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Skorokhod's topologies on path space

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Abstract. Skorokhod's J1 and M1 topologies are standard tools in proving limit theorems for stochastic processes. Motivated by applications, we extend these topologies so that they are capable of describing the convergence of a sequence of functions with varying time domains. We view both discrete and continuous time within a unified framework, requiring only that the time domains be closed sets. We base our definitions on the point of view that the graph of a cadlag function can naturally be viewed as a compact set that is equipped with a total order. We construct two natural metrics on the space of totally ordered compact subsets of a given space and show that they generate the same topology. We introduce the concept of a *betweenness* to specify how a graph is drawn at jumps of a function. This allows us to treat the J1 topology and various variants of the M1 topology in a more general framework, which streamlines the existing theory. We show in general that the resulting spaces of paths are Polish and derive compactness criteria.

1. Introduction and main results

1.1. *Introduction.* A function is *cadlag* (from the French "continue à droite, limite à gauche") if it is right continuous with left limits. The classical paper Skorokhod (1956) introduced four topologies on the space of real valued cadlag functions on a compact interval, which he called J1, J2, M1, and M2. Skorokhod's definitions have later been generalised to functions defined on unbounded intervals and, for the J1 and J2 topologies, taking values in general metrisable spaces. When restricted to continuous functions, all four topologies reduce to the topology of locally uniform convergence. Of Skorokhod's four topologies, his J1 topology has proved to be the most natural in many situations, in particular, when discussing convergence of Markov processes (Ethier and Kurtz, 1986). For this reason, Skorokhod's J1 topology is now normally known as "the Skorokhod topology". Classical textbook discussions of the J1 topology can be found in Ethier and Kurtz (1986, Section 3.5) and Billingsley (1999, Section 12). All four topologies introduced by Skorokhod are discussed in Whitt (2002, Section 11.5).

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In some applications, such as excursion theory (Rogers, 1989) or the theory of the Brownian web (Fontes et al., 2004; Schertzer et al., 2016), one requires spaces whose elements are functions that are not all defined on the same time domain. The same issue appears when simply comparing discrete and continuous time processes, and is often worked around by specifying an interpolation or extrapolation method. In the theory of the Brownian web, a central object is a space of *continuous upward paths* that are continuous functions, each defined on an interval of the form $[s,\infty)$ with $s \in \overline{\mathbb{R}} := [-\infty,\infty]$ and taking values in $\overline{\mathbb{R}}$. This space was introduced by Fontes et al. (2004), who equipped it with a topology that can roughly be described as locally uniform convergence of functions together with convergence of their starting times. They proved that this space is Polish and derived a compactness criterion, both of which are important for proving weak convergence in law of random paths, in view of Prokhorov's theorem (Billingsley, 1999, Thm 5.2).

Inspired by such developments, we introduce the *path space* over a general metrisable space. Elements of this space are *paths*, which are pairs consisting of a closed subset of the real line (representing time) and a cadlag function that is defined on that subset and takes values in the metrisable space. We introduce the concept of a *betweenness* to specify how the graph of a function is drawn at jumps of the function, which allows us to treat the J1 topology and various variants of the M1 topology in a general framework. We prove that our topologies are Polish and derive compactness criteria.

Our work is motivated by collections of cadlag paths that approximate the Brownian web (Newman et al., 2005; Berestycki et al., 2015; Etheridge et al., 2017) and more recent work in which the limiting objects are collections of cadlag paths (Mountford et al., 2019; Freeman and Swart, 2024). In our follow-up paper Freeman and Swart (2026) we use the results of the present paper to derive tightness criteria for sequences of random compact sets of cadlag paths. In the remainder of Section 1 we state our main results. Proofs are postponed till Sections 2–4.

1.2. *Paths and cadlag functions.* Let $\overline{\mathbb{R}} := [-\infty,\infty]$ be the extended real line. For any metrisable topological space $\mathcal{X}$, we set

$$\mathcal{R}(\mathcal{X}) := (\mathcal{X}\times\mathbb{R}) \cup \bigl\{(*,-\infty),(*,\infty)\bigr\}. \tag{1.1}$$

We can think of $\mathcal{R}(\mathcal{X})$ as being obtained from the product space $\mathcal{X}\times\overline{\mathbb{R}}$ by squeezing the sets $\mathcal{X}\times\{-\infty\}$ and $\mathcal{X}\times\{+\infty\}$ into single points, which we denote by $(*,-\infty)$ and $(*,+\infty)$. For this reason, we call $\mathcal{R}(\mathcal{X})$ the *squeezed space* associated with $\mathcal{X}$. We will always equip $\mathcal{R}(\mathcal{X})$ with the topology described by the following lemma. We use the (nowadays fairly standard) convention of calling a topological space $\mathcal{X}$ *Polish* if $\mathcal{X}$ is separable and there exists a complete metric generating the topology on $\mathcal{X}$.

Lemma 1.1 (Squeezed space)**.** *Let $\mathcal{X}$ be a metrisable topological space. Then there exists a unique metrisable topology on $\mathcal{R}(\mathcal{X})$ such that a sequence $(x_n,t_n)\in\mathcal{R}(\mathcal{X})$ converges to a limit $(x,t)\in\mathcal{R}(\mathcal{X})$ if and only if:*

$$\text{(i)}\quad t_n\to t, \qquad \text{(ii)}\quad \text{if } t\in\mathbb{R}, \text{ then } x_n\to x.$$

If $\mathcal{X}$ is Polish, then so is $\mathcal{R}(\mathcal{X})$, and if $\mathcal{X}$ is compact, then so is $\mathcal{R}(\mathcal{X})$.

