} =: F_q(M) \end{aligned} \tag{35}$$

holds. Then, from Theorem 4.11, we obtain

$$\begin{aligned}&[(f(Y), f(Y)')]_{\mathscr{D}^{\alpha;X}_{\Phi_\beta,q}} \\ &\lesssim^T_{\alpha,\beta,q} \left(\|f\|_{C^{2,1}_b} \vee \|f\|^2_{C^{2,1}_b}\right)\left(1 \vee |||\mathbf{X}|||_{\mathscr{D}^\alpha_{\Phi_\beta,q}}\right)\left(1 + \left([(Y, Y')]_{\mathscr{D}^{\alpha;X}_{\Phi_\beta,q}} \vee [(Y, Y')]^2_{\mathscr{D}^{\alpha;X}_{\Phi_\beta,q}}\right)\right) \\ &\leq \hat{C}_{(T)}\left(1 + (F_q(M) \vee F_q(M)^2)\right), \end{aligned} \tag{36}$$

where $\hat{C}_{(T)} = C(\|f\|_{C_b^{2,1}([0,T])}, |||\mathbf{X}|||_{\mathscr{B}^\alpha_{\Phi_{\beta,q}}([0,T])})$ for some nondecreasing positive function C captures the dependence on the respective norms of f and $\mathbf{X}$ as a function of T. Thus, as a function of T, $\hat{C}_{(T)}$ is also nondecreasing. Even though the precise value of $\hat{C}_{(T)}$, i.e. the particular function C, may increase from line to line, it will still retain the form described above. Since the Besov–Orlicz seminorm is invariant to shifts, from the Lipschitz continuity of f, it follows that

$$[f(Y) - \bar{Y}']_{B^\alpha_{\Phi_{\beta,q}}} = [f(Y) - f(y)]_{B^\alpha_{\Phi_{\beta,q}}} = [f(Y)]_{B^\alpha_{\Phi_{\beta,q}}} \le \|Df\|_{L^\infty}[Y]_{B^\alpha_{\Phi_{\beta,q}}}.$$

By using Lemma 4.10 and estimate (35), we obtain

$$\begin{aligned}
&[f(Y) - \bar{Y}']_{B^\alpha_{\Phi_{\beta,q}}} \\
&\lesssim^T_{\alpha,\beta,p,q} \|Df\|_{L^\infty}\left(\left|Y_0'\right|_{\mathbb{R}^{m\times n}}[X]_{B^\alpha_{\Phi_{\beta,q}}} + T^{\alpha-\frac{3}{p}}\left([Y']_{B^\alpha_{\Phi_{\beta,q}}}[X]_{B^\alpha_{\Phi_{\beta,q}}} + \|R^Y\|_{B^{2\alpha;2}_{\Phi_{\beta/2,q/2}}}\right)\right) \\
&\le \|f\|_{C_b^{2,1}}\left(\|f\|_{L^\infty}|||\mathbf{X}|||_{\mathscr{B}^\alpha_{\Phi_{\beta,q}}} + T^{\alpha-\frac{3}{p}}\left([Y']_{B^\alpha_{\Phi_{\beta,q}}}|||\mathbf{X}|||_{\mathscr{B}^\alpha_{\Phi_{\beta,q}}} + \|R^Y\|_{B^{2\alpha;2}_{\Phi_{\beta/2,q/2}}}\right)\right) \\
&\le \hat{C}_{(T)}(1 + T^{\alpha-\frac{3}{p}}F_q(M)).
\end{aligned}$$

Thus, by Proposition 3.6 and $\bar{Y}_0' = f(y) = f(Y_0)$, we deduce

$$\|f(Y) - \bar{Y}'\|_{L^{\Phi_\beta}} + [f(Y) - \bar{Y}']^{q\wedge 1}_{B^\alpha_{\Phi_{\beta,q}}} \lesssim_{\alpha,\beta,p,q} \hat{C}_{(T)}(1 + T^{\alpha-\frac{3}{p}}F_q(M)). \tag{37}$$

We remark that we could drop the exponent $q \wedge 1$ as $(1 + T^{\alpha-\frac{3}{p}}(F_q(M))) \ge 1$. We will use this simplification from now on without explicit notice. In a similar manner, since

$$\|R^{y+\int_0^\cdot f(Y)\,\mathrm{d}\mathbf{X}} - R^{\bar{Y}}\|_{B^{2\alpha;2}_{\Phi_{\beta/2,q/2}}} = \|R^{y+\int_0^\cdot f(Y)\,\mathrm{d}\mathbf{X}}\|_{B^{2\alpha;2}_{\Phi_{\beta/2,q/2}}} = \|R^{\int_0^\cdot f(Y)\,\mathrm{d}\mathbf{X}}\|_{B^{2\alpha;2}_{\Phi_{\beta/2,q/2}}},$$

