



*-Fuzzy martingale and Doob's maximal inequalities

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ABSTRACT

Using the *-fuzzy measure introduced by Ghafari, Saadati, and Mesiar [Fuzzy Sets and Systems 2022], we develop the concept of martingale within *-fuzzy measure spaces. In particular, we establish *-fuzzy analogues of a Doob decomposition, Doob's maximal inequality, optional stopping theorems for both bounded and unbounded stopping times, and martingale convergence results. The analysis is carried out at the level of *-fuzzy integrals, reflecting the intrinsic nonlinearity of the model. It is supported by the *-fuzzy conditional expectations and the fuzzy Radon–Nikodym type representations. These results provide a coherent dynamic framework for martingales under *-fuzzy measures.

1. Introduction and preliminaries

Fuzzy measure theory and fuzzy integration provide a powerful nonlinear framework for modeling uncertainty beyond classical probability. Foundational developments in triangular norm-based structures and decomposable fuzzy measures have led to rich integration theories and functional inequalities, particularly in *-fuzzy and t-norm-based spaces [1–3]. Recent advances include Radon–Nikodým-type theorems for monotone and *-fuzzy measures, which play a central role in defining fuzzy conditional expectations and integral representations [4–6]. Analytical and computational aspects of fuzzy integrals, especially Choquet-type integrals, have also been actively studied, enabling both theoretical development and practical applications [7,8].

On the probabilistic side, classical martingale theory and stopping results remain fundamental references [9,10], while probabilistic metric spaces provide a natural bridge between probability and fuzziness [11,12]. Explicit notions of fuzzy martingales have been previously introduced, though in limited settings, highlighting the need for a more systematic martingale framework under generalized fuzzy measures [13]. Related developments on random fuzzy measures and coherent upper conditional previsions further demonstrate the growing interaction between fuzzy measures, stochastic structures, and nonlinear expectations [14,15]. Motivated by these works, the present paper develops a comprehensive martingale theory in *-fuzzy measure spaces, extending classical results such as Doob decomposition, maximal inequalities, and optional stopping to a fully *-fuzzy integral setting.

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Classical martingale theory relies fundamentally on linear expectations induced by sigma-additive probability measures. In contrast, fuzzy probability models replace linear additivity with nonlinear aggregation governed by triangular norms, leading to substantially different analytical behavior. While several notions of fuzzy martingales have emerged in probabilistic metric spaces and possibility theory, these approaches often lack a measure-theoretic foundation that is compatible with integration and conditioning. The GSM fuzzy measure framework, introduced in [1,2,4,16], provides a robust nonlinear integration theory via triangular norm-based measures and additive generators. This structure allows fuzzy martingale problems to be systematically transferred to classical measure-theoretic settings, enabling the development of fuzzy counterparts of a Doob decomposition, maximal inequalities, optional stopping, and convergence theorems. Importantly, the results hold at the level of fuzzy integrals rather than pointwise fuzzy variables, reflecting the intrinsic nonlinearity of the model.

While the mentioned works focus on the construction of fuzzy measures and nonlinear integrals, the present paper develops a genuine dynamic theory by introducing fuzzy conditional expectations, stopping times, and martingale-type convergence results within the fuzzy measure framework.

Now, we recall the fundamental concepts that form the foundation of our work. Throughout the paper, all equalities and inequalities involving $\ast$ -fuzzy integrals are understood pointwise with respect to the parameter $\varsigma > 0$, which is fixed but arbitrary.

Definition 1.1. [3,11] A continuous triangular norm (shortly, a ct -norm) is a continuous binary operation $\ast: [0, 1]^2 \rightarrow [0, 1]$ such that for all $s_1, s_2, s_3, s_4 \in [0, 1]$:

- (a) $s_1 \ast s_2 = s_2 \ast s_1$ and $s_1 \ast (s_2 \ast s_3) = (s_1 \ast s_2) \ast s_3$;
- (b) $s_1 \ast 1 = s_1$;
- (c) $s_1 \ast s_2 \leq s_3 \ast s_4$ whenever $s_1 \leq s_3$ and $s_2 \leq s_4$.

Some examples of the ct -norms are as follows:

1. $s_1 \ast_P s_2 = s_1 s_2$ (the product t -norm);
2. $s_1 \ast_M s_2 = \min\{s_1, s_2\}$ (the minimum t -norm);
3. $s_1 \ast_L s_2 = \max\{s_1 + s_2 - 1, 0\}$ (the Lukasiewicz t -norm);
4. $s_1 \ast_H s_2 = \begin{cases} 0, & \text{if } s_1 = s_2 = 0, \\ \frac{s_1 s_2}{s_1 + s_2 - s_1 s_2}, & \text{otherwise,} \end{cases}$ (the Hamacher t -norm).

We define

$$\ast_{i=1}^k s_i = s_1 \ast s_2 \ast \cdots \ast s_k,$$

for $k \in \{2, 3, \dots\}$. Note that, a ct -norm is said to be Archimedean (cat -norm) when $s \ast s < s$ for every $s \in I^\circ$ (interior of $I = [0, 1]$) or

equivalently, for all $(x, y) \in (0, 1)^2$, there exists an integer $n \in \mathbb{N}$ such that $\overbrace{x \ast \cdots \ast x}^{n \text{ times}} < y$. For example $\ast_P, \ast_H$ and $\ast_L$ are cat -norms but $\ast_M$ is not a cat -norm. A ct -norm is Archimedean if and only if there is a strictly decreasing continuous function $h: I \rightarrow [0, +\infty]$, $h(1) = 0$, such that

$$s_1 \ast s_2 = h^{-1}(\min\{h(0), h(s_1) + h(s_2)\}),$$

for every $s_1, s_2 \in I$. The mapping h is said to be an additive generator of $\ast$. For example the additive generator for $\ast_P$ is $h_P(s) = -\ln s$, for $\ast_L$ it is $h_L(s) = 1 - s$ and for $\ast_H$ it is $h_H(s) = \frac{1-s}{s}$. the generator is strict i.e., $h(0) = +\infty$, so in this case, we have

$$s_1 \ast s_2 = h^{-1}(h(s_1) + h(s_2)),$$

and as a consequence, we have

$$h(s_1 \ast s_2) = h(s_1) + h(s_2).$$

Definition 1.2. [2] Let X be a set and $\mathcal{F}$ be a σ -algebra consisting of subsets of X . Let $\ast$ be a cat -norm. A $\ast$ -fuzzy measure on $\mathcal{F} \times (0, +\infty)$ is a probability distribution function $\pi: \mathcal{F} \times (0, +\infty) \rightarrow [0, 1]$ such that

- (i) $\pi(\emptyset, \varsigma) = 1$, for every $\varsigma \in (0, +\infty)$;
- (ii) For any $A \in \mathcal{F}$, $\pi(A, \cdot)$ is increasing, left-continuous and $\lim_{\varsigma \rightarrow +\infty} \pi(A, \varsigma) = 1$;
- (iii) If $A_i \in \mathcal{F}, i = 1, 2, \dots$, are pairwise disjoint ($A_i \cap A_j = \emptyset$, if $i \neq j$), then

$$\pi\left(\bigcup_{i=1}^{+\infty} A_i, \varsigma\right) = \ast_{i=1}^{+\infty} \pi(A_i, \varsigma), \quad \text{for all } \varsigma \in (0, +\infty).$$

The quadruple $(X, \mathcal{F}, \pi, \ast)$ is called a $\ast$ -fuzzy measure space.

Definition 1.3. Let $(X, \mathcal{F}, \pi, \ast)$ be a $\ast$ -fuzzy measure space. A set $A \in \mathcal{F}$ with $\pi(A, \varsigma) = 1$ for each $\varsigma \in (0, +\infty)$ is called a $\ast$ -fuzzy null set. If a statement about points $x \in X$ is true except for x in some null set, we say that it is true $\ast$ -fuzzy almost everywhere (abbreviated $\ast$ -fuzzy a.e.).

Definition 1.4. Let $(X, \mathcal{F}, \pi, *)$ be a $*$ -fuzzy measure space. Define

$$L_+ = \{f : X \rightarrow [0, +\infty) \mid f \text{ is measurable function}\}.$$

Let ϕ be a simple function in L_+ with standard representation $\phi = \sum_{i=1}^n a_i \chi_{E_i}$, where $a_i > 0$, $i = 1, \dots, n$, and $\{E_i\}_{i=1}^n$ is a pairwise disjoint system of non-empty sets from $\mathcal{F}$. The $*$ -fuzzy integral of ϕ is defined as

$$\int_X \phi \pi(\cdot, \varsigma) = \int_X \sum_{i=1}^n a_i \chi_{E_i} \pi(\cdot, \varsigma) = \sum_{i=1}^n \pi\left(E_i, \frac{\varsigma}{a_i}\right).$$

If $A \in \mathcal{F}$, then $\int_A \phi \pi(\cdot, \varsigma) = \int_X \phi \chi_A \pi(\cdot, \varsigma)$.