The squeezed space $\mathcal{R}(\overline{\mathbb{R}})$ plays an important role in the theory of the Brownian web, see Schertzer et al. (2016, Figure 6.2). The basic idea goes back to Fontes et al. (2004, formula (3.4)) which defines a metric generating the topology on $\mathcal{R}(\overline{\mathbb{R}})$.

We let $\Pi(\mathcal{X})$ denote the space of all pairs $(\pi,\preceq)$ where π is a nonempty compact subset of $\mathcal{R}(\mathcal{X})$ and $\preceq$ is a total order on π, such that

(i) $\pi^{\langle 2\rangle} := \bigl\{(z,z')\in\pi^2 : z\preceq z'\bigr\}$ is a closed subset of $\pi^2 := \pi\times\pi$, where we equip π with the induced topology from $\mathcal{R}(\mathcal{X})$ and π^2 with the product topology.
(ii) If $(x,s),(x',s')\in\pi$ satisfy $s<s'$, then $(x,s)\preceq(x',s')$.
(iii) For each $t\in\mathbb{R}$, the set $\{x\in\mathcal{X} : (x,t)\in\pi\}$ has at most two elements.

We call $\Pi(\mathcal{X})$ the *path space* and call its elements *paths*. We will often abuse notation and simply denote a path by π. We call the closed sets $I(\pi) \subset \mathbb{R}$ and $\hat{I}(\pi) \subset \overline{\mathbb{R}}$ defined as

$$I(\pi) := \hat{I}(\pi) \cap \mathbb{R} \quad \text{with} \quad \hat{I}(\pi) := \big\{t \in \overline{\mathbb{R}} : (x,t) \in \pi \text{ for some } x \in \mathcal{X} \cup \{*\}\big\} \tag{1.2}$$

the *domain* and *extended domain* of π, respectively. For $t \in I(\pi)$ we uniquely define $\pi(t-)$ and $\pi(t+)$ by

$$\big\{x \in \mathcal{X} : (x,t) \in \pi\big\} = \big\{\pi(t-), \pi(t+)\big\} \quad \text{with} \quad \pi(t-) \preceq \pi(t+) \quad (t \in I(\pi)). \tag{1.3}$$

If $\pm\infty \in \hat{I}(\pi)$, then we also set $\pi(\pm\infty) := *$. We call

$$\Pi_{\mathrm{c}}(\mathcal{X}) := \big\{\pi \in \Pi(\mathcal{X}) : \pi(t-) = \pi(t+)\ \forall t \in I(\pi)\big\}, \tag{1.4}$$

the space of *continuous* paths. For paths $\pi \in \Pi_{\mathrm{c}}$ we write $\pi(t) := \pi(t-) = \pi(t+)$ $(t \in I(\pi))$. We set

$$\Pi^{\mathrm{I}}(\mathcal{X}) := \big\{\pi \in \Pi(\mathcal{X}) : \hat{I}(\pi) \text{ is an interval}\big\} \tag{1.5}$$

and naturally write $\Pi^{\mathrm{I}}_{\mathrm{c}}(\mathcal{X}) := \Pi^{\mathrm{I}}(\mathcal{X}) \cap \Pi_{\mathrm{c}}(\mathcal{X})$.

The notation $\pi(t-)$ and $\pi(t+)$ is suggestive of left and right limits. Indeed, there is a close connection between paths and cadlag functions, as we now explain. We adopt the following definition. A *cadlag function* with values in a metrisable space $\mathcal{X}$ is a triple (I, f^-, f^+) such that $I \subset \mathbb{R}$ is a closed set, $f^-\colon I \to \mathcal{X}$ is left continuous, $f^+\colon I \to \mathcal{X}$ is right continuous, and

$$\begin{aligned} &\lim_{s\uparrow t} f^+(s) = f^-(t) \quad \forall t \in I^-, && \lim_{s\downarrow t} f^-(s) = f^+(t) \quad \forall t \in I^+, \\ &I^- := \big\{t \in I : (t-\varepsilon, t) \cap I \neq \emptyset\ \forall \varepsilon > 0\big\} && I^+ := \big\{t \in I : (t, t+\varepsilon) \cap I \neq \emptyset\ \forall \varepsilon > 0\big\}. \end{aligned} \tag{1.6}$$

Note that in the special case that $I = [0,\infty)$, this says that $f^+\colon I \to \mathcal{X}$ is right continuous with left limits and f^- is its left continuous modification, except that we allow for the case that $f^-(0) \neq f^+(0)$, that is, in our formulation a cadlag function defined on $I = [0,\infty)$ can jump at its initial time, contrary to the usual conventions. The following lemma explains the connection between cadlag functions and paths.

Lemma 1.2 (Cadlag functions)**.** *If $\pi \in \Pi(\mathcal{X})$ is a path, then setting $I := I(\pi)$ and $f^{\pm}(t) := \pi(t\pm)$ $(t \in I)$ defines a cadlag function (I, f^-, f^+). Conversely, if (I, f^-, f^+) is a cadlag function, then there exists a path $\pi \in \Pi(\mathcal{X})$ such that $I = I(\pi)$ and $f^{\pm}(t) = \pi(t\pm)$ $(t \in I)$. This path can be chosen such that $\{-\infty, \infty\} \subset \hat{I}(\pi)$, and with this choice, it is unique.*

Lemma 1.2 can roughly be described by saying that paths are the graphs of cadlag functions. The total order $\preceq$ on a path π plays a crucial role here. Indeed, the set $\pi \subset \mathcal{R}(\mathbb{R})$ defined as $\pi := \{(0,t) : -1 \le t \le 1\} \cup \{(1,0)\}$ is compact and satisfies condition (iii) of our definition of a path, but it is not possible to equip π with a total order satisfying conditions (i) and (ii) and indeed π is not the graph of a cadlag function.