estimate (24) from Theorem 4.16 for the remainder term $R^{\int_0^\cdot f(Y)\,\mathrm{d}\mathbf{X}}$ and estimate (36) yield

$$\begin{aligned}
&\|R^{y+\int_0^\cdot f(Y)\,\mathrm{d}\mathbf{X}} - R^{\bar{Y}}\|_{B^{2\alpha;2}_{\Phi_{\beta/2,q/2}}} \\
&\lesssim^T_{\alpha,\beta,p,q} \left|f(Y)_0'\right|_{\mathcal{L}(\mathbb{R}^n;\mathbb{R}^{m\times n})}\|\mathbb{X}\|_{B^{2\alpha;2}_{\Phi_{\beta/2,q/2}}} \\
&\qquad + T^{\alpha-\frac{1}{p}}\left(|||\mathbf{X}|||_{\mathscr{B}^\alpha_{\Phi_{\beta,q}}} \vee |||\mathbf{X}|||^2_{\mathscr{B}^\alpha_{\Phi_{\beta,q}}}\right)[(f(Y), f(Y)')]_{\mathscr{D}^{\alpha;X}_{\Phi_{\beta,q}}} \\
&\lesssim^T_{\alpha,\beta,q} \hat{C}_{(T)}(1 + T^{\alpha-\frac{1}{p}}(1 + (F_q(M) \vee F_q(M)^2))).
\end{aligned} \tag{38}$$

Combining estimates (37) and (38), we observe

$$\begin{aligned}
&\|f(Y) - \bar{Y}'\|_{L^{\Phi_\beta}([0,T])} + [f(Y) - \bar{Y}']^{q\wedge 1}_{B^\alpha_{\Phi_{\beta,q}}([0,T])} + \|R^{y+\int_0^\cdot f(Y)\,\mathrm{d}\mathbf{X}} - R^{\bar{Y}}\|^{\frac{q}{2}\wedge 1}_{B^{2\alpha;2}_{\Phi_{\beta/2,q/2}}([0,T])} \\
&\qquad \le C_1(\alpha,\beta,p,q,T)\,\hat{C}_{(T)}\left(1 + T^{\alpha-\frac{3}{p}}\left(1 + \left(F_q(M) \vee F_q(M)^2\right)\right)\right)
\end{aligned} \tag{39}$$

where C_1 is a fixed constant with nondecreasing dependence on T. Recalling (36), we may assume that C_1 and $\hat{C}_{(T)}$ also satisfy

$$[(f(Y), f(Y)')]_{\mathscr{D}^{\alpha;X}_{\Phi_{\beta,q}}} \le C_1(\alpha,\beta,p,q,T)\,\hat{C}_{(T)}\left(1 + \left(F_q(M) \vee F_q(M)^2\right)\right) \tag{40}$$

without any loss of generality. We set

$$M_0 = \left(2C_1(\alpha, \beta, p, q, T)\hat{C}_{(T)}\right) \vee |||\mathbf{X}|||_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}([0,T])} \vee \|f\|_{C_b^{2,1}([0,T])}$$

and we choose $T_1 \in (0, T]$ so that

$$T_1^{\alpha-\frac{3}{p}} \leq \frac{1}{1 + (F_q(M_0) \vee F_q(M_0)^2)}.$$

Hence, by rewriting estimate (39) for $(Y, Y') \in B_{\mathscr{Y}_{T_1}}((\bar{Y}, \bar{Y}'), M_0)$ and by recalling the nondecreasing dependence of $\hat{C}_{(T)}$ and C_1 on T, we obtain

$$\begin{aligned}
&\|f(Y) - \bar{Y}'\|_{L^{\Phi_\beta}([0,T_1])} + [f(Y) - \bar{Y}']^{q\wedge 1}_{B^{\alpha}_{\Phi_\beta,q}([0,T_1])} + \|R^{y+\int_0^{\cdot} f(Y)\,\mathrm{d}\mathbf{X}} - R^{\bar{Y}}\|^{\frac{q}{2}\wedge 1}_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}([0,T_1])} \\
&\quad \leq C_1(\alpha, \beta, p, q, T_1)\hat{C}_{(T_1)}\left(1 + T_1^{\alpha-\frac{3}{p}}\left(1 + \left(F_q(M_0) \vee F_q(M_0)^2\right)\right)\right) \\
&\quad \leq 2C_1(\alpha, \beta, p, q, T)\hat{C}_{(T)} \leq M_0,
\end{aligned}$$

which completes the proof of claim 5. Let us note that the claim also holds for any $\hat{T}_1 \in (0, T_1]$.