Definition 1.5. Let f be a function in L_+ . The $*$ -fuzzy integral of f is defined by

$$\int_X f \pi(\cdot, \varsigma) = \inf \left\{ \int_X \phi \pi(\cdot, \varsigma) \mid 0 \leq \phi \leq f, \phi \text{ is a simple function} \right\}.$$

A function $f \in L_+$ is called $*$ -fuzzy integrable if $\int_X f \pi(\cdot, \varsigma) > 0$.

Theorem 1.6. Let ϕ_1 and ϕ_2 be simple functions in L_+ with standard representations $\phi_1 = \sum_{i=1}^n a_i \chi_{E_i}$ and $\phi_2 = \sum_{j=1}^m b_j \chi_{F_j}$. If $\phi_1 \leq \phi_2$, then

$$\int_X \phi_1 \pi(\cdot, \varsigma) \geq \int_X \phi_2 \pi(\cdot, \varsigma),$$

for all $\varsigma \in (0, +\infty)$.

Corollary 1.7. [2] The $*$ -fuzzy integral is monotone decreasing with respect to the integrated function. Specifically, for any measurable set $A \in \mathcal{F}$, $\varsigma > 0$ and functions $f, g \in L_+$ we have

1. $0 < \int_A f \pi(\cdot, \varsigma) \leq 1$,
2. If $f \leq g$ on A , then $\int_A f \pi(\cdot, \varsigma) \geq \int_A g \pi(\cdot, \varsigma)$,
3. For any constant $c > 0$, $\int_X c \pi(\cdot, \varsigma) = \pi\left(X, \frac{\varsigma}{c}\right)$,
4. If $f = 0$ almost everywhere, then $\int_X f \pi(\cdot, \varsigma) = 1$.

Theorem 1.8. [2] If $\{f_n\}$ is a sequence in L_+ such that $f_n \leq f_{n+1}$ for every $n \in \mathbb{N}$, and $f = \lim_{n \rightarrow +\infty} f_n = \sup_{n \in \mathbb{N}} f_n$, then

$$\int_X f \pi(\cdot, \varsigma) = \lim_{n \rightarrow +\infty} \int_X f_n \pi(\cdot, \varsigma).$$

Lemma 1.9. [2] Let $\{f_n\}$ be a sequence of fuzzy functions/variables in L_+ , then

$$\int_X \liminf_{n \rightarrow +\infty} f_n \pi(\cdot, \varsigma) \geq \liminf_{n \rightarrow +\infty} \int_X f_n \pi(\cdot, \varsigma).$$

Theorem 1.10. [2] Let $\{f_n\}$ be a sequence of fuzzy functions/variables in L_+ such that

$$f_n(\cdot, \varsigma) \rightarrow f(\cdot, \varsigma) \quad \text{almost everywhere.}$$

Then $f \in L_+$ and

$$\int_X f \pi(\cdot, \varsigma) = \lim_{n \rightarrow +\infty} \int_X f_n \pi(\cdot, \varsigma).$$

Example 1.11. [2] Let $(X, \mathcal{F}, m)$ be a classical measure space.

1. For $*$ -norm H (Hamacher t -norm), $\pi(A, \varsigma) = \frac{\varsigma}{\varsigma + m(A)}$ is a $*$ -fuzzy measure.
2. For $*$ -norm p (product t -norm), $\pi(A, \varsigma) = e^{-\frac{m(A)}{\varsigma}}$ is a $*$ -fuzzy measure.
3. More generally, if $*$ is a cat -norm generated by h , then

$$\pi(A, \varsigma) = h^{-1} \left(\min \left\{ h(0), \frac{m(A)}{\varsigma} \right\} \right), \quad (1)$$

is a $*$ -fuzzy measure named canonical $*$ -fuzzy measure.

Lemma 1.12. Consider $f \in L_+$ and disjoint sets $C, D \in \mathcal{F}$ with $C \cap D = \emptyset$, then

$$\int_{C \cup D} f d\pi(\cdot, \varsigma) = \int_C f d\pi(\cdot, \varsigma) * \int_D f d\pi(\cdot, \varsigma), \quad \text{for all } \varsigma > 0.$$

Proof. Using definition of the $*$ -fuzzy integral for general functions, we have

$$\begin{aligned}\int_{C \cup D} f d\pi &= \inf \left\{ \int_{C \cup D} \phi \pi(\cdot, \varsigma) : 0 \leq \phi \leq f, \phi \text{ is a simple function} \right\}, \\ \int_C f d\pi &= \inf \left\{ \int_C \psi \pi(\cdot, \varsigma) : 0 \leq \psi \leq f, \psi \text{ is a simple function} \right\}, \\ \int_D f d\pi &= \inf \left\{ \int_D A \pi(\cdot, \varsigma) : 0 \leq A \leq f, A \text{ is a simple function} \right\}.\end{aligned}$$

First, we prove the following inequality

$$\int_{C \cup D} f \pi(\cdot, \varsigma) \leq \int_C f \pi(\cdot, \varsigma) * \int_D f \pi(\cdot, \varsigma).$$

Consider simple functions ψ, θ with $0 \leq \psi \leq f \chi_C$, $0 \leq \theta \leq f \chi_D$, define $\phi = \psi + \theta$. Then $0 \leq \phi \leq f \chi_{C \cup D}$ and ϕ is simple. Since C and D are disjoint, by Definition 1.4 we have

$$\int_{C \cup D} \phi \pi(\cdot, \varsigma) = \int_C \psi \pi(\cdot, \varsigma) * \int_D \theta \pi(\cdot, \varsigma).$$

Thus, by the decreasing property of the $*$ -fuzzy integral, we get

$$\int_{C \cup D} f \pi(\cdot, \varsigma) \leq \int_C \psi \pi(\cdot, \varsigma) * \int_D \theta \pi(\cdot, \varsigma).$$

Taking the infimum on ψ, θ

$$\int_{C \cup D} f \pi(\cdot, \varsigma) \leq \inf_{\psi} \int_C \psi \pi(\cdot, \varsigma) * \inf_{\theta} \int_D \theta \pi(\cdot, \varsigma) = \int_C f d\pi * \int_D f \pi(\cdot, \varsigma). \quad (2)$$

Now, we show the reverse inequality

$$\int_C f \pi(\cdot, \varsigma) * \int_D f \pi(\cdot, \varsigma) \leq \int_{C \cup D} f \pi(\cdot, \varsigma).$$

Consider simple function ϕ with $0 \leq \phi \leq f \chi_{C \cup D}$, define $\psi = \phi \chi_C$, $\theta = \phi \chi_D$. Then ψ, θ are simple with $0 \leq \psi \leq f \chi_C$, $0 \leq \theta \leq f \chi_D$, and

$$\int_C \psi \pi(\cdot, \varsigma) * \int_D \theta \pi(\cdot, \varsigma) = \int_{C \cup D} \phi \pi(\cdot, \varsigma).$$

Thus

$$\int_C f \pi(\cdot, \varsigma) * \int_D f \pi(\cdot, \varsigma) \leq \int_C \psi \pi(\cdot, \varsigma) * \int_D A \pi(\cdot, \varsigma) = \int_{C \cup D} \phi \pi(\cdot, \varsigma).$$

Taking the infimum on ϕ

$$\int_C f d\pi * \int_D f \pi(\cdot, \varsigma) \leq \int_{C \cup D} f \pi(\cdot, \varsigma), \quad (3)$$

From inequalities (2) and (3), we conclude that

$$\int_{C \cup D} f \pi(\cdot, \varsigma) = \int_C f \pi(\cdot, \varsigma) * \int_D f \pi(\cdot, \varsigma). \quad \square$$

Remark 1.13. Using Lemma 1.12 and for pairwise disjoint sets $C_1, C_2, \dots, C_n \in \mathcal{F}$ we get

$$\int_{\bigcup_{i=1}^n C_i} f d\pi = *_{i=1}^n \int_{C_i} f d\pi.$$

Lemma 1.14. Let $(X, \mathcal{F}, \pi, *)$ be a $*$ -fuzzy measure space, where $*$ is a cat-norm generated by the additive generator h . Then for every $A \in \mathcal{F}$, every $\varsigma > 0$, and all $a, b > 0$, we have

$$\pi\left(A, \frac{\varsigma}{a+b}\right) \geq \pi\left(A, \frac{\varsigma}{a}\right) * \pi\left(A, \frac{\varsigma}{b}\right). \quad (4)$$

Proof. Note that $\pi(A, \cdot)$ is increasing and h and h^{-1} are strictly decreasing. So, we can write

$$\begin{aligned}h\left(\pi\left(A, \frac{\varsigma}{a}\right) * \pi\left(A, \frac{\varsigma}{b}\right)\right) &= \min \left\{ h(0), h\left(\pi\left(A, \frac{\varsigma}{a}\right)\right) + h\left(\pi\left(A, \frac{\varsigma}{b}\right)\right) \right\} \\ &\geq \min \left\{ h(0), 2h\left(\pi\left(A, \frac{\varsigma}{a+b}\right)\right) \right\} \\ &\geq h\left(\pi\left(A, \frac{\varsigma}{a+b}\right)\right).\end{aligned} \quad (5)$$

