Invariants of Color Images to N-Fold Symmetric Out-of-Focus Blur

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***Abstract*—The paper deals with the recognition of color images, blurred by N-fold symmetric out-of-focus blur, directly without any deblurring. We adopt the general theory of blur invariants of color images and modify it to this specific kind of blur. The main original contribution of the paper lies in introducing cross-channel blur invariants for N-fold symmetric kernels, which substantially improve the recognition power. The performance of the new invariants is illustrated by an experiment on blurred template matching.**

***Index Terms*—Color image, out-of-focus blur, blur invariants, cross-channel invariants, complex moments, template matching.**

I. Introduction

Recognition of blurred images is a serious challenge in many image analysis applications since blurred images suffer from smoothing and loss of high-frequency details. Blur may arise due to camera shake, defocus, lens aberration, object motion, media turbulence, and other factors. Recognition algorithms either must be able to suppress the blur or they should be sufficiently robust with respect to it. The blurring process can be approximately modeled as a convolution of the original scene f with a point-spread function (PSF) h, which leads to a capturing of image g:

$$g(\mathbf{x}) = (f * h)(\mathbf{x})\,. \tag{1}$$

Blur removal requires blind deconvolution techniques, that are generally ill-posed, unstable and time consuming [1], [2]. In image recognition, robustness to blur without restoration of the query image can be achieved by a brute-force approach, which means that the training set is massively augmented by many blurred copies of each training sample. This strategy was adopted in deep learning because the ability of a convolutional neural network (CNN) or a transformer to recognize blurred images is very low [3]–[5]. However, regardless of the classification technique, data augmentation is time-consuming and introduces redundancy, which increases the complexity of deep learning models. Realizing this, many researchers have tried to overcome the deconvolution and augmentation by using so-called *blur invariants*. Blur invariants are hand-crafted features that can be directly used as an input to any standard image classifier. Blur invariant I is an image descriptor fulfilling the constraint $I(f) = I(f * h)$ for any h from a certain set of admissible PSFs.

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Blur invariants of graylevel images are well-established descriptors. The basic concept appeared 30 years ago in the series of papers by Flusser et al. [6]–[8] followed by more than 100 theoretical as well as application papers, for instance [9]–[18] (see [19], Chapter 6 for a survey and further references). The latest development in this area was presented in [20] and [21], where a general construction of blur invariants independent of the particular blur type was proposed. However, all these papers have been dealing solely with blur invariants of graylevel (i.e. single-channel) images only.

The first attempt to design blur invariants of color images has been published recently in [22]. The authors have proposed a general theory of blur invariants of color images but have explicitly constructed the invariants only for the cases of *centrosymmetric* and *unconstrained* PSFs. The mathematical tools used in [22] have only limited possibilities, which do not allow to design blur invariants to other types of blurs.

In this paper, we propose for the first time invariants of color images with respect to real out-of-focus blur. In this case, the blurring PSF is defined by a shape of the aperture of the optical system. In common cameras, the aperture is formed by an adjustable diaphragm. Its size and shape are controlled by diaphragm blades, which usually leads to a polygonal or close-to-polygonal shape. If the diaphragm is fully open, the aperture is circular. Mathematically, the blurring PSF is an N-fold symmetric function, where N is the number of the diaphragm blades (see Fig. 1).

The key assumption, which links the individual channels together, is that the blur acts channel-wise (i.e. there is no channel mixing) and it is *the same* in all color channels. This is well in accordance with our experimental measurements described in [22] and with recent observations made by other authors [23], [24]. This assumption is justified whenever we consider a common camera with a single objective and a sensor shared by all spectral bands.

The newly designed invariants proposed in this paper are based on two core ideas. We build on the out-of-focus blur invariants of graylevel images introduced in [20]. To deal with color images, we adopt the approach originally proposed in [22] for centrosymmetric blurs and we design *cross-channel* invariants. As we demonstrate experimentally, the new invariants improve the recognition power of the system. This is the main novelty of the paper with a noticeable practical impact.