It remains to establish claim 5. We first observe that, for all $(Y, Y') \in B_{\mathscr{Y}_{T_1}}((\bar{Y}, \bar{Y}'), M_0)$, estimate (36) together with the choice of C_1 and $\hat{C}_{(T)}$ so that (40) holds as well, implies

$$\begin{aligned}
&|f(Y)_0|_{\mathbb{R}^{m\times n}} \vee \left|f(Y)'_0\right|_{\mathcal{L}(\mathbb{R}^n;\mathbb{R}^{m\times n})} \vee [(f(Y), f(Y)')]_{\mathscr{D}^{\alpha;X}_{\Phi_\beta,q}} \\
&\quad \lesssim^{T_1}_{\alpha,\beta,q} (F_q(M_0) \vee F_q(M_0)^2)(1 \vee F_q(M_0))\left(1 + \left(F_q(M_0) \vee F_q(M_0)^2\right)\right) \lesssim^{T_1,M_0}_{\alpha,\beta,p,q} 1.
\end{aligned}$$

Let us fix $(Y, Y'), (\tilde{Y}, \tilde{Y}') \in B_{\mathscr{Y}_{T_1}}((\bar{Y}, \bar{Y}'), M_0)$. With the bound above, Theorem 4.18 yields

$$\begin{aligned}
&d_{X,\mathscr{D}^{\alpha}_{\Phi_\beta,q}([0,T_1])}(\mathscr{Z}(Y, Y'), \mathscr{Z}(\tilde{Y}, \tilde{Y}')) \\
&\quad \lesssim^{T_1,M_0}_{\alpha,\beta,p,q} T_1^{(\alpha-\frac{4}{p})(\frac{q}{2}\wedge 1)}\left(\|R^{f(Y)} - R^{f(\tilde{Y})}\|^{\frac{q}{2}\wedge 1}_{B^{\gamma;2}_{\Phi_{\beta/2},q/2}([0,T_1])} + [f(Y)' - f(\tilde{Y})']^{\frac{q}{2}\wedge 1}_{B^{\alpha}_{\Phi_\beta,q}([0,T_1])}\right).
\end{aligned} \tag{41}$$

Hence, from Theorem 4.11, we obtain

$$\begin{aligned}
&d_{X,\mathscr{D}^{\alpha}_{\Phi_\beta,q}([0,T_1])}(\mathscr{Z}(Y, Y'), \mathscr{Z}(\tilde{Y}, \tilde{Y}')) \\
&\quad \leq C_2(\alpha, \beta, p, q, M_0, T_1)\bar{C}_{(T_1)}T_1^{(\alpha-\frac{4}{p})(\frac{q}{2}\wedge 1)} d_{X,\mathscr{D}^{\alpha}_{\Phi_\beta,q}([0,T_1])}((Y, Y'), (\tilde{Y}, \tilde{Y}')),
\end{aligned} \tag{42}$$

where $\bar{C}_{(T_1)} = C(\|f\|_{C_b^{2,1}([0,T_1])}, |||\mathbf{X}|||_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}([0,T_1])})$ denotes a constant of the same form as $\hat{C}_{(T)}$ and $C_2(\alpha, \beta, p, q, M_0, T_1)$ is a constant nondecreasing in the last argument. Let $T_0 \in (0, T_1]$ be such that

$$T_0^{(\alpha-\frac{4}{p})(\frac{q}{2}\wedge 1)} < \frac{1}{2C_2(\alpha, \beta, p, q, M_0, T_1)\bar{C}_{(T_1)}}.$$

We may now consider estimate (42) on $[0, T_0]$ instead of $[0, T_1]$. By the nondecreasing dependence of both C_2 and $\bar{C}_{(T_1)}$ on T_1, we deduce

$$\begin{aligned}
&d_{X,\mathscr{D}^{\alpha}_{\Phi_\beta,q}([0,T_0])}(\mathscr{Z}(Y,Y'),\mathscr{Z}(\tilde{Y},\tilde{Y}'))\\
&\quad\le C_2(\alpha,\beta,p,q,M_0,T_0)\bar{C}_{(T_0)}T_0^{(\alpha-\frac{4}{p})(\frac{q}{2}\wedge 1)}d_{X,\mathscr{D}^{\alpha}_{\Phi_\beta,q}([0,T_0])}((Y,Y'),(\tilde{Y},\tilde{Y}'))\\
&\quad\le C_2(\alpha,\beta,p,q,M_0,T_1)\bar{C}_{(T_1)}T_0^{(\alpha-\frac{4}{p})(\frac{q}{2}\wedge 1)}d_{X,\mathscr{D}^{\alpha}_{\Phi_\beta,q}([0,T_0])}((Y,Y'),(\tilde{Y},\tilde{Y}'))\\
&\quad< \frac{1}{2}d_{X,\mathscr{D}^{\alpha}_{\Phi_\beta,q}([0,T_0])}((Y,Y'),(\tilde{Y},\tilde{Y}')),
\end{aligned}$$

which concludes the proof of Theorem 5.2. □

We conclude this section by discussing the continuity of the solution map $(y_0, f, \mathbf{X}) \mapsto Y$ of the RDE (30).