From (5), we take h^{-1} and we obtain

$$\pi\left(A, \frac{\varsigma}{a+b}\right) \geq \pi\left(A, \frac{\varsigma}{a}\right) * \pi\left(A, \frac{\varsigma}{b}\right),$$

which is exactly (4). This completes the proof. $\square$

Lemma 1.15. [1] Let $(X, \mathcal{F}, \pi, *)$ be a $*$ -fuzzy measure space. Then for every $A \in \mathcal{F}$, every $\varsigma > 0$, and all $f, g \in L_+$, we have

$$\int_A (f + g) d\pi(\cdot, \varsigma) \geq \int_A f d\pi(\cdot, \varsigma) * \int_A g d\pi(\cdot, \varsigma). \quad (6)$$

Lemma 1.16. Consider $*$ -fuzzy measure space $(X, \mathcal{F}, \pi, *)$, where $*$ is a cat-norm with a strict additive generator h (i.e., $h(0) = +\infty$). Then,

- (1) For any $Y \in L_+$, $A \in \mathcal{F}$, and $\varsigma > 0$, the function $h\left(\int_A Y d\pi(\cdot, \varsigma)\right)$ is a classical measure on $\mathcal{F}$.
 (2) For any set $A \in \mathcal{F}$, any $\varsigma > 0$, and any real-valued function Z_ς that is decreasing and continuous in ς with a positive integral, satisfying the condition $\lim_{\varsigma \rightarrow +\infty} Z_\varsigma = 0$, the function

$$h^{-1}\left(\int_A Z_\varsigma d\nu\right)$$

is a $*$ -fuzzy measure on the sigma-algebra $\mathcal{F}$, where ν represents a classical measure.

Proof. (1). Fix $\varsigma > 0$. For $A \in \mathcal{F}$, define the set function

$$\mu_Y(A) := h\left(\int_A Y d\pi(\cdot, \varsigma)\right).$$

Since $\int_\emptyset Y d\pi = 1$ and $h(1) = 0$, we have

$$\mu_Y(\emptyset) = 0.$$

Because $0 < \int_A Y d\pi \leq 1$ and h is decreasing, it follows that

$$0 \leq \mu_Y(A) < +\infty \quad \text{for all } A \in \mathcal{F}.$$

Let $A, B \in \mathcal{F}$ with $A \cap B = \emptyset$. By Lemma 1.12, we have

$$\int_{A \cup B} Y d\pi = \int_A Y d\pi * \int_B Y d\pi.$$

Applying the additive generator h and using $h(a * b) = h(a) + h(b)$, we obtain

$$\mu_Y(A \cup B) = \mu_Y(A) + \mu_Y(B).$$

Let $\{A_k\}_{k \geq 1} \subset \mathcal{F}$ be pairwise disjoint and define $B_m = \bigcup_{k=1}^m A_k$. By finite additivity,

$$\mu_Y(B_m) = \sum_{k=1}^m \mu_Y(A_k).$$

Using the monotone convergence property of the $*$ -fuzzy integral,

$$\int_{\bigcup_{k=1}^\infty A_k} Y d\pi = \lim_{m \rightarrow \infty} \int_{B_m} Y d\pi,$$

and the continuity of h , we conclude that

$$\mu_Y\left(\bigcup_{k=1}^\infty A_k\right) = \sum_{k=1}^\infty \mu_Y(A_k).$$

Therefore, μ_Y is a classical measure on $\mathcal{F}$.

(2). Fix $\varsigma > 0$ and the classical measure ν . For $A \in \mathcal{F}$, define the set function

$$\mu_{Z_\varsigma}(A) := h^{-1}\left(\int_A Z_\varsigma d\nu\right).$$

Since $\int_\emptyset Z d\nu = 0$ and $h^{-1}(0) = 1$, we have

$$\mu_{Z_\varsigma}(\emptyset) = 1.$$

Because $0 \leq \int_A Z d\nu < +\infty$ and h^{-1} is decreasing, it follows that

$$0 < \mu_{Z_\varsigma}(A) \leq 1 \quad \text{for all } A \in \mathcal{F}.$$

Let $A, B \in \mathcal{F}$ with $A \cap B = \emptyset$. By Lemma 1.12, we have

$$\int_{A \cup B} Z d\nu = \int_A Z d\nu + \int_B Z d\nu.$$

Applying the additive generator h^{-1} and using $h^{-1}(a) * h^{-1}(b) = h^{-1}(a + b)$, we obtain

$$\mu_{Z_\varsigma}(A \cup B) = \mu_{Z_\varsigma}(A) * \mu_{Z_\varsigma}(B).$$

Let $\{A_k\}_{k \geq 1} \subset \mathcal{F}$ be pairwise disjoint and define $B_m = \bigcup_{k=1}^m A_k$. By finite additivity,

$$\mu_{Z_\zeta}(B_m) = \mu_{Z_\zeta}^*(A_k).$$

Using the monotone convergence property of the classical integral,

$$\int_{\bigcup_{k=1}^\infty A_k} Z \, d\nu = \lim_{m \rightarrow \infty} \int_{B_m} Z \, d\nu,$$

and the continuity of h , we conclude that

$$\mu_{Z_\zeta}\left(\bigcup_{k=1}^\infty A_k\right) = \mu_{Z_\zeta}^*(A_k).$$

Finally, we find that as $\lim_{\zeta \rightarrow +\infty} Z_\zeta = 0$, coupled with the arbitrariness of $A \in \mathcal{F}$ and the continuity of h^{-1} , we conclude that $\lim_{\zeta \rightarrow +\infty} \pi(A, \zeta) = 1$. Additionally, continuity is preserved, and since h^{-1} is a decreasing function, it follows that μ_{Z_ζ} is increasing as ζ increases. Thus, μ_{Z_ζ} qualifies as a $*$ -fuzzy measure on $\mathcal{F}$. This completes the proof. $\square$

Lemma 1.17. [4] Let $(X, \mathcal{F}, \pi, *)$ be a $*$ -fuzzy measure space and let $f, g \in L_+$. Then

$$\int_A f \, d\pi(\cdot, \zeta) = \int_A g \, d\pi(\cdot, \zeta), \quad \text{for all } A \in \mathcal{F}, \quad \zeta > 0,$$

if and only if

$$f = g \quad * \text{-fuzzy a.e..}$$

1.1. Fuzzy Radon–Nikodým theorem

Definition 1.18. [4] [Absolute Continuity] Consider $(X, \mathcal{F}, \pi_1, *)$ and $(X, \mathcal{F}, \pi_2, *)$ two $*$ -fuzzy measure spaces. If $\pi_2(A, \zeta) = 1$ for every $A \in \mathcal{F}$ whenever $\pi_1(A, \zeta) = 1$, then we say that π_2 is absolutely continuous with respect to π_1 and write $\pi_2 \ll \pi_1$.

Definition 1.19. [4] [Mutual Singularity] Consider $(X, \mathcal{F}, \pi_1, *)$ and $(X, \mathcal{F}, \pi_2, *)$ two $*$ -fuzzy measure spaces. If there exist $A, B \in \mathcal{F}$ such that $A \cap B = \emptyset$, $A \cup B = X$, $\pi_1(A, \zeta) = 1$ and $\pi_2(B, \zeta) = 1$, then we say π_1, π_2 are mutually singular and write $\pi_2 \perp \pi_1$.

Definition 1.20. [2] [Bounded Measures] A $*$ -fuzzy measure π is a *bounded $*$ -fuzzy measure* if

$$\pi(X, \zeta) > 0 \quad \text{for each } \zeta > 0.$$

A $*$ -fuzzy measure π is σ -bounded if there exist sets $A_i \in \mathcal{F}$ for $i = 1, 2, \dots$ such that

$$\pi(A_i, \zeta) > 0 \quad \text{for each } i \quad \text{and} \quad X = \bigcup_{i=1}^\infty A_i.$$

If π is a bounded $*$ -fuzzy measure, then $(X, \mathcal{F}, \pi, *)$ is called a *bounded $*$ -fuzzy measurable set*. Similarly, if π is a σ -bounded $*$ -fuzzy measure, then $(X, \mathcal{F}, \pi, *)$ is called a σ -bounded $*$ -FMS.

Theorem 1.21. [4] [$*$ -Fuzzy Lebesgue–Radon–Nikodym Theorem] If π_2, π_1 are σ -bounded $*$ -fuzzy measures on $(X, \mathcal{F}, *)$, then there exist unique σ -bounded $*$ -fuzzy measures λ, ρ on $(X, \mathcal{F}, *)$ such that:

1. $\rho \ll \pi_1$
2. $\lambda \perp \pi_1$
3. $\pi_2 = \lambda * \rho$

The decomposition $\pi_2 = \lambda * \rho$ with properties $\lambda \perp \pi_1$ and $\rho \ll \pi_1$ is called the $*$ -fuzzy measure space Lebesgue decomposition of π_2 with respect to π_1 .