Let $I \subset \mathbb{R}$ be a closed real interval of positive length and let ∂I be the set of its finite endpoints. Then we let $\mathcal{D}_I(\mathcal{X})$ denote the space of pairs (f^-, f^+) such that (I, f^-, f^+) is a cadlag function with values in $\mathcal{X}$ and $f^-(t) = f^+(t)$ for all $t \in \partial I$. By Lemma 1.2, we have the natural identification

$$\mathcal{D}_I(\mathcal{X}) \cong \big\{\pi \in \Pi(\mathcal{X}) : \hat{I}(\pi) = I \cup \{-\infty, \infty\},\ \pi(t-) = \pi(t+)\ \forall t \in \partial I\big\}. \tag{1.7}$$

An element (f^-, f^+) of $\mathcal{D}_I(\mathcal{X})$ is uniquely determined by either of the functions f^- or f^+. The convention is to choose f^+. Thus, we can also naturally identify $\mathcal{D}_I(\mathcal{X})$ with the space of right continuous functions $f\colon I \to \mathcal{X}$ such that

$$\begin{gathered} f(t-) := \lim_{s\uparrow t} f(s) \text{ exists for all } t \in I\backslash\{l\} \quad \text{and} \quad f(r-) = f(r) \quad \text{if } r < \infty, \\ \text{where} \quad l := \inf I \text{ and } r := \sup I. \end{gathered} \tag{1.8}$$

1.3. *Skorokhod's J1 topology on path space.* A *correspondence* between two sets A, B is a surjective multi-valued function from A to B. More precisely, a correspondence is a set $C \subset A \times B$ such that

$$\forall a \in A\ \exists b \in B \text{ s.t. } (a,b) \in C \quad \text{and} \quad \forall b \in B\ \exists a \in A \text{ s.t. } (a,b) \in C. \tag{1.9}$$

We let $\mathrm{Corr}(A,B)$ denote the set of all correspondences between A and B. If $\mathcal{X}$ is a metrisable topological space, then we let $\mathcal{K}_+(\mathcal{X})$ denote the space of nonempty compact subsets of $\mathcal{X}$. If d is a metric generating the topology on $\mathcal{X}$, then the corresponding *Hausdorff metric* on $\mathcal{K}_+(\mathcal{X})$ is defined as

$$d_{\mathrm{H}}(K_1,K_2) := \sup_{z_1\in K_1} d(z_1,K_2) \vee \sup_{z_2\in K_2} d(z_2,K_1) = \inf_{C\in\mathrm{Corr}(K_1,K_2)} \sup_{(z_1,z_2)\in C} d(z_1,z_2), \tag{1.10}$$

where as usual $d(z,K) := \inf_{z'\in K} d(z,z')$ denotes the distance of a point to a set. The topology generated by d_{H} is known as the *Hausdorff topology* on $\mathcal{K}_+(\mathcal{X})$. It coincides with the induced topology from the embedding $\mathcal{K}_+(\mathcal{X}) \subset \mathrm{Clos}(\mathcal{X})$, where $\mathrm{Clos}(\mathcal{X})$ is the space of closed subsets of $\mathcal{X}$ equipped with the well-known Vietoris topology (Srivastava, 1998, P2.4.14). The following facts are proved in Schertzer et al. (2014, Lemmas B.1–B.3).

Lemma 1.3 (The Hausdorff topology)**.** *The topology on $\mathcal{K}_+(\mathcal{X})$ generated by d_{H} only depends on the topology on $\mathcal{X}$ and not on the choice of the metric d. The space $\mathcal{K}_+(\mathcal{X})$ is Polish if $\mathcal{X}$ is Polish and compact if $\mathcal{X}$ is compact.*

We fix a metric d_{sqz} generating the topology on the squeezed space $\mathcal{R}(\mathcal{X})$ and let

$$d_{\mathrm{H}}(\pi_1,\pi_2) := \inf_{C\in\mathrm{Corr}(\pi_1,\pi_2)} \sup_{(z_1,z_2)\in C} d_{\mathrm{sqz}}(z_1,z_2) \qquad \big(\pi_1,\pi_2 \in \Pi(\mathcal{X})\big) \tag{1.11}$$

denote the corresponding distance in Hausdorff metric between two paths $\pi_1, \pi_2 \in \Pi(\mathcal{X})$. Then d_{H} is a metric on the space $\Pi_{\mathrm{c}}(\mathcal{X})$ of continuous paths defined in (1.4), but only a pseudometric on the larger space $\Pi(\mathcal{X})$, since in general to characterise a path one needs both the set π and the total order $\preceq$ on π. There are two natural ways to fix this, which we describe now.

For paths $\pi_1, \pi_2 \in \Pi(\mathcal{X})$, we let $\mathrm{Cor}_+(\pi_1,\pi_2)$ denote the set of correspondences C between π_1 and π_2 that are *monotone* in the sense that

$$\text{there are no } (z_1,z_2),(z_1',z_2') \in C \text{ such that } z_1 \prec_1 z_1' \text{ and } z_2' \prec_2 z_2, \tag{1.12}$$

where $z \prec z'$ means $z \preceq z'$ and $z \neq z'$. If d_{sqz} is a metric generating the topology on the squeezed space $\mathcal{R}(\mathcal{X})$, then we define a corresponding distance d_{tot} on the path space $\Pi(\mathcal{X})$ by

$$d_{\mathrm{tot}}(\pi_1,\pi_2) := \inf_{C\in\mathrm{Cor}_+(\pi_1,\pi_2)} \sup_{(z_1,z_2)\in C} d_{\mathrm{sqz}}(z_1,z_2) \qquad \big(\pi_1,\pi_2 \in \Pi(\mathcal{X})\big), \tag{1.13}$$

which differs from (1.11) only in the fact that we restrict the infimum to monotone correspondences. We will shortly see that d_{tot} is, indeed, a metric.