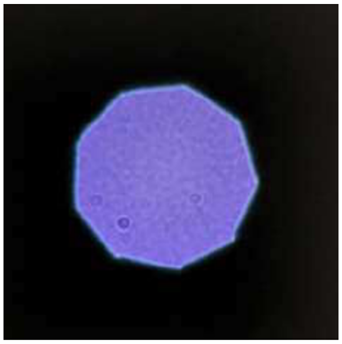

Fig. 1. The ground-truth PSF of out-of-focus blur obtained as a photo of a single bright point observed through a partially open aperture. 9-fold symmetric shape is determined by diaphragm blades. This PSF is approximately the same in all three color channels.

II. Blur invariants in Fourier domain

In this Section, we show how blur invariants of color images can be constructed in the Fourier domain. This general construction is independent of a particular blur type. We use similar mathematical tools based on projection operators that were employed in [21] for the design of graylevel invariants.

By a *color image* we understand a triplet $\mathbf{f}(\mathbf{x}) = (f_R(\mathbf{x}), f_G(\mathbf{x}), f_B(\mathbf{x}))$, where any function f_j has a compact support and $f_j \in L_1(\mathbb{R}^2) \cap L_2(\mathbb{R}^2)$.

The set of all color images is denoted as $\mathcal{I}$. Fourier transform and convolution on $\mathcal{I}$ are defined channel-wise, which means $\mathcal{F}(\mathbf{f}) = (\mathcal{F}(f_R), \mathcal{F}(f_G), \mathcal{F}(f_B))$ and $\mathbf{f} * \mathbf{h} = (f_R * h_R, f_G * h_G, f_B * h_B)$.

Consider a projection operator[1] P that acts on $\mathcal{I}$ channel-wise, i.e. $P\mathbf{f} = (Pf_R, Pf_G, Pf_B)$. Let us denote $\mathbf{S} = P(\mathcal{I})$. Then, $\mathcal{I}$ can be expressed as a direct sum $\mathcal{I} = \mathbf{S} \oplus \mathbf{A}$, where $\mathbf{A}$ is called the complement of $\mathbf{S}$. Hence, any $\mathbf{f}$ can be unambiguously written as $\mathbf{f} = P\mathbf{f} + \mathbf{f}_A$.

Since P is the same in all channels, $\mathbf{S}$ and $\mathbf{A}$ are "stacks" of three-times repeated scalar sets, $\mathbf{S} = (S, S, S)$, $\mathbf{A} = (A, A, A)$. Let $\mathbf{H} \subset \mathbf{S}$ be a set of functions that are the same in all channels, so any $\mathbf{h} \in \mathbf{H}$ is in fact a repeatedly applied scalar function $\mathbf{h}(\mathbf{x}) = (h(\mathbf{x}), h(\mathbf{x}), h(\mathbf{x}))$. Let $\mathbf{H}$ be, from now on, the set of blurring functions (PSFs) with respect to which we want to design the invariants. Our blurring model obtains the form

$$\mathbf{g}(\mathbf{x}) = (\mathbf{f} * \mathbf{h})(\mathbf{x}) = (\mathbf{f} * h)(\mathbf{x}). \tag{2}$$

Any meaningful $\mathbf{H}$ must contain at least one non-zero function and must be closed under convolution. In other words, for any $\mathbf{h}_1, \mathbf{h}_2 \in \mathbf{H}$ it must be $\mathbf{h}_1 * \mathbf{h}_2 \in \mathbf{H}$. This is the basic assumption without which the question of invariance does not make sense.

Now we are ready to formulate the following *General theorem of color blur invariants* (GTCBI).