Corollary 1.22. [4] [$*$ -Fuzzy Radon–Nikodym Derivative] If $\pi_2 \ll \pi_1$, then there exists a π_1 -integrable function $f : X \rightarrow [0, \infty)$ such that

$$d\pi_2 = f \, d\pi_1,$$

or equivalently,

$$\pi_2(A, \zeta) = \int_A f \, d\pi_1(\cdot, \zeta) \quad \text{for all } A \in \mathcal{F}.$$

The function f is called the $*$ -fuzzy Radon–Nikodym derivative of π_2 with respect to π_1 , denoted by $\frac{d\pi_2}{d\pi_1}$.

2. *-Fuzzy expectation

Now, we define the fundamental concept of expectation and related topics in the sense of *-fuzzy measure space.

Definition 2.1. Consider *-fuzzy measure space $(X, \mathcal{F}, \pi, *)$. For a *-fuzzy integrable function $f \in L_+$, the *-fuzzy expectation of f is defined as its *-fuzzy integral over X

$$E_*[f](\varsigma) = \int_X f d\pi(\cdot, \varsigma), \quad \forall \varsigma > 0.$$

A function $f \in L_+$ is called *-fuzzy integrable if $E_*[f](\varsigma) > 0$ for all $\varsigma > 0$.

Now, we are ready to define the conditional expectation in the sense of *-fuzzy measure.

Definition 2.2. Consider the *-fuzzy measure space $(X, \mathcal{F}, \pi, *)$. Let $\mathcal{G} \subseteq \mathcal{F}$ be a sub- σ -algebra. A *-conditional expectation of $f \in L_+$ given $\mathcal{G}$ is a $\mathcal{G}$ -measurable function g in L_+ such that

$$\int_A g d\pi(\cdot, \varsigma) = \int_A f d\pi(\cdot, \varsigma)$$

for $A \in \mathcal{G}$ and $\varsigma > 0$. If such a g exists, then we write $g = E_*(f | \mathcal{G})$.

Theorem 2.3 (Existence and Uniqueness of the *-Fuzzy Conditional Expectation). *Consider the *-fuzzy measure space $(X, \mathcal{F}, \pi, *)$, in which $*$ is a cat-norm with a strict additive generator h . Let $\mathcal{G} \subseteq \mathcal{F}$ be a sub- σ -algebra. Then, for every nonnegative *-fuzzy integrable function $f \in L_+$, there exists a unique *-conditional expectation $E_*(f | \mathcal{G})$ such that*

$$E_*(f | \mathcal{G}) = f \quad \text{fuzzy a.e.}$$

Proof. Fix $\varsigma > 0$. For $A \in \mathcal{F}$, define the set function

$$\mu_f(A) := h\left(\int_A f d\pi(\cdot, \varsigma)\right).$$

By Lemma 1.16, μ_f is a classical finite measure on $\mathcal{F}$, and in particular, $\mu_f|_{\mathcal{G}}$ is a finite measure on $\mathcal{G}$. Let $\nu := \mu_1$. Clearly, $\mu_f|_{\mathcal{G}} \ll \nu|_{\mathcal{G}}$. Hence, by the classical Radon-Nikodým theorem, there exists a nonnegative $\mathcal{G}$ -measurable function Y such that

$$\mu_f(A) = \int_A Y d\nu, \quad \forall A \in \mathcal{G}. \quad (7)$$

Denote by $\pi_1(A, \varsigma) := h^{-1}\left(\int_A Y d\nu\right)$. If $\pi(A, \varsigma) = 1$ for some $A \in \mathcal{F}$, then by the definition of fuzzy null sets and from (7) we obtain

$$\int_A f d\pi(\cdot, \varsigma) = 1 = h^{-1}\left(\int_A Y d\nu\right),$$

and we conclude $\pi_1(A, \varsigma) = 1$, which shows $\pi_1 \ll \pi$. By Corollary 1.22, there exists a $\mathcal{G}$ -measurable function $E_*(f | \mathcal{G}) \in L_+$ satisfying

$$h\left(\int_A E_*(f | \mathcal{G}) d\pi(\cdot, \varsigma)\right) = \int_A Y d\nu, \quad \forall A \in \mathcal{G}.$$

Combining the above equations yields

$$\int_A E_*(f | \mathcal{G}) d\pi(\cdot, \varsigma) = \int_A f d\pi(\cdot, \varsigma), \quad \forall A \in \mathcal{G}.$$

From Lemma 1.17, we conclude

$$E_*(f | \mathcal{G}) = f \quad \text{*-fuzzy a.e.}$$

Finally, suppose that $E_*^1(f | \mathcal{G})$ and $E_*^2(f | \mathcal{G})$ both satisfy the defining property. Then, for all $A \in \mathcal{G}$

$$\int_A E_*^1(f | \mathcal{G}) d\pi(\cdot, \varsigma) = \int_A E_*^2(f | \mathcal{G}) d\pi(\cdot, \varsigma),$$

which shows that $E_*^1(f | \mathcal{G})$ and $E_*^2(f | \mathcal{G})$ are $\mathcal{G}$ -integrally equivalent (Their integrals over all $A \in \mathcal{G}$ are equal). This completes the proof. $\square$

Definition 2.4. Assume that $(Y, \mathcal{F}, \pi, *)$ is a *-fuzzy measure space. Let $\{\mathcal{F}_n\}_{n \geq 0}$ be an increasing sequence of sub- σ -algebras of $\mathcal{F}$ (a filtration), and $\{f_n\}_{n \geq 0} \subset L_+$ a sequence of nonnegative measurable functions.

1. The sequence $\{f_n\}$ is called a *-fuzzy martingale if

- (a) f_n is $\mathcal{F}_n$ -measurable for each $n \geq 0$;
- (b) for all $n \geq 0$ and $\varsigma > 0$,

$$\int_Y f_n d\pi(x, \varsigma) > 0;$$

(c) the fuzzy conditional expectation satisfies

$$f_n = E_*(f_{n+1} | \mathcal{F}_n) \quad \text{*--fuzzy a.e. in } \mathcal{F}_n.$$

2. The sequence $\{f_n\}$ is called a **-fuzzy submartingale* if

$$f_n \leq E_*(f_{n+1} | \mathcal{F}_n) \quad \text{*--fuzzy a.e. in } \mathcal{F}_n.$$

3. The sequence $\{f_n\}$ is called a **-fuzzy supermartingale* if

$$f_n \geq E_*(f_{n+1} | \mathcal{F}_n) \quad \text{*--fuzzy a.e. in } \mathcal{F}_n.$$

Remark 2.5. Assume that $\{f_n\}_{n \geq 0} \subset L_+$ is a **-fuzzy martingale* sequence, then

$$\int_A f_n d\pi(\cdot, \varsigma) = \int_A f_{n+1} d\pi(\cdot, \varsigma), \quad \forall A \in \mathcal{F}_n, \varsigma > 0.$$

If $\{f_n\}_{n \geq 0} \subset L_+$ is a **-fuzzy submartingale* sequence, then by the decreasing property of the fuzzy integral, we have

$$\int_A f_n d\pi(\cdot, \varsigma) \geq \int_A f_{n+1} d\pi(\cdot, \varsigma), \quad \forall A \in \mathcal{F}_n, \varsigma > 0,$$

If $\{f_n\}_{n \geq 0} \subset L_+$ is a **-fuzzy supermartingale* sequence, then by the decreasing property of the fuzzy integral, we have

$$\int_A f_n d\pi(\cdot, \varsigma) \leq \int_A f_{n+1} d\pi(\cdot, \varsigma), \quad \forall A \in \mathcal{F}_n, \varsigma > 0,$$

for all $n \geq 0$.

Note: Let $A \in \mathcal{F}_n$, since $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$, we get, $A \in \mathcal{F}_{n+1}$ and so the integral $\int_A f_{n+1}(x) d\pi(x, \varsigma)$ is well defined. In the following, we require to show

$$\int_A f_{n+1} d\pi(\cdot, \varsigma) = \int_A f_n d\pi(\cdot, \varsigma) \quad \text{for all } A \in \mathcal{F}_n,$$

that, by the definition of **-fuzzy conditional expectation*, is equivalent to

$$E_*(f_{n+1} | \mathcal{F}_n) = f_n \quad \text{*--fuzzy a.e.}$$

Definition 2.6. A sequence $\{f_n\}_{n \geq 0} \subset L_+$ is *predictable* if f_0 is constant and f_n is $\mathcal{F}_{n-1}$ -measurable for $n \geq 1$. It is *increasing* if $f_{n+1} \geq f_n$ for all n .

Definition 2.7. A map $T : X \rightarrow \{0, 1, 2, \dots\}$ is called *stopping time* with respect to $\{\mathcal{F}_n\}$ if for each $n \geq 0$ the set $\{x \in X : T(x) = n\} \in \mathcal{F}_n$. Stopping times are defined relative to the underlying sigma-algebra, not the **-fuzzy measure*.