Theorem 5.4 *Let $y_0, \tilde{y}_0 \in \mathbb{R}^m$, $f, \tilde{f} \in C_b^{2,1}(\mathbb{R}^m;\mathbb{R}^{m\times n})$, and $\mathbf{X}, \tilde{\mathbf{X}} \in \mathscr{B}^{\alpha}_{\Phi_\beta,q}([0,T];\mathbb{R}^n)$. Moreover, let $(Y, f(Y)) \in \mathscr{D}^{\alpha;X}_{\Phi_\beta,q}([0,T];\mathbb{R}^m)$ and $(\tilde{Y}, \tilde{f}(\tilde{Y})) \in \mathscr{D}^{\alpha;\tilde{X}}_{\Phi_\beta,q}([0,T];\mathbb{R}^m)$ be the solutions to the RDEs*

$$Y_t = y_0 + \int_0^t f(Y_s)\,\mathrm{d}\mathbf{X}_s \quad \text{and} \quad \tilde{Y}_t = \tilde{y}_0 + \int_0^t \tilde{f}(\tilde{Y}_s)\,\mathrm{d}\tilde{\mathbf{X}}_s,$$

respectively. Assuming that

$$|y_0|_{\mathbb{R}^m} \vee \|f\|_{C_b^{2,1}} \vee |||\mathbf{X}|||_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}} \vee |\tilde{y}_0|_{\mathbb{R}^m} \vee \|\tilde{f}\|_{C_b^{2,1}} \vee |||\tilde{\mathbf{X}}|||_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}} \le M \tag{43}$$

holds for some $M \in (0,\infty)$, we have the local Hölder (Lipschitz if $q \ge 2$) estimates

$$d_{X,\tilde{X},\mathscr{D}^{\alpha}_{\Phi_\beta,q}}((Y,f(Y)),(\tilde{Y},\tilde{f}(\tilde{Y}))) \lesssim^{M,T}_{\alpha,\beta,q} |y_0-\tilde{y}_0|_{\mathbb{R}^m}^{\frac{q}{2}\wedge 1} + \|f-\tilde{f}\|_{C_b^{2,1}}^{\frac{q}{2}\wedge 1} + \rho_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}}(\mathbf{X},\tilde{\mathbf{X}})^{\frac{q}{2}\wedge 1}$$

and

$$[Y-\tilde{Y}]_{B^{\alpha}_{\Phi_\beta,q}} \lesssim^{M,T}_{\alpha,\beta,q} |y_0-\tilde{y}_0|_{\mathbb{R}^m}^{\frac{q}{2}\wedge 1} + \|f-\tilde{f}\|_{C_b^{2,1}}^{\frac{q}{2}\wedge 1} + \rho_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}}(\mathbf{X},\tilde{\mathbf{X}})^{\frac{q}{2}\wedge 1}.$$

Proof By Theorem 5.2, there exist constants

$$M_0 = M_0(\alpha,\beta,q,T,|||\mathbf{X}|||_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}},\|f\|_{C_b^{2,1}}) \quad \text{and} \quad \tilde{M}_0 = \tilde{M}_0(\alpha,\beta,q,T,|||\tilde{\mathbf{X}}|||_{\mathscr{B}^{\alpha}_{\Phi_\beta,q}},\|\tilde{f}\|_{C_b^{2,1}})$$

such that

$$[(Y,f(Y))]_{\mathscr{D}^{\alpha;X}_{\Phi_\beta,q}} \le M_0 \quad \text{and} \quad [(\tilde{Y},\tilde{f}(\tilde{Y}))]_{\mathscr{D}^{\alpha;\tilde{X}}_{\Phi_\beta,q}} \le \tilde{M}_0.$$

Without loss of generality, we assume that $M \ge M_0 \vee \tilde{M}_0$, otherwise we consider $M' = M \vee M_0 \vee \tilde{M}_0$ instead of M. Since $R^{y_0+\int_0^\cdot f(Y)\,\mathrm{d}\mathbf{X}} = R^{\int_0^\cdot f(Y)\,\mathrm{d}\mathbf{X}}$, it holds that

$$d_{X,\tilde{X},\mathscr{D}^{\alpha}_{\Phi_\beta,q}}((Y,f(Y)),(\tilde{Y},\tilde{f}(\tilde{Y}))) = d_{X,\tilde{X},\mathscr{D}^{\alpha}_{\Phi_\beta,q}}\left(\left(\int_0^\cdot f(Y)\,\mathrm{d}\mathbf{X}, f(Y)\right),\left(\int_0^\cdot \tilde{f}(\tilde{Y})\,\mathrm{d}\tilde{\mathbf{X}}, \tilde{f}(\tilde{Y})\right)\right).$$