Theorem 1 (GTCBI): Let $\mathcal{I}$, P, $\mathbf{S}$, and $\mathbf{H}$ be the symbols defined in the above text and let $\mathcal{F}$ denote the operator of the Fourier transform. Let P be "distributive" over a convolution with functions from $\mathbf{H}$, which means $P(\mathbf{f} * \mathbf{h}) = P\mathbf{f} * P\mathbf{h} = P\mathbf{f} * \mathbf{h}$ for any $\mathbf{f} \in \mathcal{I}$ and $\mathbf{h} \in \mathbf{H}$. Then for any $j, k = R, G, B$

$$I_{jk}(\mathbf{f})(\mathbf{u}) \equiv \frac{\mathcal{F}(f_j)(\mathbf{u})}{\mathcal{F}(Pf_k)(\mathbf{u})} \tag{3}$$

is an invariant w.r.t. a convolution with arbitrary $\mathbf{h} \in \mathbf{H}$ at all frequencies $\mathbf{u}$ where $I_{jk}(\mathbf{f})$ is well defined, i.e. $I_{jk}(\mathbf{f})(\mathbf{u}) = I_{jk}(\mathbf{f} * \mathbf{h})(\mathbf{u})$.

To prove this Theorem, we just use the basic properties of the Fourier transform:

$$I_{jk}(\mathbf{f} * \mathbf{h}) \equiv \frac{\mathcal{F}(f_j * h)}{\mathcal{F}(P(f_k * h))} = \tag{4}$$

$$= \frac{\mathcal{F}(f_j) \cdot \mathcal{F}(h)}{\mathcal{F}(Pf_k * h)} = \frac{\mathcal{F}(f_j) \cdot \mathcal{F}(h)}{\mathcal{F}(Pf_k) \cdot \mathcal{F}(h)} = I_{jk}(\mathbf{f})\,. \tag{5}$$

The "distributive property" of P in the assumption is not much restrictive and is naturally fulfilled in most practically important cases.[2]

The invariants $I_{jj}(\mathbf{f})$ are equivalent to graylevel invariants known from [21] applied channel-wise. The invariants $I_{jk}(\mathbf{f}), j \neq k$, are called *cross-channel* invariants and do not have a counterpart in the single-channel theory. However, only two cross-channel invariants are independent together with the single channels ones, the other cross-channel invariants are their functions and do not carry any new information about the image. For recognition purposes, we take the following *maximum independent set* of the invariants $\mathcal{B} = \{I_R, I_G, I_B, I_{RG}, I_{GB}\}$ (we simply write I_R instead of I_{RR}, etc.). Fig. 2 shows the outline of the method.

III. Invariants in Moment Domain

The blur invariants defined in the frequency domain by GTCBI are not suitable for a direct usage in practical object recognition tasks. They are numerically unstable, especially if the input image is noisy. Then the high-frequency components of $I_{jk}(\mathbf{f})$ may be significantly corrupted. When computing the invariants, there is a possibility of dividing by very small numbers, which again requires careful numerical treatment. Last but not least, using Eq. (3) needs to explicitly construct the projections Pf_k, which brings another computational burden. For all the above reasons, it is more efficient to work directly in the image domain, particularly with moment-based representation of images. The idea of moving from the frequency domain to the moment domain was already proposed for graylevel images in [20]. It avoids the unstable numerics and also the calculation of the projections. In this Section, we basically adapt the theory from [20] to color images.

Blur invariants in the moment domain can be constructed for an arbitrary P and $\mathbf{S}$ that fulfill the assumptions of

[1] Projection operator is linear and idempotent, i.e. $P^2 = P$.

[2] It is, however, not fulfilled automatically and should be tested for every particular P. As was proved in [21], a sufficient condition is that P is orthogonal and S is closed not only to convolution but also to cross-correlation.