Definition 2.8. Let $(X, \mathcal{F}, \pi, *)$ be a **-fuzzy measure space* and $\{\mathcal{F}_n\}_{n \geq 0}$ be a filtration. Let $\{f_n\}_{n \geq 0} \subset L_+$ be a process and $T : X \rightarrow \{0, 1, 2, \dots\}$ be a stopping time with respect to $\{\mathcal{F}_n\}_{n \geq 0}$. For each $n \geq 0$, and $x \in X$, the stopped process $\{f_n^T\}_{n \geq 0}$ is defined by

$$f_n^T := f_{T \wedge n}.$$

Note that since T is a stopping time, $\{x \in X : T(x) = n\} \in \mathcal{F}_n$, and so f_n^T is $\mathcal{F}_n$ -measurable.

Lemma 2.9. Consider **-fuzzy measure space* $(X, \mathcal{F}, \pi, *)$ and a filtration $\{\mathcal{F}_n\}$ with $\mathcal{F}_m \subseteq \mathcal{F}_n$ for $m \leq n$. For any $f \in L_+$, we have

$$E_*(E_*(f | \mathcal{F}_n) | \mathcal{F}_m) = E_*(f | \mathcal{F}_m) \quad \text{*--fuzzy a.e.}$$

Proof. Let $f_n = E_*(f | \mathcal{F}_n)$ and $f_m = E_*(f | \mathcal{F}_m)$. Note, f_n is $\mathcal{F}_n$ -measurable and satisfies

$$\int_A f_n d\pi(\cdot, \varsigma) = \int_A f d\pi(\cdot, \varsigma), \quad \forall A \in \mathcal{F}_n, \varsigma > 0.$$

Now, let $k_m = E_*(f_n | \mathcal{F}_m)$. Then k_m is $\mathcal{F}_m$ -measurable and satisfies

$$\int_A k_m d\pi(\cdot, \varsigma) = \int_A f_n d\pi(\cdot, \varsigma), \quad \forall A \in \mathcal{F}_m, \varsigma > 0.$$

Since $\mathcal{F}_m \subseteq \mathcal{F}_n$, any $A \in \mathcal{F}_m$ is also in $\mathcal{F}_n$, so from the definition of f_n we have

$$\int_A f_n d\pi(\cdot, \varsigma) = \int_A f d\pi(\cdot, \varsigma), \quad \forall A \in \mathcal{F}_m, \varsigma > 0.$$

Combining the above, we obtain

$$\int_A k_m d\pi(\cdot, \varsigma) = \int_A f d\pi(\cdot, \varsigma), \quad \forall A \in \mathcal{F}_m, \varsigma > 0.$$

But this is exactly the defining property of $f_m = E_*(f | \mathcal{F}_m)$. It follows that

$$E_*(E_*(f | \mathcal{F}_n) | \mathcal{F}_m) = E_*(f_n | \mathcal{F}_m) = k_m = E_*(f | \mathcal{F}_m) \quad \text{*--fuzzy a.e.}$$

This completes the proof. $\square$

Corollary 2.10. If $\{f_n\}_{n \geq 0}$ is a $*$ -fuzzy martingale with respect to $\{\mathcal{F}_n\}_{n \geq 0}$, i.e.,

$$f_n = E_*(f_{n+1} | \mathcal{F}_n) \quad * \text{-fuzzy a.e., } \forall n \geq 0,$$

then for all $m \geq n$,

$$f_n = E_*(f_m | \mathcal{F}_n) \quad * \text{-fuzzy a.e.}$$

Proof. We proceed by induction on m . For $(m = n + 1)$, this is the martingale property itself. Now, for $m = n + 2$, we apply Lemma 2.9 to get

$$\begin{aligned} f_n &= E_*(f_{n+1} | \mathcal{F}_n) \\ &= E_*(E_*(f_{n+2} | \mathcal{F}_{n+1}) | \mathcal{F}_n) \\ &= E_*(f_{n+2} | \mathcal{F}_n). \end{aligned}$$

This completes the induction and the proof. $\square$

3. Main theorems

All results in this paper are formulated at the level of $*$ -fuzzy integrals, reflecting the intrinsic nonlinearity of $*$ -fuzzy measures. Pointwise ($*$ -fuzzy a.e.) statements are not addressed unless explicitly stated.

Theorem 3.1. Let $(X, \mathcal{F}, \pi, *)$ be a $*$ -fuzzy measure space with strict generator h , $\{\mathcal{F}_n\}_{n \geq 0}$ be a filtration, and $\{X_n\}_{n \geq 0}$ be a sequence of positive $*$ -fuzzy supermartingales with respect to $\{\mathcal{F}_n\}_{n \geq 0}$. Then, there exists a unique decomposition

$$\int_A M_n d\pi(\cdot, \varsigma) = \int_A X_n d\pi(\cdot, \varsigma) * \int_A B_n d\pi(\cdot, \varsigma), \quad \forall A \in \mathcal{F}_n, \quad \forall \varsigma > 0, \quad (8)$$

such that

- (1) $\{M_n\}_{n \geq 0}$ is a $*$ -fuzzy martingale with respect to $\{\mathcal{F}_n\}_{n \geq 0}$,
- (2) $\{B_n\}_{n \geq 0}$ is predictable and decreasing in the $*$ -fuzzy integral level with $B_0 = 0$ $*$ -fuzzy a.e.

Proof. Let $(X, \mathcal{F}, \pi, *)$ be a $*$ -fuzzy measure space, where $*$ is a cat-norm with additive generator $h : [0, 1] \rightarrow [0, +\infty]$, and let $\{X_n\}_{n \geq 0}$ be a positive $*$ -fuzzy super martingale with respect to a filtration $\{\mathcal{F}_n\}_{n \geq 0}$. For each $n \geq 0$, define the set function

$$\mu_n(A) := h\left(\int_A X_n d\pi(\cdot, \varsigma)\right), \quad A \in \mathcal{F}_n,$$

where $\varsigma > 0$ is fixed but arbitrary and the generator h is assumed to be strict i.e., $h(0) = +\infty$. By Lemma 1.16, μ_n is a classical measure on $\mathcal{F}_n$. For $A \in \mathcal{F}_n$, since $\{X_n\}$ is a $*$ -fuzzy supermartingale and h is decreasing,

$$\mu_n(A) \geq \mu_{n+1}(A),$$

so $\{\mu_n(A)\}_{n \geq 0}$ is decreasing for each $A \in \mathcal{F}_n$. Define a finite measure ν on $(X, \mathcal{F})$ by

$$\nu := \sum_{n=0}^{\infty} 2^{-(n+1)} \mu_n.$$

If $\nu = 0$, then from the fact that $\mu_n \geq 0$ for all $n \geq 0$, it immediately follows that $\mu_n = 0$ for all $n \geq 0$. Then $\mu_n \ll \nu$ for all $n \geq 0$. By the Radon–Nikodým theorem, there exists an $\mathcal{F}_n$ -measurable fuzzy variable $Y_n \in L^1(\nu)$ such that

$$\mu_n(A) = \int_A Y_n d\nu, \quad A \in \mathcal{F}_n.$$

The monotonicity of (μ_n) implies

$$\int_A Y_n d\nu \geq \int_A Y_{n+1} d\nu, \quad A \in \mathcal{F}_n,$$

so $\{Y_n\}_{n \geq 0}$ is a classical supermartingale with respect to $\{\mathcal{F}_n\}_{n \geq 0}$ and ν . By the discrete-time Doob decomposition theorem (see [9,10]), there exist unique processes $\{\widetilde{M}_n\}_{n \geq 0}$ and $\{\widetilde{B}_n\}_{n \geq 0}$ such that

$$\widetilde{M}_n = Y_n + \widetilde{B}_n,$$

where

$$\widetilde{B}_n = \sum_{j=1}^n (\mathbb{E}_\nu(Y_{j-1} | \mathcal{F}_{j-1}) - \mathbb{E}_\nu(Y_j | \mathcal{F}_{j-1})),$$

where $\mathbb{E}_\nu$ is the classical conditional expectation with respect to ν , $\{\widetilde{M}_n\}$ is a ν -martingale, $\{\widetilde{B}_n\}$ is predictable and increasing, and $\widetilde{B}_0 = 0$. Since $Y_n \geq 0$ for all $n \geq 0$, then from the decomposition it can be concluded that $\widetilde{M}_n \geq 0$ for every $n \geq 0$.