By the continuity of the rough integral from Theorem 4.18, we estimate

$$\begin{aligned}
&d_{X,\tilde{X},\mathscr{D}^{\alpha}_{\Phi_\beta,q}}\left(\left(\int_0^{\cdot} f(Y)\,\mathrm{d}\mathbf{X},\, f(Y)\right),\left(\int_0^{\cdot} \tilde{f}(\tilde{Y})\,\mathrm{d}\tilde{\mathbf{X}},\, \tilde{f}(\tilde{Y})\right)\right)\\
&\quad\lesssim^{T,M}_{\alpha,\beta,p,q} |f(Y_0)-\tilde{f}(\tilde{Y}_0)|^{\frac{q}{2}\wedge 1}_{\mathbb{R}^{m\times n}} + |f(Y_0)'-\tilde{f}(\tilde{Y}_0)'|^{\frac{q}{2}\wedge 1}_{\mathcal{L}(\mathbb{R}^n;\mathbb{R}^{m\times n})} + \rho^{\alpha}_{\mathscr{B}_{\Phi_\beta,q}}(\mathbf{X},\tilde{\mathbf{X}})^{\frac{q}{2}\wedge 1}\\
&\qquad + (T^{\alpha-\frac{4}{p}})^{\frac{q}{2}\wedge 1}\left(\|R^{f(Y)}-R^{\tilde{f}(\tilde{Y})}\|^{\frac{q}{2}\wedge 1}_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} + [f(Y)'-\tilde{f}(\tilde{Y})']^{\frac{q}{2}\wedge 1}_{B^{\alpha}_{\Phi_\beta,q}}\right)\\
&\quad=: I_1^{\frac{q}{2}\wedge 1} + I_2^{\frac{q}{2}\wedge 1} + \rho^{\alpha}_{\mathscr{B}_{\Phi_\beta,q}}(\mathbf{X},\tilde{\mathbf{X}})^{\frac{q}{2}\wedge 1} + (T^{\alpha-\frac{4}{p}})^{\frac{q}{2}\wedge 1}\left(I_3^{\frac{q}{2}\wedge 1} + I_4^{\frac{q}{2}\wedge 1}\right).
\end{aligned}$$

By the triangle inequality and assumption (43), it holds that

$$I_1 \le |f(y_0)-f(\tilde{y}_0)|_{\mathbb{R}^{m\times n}} + |f(\tilde{y}_0)-\tilde{f}(\tilde{y}_0)|_{\mathbb{R}^{m\times n}} \lesssim^{M} |y_0-\tilde{y}_0|_{\mathbb{R}^m} + \|f-\tilde{f}\|_{C_b^{2,1}}$$

and

$$\begin{aligned}
I_2 &\le |Df(y_0)(f(y_0)-\tilde{f}(\tilde{y}_0))|_{\mathcal{L}(\mathbb{R}^n;\mathbb{R}^{m\times n})} + |(Df(y_0)-Df(\tilde{y}_0))(\tilde{f}(\tilde{y}_0))|_{\mathcal{L}(\mathbb{R}^n;\mathbb{R}^{m\times n})}\\
&\quad + |(Df(\tilde{y}_0)-D\tilde{f}(\tilde{y}_0))(\tilde{f}(\tilde{y}_0))|_{\mathcal{L}(\mathbb{R}^n;\mathbb{R}^{m\times n})}\\
&\lesssim^{M} |y_0-\tilde{y}_0|_{\mathbb{R}^m} + \|f-\tilde{f}\|_{C_b^{2,1}}.
\end{aligned}$$

Let $f(\tilde{Y})'_{\cdot} = ((Df)(\tilde{Y}_{\cdot}))\tilde{Y}'_{\cdot}$. Theorem 4.11 implies that $(f(\tilde{Y}), f(\tilde{Y})') \in \mathscr{D}^{\alpha;\tilde{X}}_{\Phi_\beta,q}([0,T];\mathbb{R}^{m\times n})$. Hence, by the triangle inequality, we get

$$I_3^{\frac{q}{2}\wedge 1} \le \|R^{f(Y)}-R^{f(\tilde{Y})}\|^{\frac{q}{2}\wedge 1}_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} + \|R^{f(\tilde{Y})}-R^{\tilde{f}(\tilde{Y})}\|^{\frac{q}{2}\wedge 1}_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} =: I_{3,1}^{\frac{q}{2}\wedge 1} + I_{3,2}^{\frac{q}{2}\wedge 1}.$$