Another natural way to define a metric on path space is as follows. First, we define a metric d_{sqz}^2 generating the product topology on $\mathcal{R}(\mathcal{X})^2 := \mathcal{R}(\mathcal{X}) \times \mathcal{R}(\mathcal{X})$ by

$$d_{\mathrm{sqz}}^2\big((z_1,z_1'),(z_2,z_2')\big) := d_{\mathrm{sqz}}(z_1,z_2) \vee d_{\mathrm{sqz}}(z_1',z_2') \qquad \big(z_1,z_1',z_2,z_2' \in \mathcal{R}(\mathcal{X})\big), \tag{1.14}$$

and we let d_{H}^2 denote the corresponding Hausdorff metric on $\mathcal{K}_+(\mathcal{R}(\mathcal{X})^2)$. It is then easy to see that we can define a metric d_{part} on the path space $\Pi(\mathcal{X})$ by setting

$$d_{\mathrm{part}}(\pi_1,\pi_2) := d_{\mathrm{H}}^2(\pi_1^{\langle 2\rangle},\pi_2^{\langle 2\rangle}) \qquad \big(\pi_1,\pi_2 \in \Pi(\mathcal{X})\big), \tag{1.15}$$

where $\pi^{\langle 2\rangle}$ is defined below Lemma 1.1. Below is our first main result. We recall that the statement that $\Pi(\mathcal{X})$ is Polish merely means that $\Pi(\mathcal{X})$ is separable and there exists a complete metric generating the topology. This does not imply that the metrics d_{tot} and d_{part} are complete, which, indeed, they are not.

Theorem 1.4 (Skorokhod's J1 topology on path space)**.** *Let $\mathcal{X}$ be a metrisable space and let d_{H}, d_{tot}, and d_{part} be defined as in (1.11), (1.13), and (1.15) in terms of some metric d_{sqz} on the squeezed space $\mathcal{R}(\mathcal{X})$. Then*

$$d_{\mathrm{H}}(\pi_1,\pi_2) \le d_{\mathrm{part}}(\pi_1,\pi_2) \le d_{\mathrm{tot}}(\pi_1,\pi_2) \qquad \big(\pi_1,\pi_2 \in \Pi(\mathcal{X})\big). \tag{1.16}$$

The functions d_{tot} and d_{part} are metrics on $\Pi(\mathcal{X})$ that generate the same topology. This topology does not depend on the choice of the metric d_{sqz} on the squeezed space $\mathcal{R}(\mathcal{X})$. On the space $\Pi_{\mathrm{c}}(\mathcal{X})$, all three metrics d_{H}, d_{tot}, and d_{part} generate the same topology. The space $\Pi_{\mathrm{c}}^{|}(\mathcal{X})$ defined in (1.5) is a closed subset of $\Pi(\mathcal{X})$. If $\mathcal{X}$ is Polish, then so are $\Pi(\mathcal{X})$, $\Pi_{\mathrm{c}}(\mathcal{X})$, and the space $\mathcal{D}_I(\mathcal{X})$ defined in (1.7). On $\mathcal{D}_I(\mathcal{X})$, the metrics d_{tot} and d_{part} generate Skorokhod's J1 topology while d_{H} is a metric that generates the J2 topology.

We call the topology on $\Pi(\mathcal{X})$ generated by d_{tot} or d_{part} the *J1 topology* on $\Pi(\mathcal{X})$. We will derive a compactness criterion for this topology. Before we do this, we discuss a more general class of topologies on path space which includes (a generalisation of) Skorokhod's M1 topology. This will allow us to formulate a compactness criterion that applies to all these topologies at once.

1.4. *M1 and other topologies.* There are great similarities between Skorokhod's J1 and M1 topologies. In his Ph.D. thesis, Pomarede (1976) treated both in a unified way based upon parametric representations of the graphs of functions, where in case of the M1 topology for each jump of a function the corresponding gap in the graph is filled in. We will not use parametric representations but otherwise our approach is related, though a bit more general. We will introduce a concept that we call "betweenness" to specify to what degree the gaps in a graph due to jumps of the function should be filled in. The reader wishing to concentrate on J1 may just read the first few definitions until the trivial betweenness is introduced and then skip to the next subsection.

The concept we call "betweenness" is a natural one and it seems conceivable that it may have been invented in other contexts, but we have not been able to find a reference. If $\mathcal{X}$ is a set, then we define a *betweenness* on $\mathcal{X}$ to be a function that assigns to each pair x, z of elements of $\mathcal{X}$ a subset $\langle x,z\rangle$ of $\mathcal{X}$, such that the following axioms hold for all $x,y,z \in \mathcal{X}$:

(i) $\langle x,z\rangle = \langle z,x\rangle$,
(ii) $x \in \langle x,z\rangle$,
(iii) $y \in \langle x,z\rangle \;\Rightarrow\; \langle x,y\rangle \cap \langle y,z\rangle = \{y\}$,
(iv) $y \in \langle x,z\rangle \;\Rightarrow\; \langle x,y\rangle \cup \langle y,z\rangle = \langle x,z\rangle$.

If $y \in \langle x,z\rangle$, then we say that y lies *between* x and z. We call $\langle x,z\rangle$ the *segment* with *endpoints* x and z. The following lemma says that segments of a betweenness are equipped with a natural total order.