Now, from Lemma 1.16 (item 2), we know that $h^{-1}\left(\int_A \widetilde{M}_n d\nu\right)$ and $h^{-1}\left(\int_A \widetilde{B}_n d\nu\right)$ are $*$ -fuzzy measures. Denote by $\pi_1(A, \varsigma) := h^{-1}\left(\int_A \widetilde{M}_n d\nu\right)$ and $\pi_2(A, \varsigma) := h^{-1}\left(\int_A \widetilde{B}_n d\nu\right)$. Note that

$$\int_A \widetilde{M}_n d\nu = \int_A Y_n d\nu + \int_A \widetilde{B}_n d\nu.$$

If $\pi_1(A, \varsigma) = 1$ for some $A \in \mathcal{F}$, then

$$0 = h(\pi_1(A, \varsigma)) = h\left(\int_A X_n d\pi(\cdot, \varsigma)\right) + h(\pi_2(A, \varsigma)),$$

implying $\pi(A, \varsigma) = \pi_2(A, \varsigma) = 1$, and we conclude $\pi, \pi_2 \ll \pi_1$. Next we show that $\pi_1, \pi_2 \ll \pi$. If $\pi(A, \varsigma) = 1$ then we have $\int_A X_n d(\cdot, \varsigma) = 1$ for all $\varsigma > 0$. Therefore $\mu_n(A) = h\left(\int_A X_n d\pi(\cdot, \varsigma)\right) = 0$ and so $\int_A Y_n d\nu = 0$ implying $Y_n = 0$ ν -a.e. on A for all $n \geq 0$. Therefore $\widetilde{B}_n = 0$ ν -a.e. on A for all $n \geq 0$ and so $\widetilde{M}_n = 0$ ν -a.e. on A for all $n \geq 0$. Hence $\pi_1(A, \varsigma) = 1$ and $\pi_2(A, \varsigma) = 1$.

By Corollary 1.22, there exist the unique processes $\{M_n\}_{n \geq 0}$ and $\{B_n\}_{n \geq 0}$ by specifying their $*$ -fuzzy integrals for each $A \in \mathcal{F}_n$,

$$h\left(\int_A M_n d\pi(\cdot, \varsigma)\right) = \int_A \widetilde{M}_n d\nu, \quad h\left(\int_A B_n d\pi(\cdot, \varsigma)\right) = \int_A \widetilde{B}_n d\nu.$$

By the existence result for the $*$ -fuzzy integral, such M_n and B_n belong to L_+ for each $A \in \mathcal{F}_n$. For all $A \in \mathcal{F}_n$, we have

$$\begin{aligned} h\left(\int_A M_n d\pi(\cdot, \varsigma)\right) &= \int_A \widetilde{M}_n d\nu \\ &= \int_A Y_n d\nu + \int_A \widetilde{B}_n d\nu \\ &= h\left(\int_A X_n d\pi(\cdot, \varsigma)\right) + h\left(\int_A B_n d\pi(\cdot, \varsigma)\right) \\ &= h\left(\int_A X_n d\pi(\cdot, \varsigma) * \int_A B_n d\pi(\cdot, \varsigma)\right), \end{aligned}$$

which implies

$$\int_A M_n d\pi(\cdot, \varsigma) = \int_A X_n d\pi(\cdot, \varsigma) * \int_A B_n d\pi(\cdot, \varsigma),$$

which is (8). Since $\{\widetilde{M}_n\}$ is a classical martingale, for all $A \in \mathcal{F}_n$, we have

$$\int_A \widetilde{M}_{n+1} d\nu = \int_A \widetilde{M}_n d\nu,$$

and taking the inverse function h^{-1} implies that

$$\int_A M_{n+1} d\pi(\cdot, \varsigma) = \int_A M_n d\pi(\cdot, \varsigma), \quad A \in \mathcal{F}_n,$$

and so, $\{M_n\}$ are $*$ -fuzzy martingales. Because $\{\widetilde{B}_n\}$ is increasing in the classical sense,

$$\widetilde{B}_n \leq \widetilde{B}_{n+1} \quad \nu\text{-a.e. for all } n \geq 0.$$

Thus, for any $A \in \mathcal{F}_n$,

$$\begin{aligned} \int_A \widetilde{B}_n d\nu \leq \int_A \widetilde{B}_{n+1} d\nu &\implies h\left(\int_A B_n d\pi(\cdot, \varsigma)\right) \leq h\left(\int_A B_{n+1} d\pi(\cdot, \varsigma)\right) \\ &\implies \int_A B_n d\pi(\cdot, \varsigma) \geq \int_A B_{n+1} d\pi(\cdot, \varsigma), \end{aligned}$$

so $\{B_n\}_{n \geq 0}$ is a decreasing process in integral level. Classically, $\{\widetilde{B}_n\}_{n \geq 0}$ is predictable, i.e. $\widetilde{B}_n$ is $\mathcal{F}_{n-1}$ -measurable for $n \geq 1$. Thus, for every $A \in \mathcal{F}_{n-1}$, $\int_A \widetilde{B}_n d\nu$ depends only on $\mathcal{F}_{n-1}$. Note,

$$h\left(\int_A B_n d\pi(\cdot, \varsigma)\right) = \int_A \widetilde{B}_n d\nu, \quad A \in \mathcal{F}_{n-1},$$

and so the $*$ -fuzzy integrals of B_n over sets in $\mathcal{F}_{n-1}$ are determined by $\mathcal{F}_{n-1}$. Hence $\{B_n\}_{n \geq 0}$ are predictable. Note that $\widetilde{B}_0 = 0$, so for all $A \in \mathcal{F}$, we have

$$h\left(\int_A B_0 d\pi(\cdot, \varsigma)\right) = \int_A \widetilde{B}_0 d\nu = 0 \implies \int_A B_0 d\pi(\cdot, \varsigma) = 1 \implies B_0 = 0, \quad * \text{-fuzzy a.e.}$$

This argument can be easily verified with the contradiction argument. Finally, suppose that two $\ast$ -fuzzy Doob decompositions satisfy

$$\int_A M_n^{(i)} d\pi(\cdot, \varsigma) = \int_A X_n d\pi(\cdot, \varsigma) \ast \int_A B_n^{(i)} d\pi(\cdot, \varsigma), \quad i = 1, 2,$$

for all $A \in \mathcal{F}_n$. Applying the generator h yields

$$\int_A \widetilde{M}_n^{(i)} d\nu = \mu_n(A) + \int_A \widetilde{B}_n^{(i)} d\nu \quad i = 1, 2,$$

or equivalently

$$\mu_n(A) = \int_A \widetilde{M}_n^{(1)} d\nu - \int_A \widetilde{B}_n^{(1)} d\nu = \int_A \widetilde{M}_n^{(2)} d\nu - \int_A \widetilde{B}_n^{(2)} d\nu.$$

Since the family $\{\widetilde{M}_n^{(i)}\}_{n \geq 0}$ is a classical martingale, and $\{\widetilde{B}_n^{(i)}\}_{n \geq 0}$ is a predictable increasing process ($i = 1, 2$), therefore, by the uniqueness of the classical Doob decomposition,

$$\int_A \widetilde{M}_n^{(1)} d\nu = \int_A \widetilde{M}_n^{(2)} d\nu, \quad \int_A \widetilde{B}_n^{(1)} d\nu = \int_A \widetilde{B}_n^{(2)} d\nu.$$

Finally, since h is strictly monotone, this implies

$$\int_A M_n^{(1)} d\pi(\cdot, \varsigma) = \int_A M_n^{(2)} d\pi(\cdot, \varsigma), \quad \int_A B_n^{(1)} d\pi(\cdot, \varsigma) = \int_A B_n^{(2)} d\pi(\cdot, \varsigma),$$

for all $A \in \mathcal{F}_n$. Hence, the $\ast$ -fuzzy Doob inequality is unique up to $\mathcal{F}_n$ -integral equivalence. $\square$

Now, we will state and prove Doob's maximal inequality for the $\ast$ -fuzzy setting. We present it in the following theorem.

Theorem 3.2. Let $(X, \mathcal{F}, \pi, \ast)$ be a $\ast$ -fuzzy measure space, $\{\mathcal{F}_n\}_{n \geq 0}$ be a filtration, and $\{E_n\}_{n \geq 0}$ be a sequence of positive $\ast$ -fuzzy submartingales. Then for any $\varsigma, \lambda > 0$, we have the following inequality

$$\pi\left(\left\{\max_{1 \leq k \leq n} E_k \geq \lambda\right\}, \frac{\varsigma}{\lambda}\right) \geq \int_X E_n d\pi(\cdot, \varsigma). \quad (9)$$