From Theorem 4.11 and (43), we obtain

$$\begin{aligned}
I_{3,1} &\lesssim^{T,M}_{\alpha,\beta,q} |y_0-\tilde{y}_0|_{\mathbb{R}^m} + |f(y_0)-f(\tilde{y}_0)|_{\mathbb{R}^{m\times n}} + [X-\tilde{X}]_{B^{\alpha}_{\Phi_\beta,q}} + d_{X,\tilde{X},\mathscr{D}^{\alpha}_{\Phi_\beta,q}}((Y,Y'),(\tilde{Y},\tilde{Y}'))\\
&\lesssim^{M} |y_0-\tilde{y}_0|_{\mathbb{R}^m} + \rho^{\alpha}_{\mathscr{B}_{\Phi_\beta,q}}(\mathbf{X},\tilde{\mathbf{X}}) + d_{X,\tilde{X},\mathscr{D}^{\alpha}_{\Phi_\beta,q}}((Y,Y'),(\tilde{Y},\tilde{Y}')).
\end{aligned}$$

By a direct calculation, one can check that

$$R^{f(\tilde{Y})}_{s,t} - R^{\tilde{f}(\tilde{Y})}_{s,t} = R^{(f-\tilde{f})(\tilde{Y})}_{s,t}.$$

Since $(f-\tilde{f})(\tilde{Y})$ is controlled by $\tilde{X}$, Theorem 4.11 and (43) yield

$$I_{3,2} \lesssim^{T,M}_{\alpha,\beta,q} \|f-\tilde{f}\|_{C_b^{2,1}}(1+[\tilde{X}]_{B^{\alpha}_{\Phi_\beta,q}}) \lesssim^{M} \|f-\tilde{f}\|_{C_b^{2,1}}.$$

Next, we estimate

$$I_4 \le [f(Y)'-f(\tilde{Y})']^{\frac{q}{2}\wedge 1}_{B^{\alpha}_{\Phi_\beta,q}} + [f(\tilde{Y})'-\tilde{f}(\tilde{Y})']^{\frac{q}{2}\wedge 1}_{B^{\alpha}_{\Phi_\beta,q}} =: I_{4,1}^{\frac{q}{2}\wedge 1} + I_{4,2}^{\frac{q}{2}\wedge 1}.$$

We deal with $I_{4,1}$ and $I_{4,2}$ by the means of Theorem 4.11 and (43) similarly as with $I_{3,1}$ and $I_{3,2}$, respectively. We obtain

$$I_{4,1} \lesssim^{T,M}_{\alpha,\beta,q} |y_0-\tilde{y}_0|_{\mathbb{R}^m} + |y_0-\tilde{y}_0|_{\mathbb{R}^m} + \rho^{\alpha}_{\mathscr{B}_{\Phi_\beta,q}}(\mathbf{X},\tilde{\mathbf{X}}) + d_{X,\tilde{X},\mathscr{D}^{\alpha}_{\Phi_\beta,q}}((Y,Y'),(\tilde{Y},\tilde{Y}'))$$

and

$$I_{4,2} \le \|D(f-\tilde{f})\|_{L^\infty}[\tilde{Y}]_{B^{\alpha}_{\Phi_\beta,q}} \lesssim^{T,M}_{\alpha,\beta,q} \|f-\tilde{f}\|_{C_b^{2,1}},$$

where we have used $[\tilde{Y}]_{B^\alpha_{\Phi_{\beta,q}}} \lesssim^{T,M}_{\alpha,\beta,q} 1$ which follows from Lemma 4.10 and assumption (43). Combining the estimates above, we deduce

$$d_{X,\tilde{X},\mathscr{D}^\alpha_{\Phi_{\beta,q}}}((Y, f(Y)), (\tilde{Y}, \tilde{f}(\tilde{Y}))) \lesssim^{T,M}_{\alpha,\beta,p,q} |y_0 - \tilde{y}_0|^{\frac{q}{2}\wedge 1}_{\mathbb{R}^m} + \|f - \tilde{f}\|^{\frac{q}{2}\wedge 1}_{C^{2,1}_b} + \rho_{\mathscr{B}^\alpha_{\Phi_{\beta,q}}}(\mathbf{X}, \tilde{\mathbf{X}})^{\frac{q}{2}\wedge 1} + (T^{\alpha-\frac{4}{p}})^{\frac{q}{2}\wedge 1} d_{X,\tilde{X},\mathscr{D}^\alpha_{\Phi_{\beta,q}}}((Y, Y'), (\tilde{Y}, \tilde{Y}'))^{\frac{q}{2}\wedge 1}. \tag{44}$$

Recalling the convention from Section 2.1, let $C(T, M, \alpha, \beta, p, q)$ be the constant non-decreasing in T and M verifying (44). Let $T_0 \in (0, T]$ be such that

$$T_0 < \frac{1}{2} \wedge \left(\frac{1}{2C(T, M, \alpha, \beta, p, q)}\right)^{\frac{1}{(\alpha-\frac{4}{p})(\frac{q}{2}\wedge 1)}}$$