Lemma 1.5 (Total order on segments)**.** *For any betweenness on $\mathcal{X}$, for each $x,z \in \mathcal{X}$ and $y,y' \in \langle x,z\rangle$, one has*

$$\langle x,y\rangle \subset \langle x,y'\rangle \;\Leftrightarrow\; y \in \langle x,y'\rangle \;\Leftrightarrow\; y' \in \langle y,z\rangle \;\Leftrightarrow\; \langle y,z\rangle \supset \langle y',z\rangle. \tag{1.17}$$

Setting $y \le_{x,z} y'$ if any of these equivalent conditions holds defines a total order on $\langle x,z\rangle$.

We next give some examples of betweennesses. For any set $\mathcal{X}$, it is straightforward to check that setting $\langle x,z\rangle := \{x,z\}$ defines a betweenness. We call this the *trivial betweenness*. If $\mathcal{X}$ is a linear space, then it is easy to see that

$$\langle x,z\rangle := \big\{(1-p)x + pz : p \in [0,1]\big\} \qquad (x,z \in \mathcal{X}) \tag{1.18}$$

defines a betweenness on $\mathcal{X}$. We call this the *linear betweenness*. If $(\mathcal{X},\le)$ is a totally ordered space, then one can check that setting

$$\langle x,z\rangle := \big\{y \in \mathcal{X} : x \le y \le z \text{ or } z \le y \le x\big\} \qquad (x,z \in \mathcal{X}) \tag{1.19}$$

defines a betweenness on $\mathcal{X}$. We call this the *order betweenness*. If $(\mathcal{X}, d)$ is a metric space, then we recall that a *geodesic* in $(\mathcal{X}, d)$ is a subset Γ of $\mathcal{X}$ that is isometric to a compact real interval, i.e., there exists an isometry $\gamma\colon [s,u] \to \mathcal{X}$ (with $s, u \in \mathbb{R},\ s \le u$) such that Γ is the image of $[s,u]$ under γ. Clearly, γ is uniquely determined by Γ up to translations and mirror images of the interval $[s,u]$. The points $\gamma(s), \gamma(u)$ are called the *endpoints* of the geodesic Γ. We say that a metric space has *unique geodesics* if for each $x, z \in \mathcal{X}$, there exists a unique geodesic Γ with endpoints x, z. We call the betweenness defined in the following lemma the *geodesic betweenness*.

Lemma 1.6 (Geodesic betweenness)**.** *Let $(\mathcal{X}, d)$ be a metric space with unique geodesics. Then letting $\langle x, z\rangle$ denote the unique geodesic with endpoints x, z defines a betweenness on $\mathcal{X}$.*

As an example of spaces without a linear structure where Lemma 1.6 is applicable we mention d-dimensional hyperbolic space, and real-trees (Dress and Terhalle, 1996). If $\mathcal{X}$ is a metrisable space, then we say that a betweenness on $\mathcal{X}$ is *proper* if $\langle x, z\rangle$ is compact for each $x, z \in \mathcal{X}$, and the map $\mathcal{X}^2 \ni (x, z) \mapsto \langle x, z\rangle \in \mathcal{K}_+(\mathcal{X})$ is continuous with respect to the product topology on $\mathcal{X}^2$ and the Hausdorff topology on $\mathcal{K}_+(\mathcal{X})$. We recall that a metric space is called *proper* if for each $x \in \mathcal{X}$ and $r \ge 0$, the closed ball $\{y \in \mathcal{X} : d(x,y) \le r\}$ is a compact subset of $\mathcal{X}$.

Lemma 1.7 (Proper betweennesses)**.** *The following betweennesses are proper:*

(i) *The trivial betweenness on any metrisable space $\mathcal{X}$,*
(ii) *The linear betweenness on any normed linear space $\mathcal{X}$,*
(iii) *The order betweenness on any closed subset $\mathcal{X}$ of $\mathbb{R}$,*
(iv) *The geodesic betweenness on any proper metric space $\mathcal{X}$.*

We do not know if the assumption in part (iv) of Lemma 1.7 that $\mathcal{X}$ be proper is necessary, but it is certainly sufficient.

If $\mathcal{X}$ is a metrisable space that is equipped with a betweenness, then for each path $\pi \in \Pi(\mathcal{X})$ we define a *filled path* by

$$\overline{\pi} := \big\{(x,t) : t \in I(\pi),\ x \in \langle \pi(t-), \pi(t+)\rangle\big\} \cup \big\{(*,t) : t \in \hat{I}(\pi)\backslash I(\pi)\big\}. \tag{1.20}$$

The intuition of (1.20) is that the betweenness specifies which parts of the space the path π should be regarded as passing through, during its instantaneous jumps. Note that for the trivial betweenness, $\overline{\pi} = \pi$. Also, for continuous paths $\pi \in \Pi_{\mathrm{c}}(\mathcal{X})$ one has $\overline{\pi} = \pi$ regardless of the choice of the betweenness. We equip $\overline{\pi}$ with the total order defined by setting $(x,s) \preceq (y,t)$ if either $s < t$ and x, y are arbitrary, or $s = t \in \mathbb{R}$ and $x \le_{\pi(t-),\pi(t+)} y$, where $\le_{\pi(t-),\pi(t+)}$ is the total order on the segment $\langle \pi(t-), \pi(t+)\rangle$ defined in Lemma 1.5. Informally, the total order $\preceq$ corresponds to the direction of time. We need the following simple lemma.