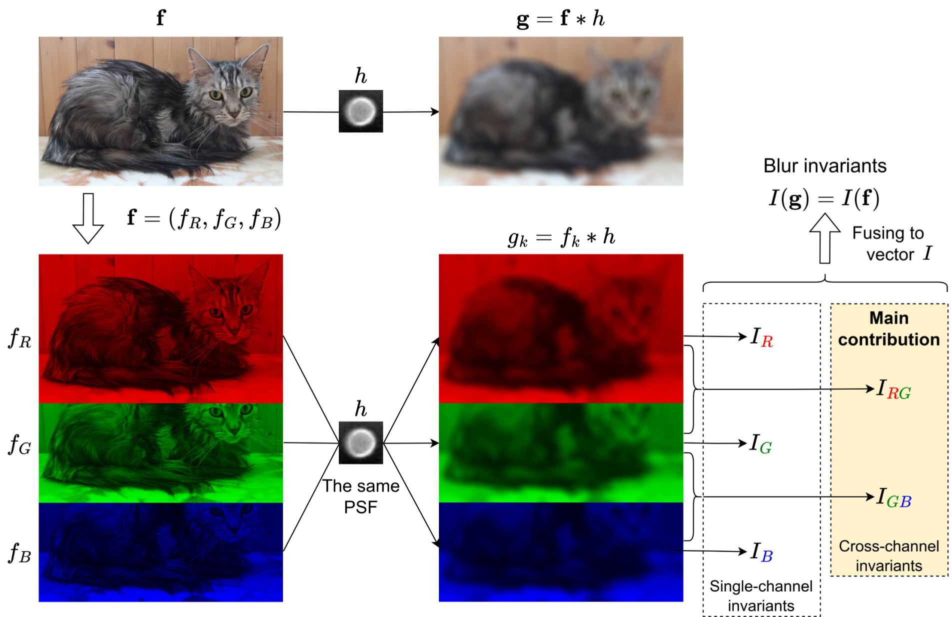

Fig. 2. The key idea of the blur invariants of color images. Single-channel invariants I_R, I_G, I_B are augmented by cross-channel invariants I_{RG} and I_{GB} that carry complementary information.

GTCBI. In this paper, we focus on blur kernels with N-fold rotation symmetry, i.e. kernels which are equal to their copy rotated around the origin by $2\pi/N$. As explained above, N-fold symmetric PSFs model the out-of-focus blur on partially open aperture. The theory presented below comprises also $N = 1$, which is unconstrained blur, and $N = 2$, which is centrosymmetric blur. Although these two cases are particular instances of N-fold symmetric blurs, they are not directly relevant to out-of-focus blur and the invariants w.r.t. them can be designed in an easier way that employs their specific properties [22].

Let S_N be a set of all N-fold symmetric functions and let P_N be a projector, defined in polar coordinates as

$$(P_N f)(r, \theta) = \frac{1}{N} \sum_{j=0}^{N-1} f(r, \theta + 2\pi j/N)$$

for finite N and

$$(P_\infty f)(r) = \frac{1}{2\pi} \int_0^{2\pi} f(r, \theta)\, \mathrm{d}\theta.$$

For arbitrary N, the set S_N is closed under convolution and P_N fulfills the distributivity assumption of GTCBI (the latter statement follows from the orthogonality of P_N, but can also be easily verified directly).

Now we take advantage of the fact that any function from $\mathcal{I}$ can be unambiguously expressed by its moments. Image moments can be understood as the "coordinates" of the image in a polynomial basis. The choice of the polynomials is conditioned by the purpose for which we want to use the moments. Here we employ the basis of complex monomials, that exhibit several nice properties when applied to N-fold symmetric functions. The corresponding moments are called *complex moments*

$$c_{pq}^{(f)} = \iint (x + iy)^p (x - iy)^q f(x, y)\, \mathrm{d}x\, \mathrm{d}y \tag{6}$$

(see [19] for details on complex moments). The moment of a color image $\mathbf{f}$ is just a vector $c_{pq}^{(\mathbf{f})} = (c_{pq}^{(R)}, c_{pq}^{(G)}, c_{pq}^{(B)})$.