Restricting ourselves to $[0, T_0]$, (44) and the particular choice of T_0 implies

$$d_{X,\tilde{X},\mathscr{D}^\alpha_{\Phi_{\beta,q}}}((Y, f(Y)), (\tilde{Y}, \tilde{f}(\tilde{Y}))) \leq 2C(T, M, \alpha, \beta, p, q)\left(|y_0 - \tilde{y}_0|^{\frac{q}{2}\wedge 1}_{\mathbb{R}^m} + \|f - \tilde{f}\|^{\frac{q}{2}\wedge 1}_{C^{2,1}_b} + \rho_{\mathscr{B}^\alpha_{\Phi_{\beta,q}}}(\mathbf{X}, \tilde{\mathbf{X}})^{\frac{q}{2}\wedge 1}\right).$$

Lemma 4.10 then yields

$$\begin{aligned}
[Y - \tilde{Y}]_{B^\alpha_{\Phi_{\beta,q}}} &\lesssim^{T,M}_{\alpha,\beta,q} |y_0 - \tilde{y}_0|_{\mathbb{R}^m} + \rho_{\mathscr{B}^\alpha_{\Phi_{\beta,q}}}(\mathbf{X}, \tilde{\mathbf{X}}) + [f(Y) - \tilde{f}(\tilde{Y})]_{B^\alpha_{\Phi_{\beta,q}}} \\
&\quad + \rho_{\mathscr{B}^\alpha_{\Phi_{\beta,q}}}(\mathbf{X}, \tilde{\mathbf{X}}) + \|R^Y - R^{\tilde{Y}}\|_{B^{2\alpha;2}_{\Phi_{\beta/2},q/2}} \\
&\lesssim^{T,M}_{\alpha,\beta,q} |y_0 - \tilde{y}_0|_{\mathbb{R}^m} + \rho_{\mathscr{B}^\alpha_{\Phi_{\beta,q}}}(\mathbf{X}, \tilde{\mathbf{X}}) + d_{X,\tilde{X},\mathscr{D}^\alpha_{\Phi_{\beta,q}}}((Y, Y'), (\tilde{Y}, \tilde{Y}')) \\
&\lesssim^{T,M}_{\alpha,\beta,q} |y_0 - \tilde{y}_0|^{\frac{q}{2}\wedge 1}_{\mathbb{R}^m} + \|f - \tilde{f}\|^{\frac{q}{2}\wedge 1}_{C^{2,1}_b} + \rho_{\mathscr{B}^\alpha_{\Phi_{\beta,q}}}(\mathbf{X}, \tilde{\mathbf{X}})^{\frac{q}{2}\wedge 1}.
\end{aligned}$$

The bound can be extended to the whole interval $[0, T]$ in a finite number of steps similarly as in Theorem 5.2. □

6 Concluding Remark

The class of (exponential) Besov–Orlicz spaces forms a natural scale of function spaces in which various stochastic processes can be analyzed [4, 7, 23, 34, 35]. As these processes are commonly used to model random fluctuations influencing the dynamics of a physical system, it is natural to analyze differential equations of the form

$$Y_t = y_0 + \int_0^t f(Y_s)\mathrm{d}X_s, \quad t \in [0, T], \tag{45}$$

for $T \in (0, \infty)$, $y \in \mathbb{R}^m$, $f : \mathbb{R}^m \to \mathbb{R}^{m\times n}$ sufficiently smooth, and a function $X \in B^\alpha_{\Phi_{\beta,q}}([0, T]; \mathbb{R}^n)$ for $\alpha \in (0, 1)$, $\beta \in (0, \infty)$, $q \in (0, \infty]$, and $m, n \in \mathbb{N}$.

We have recently shown in [6] that if $\alpha \in (1/2, 1)$ and if $f \in C^{1,1}_b(\mathbb{R}^m; \mathbb{R}^{m\times n})$, the integral in equation (45) can be given meaning as a Young-type integral and the equation admits a unique solution of Besov–Orlicz regularity. The result, for example, applies to SDEs

driven by scalar fractional Brownian motions with Hurst parameter $H \in (1/2, 1)$ or, more generally, to Hermite processes [32, 33].

In this article, we have treated the rough case of $\alpha \in (1/3, 1/2]$. In this case, path X has to be endowed with a two-parametric map $\mathbb{X}$, that plays the role of the iterated integral. We have defined the space of Besov–Orlicz rough paths, the notion of a Besov–Orlicz rough integral, the space of controlled Besov–Orlicz rough paths, shown that the rough integral is a controlled Besov–Orlicz rough path, established the stability of controlled Besov–Orlicz rough paths under composition with regular maps, and used these results to show that if $f \in C_b^{2,1}(\mathbb{R}^m; \mathbb{R}^{m\times n})$, equation (45) in which the integral is understood as the Besov–Orlicz rough integral admits a unique solution that is again a controlled Besov–Orlicz rough path. This result, for example, applies to SDEs driven by scalar continuous local martingales with Lipschitz continuous quadratic variation, fractional Brownian motions with Hurst parameter $H \in (1/3, 1/2]$ or, more generally, fractionally filtered Hermite processes with self-similarity parameter $H \in (1/3, 1/2]$ [1].