Proof. Define

$$\tau := \min\{k \leq n : E_k \geq \lambda\}, \quad \tau(\emptyset) = n + 1,$$

and

$$A := \left\{\max_{1 \leq k \leq n} E_k \geq \lambda\right\}.$$

Note that $\{\tau = k\}$ is $\mathcal{F}_k$ -measurable (τ is a function of $x \in X$). We show that

$$A = \bigcup_{k=1}^n \{\tau = k\}.$$

Indeed, if $x \in A$, there are some $1 \leq k \leq n$ such that $E_k \geq \lambda$. Let k' be the smallest k . By definition $x \in \{\tau = k'\}$. So $x \in \bigcup_{k=1}^n \{\tau = k\}$. If $x \in \bigcup_{k=1}^n \{\tau = k\}$, there are some $1 \leq k \leq n$ such that $\tau = k$, or equivalently $\max_{1 \leq k \leq n} E_k \geq E_k \geq \lambda$, which implies $x \in A$. Now, using the assumption that $\{E_n\}_{n \geq 0}$ is a positive $\ast$ -fuzzy submartingale and the decreasing property of the $\ast$ -fuzzy integral, we obtain

$$\begin{aligned} \int_{\{\tau=k\}} \lambda d\pi(\cdot, \varsigma) &\geq \int_{\{\tau=k\}} E_k d\pi(\cdot, \varsigma) \\ &\geq \int_{\{\tau=k\}} E_n d\pi(\cdot, \varsigma), \end{aligned}$$

which implies

$$\pi\left(\{\tau = k\}, \frac{\varsigma}{\lambda}\right) \geq \int_{\{\tau=k\}} E_n d\pi(\cdot, \varsigma). \quad (10)$$

Note that $\{\tau = k\}$ with $1 \leq k \leq n$ are disjoint sets. From (10), we multiply over $k = 1, \dots, n$ (with $\ast$) and apply Lemma 1.12, and we obtain

$$\begin{aligned} \pi\left(A, \frac{\varsigma}{\lambda}\right) &\geq \int_A E_n d\pi(\cdot, \varsigma) \\ &\geq \int_X E_n d\pi(\cdot, \varsigma), \end{aligned}$$

since $A \subseteq X$ and the $\ast$ -fuzzy integral is decreasing in the domain. Therefore (9) is valid. This completes the proof. $\square$

Next, we proceed with the optional stopping theorem in the bounded case. First, we need the following proposition.

Proposition 3.3. Let $(X, \mathcal{F}, \pi, \ast)$ be a $\ast$ -fuzzy measure space, $\{\mathcal{F}_n\}_{n \geq 0}$ be a filtration, and $\{M_n\}_{n \geq 0}$ be a $\ast$ -fuzzy martingale with respect to $\{\mathcal{F}_n\}$. Let T represents a stopping time. Then, $\{M_n^T\}_{n \geq 0}$ is a $\ast$ -fuzzy martingale with respect to $\mathcal{F}_n$.

Proof. Recall

$$M_n^T := M_{T \wedge n},$$

where $T \wedge n = \min\{T, n\}$ and $\{x \in X : T(x) = n\} \in \mathcal{F}_n$, so M_n^T is $\mathcal{F}_n$ *-fuzzy measurable. We have to show that

$$\int_A M_{T \wedge (n+1)} d\pi(\cdot, \varsigma) = \int_A M_{T \wedge n} d\pi(\cdot, \varsigma), \quad \forall A \in \mathcal{F}_n, \quad \forall \varsigma > 0.$$

Let $\varsigma > 0$ and $A \in \mathcal{F}_n$ and note that,

$$A = (A \cap \{T \leq n\}) \cup (A \cap \{T \geq n+1\}).$$

From [Corollary 2.10](#) and since $\{M_n\}_{n \geq 0}$ is a *-fuzzy martingale, we obtain

$$\begin{aligned} \int_A M_{T \wedge (n+1)} d\pi(\cdot, \varsigma) &= \int_{(A \cap \{T \leq n\})} M_{T \wedge (n+1)} d\pi(\cdot, \varsigma) * \int_{(A \cap \{T \geq n+1\})} M_{T \wedge (n+1)} d\pi(\cdot, \varsigma) \\ &= \int_{(A \cap \{T \leq n\})} M_T d\pi(\cdot, \varsigma) * \int_{(A \cap \{T \geq n+1\})} M_{n+1} d\pi(\cdot, \varsigma) \\ &= \int_{(A \cap \{T \leq n\})} M_{T \wedge n} d\pi(\cdot, \varsigma) * \int_{(A \cap \{T \geq n+1\})} M_n d\pi(\cdot, \varsigma) \\ &= \int_{(A \cap \{T \leq n\})} M_{T \wedge n} d\pi(\cdot, \varsigma) * \int_{(A \cap \{T \geq n+1\})} M_{T \wedge n} d\pi(\cdot, \varsigma) \\ &= \int_A M_{T \wedge n} d\pi(\cdot, \varsigma). \end{aligned}$$

□

Theorem 3.4. Let $(X, \mathcal{F}, \pi, *)$ be a *-fuzzy measure space with strict generator h ($h(0) = +\infty$), $\{\mathcal{F}_n\}_{n \geq 0}$ be a filtration, and $\{M_n\}_{n \geq 0}$ be a *-fuzzy martingale, and T be a bounded stopping time. Then, for any $\varsigma > 0$, we have

$$\int_X M_T d\pi(\cdot, \varsigma) = \int_X M_0 d\pi(\cdot, \varsigma).$$

Proof. Recall from [Proposition 3.3](#) that for any $n \geq 0$ and $\varsigma > 0$,

$$\int_X M_{n+1}^T d\pi(\cdot, \varsigma) = \int_X M_n^T d\pi(\cdot, \varsigma).$$

By induction on $n \in \mathbb{N}$, we obtain

$$\int_X M_n^T d\pi(\cdot, \varsigma) = \int_X M_0^T d\pi(\cdot, \varsigma).$$

Note that $T \wedge 0 = 0$, so $M_0^T = M_0$. Since T is bounded, there exists $N \in \mathbb{N}$ such that $T \leq N$. Therefore, $T \wedge N = T$. Taking $n = N$, we obtain

$$\begin{aligned} \int_X M_0 d\pi(\cdot, \varsigma) &= \int_X M_N^T d\pi(\cdot, \varsigma) \\ &= \int_X M_{T \wedge N} d\pi(\cdot, \varsigma) \\ &= \int_X M_T d\pi(\cdot, \varsigma), \end{aligned}$$

for any $\varsigma > 0$. This completes the proof. □

Now, we proceed with the optional stopping theorem in the bounded case for *-fuzzy super martingales.

Corollary 3.5. Let $(X, \mathcal{F}, \pi, *)$ be a *-fuzzy measure space with strict generator h ($h(0) = +\infty$), $\{\mathcal{F}_n\}_{n \geq 0}$ be a filtration, and $\{X_n\}_{n \geq 0}$ be a *-fuzzy super martingale, and T be a bounded stopping time. Then, for every $\varsigma > 0$, we have

$$X_T \leq X_0 \quad \text{-fuzzy a.e.}$$

Proof. By [Theorem 3.1](#), there exists a *-fuzzy a.e. decomposition

$$\int_A M_n d\pi(\cdot, \varsigma) = \int_A X_n d\pi(\cdot, \varsigma) * \int_A B_n d\pi(\cdot, \varsigma), \quad \forall A \in \mathcal{F}_n, \quad \forall \varsigma > 0, \quad (11)$$

such that $\{M_n\}$ is a *-fuzzy martingale with respect to $\{\mathcal{F}_n\}_{n \geq 0}$, and $\{B_n\}_{n \geq 0}$ is predictable and decreasing (in the *-fuzzy integral sense) with $B_0 = 0$. Define

$$M_n^T(x) := M_{T \wedge n}(x),$$

where $T \wedge n = \min\{T, n\}$ and $\{x \in X : T(x) = n\} \in \mathcal{F}_n$, so M_n^T is $\mathcal{F}_n$ measurable. By Proposition 3.3, $\{M_n^T\}_{n \geq 0}$ is a $*$ -fuzzy martingale and by Theorem 3.4 for any $\varsigma > 0$, we have

$$\int_X M_T d\pi(\cdot, \varsigma) = \int_X M_0 d\pi(\cdot, \varsigma). \quad (12)$$

Take $A = X$. For any $\varsigma > 0$, we use (11) and we obtain

$$\begin{cases} \int_X M_T d\pi(\cdot, \varsigma) = \int_X X_T d\pi(\cdot, \varsigma) * \int_X B_T d\pi(\cdot, \varsigma), \\ \int_X M_0 d\pi(\cdot, \varsigma) = \int_X X_0 d\pi(\cdot, \varsigma) * \int_X B_0 d\pi(\cdot, \varsigma). \end{cases}$$

Now, we apply (12) and we obtain

$$\begin{aligned} \int_X X_0 d\pi(\cdot, \varsigma) &= \int_X X_T d\pi(\cdot, \varsigma) * \int_X B_T d\pi(\cdot, \varsigma) \\ &\leq \int_X X_T d\pi(\cdot, \varsigma), \end{aligned}$$

or equivalently

$$X_T \leq X_0 \quad * \text{-fuzzy a.e.}$$

This completes the proof. $\square$

Recall: $*$ -fuzzy integral is a decreasing operator. (see Corollary 1.7.)