The reason we employ complex moments is that they allow us to compute moments of $P_N\mathbf{f}$ directly from $\mathbf{f}$, even without constructing $P_N\mathbf{f}$ itself. This is because they exhibit the following *separation property*. Let D be a set of indices (p, q) such that $(p - q) \mod N = 0$. Then $c_{pq}^{(P\mathbf{f})} = c_{pq}^{(\mathbf{f})}$ if $(p, q) \in D$ and $c_{pq}^{(P\mathbf{f})} = 0$ otherwise. We say that D *separates* the moments. This is implied by the fact that complex moments always vanish on N-fold symmetric functions if $(p - q)$ is not divisible by N (see [19] for the proof). Geometric moments, which are simpler and more common in image processing and which were used in [22] for centrosymmetric kernels ($N = 2$), do not have this property for $N > 2$ and are thus not suitable here.

To get the link between $I_{jk}(\mathbf{f})$ and the moments $c_{pq}^{(\mathbf{f})}$, we recall that the Taylor expansion of the Fourier transform of a scalar function f is

$$\mathcal{F}(f)(u + v, i(u - v)) = \sum_{p,q=0}^{\infty} \frac{(-2\pi i)^{p+q}}{p!q!} c_{pq}^{(f)} u^p v^q. \tag{7}$$

Let us substitute $U = u + v, V = i(u - v)$. The GTCBI can be rewritten as

$$\mathcal{F}(Pf_k)(U, V) \cdot I_{jk}(\mathbf{f})(U, V) = \mathcal{F}(f_j)(U, V).$$

All these three functions can be expanded according to (7) into absolutely convergent Taylor series. Due to the moment separability, we write, for any $(p,q) \in D$, $c_{pq}^{(P\mathbf{f})} = c_{pq}^{(\mathbf{f})}$. Therefore, we have

$$\sum_{(p,q)\in D} \frac{(-2\pi i)^{p+q}}{p!q!} c_{pq}^{(f_k)} u^p v^q \cdot \sum_{p,q=0}^{\infty} \frac{(-2\pi i)^{p+q}}{p!q!} C_{pq}^{(jk)} u^p v^q = \tag{8}$$

$$= \sum_{p,q=0}^{\infty} \frac{(-2\pi i)^{p+q}}{p!q!} c_{pq}^{(f_j)} u^p v^q \,, \tag{9}$$

where $C_{pq}^{(jk)}$ are Taylor coefficients of $I_{jk}(\mathbf{f})$. Comparing the coefficients of the same powers of u and v we obtain, for any p, q,

$$\sum_{\substack{m=0\\(m,n)\in D}}^{p} \sum_{n=0}^{q} \frac{(-2\pi i)^{m+n}}{m!n!} \frac{(-2\pi i)^{p+q-m-n}}{(p-m)!(q-n)!} c_{mn}^{(f_k)} C_{p-m,q-n}^{(jk)} = \tag{10}$$

$$= \frac{(-2\pi i)^{p+q}}{p!q!} c_{pq}^{(f_j)} \,, \tag{11}$$

which can be read as

$$\sum_{\substack{m=0\\(m,n)\in D}}^{p} \sum_{n=0}^{q} \binom{p}{m} \binom{q}{n} c_{mn}^{(f_k)} C_{p-m,q-n}^{(jk)} = c_{pq}^{(f_j)} \,. \tag{12}$$

After isolating $C_{pq}^{(jk)}$ on the left-hand side, we obtain the final recurrence

$$c_{00}^{(f_k)} C_{pq}^{(jk)} = c_{pq}^{(f_j)} - \sum_{\substack{m=0\\(m,n)\in D\\m+n\neq 0}}^{p} \sum_{n=0}^{q} \binom{p}{m} \binom{q}{n} c_{mn}^{(f_k)} C_{p-m,q-n}^{(jk)} \,. \tag{13}$$

Since $I_{jk}(\mathbf{f})$ has been proven to be invariant to blur belonging to $\mathbf{H}$, all its Taylor coefficients $C_{pq}^{(jk)}$ must also be blur invariants. The beauty of Eq. (13) lies in the fact that we can calculate the invariants from the moments of $\mathbf{f}$ directly and in a numerically stable way.