In principle, the case $\alpha \leq 1/3$ can also be treated and indeed, many of the steps described above extend to this case in a routine manner. There is however one notable difference - the branched iterated integrals appear. For a formal illustration, we follow [17] and assume that $\alpha \in (1/4, 1/3]$. Consider the equation

$$\delta Y_{s,t} = \sum_{i=1}^{n} \int_s^t f_i(Y_u) \mathrm{d}X_u^i, \quad s, t \in [0, T],$$

where $f_i = (f_i^1, f_i^2, \ldots, f_i^m) : \mathbb{R}^m \to \mathbb{R}^{m\times n}$, $i = 1, 2, \ldots, n$, are smooth vector fields and $X = (X^1, X^2, \ldots X^n) \in C^\alpha([0, T]; \mathbb{R}^n)$ for simplicity. By using Taylor's expansion and iterating the above equation in itself, we obtain the approximation

$$\begin{aligned}
\delta Y_{s,t} \approx & \sum_{i_1=1}^{n} f_{i_1}(Y_s) \int_s^t \mathrm{d}X_{v_1}^{i_1} \\
& + \sum_{j_1=1}^{m} \sum_{i_1,i_2=1}^{n} f_{i_2}^{j_1}(Y_s)(\partial_{j_1} f_{i_1})(Y_s) \int_s^t \int_s^{v_1} \mathrm{d}X_{v_2}^{i_2} \mathrm{d}X_{v_1}^{i_1} \\
& + \sum_{j_1,j_2=1}^{m} \sum_{i_1,i_2,i_3=1}^{n} f_{i_3}^{j_1}(Y_s)(\partial_{j_1} f_{i_2}^{j_2})(Y_s)(\partial_{j_2} f_{i_1})(Y_s) \int_s^t \int_s^{v_2} \int_s^{v_1} \mathrm{d}X_{v_3}^{i_3} \mathrm{d}X_{v_2}^{i_2} \mathrm{d}X_{v_1}^{i_1} \\
& + \frac{1}{2} \sum_{j_1,j_2=1}^{m} \sum_{i_1,i_2,i_3=1}^{n} f_{i_3}^{j_1}(Y_s) f_{i_2}^{j_2}(Y_s)(\partial_{j_1,j_2}^2 f_{i_1})(Y_s) \int_s^t \left(\int_s^{v_3} \mathrm{d}X_{v_1}^{i_3} \right) \left(\int_s^{v_3} \mathrm{d}X_{v_2}^{i_2} \right) \mathrm{d}X_{v_3}^{i_1}
\end{aligned}$$

with the remainder being of order $|t - s|^{4\alpha}$. If we understand a rough path as a function with values in the (level 3) truncated tensor algebra $T_1^{(3)}(\mathbb{R}^n)$ [13, Chapter 7], then we would be enhancing the original path X by the first three iterated integrals but it would not be possible to encode the branched integral

$$\int_s^t \left(\int_s^{v_3} \mathrm{d}X_{v_1}^{i_3} \right) \left(\int_s^{v_3} \mathrm{d}X_{v_2}^{i_2} \right) \mathrm{d}X_{v_3}^{i_1}.$$

The situation can be resolved in two ways: Either we impose a rule that allows to evaluate the product $\delta X^{i_3} \delta X^{i_2}$ in terms of the already given iterated integrals (such rule is usually called the *shuffle product formula*) by requiring that the state space for the considered rough path is the (level 3) free nilpotent group $G^{(3)}(\mathbb{R}^n) \subseteq T_1^{(3)}(\mathbb{R}^n)$; cf., e.g., [13, Chapter 7]. This approach leads to the notion of *(weak) geometric* rough paths [13, Chapter 9]. Alternatively,

we can replace the state space $T_1^{(3)}(\mathbb{R}^n)$ for the considered rough path by the (truncated) Hopf algebra generated by the set of labeled rooted trees (the so-called *Connes–Kreimer* Hopf algebra) [16, 17]. This allows to postulate the behavior of the branched integral as well as the iterated integrals. The two approaches are intimately connected [2, 17] but while the former approach is more classical (see, e.g., [22]), the latter allows for some simplifications, for example, in establishing the stability of controlled paths under composition with regular functions. We refer to, e.g., [8, 28] for some further developments of branched rough paths and to, e.g., [14] for further example of their use.

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Declarations

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