Lemma 1.8 (Compatibility with the topology)**.** *Let $\mathcal{X}$ be a metrisable space that is equipped with a proper betweenness. Then for each $\pi \in \Pi(\mathcal{X})$, the set $\overline{\pi}^{\langle 2\rangle} := \big\{(z,z') \in \overline{\pi}^2 : z \preceq z'\big\}$ is a closed subset of $\overline{\pi}^2$.*

It is easy to see that a path π is uniquely determined by the corresponding filled path $\overline{\pi}$. We generalise (1.11) by setting

$$d_{\mathrm{H}}(\pi_1, \pi_2) := \inf_{C \in \mathrm{Corr}(\overline{\pi}_1, \overline{\pi}_2)} \sup_{(z_1,z_2)\in C} d_{\mathrm{sqz}}(z_1, z_2). \tag{1.21}$$

In general, this is only a pseudometric. We define monotone correspondences between filled paths in the same way as for paths. Generalising (1.13) and (1.15), we define metrics on $\Pi(\mathcal{X})$ by

$$\begin{aligned} d_{\mathrm{tot}}(\pi_1, \pi_2) &:= \inf_{C \in \mathrm{Cor}_+(\overline{\pi}_1, \overline{\pi}_2)} \sup_{(z_1,z_2)\in C} d_{\mathrm{sqz}}(z_1, z_2), \\ d_{\mathrm{part}}(\pi_1, \pi_2) &:= d_{\mathrm{H}}^2(\overline{\pi}_1^{\langle 2\rangle}, \overline{\pi}_2^{\langle 2\rangle}). \end{aligned} \tag{1.22}$$

In particular, for the trivial betweenness, these equations reduce to precisely the J1 metrics (1.13) and (1.15). The following theorem is our second main result, which extends the major parts of Theorem 1.4 to general betweennesses.

Theorem 1.9 (Skorokhod-type topologies on path space)**.** *Assume that $\mathcal{X}$ is a metrisable space that is equipped with a proper betweenness. Let d_{H}, d_{tot}, and d_{part} be defined as in (1.21) and (1.22). Then*

$$d_{\mathrm{H}}(\pi_1,\pi_2) \le d_{\mathrm{part}}(\pi_1,\pi_2) \le d_{\mathrm{tot}}(\pi_1,\pi_2) \qquad \big(\pi_1,\pi_2 \in \Pi(\mathcal{X})\big). \tag{1.23}$$

The functions d_{tot} and d_{part} are metrics on $\Pi(\mathcal{X})$ that generate the same topology. This topology does not depend on the choice of the metric d_{sqz} on the squeezed space $\mathcal{R}(\mathcal{X})$. If $\mathcal{X}$ is Polish, then so is $\Pi(\mathcal{X})$ and the same is true for the space $\mathcal{D}_I(\mathcal{X})$ from (1.7). In the special case that $\mathcal{X} = \mathbb{R}$ equipped with the linear betweenness, the metrics d_{tot} and d_{part} generate Skorokhod's M1 topology on $\mathcal{D}_I(\mathbb{R})$ while d_{H} is a metric that generates the M2 topology.

In general, we call the topology generated by d_{tot} or d_{part} from (1.22) the *Skorokhod topology* corresponding to the betweenness on $\mathcal{X}$. Choosing the trivial betweenness then yields Skorokhod's J1 topology while the linear betweenness on $\mathbb{R}$ yields Skorokhod's M1 topology. Various ways to extend the M1 topology to more general spaces than $\mathcal{X} = \mathbb{R}$ have been proposed in the literature, not all of which are covered by Theorem 1.9. In particular, it is natural to identify $\mathcal{D}_I(\mathbb{R}^n)$ with the product space $\mathcal{D}_I(\mathbb{R})^n$, and to equip $\mathcal{D}_I(\mathbb{R})$ with the M1 topology and $\mathcal{D}_I(\mathbb{R})^n$ with the product topology, but the topology one obtains in this way does not come from a betweenness. In view of this, we will reserve the term "M1 topology" for the one-dimensional setting only. We note that the Skorokhod topology corresponding to the linear betweenness on a Banach space $\mathcal{X}$ is called the *strong M1 topology* in Whitt (2002) and Pavlyukevich and Riedle (2015). See also Ledger (2016) for an infinite dimensional version of the M1 topology.

The following lemma implies that the J1 topology is stronger than any other Skorokhod topology, defined by a betweenness other than the trivial betweenness. As a special case we recover that convergence in the J1 topology implies convergence in the M1 topology.

Lemma 1.10 (Comparison of Skorokhod topologies)**.** *Let $\mathcal{X}$ be a metrisable space on which there are defined two proper betweennesses $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle'$ such that $\langle x,z\rangle \subset \langle x,z\rangle'$ for all $x,z \in \mathcal{X}$. Then the Skorokhod topology defined by the first betweenness is stronger than the Skorokhod topology defined by the second betweenness.*

1.5. *Compactness criteria.* In this subsection, we give a compactness criterion for subsets of the path space $\Pi(\mathcal{X})$ equipped with the Skorokhod topology corresponding to a betweenness. The criterion is similar in structure to well known results for functions with fixed time domains. We first state an analogous compactness criterion for sets of continuous paths (Fontes et al., 2004; Schertzer et al., 2016). Let $(\mathcal{X},d)$ be a metric space. We say that a set $\mathcal{A} \subset \Pi(\mathcal{X})$ satisfies the *compact containment condition* if

$$\forall T<\infty\ \exists \text{ compact } C \subset \mathcal{X} \text{ s.t. } \pi(t\pm) \in C\ \forall \pi \in \mathcal{A} \text{ and } t \in I(\pi) \cap [-T,T]. \tag{1.24}$$

For each $0 < T < \infty$ and $\delta > 0$, we define the (traditional) *modulus of continuity* of a continuous path $\pi \in \Pi_{\mathrm{c}}(\mathcal{X})$ as

$$m_{T,\delta}(\pi) := \sup\big\{d\big(\pi(t_1),\pi(t_2)\big) : t_1,t_2 \in I(\pi),\ -T \le t_1 < t_2 \le T,\ t_2 - t_1 \le \delta\big\}. \tag{1.25}$$

We say that a set $\mathcal{A} \subset \Pi_{\mathrm{c}}(\mathcal{X})$ is *equicontinuous* if

$$\lim_{\delta\to 0} \sup_{\pi\in\mathcal{A}} m_{T,\delta}(\pi) = 0 \qquad \forall T<\infty. \tag{1.26}$$

The following theorem generalises the classical Arzelà–Ascoli theorem to sets of functions that are not necessarily all defined on the same domain, which moreover does not need to be an interval.