Analyzing Eq. (13), one can realize that among single-channel invariants $C_{pq}^{(jj)}$ only those with $(p,q) \notin D$ are valid. If $(p,q) \in D$, then $C_{pq}^{(jj)} = 0$ for arbitrary $\mathbf{f}$. Here we clearly see the important role played by the cross-channel invariants $C_{pq}^{(jk)}, j \neq k$. They are valid for all $(p,q) \in D$ while for $(p,q) \notin D$ they are either trivial or dependent on the single-channel invariants $C_{pq}^{(jj)}$. In other words, single-channel and cross-channel invariants provide mutually complementary information (see Fig. 3).

The amount of new information provided by the cross-channel invariants decreases with increasing N because the set D becomes more sparse. For $N = \infty$, only the cross-channel invariants on the main diagonal $C_{pp}^{(jk)}$ contribute with new information. Naturally, we do not want to restrict the set $\mathbf{H}$ too much as it might not include all possible PSFs we are likely going to encounter. The $N = \infty$ case may be too restrictive and we might rather want to opt for $N = 2$ or $N = 4$ to be on the safe side. However, before the introduction of the cross-channel invariants, this choice meant a significant decrease of the number of invariants and, therefore, a reduction of the recognition power. Cross-channel invariants diminish this disadvantage by adding moment invariants of orders $(p,q) \in D$ that are missing in the set of single-channel invariants.

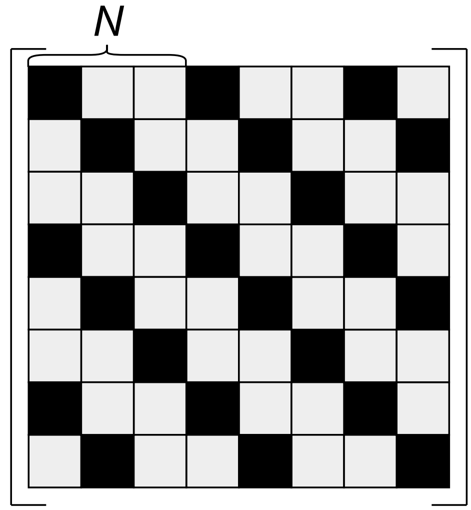

Fig. 3. Typical strip structure of the set of the blur moment invariants w.r.t. N-fold symmetric blur (here shown for the $N = 3$ case). Horizontal direction - index p, vertical direction - index q, (0,0) is in the upper left corner. White diagonals – single-channel invariants exist in R, G, and B. Cross-channel invariants also exist but they are dependent on the single-channel ones. Black diagonals – all single-channel invariants are zero. Cross-channel invariants RG and GB exist and provide independent information. Any black diagonal is always followed by $N - 1$ white diagonals.

IV. EXPERIMENTS

In this experiment, we worked in a controlled environment, which enables a detailed quantitative evaluation of the performance. We used real color photographs of indoor as well as outdoor scenes that had been taken as sharp as possible. In each image, 100 templates of the size 100×100 pixels were selected such that they contain some edges or other high-frequency patterns, see Fig. 4 for a sample scene. Then we artificially blurred the images by a triangular kernel having 3-fold rotation symmetry. The size of the kernel ranged from 9×9 to 63×63 pixels. The templates were extracted from the blurred images (see Fig. 5 for a sample template) and matched against the original sharp scene.

The way we created the blurred templates is realistic but violates the convolution relationship between the blurred and sharp patches. We face the so-called boundary effect, where the blurred template is influenced by the objects lying outside it. The boundary effect becomes more apparent as the blur size increases.