Definition 3.6. A sequence $\{f_n\}$ in L_+ is uniformly $*$ -fuzzy integrable if

1. $\inf_{n \geq 0} \int_X f_n(x) d\pi(x, \varsigma) > 0$ for all $\varsigma > 0$;
2. For every $\epsilon > 0$, there exists $\delta > 0$ such that for all $A \in \mathcal{F}$ with $\pi(A, \varsigma) > 1 - \delta$ we have

$$\int_A f_n(x) d\pi(x, \varsigma) > 1 - \epsilon \quad \text{for all } n \geq 0.$$

Theorem 3.7. Let $(X, \mathcal{F}, \pi, *)$ be a $*$ -fuzzy measure space with strict continuous generator h , and fix $\varsigma > 0$. Let $\{M_n\}_{n \geq 0}$ be a $*$ -fuzzy martingale with respect to the filtration $\{\mathcal{F}_n\}_{n \geq 0}$. Assume that $\{M_n\}_{n \geq 0}$ is uniformly $*$ -fuzzy integrable. Let T be an unbounded stopping time with $\int_X M_T d\pi(\cdot, \varsigma) > 0$. Then,

$$\int_X M_T d\pi(\cdot, \varsigma) = \int_X M_0 d\pi(\cdot, \varsigma).$$

Proof. From Proposition 3.3, we know that $\{M_{T \wedge n}\}_{n \geq 0}$ is a $*$ -fuzzy martingale, where $T \wedge n = \min\{T, n\}$. Since $T \wedge n$ is bounded, Theorem 3.4 yields

$$\int_X M_{T \wedge n} d\pi(\cdot, \varsigma) = \int_X M_0 d\pi(\cdot, \varsigma), \quad \forall n \geq 0.$$

Note that $T \wedge n \uparrow T$ as $n \rightarrow \infty$, so

$$M_{T \wedge n} \longrightarrow M_T \quad * \text{-fuzzy a.e.}$$

Since $\{M_n\}_{n \geq 0}$ is uniformly $*$ -fuzzy integrable, the stopped sequence $\{M_{T \wedge n}\}_{n \geq 0}$ is also uniformly $*$ -fuzzy integrable. Fix $\varsigma > 0$. We apply Theorem 1.10 to the $*$ -fuzzy martingale $\{M_{T \wedge n}\}_{n \geq 0}$, and we obtain

$$\lim_{n \rightarrow \infty} \int_X M_{T \wedge n} d\pi(\cdot, \varsigma) = \int_X M_T d\pi(\cdot, \varsigma).$$

Since the left-hand side is equal to $\int_X M_0 d\pi(\cdot, \varsigma)$ for all $n \geq 0$, it follows that

$$\int_X M_T d\pi(\cdot, \varsigma) = \int_X M_0 d\pi(\cdot, \varsigma),$$

which completes the proof. $\square$

Corollary 3.8. Let $(X, \mathcal{F}, \pi, *)$ be a $*$ -fuzzy measure space with strict generator h ($h(0) = +\infty$), $\{\mathcal{F}_n\}_{n \geq 0}$ be a filtration, and $\{X_n\}_{n \geq 0}$ be a $*$ -fuzzy super martingale, and T be a unbounded stopping time. Then, for every $\varsigma > 0$, we have

$$X_T \leq X_0 \quad * \text{-fuzzy a.e.}$$

Proof. The proof is similar to that of Corollary 3.5. $\square$

Now, we state and prove the existence of integral limits for the uniformly $*$ -fuzzy integrable martingales.

Theorem 3.9. Let $(X, \mathcal{F}, \pi, *)$ be a $*$ -fuzzy measure space with strict continuous generator h , and fix $\varsigma > 0$. Let $\{M_n\}_{n \geq 0}$ be a uniformly $*$ -fuzzy integrable martingale with respect to filtration $\{\mathcal{F}_n\}_{n \geq 0}$. Then, there exists a $M \in L_+$ such that for any $A \in \mathcal{F}$ and $\varsigma > 0$,

$$\lim_{n \rightarrow \infty} \int_A M_n d\pi(\cdot, \varsigma) = \int_A M d\pi(\cdot, \varsigma).$$

Proof. Fix $\varsigma > 0$. For each $n \geq 0$, define the set function $\mu_n : \mathcal{F}_n \rightarrow [0, +\infty)$ by

$$\mu_n(A) := h\left(\int_A M_n d\pi(\cdot, \varsigma)\right), \quad A \in \mathcal{F}_n.$$

By Lemma 1.16 and the strict additivity of h , each μ_n is a finite classical measure on $\mathcal{F}$. Define a finite measure ν on $(X, \mathcal{F})$ by

$$\nu := \sum_{n=0}^{\infty} 2^{-(n+1)} \mu_n.$$

If $\nu = 0$, then from the fact that $\mu_n \geq 0$ for all $n \geq 0$, it immediately follows that $\mu_n = 0$ for all $n \geq 0$. Now, the Radon–Nikodým theorem yields nonnegative ν -integrable functions $Y_n \in L^1(\nu)$ such that

$$\mu_n(A) = \int_A Y_n d\nu, \quad \forall A \in \mathcal{F}, \quad \forall n \geq 0.$$

In particular, for any $A \in \mathcal{F}_n$,

$$h\left(\int_A M_n d\pi(\cdot, \varsigma)\right) = \mu_n(A) = \int_A Y_n d\nu.$$

Note that since $\{M_n\}_{n \geq 0}$ a $*$ -fuzzy martingale and h is strictly decreasing, $\{Y_n\}_{n \geq 0}$ is a classical martingale. We now show that the family $\{Y_n\}_{n \geq 0}$ is uniformly integrable in $L^1(\nu)$. By Definition 3.6 and the strict monotonicity of h , the quantities $\mu_n(X) = \int_X Y_n d\nu$ are uniformly bounded in n , so $\{Y_n\}$ is bounded in $L^1(\nu)$. Moreover, Definition 3.6 implies that for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $\nu(X \setminus A) < \delta$ we have

$$\mu_n(A) = h\left(\int_A M_n d\pi(\cdot, \varsigma)\right) = \int_A Y_n d\nu > \mu_n(X) - \varepsilon, \quad \forall n \geq 0.$$

Equivalently,

$$\int_{X \setminus A} Y_n d\nu < \varepsilon, \quad \forall n \geq 0, \quad \forall A \in \mathcal{F}_n.$$

Hence $\{Y_n\}$ is uniformly integrable. Now, by the classical martingale convergence theorem, there exists a random variable Y such that

$$Y_n \longrightarrow Y \quad \nu\text{-a.e.}, \quad \int_A Y_n d\nu \longrightarrow \int_A Y d\nu, \quad \forall A \in \mathcal{F},$$

or equivalently

$$\lim_{n \rightarrow +\infty} h\left(\int_A M_n d\pi(\cdot, \varsigma)\right) = \int_A Y d\nu < +\infty.$$

Since h is continuous, we obtain

$$\lim_{n \rightarrow +\infty} \int_A M_n d\pi(\cdot, \varsigma) = h^{-1}\left(\int_A Y d\nu\right) > 0. \quad (13)$$

From Lemma 1.16 (item 2), we know that $h^{-1}\left(\int_A Y d\nu\right)$ is a $*$ -fuzzy measure. Denote by $\pi_1(A, \varsigma) := h^{-1}\left(\int_A Y d\nu\right)$. If $\pi(A, \varsigma) = 1$ for some $A \in \mathcal{F}$, then by the definition of fuzzy null sets and from (13) we obtain

$$\lim_{n \rightarrow +\infty} \int_A M_n d\pi(\cdot, \varsigma) = 1 = h^{-1}\left(\int_A Y d\nu\right),$$

and we conclude $\pi_1(A, \varsigma) = 1$, which shows $\pi_1 \ll \pi$. Finally, by Corollary 1.22, we have a $M \in L_+$ such that

$$h^{-1}\left(\int_A Y d\nu\right) = \int_A M d\pi(\cdot, \varsigma), \quad \forall A \in \mathcal{F}, \quad \varsigma > 0,$$

implying

$$\lim_{n \rightarrow +\infty} \int_A M_n d\pi(\cdot, \varsigma) = \int_A M d\pi(\cdot, \varsigma), \quad \forall A \in \mathcal{F}, \quad \varsigma > 0.$$

This completes the proof. $\square$

4. Conclusion

In this paper, we developed a systematic martingale theory within $\ast$ -fuzzy measure spaces. $\ast$ -fuzzy conditional expectations and stopping times were introduced, allowing the formulation of martingale, submartingale, and supermartingale concepts in the fuzzy setting. We established $\ast$ -fuzzy versions of a Doob decomposition, maximal inequality, optional stopping theorems for both bounded and unbounded stopping times, and martingale convergence results. All results were formulated at the level of $\ast$ -fuzzy integrals, which naturally capture the nonlinear structure induced by triangular norms. Together, these results provide an integral-level extension of classical martingale theory to fuzzy measure spaces and lay the groundwork for further developments in fuzzy stochastic analysis.

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Mohammad Saeid Abolhassanifar: Writing – review & editing, Writing – original draft; **Reza Saadati:** Writing – review & editing, Supervision; **Mohammad Bagher Ghaemi:** Writing – review & editing; **Radko Mesiar:** Writing – review & editing, Supervision.

Data availability

No data was used for the research described in the article.

Declaration of competing interest

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