Theorem 1.11 (Arzelà–Ascoli). *Let $\mathcal{X}$ be a metric space. Then a set $\mathcal{A} \subset \Pi_{\mathrm{c}}(\mathcal{X})$ is precompact if and only if it is equicontinuous and satisfies the compact containment condition.*

For paths with jumps, it is possible to give a very similar compactness criterion. We first need to introduce a bit of notation. For any $I \subset \mathbb{R}$, we let

$$I_{\mathfrak{s}} := \bigl\{t\star : t \in I,\ \star \in \{-,+\}\bigr\}. \tag{1.27}$$

denote the set that consists of all words of the form $t\star$ with $t \in I$ and $\star \in \{-,+\}$. We think of $I_{\mathfrak{s}}$ as obtained by cutting each point of I into two. Consequently, we call $\mathbb{R}_{\mathfrak{s}}$ the *split real line* and call elements of $\mathbb{R}_{\mathfrak{s}}$ *split real numbers*. We denote split real numbers either by words $t\star$ consisting of a Roman letter and a sign, or by a single Greek letter. In this case, if $\tau = t\star$, then we call $\underline{\tau} := t$ the *real part* of τ and we call $\mathfrak{s}(\tau) := \star$ its *sign*. We equip $\mathbb{R}_{\mathfrak{s}}$ with the lexicographic order, i.e., we set $\sigma \leq \tau$ if and only if either $\underline{\sigma} < \underline{\tau}$ or $\underline{\sigma} = \underline{\tau}$ and $\mathfrak{s}(\sigma) \leq \mathfrak{s}(\tau)$, where $\{-,+\}$ is equipped with the natural total order in which $- \leq +$. We write $\sigma < \tau$ if $\sigma \leq \tau$ and $\sigma \neq \tau$. Although we do not need this here, we note that Kolmogorov (1956) observed that it is possible to equip $\mathbb{R}_{\mathfrak{s}}$ with a topology such that continuous functions on $\mathbb{R}_{\mathfrak{s}}$ correspond to cadlag functions on $\mathbb{R}$, see Subsection 2.1 below.

Let $\mathcal{X}$ be a metrisable space that is equipped with a proper betweenness. For any path $\pi \in \Pi(\mathcal{X})$ with domain $I(\pi)$ we define $I_{\mathfrak{s}}(\pi) \subset \mathbb{R}_{\mathfrak{s}}$ as in (1.27). Using the notation for split real numbers we have just introduced, for each $0 < T < \infty$ and $\delta > 0$, we define the *Skorokhod modulus of continuity* as

$$\begin{aligned} m^{\mathrm{S}}_{T,\delta}(\pi) := \sup\bigl\{d\bigl(\pi(\tau_2), \langle \pi(\tau_1), \pi(\tau_3)\rangle\bigr) : \tau_1,\tau_2,\tau_3 \in I_{\mathfrak{s}}(\pi),\ \tau_1 \leq \tau_2 \leq \tau_3,\\ -T \leq \underline{\tau}_1,\ \underline{\tau}_3 \leq T,\ \underline{\tau}_3 - \underline{\tau}_1 \leq \delta\bigr\}, \end{aligned} \tag{1.28}$$

where as before $d(x,A)$ denotes the distance of a point x to a set A and $\langle x,y\rangle$ is the segment with endpoints x and y. We say that a set $\mathcal{A} \subset \Pi(\mathcal{X})$ is *Skorokhod-equicontinuous* if

$$\lim_{\delta\to 0} \sup_{\pi\in\mathcal{A}} m^{\mathrm{S}}_{T,\delta}(\pi) = 0 \qquad \forall T < \infty. \tag{1.29}$$

Specialising these definitions to the trivial betweenness, for which $\langle \pi(\tau_1), \pi(\tau_3)\rangle = \{\pi(\tau_1), \pi(\tau_3)\}$, yields the definitions of the *J1-modulus of continuity* and *J1-equicontinuity*. If $\mathcal{X} = \mathbb{R}$, equipped with the linear betweenness, then we speak of the *M1-modulus of continuity* and *M1-equicontinuity*.

Theorem 1.12 (Compactness criterion). *Let $(\mathcal{X},d)$ be a metric space that is equipped with a proper betweenness. Then a set $\mathcal{A} \subset \Pi(\mathcal{X})$ is precompact in the Skorokhod topology corresponding to this betweenness if and only if it is Skorokhod-equicontinuous and satisfies the compact containment condition.*

For functions $f \in \mathcal{D}_I(\mathcal{X})$ as defined in (1.8), one can check that the Skorokhod modulus of continuity can more simply be defined as

$$\begin{aligned} m^{\mathrm{S}}_{T,\delta}(f) = \sup\bigl\{d\bigl(f(t_2), \langle f(t_1), f(t_3)\rangle\bigr) : t_1,t_2,t_3 \in I,\ t_1 \leq t_2 \leq t_3,\\ -T \leq t_1,\ t_3 \leq T,\ t_3 - t_1 \leq \delta\bigr\}. \end{aligned} \tag{1.30}$$

As a fairly simple consequence of Theorem 1.12, we can derive the following compactness criterion for $\mathcal{D}_I(\mathcal{X})$. As before ∂I denotes the set of finite endpoints of a closed real interval I.