The matching was performed by exhaustive search over the entire scene. In each position, we matched a feature vector $\mathbf{a}$ of the scene patch against a feature vector $\mathbf{b}$ of the template. The matching position was determined as the minimum of $\|\frac{\mathbf{a}-\mathbf{b}}{\mathbf{a}}\|^2$. As the features, we used two families: single-channel invariants I_R, I_G, I_B applied channel-wise, and the

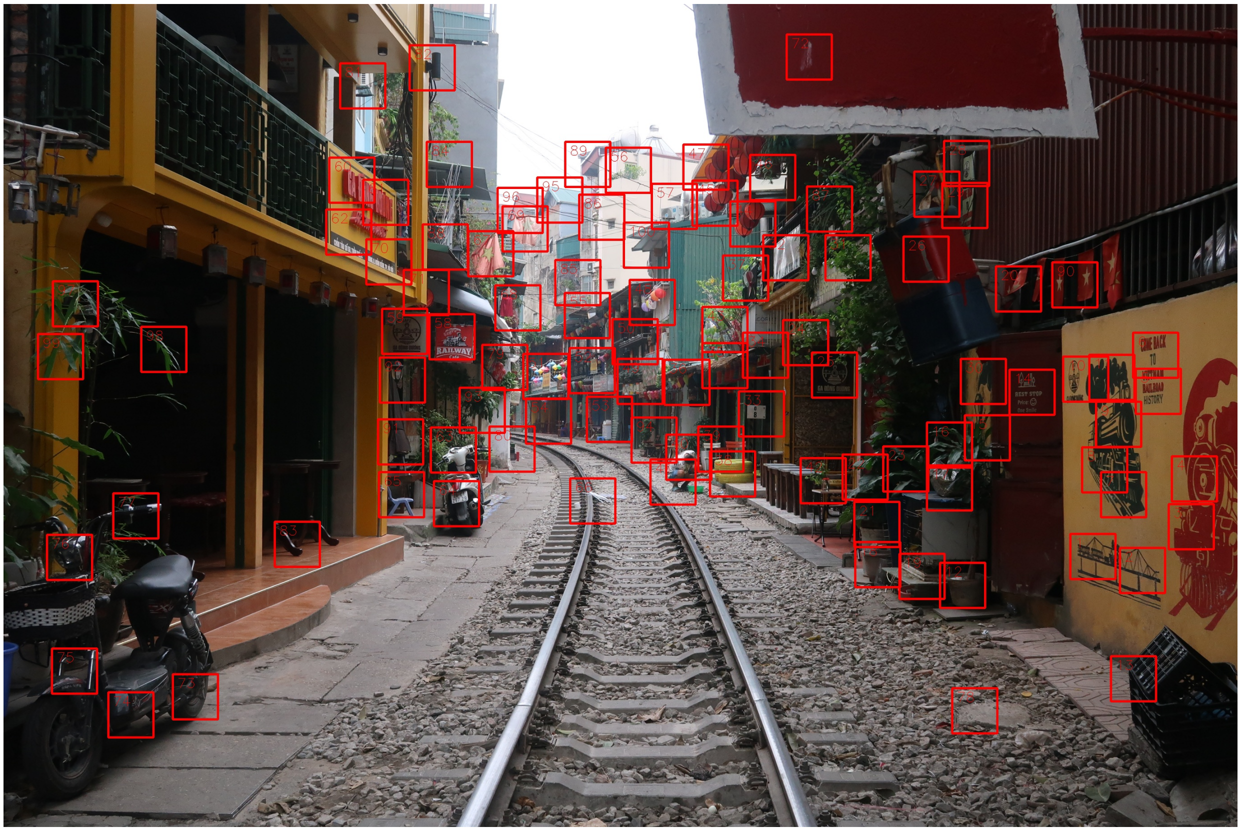

Fig. 4. A sample scene of the size 1632×2456 pixels with 100 selected templates 100×100 pixels.

Fig. 5. An example of one template from Fig. 4 and its artificial blurring. Triangle blur of the size 15, 27, 39, 51, respectively, was applied.

same single-channel invariants augmented by cross-channel ones I_{GR} and I_{BG}. Both feature sets were computed with invariants of $N = 2$ (which does not correspond to the blur shape) and with "true" invariants of $N = 3$.

All these invariants were calculated in the moment domain by Eq. (13). Centralized complex moments up to the 5th order were employed.

