# Existence and Stability of Solutions of Quasilinear Parabolic Equations with Delays

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## 1. EXISTENCE OF SOLUTIONS OF QUASILINEAR DELAYED PARABOLIC EQUATIONS

Partial differential equations with delays are a topic that has not been so far sufficiently explored, see, e.g., Aibinu et al. (2021); Kryspin and Mierczyński (2024); Lv et al. (2019).

This contribution is focused at investigation of the existence of solutions and their stability for quasilinear parabolic partial differential equations, i.e., for equations of type

$$\frac{\partial}{\partial t}u - \text{div }a(x,u,\nabla u,u_\tau) + c(x,u,u_\tau) = f(t,x) \qquad (1)$$

for $t \in [0,\infty)$, $x \in \Omega$ where $N = 1,2,3$, $\Omega \subset \mathbb{R}^N$ is a (fixed) bounded domain with Lipschitz booundary and $\tau > 0$ is a fixed time delay and, finally, $u_\tau(t,x) = u(t-\tau,x)$. The initial conditions are

$$u(t) = u_0(t) \in L^2(\Omega), t \in [-\tau, 0]$$

The main tools to prove existence of the solutions are the weak formulation in suitable Sobolev spaces. For this reason the following weak formulation is of importance:

$$\langle \frac{\partial}{\partial t}u, v\rangle - \langle a(x,u,\nabla u,u_\tau),\nabla v\rangle + \langle c(x,u,u_\tau),v\rangle = \langle f,v\rangle \qquad (2)$$

with $v$ being a function from a suitable space of functions defined on $\Omega$ (to be precised in the paper) and the $\langle .,. \rangle$ is the generic symbol that denotes the duality pairing between a space and its dual (which may be different in each term).

Moreover, another tool to the existence of the solutions is the Galerkin method, modified from the theory of nonlinear parabolic equations without delay, see, e.g., Roubicek (2013). This method, based on the weak formulation, allows us to approximate the solution by the sequence of ordinary differential equations with delay. First, let there exist a sequence of $k$-dimensional Banach spaces $H_k \subset H_0^1(\Omega)$, $k \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ holds $H_k \subset H_{k+1}$ and $\cup_{k=1}^{\infty} H_k$ is dense in $H_0^1(\Omega)$. The finite-dimensional approximations of the function $u$, denoted by $u_k$, satisfying $u_k \in H_k$, obey the equation

$$\dot{u}_k = A_k(u_k, u_{k,\tau}) + f_k \qquad (3)$$

where $u_k$ is the $k$-dimensional finite-dimensional (in the space domain) approximation of function $u$, $A_k$ is the approximation of the differential operator and $f_k$ is the discretized right-hand side (constructed as a projection of function $f$ on the finite-dimensional space $H_k$).

Assume the functional $A_k$ is Lipschitz continuous in the first variable. Then, one can see that the solution of (3) exists on the interval $[0,\tau]$. Continuing this procedure subsequently on intervals $[\tau, 2\tau]$, $[2\tau, 3\tau]$,... yields conditions guaranteeing existence of solutions on $[0,\infty)$.

Let $I = [k\tau, (k+1)\tau]$ for some $k \in \mathbb{N}$. The Galerkin method thus guarantees that

$$u_k \rightharpoonup u$$

in $H_0^1(I,\Omega)$ $u$ is the solution of the original equation.

Hence, existence of a solution of the original quasilinear parabolic equation with delays can be inferred from existence of the finite-dimensional solutions in $H_k$ by a limit $k \to \infty$. Conditions guaranteeing correctness of this limit passage will also be presented.

## 2. RAZUMIKHIN FUNCTIONAL

The second part of the contribution is devoted to investigation of stability of the quasilinear parabolic equations.

To prove stability, a Razumikhin functional $V : H_0^1 \to [0,\infty)$ for such equations is presented. This functional is supposed to be everywhere Gateaux-differentiable, moreover, V(0)=0.

Note that weak convergence of approximations $u_k(t)$ in $H_0^1$ implies a strong convergence in $L^2(\Omega)$, hence, one can also apply these approximations to the investigation of stability of the function $u$. Namely, we can substitute the discrete solutions $u_k$ into the functional $V$.

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To facilitate the considerations, namely to apply the well-known theory of Razumikhin functionals for ordinary differential equations, one considers a discretized (in the space domain) functional $V_k$ defined on the $k$-dimensional space in which the functions $u_k$ live. To be specific, the functional $V_k$ satisfies

$$V_k(v) = V(v) \text{ for all } v \in H_k.$$

Let $\Pi : H_O^1 \to H_k$ be the projection from $H_0^1$ on $H_k$. Then we define $V_k(v) = V_k(\Pi(v))$.

To prove stability of the solution of the original equation, one needs a sort of uniqueness with respect to $k$. Under this assumption and the fact that the discretized (in space) solutions converge to the solution of the original equation strongly in $L^2(\Omega)$ yields stability of the functions $u$.

Now, using the well-known Razumikhin theorem (Hale and Verdyun-Lunel (1993); Fridman (2015) or others), one can infer asymptotic stability of the discrete systems.

To be specific, we prove:

**Theorem** let $\psi_1, \psi_2 : [0, \infty) \to [0, \infty)$ be strictly increasing continuous functions such that $\psi_i(0) = 0$, let $w : [0, \infty) \to [0, \infty)$ be a nondecreasing function.

Assume $\omega > 1$ and let the functionals $V_k$ satisfy

$$\begin{aligned} &\psi_1(\|\eta\|) \leq V_k(\eta) \leq \psi_2(\|\eta\|), \\ &V_k(u_k(t+h)) \leq \omega V_k(u_k(t)) \forall h \in [-\tau, 0] \\ &\Rightarrow \dot{V}_k(u_k) < w(\|u_k(t)\|) \end{aligned}$$

then the solution $u = 0$ of the original equation with $f = 0$ is asymptotically stable.

## 3. INPUT-TO-STATE STABILITY

Finally, robustness issues will be tackled. Using standard arguments from the (finite-dimensional) control theory, we obtain input-to state stability of the approximations according to the following formula:

$$\|u_k\|^2 \leq \beta_k(\|u_k(0)\|, t) + \gamma_k(\|f_k\|)$$

for some $\beta_k \in \mathcal{KL}$, $\gamma_k \in \mathcal{K}$.

Thanks to uniformness of the estimate above, one can see that there exist one par of functions $\beta, \gamma : L^2(\Omega) \to [0, \infty)$ so that

$$\|u_k\|^2 \leq \beta(\|u_k(0)\|, t) + \gamma(\|f_k\|)$$

Under these conditions, one has

$$\begin{aligned} \psi_1(\|\eta\|) \leq V(\eta) \leq &\psi_2(\eta), \\ \dot{V}(u(t)) < &-\chi(\|\phi(t)\|) \end{aligned}$$

for some class-$\mathcal{K}$ functions $\psi_i$ and $\chi$. where the symbol $\phi$ denotes the flow of the solution of the original partial differential equation.

Thus, the quasilinear parabolic equation is Input-to-state stable.

As already known, the interconnection of two ISS systems is also ISS provided $\gamma_1(\gamma_2(s)) < s$ for all $s > 0$ (equivalently, $\gamma_2(\gamma_1(s)) < s$. This statement gives a condition for the stability of the interconnection of two systems described by quasilinear parabolic equations.

The contribution is augmented by practical numerically solved examples demonstrating how the theoretical results can be applied in practice.

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