As a performance measure of all methods, we count the number of erroneously matched templates. The matching is considered to be erroneous if the matched position is more than half of the template size from the ground-truth position. The results over 15 scenes are summarized in Fig. 6. Probably the most important conclusion of this experiment is that inclusion of the cross-channel invariants substantially improves the recognition power. Since the main source of errors is the boundary effect, the choice of N surprisingly plays a secondary role only. Even if N has been chosen incorrectly, blur invariants work reasonably and cross-channel invariants improve the performance of single-channel ones.

To illustrate the influence of the boundary effect, we repeated this experiment with the same features but the templates were extracted from the sharp frame, zero-padded and then blurred. This scenario (which is, of course, non-realistic in practice) eliminates the boundary effect completely. Then both sets of $N = 3$ invariants perform without errors even for large blurs (which demonstrates the correctness of the theory), while both $N = 2$ sets behave similarly as in Fig. 6. In this case, the only source of errors is an incorrect choice of N.

Both experiments (along with blur invariants for any N) were added to our general repository of blur invariants for color images, the link is https://github.com/Vakosik/Blur_invariants_for_color_images.

V. Conclusion

In this paper, we applied the general theory of invariants of color images with respect to blur [22] to N-fold symmetric blur kernels. This kind of blur appears in out-of-focused images which were taken with a partially open diaphragm.

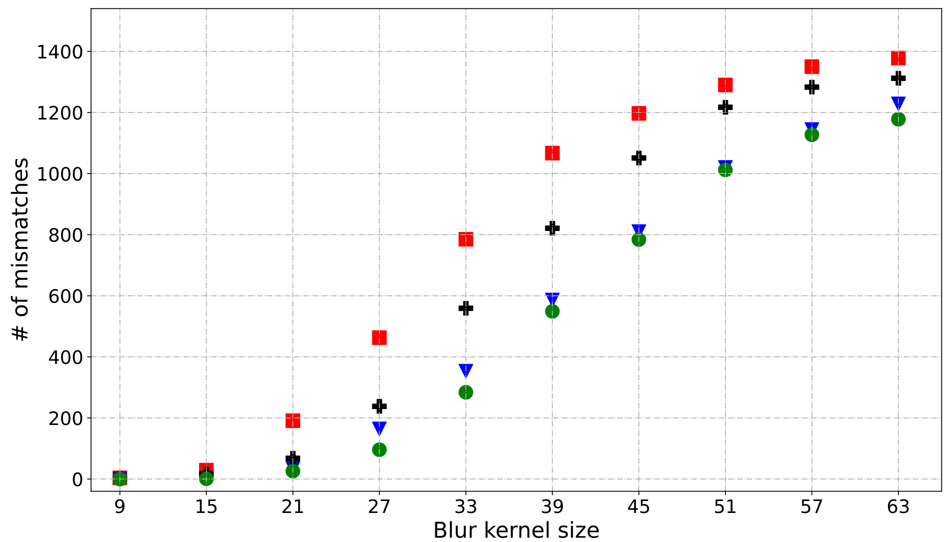

Fig. 6. Template recognition rate as a function of the blur size for different invariants. The templates were extracted from the scenes blurred artificially by a triangular blur without any padding, so the convolution is not exact and we face the boundary effect. Horizontal axis: The blur size in pixels. Vertical axis: The number of mismatched templates (out of 1500). Green dots: $N = 3$ complete set $(I_R, I_G, I_B, I_{GR}, I_{BG})$; blue triangles: the same complete set composed of $N = 2$ invariants; black marks: (I_R, I_G, I_B) for $N = 3$ applied channel-wise; red squares: the same single-channel invariants for $N = 2$. Matching errors are mainly due to boundary effect. Cross-channel invariants clearly increase the recognition power.

Mathematical tools required for the design of moment invariants are different from those used for centrosymmetric blur. We proved experimentally that the newly introduced cross-channel blur invariants improve the recognition power in blurred template matching.

Acknowledgement

This work has been supported by the Czech Science Foundation under the grant GA24-10069S.

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