Theorem 1.13 (Functions on a fixed interval). *Let $(\mathcal{X},d)$ be a metric space that is equipped with a proper betweenness and let I be a closed real interval of positive length. Then a set $\mathcal{F} \subset \mathcal{D}_I(\mathcal{X})$ is precompact in the Skorokhod topology corresponding to the betweenness if and only if:*

(i) *the compact containment condition holds,*

(ii) $\lim_{\delta\to 0} \sup_{f\in\mathcal{F}} m^{\mathrm{S}}_{T,\delta}(f) = 0$ *for all* $T < \infty$,

(iii) $\lim_{\delta\to 0} \sup_{f\in\mathcal{F}} \sup\bigl\{d\bigl(f(s), f(t)\bigr) : s \in I,\ |s-t| \leq \delta\bigr\} = 0$ *for all* $t \in \partial I$.

Note that compared to Theorem 1.12, we need the extra condition (iii) to rule out the scenario that a sequence of functions in $\mathcal{F}$ converges to a function with a discontinuity at a time $t \in \partial I$. For the J1 topology on $\mathcal{D}_{[0,1]}$, Theorem 1.13 was first proved in Kolmogorov (1956, Thm IV). The analogous statement for the M1 topology was proved in Skorokhod (1956, 2.7.3).

1.6. *The ordered Hausdorff topology.* The metrics d_{tot} and d_{part} defined first in (1.13) and (1.15) and then more generally in (1.22) are special cases of two metrics that can be defined on an even more general space of totally ordered compact sets. In the present subsection, we investigate the topology generated by d_{tot} and d_{part} on this general space. The results in this subsection are more theoretical in nature and can be skipped by readers who are interested in cadlag functions only.

Throughout this subsection $\mathcal{X}$ is a metrisable space. We let $\mathcal{K}_{\mathrm{part}}(\mathcal{X})$ denote the space of all pairs $(K, \preceq)$ where K is a nonempty compact subset of $\mathcal{X}$ and $\preceq$ is a partial order on K that is *compatible with the topology* in the sense that the set

$$K^{\langle 2\rangle} := \big\{(x, y) \in K^2 : x \preceq y\big\} \tag{1.31}$$

is a closed subset of K^2, where we equip K with the induced topology from $\mathcal{K}$ and K^2 with the product topology. We let $\mathcal{K}_{\mathrm{tot}}(\mathcal{X})$ denote the space of pairs $(K, \preceq) \in \mathcal{K}_{\mathrm{part}}(\mathcal{X})$ for which $\preceq$ is a total order on K. We will sometimes be sloppy and denote elements of $\mathcal{K}_{\mathrm{part}}(\mathcal{X})$ simply as K. We equip the space $\mathcal{X}^2$ with the metric

$$d^2\big((x_1, y_1), (x_2, y_2)\big) := d(x_1, x_2) \vee d(y_1, y_2), \tag{1.32}$$

which generates the product topology and let d_{H}^2 denote the associated Hausdorff metric on $\mathcal{K}_+(\mathcal{X}^2)$. An element $(K, \preceq)$ of $\mathcal{K}_{\mathrm{part}}(\mathcal{X})$ is clearly uniquely determined by the compact set $K^{\langle 2\rangle} \subset \mathcal{X}^2$ defined in (1.31), so setting

$$d_{\mathrm{part}}(K_1, K_2) := d_{\mathrm{H}}^2(K_1^{\langle 2\rangle}, K_2^{\langle 2\rangle}) \qquad \big(K_1, K_2 \in \mathcal{K}_{\mathrm{part}}(\mathcal{X})\big) \tag{1.33}$$

defines a metric d_{part} on $\mathcal{K}_{\mathrm{part}}(\mathcal{X})$. For $K_1, K_2 \in \mathcal{K}_{\mathrm{tot}}(\mathcal{X})$, we let $\mathrm{Cor}_+(K_1, K_2)$ denote the set of monotone correspondences between K_1 and K_2, defined as in (1.12), and define a metric d_{tot} on $\mathcal{K}_{\mathrm{tot}}(\mathcal{X})$ by

$$d_{\mathrm{tot}}(K_1, K_2) := \inf_{R \in \mathrm{Cor}_+(K_1, K_2)} \sup_{(x_1, x_2) \in R} d(x_1, x_2) \qquad \big(K_1, K_2 \in \mathcal{K}_{\mathrm{tot}}(\mathcal{X})\big). \tag{1.34}$$

The following theorem is a major ingredient in the proof of Theorems 1.4 and 1.9. Formula (1.35) generalises (1.16) and (1.23). The final statement shows that in general no inequality of the form $d_{\mathrm{tot}} \le C d_{\mathrm{part}}$ holds for any $C < \infty$.

Theorem 1.14 (The ordered Hausdorff topology)**.** *Let $\mathcal{X}$ be a metrisable space, let d be a metric on $\mathcal{X}$ and let d_{part} and d_{tot} be defined in terms of d as in (1.33) and (1.34). Then d_{part} and d_{tot} are metrics that generate the same topology on $\mathcal{K}_{\mathrm{tot}}(\mathcal{X})$, and this topology does not depend on the choice of the metric d on $\mathcal{X}$. If $\mathcal{X}$ is Polish, then so is $\mathcal{K}_{\mathrm{tot}}(\mathcal{X})$. One has*

$$d_{\mathrm{H}}(K_1, K_2) \le d_{\mathrm{part}}(K_1, K_2) \le d_{\mathrm{tot}}(K_1, K_2) \qquad \big(K_1, K_2 \in \mathcal{K}_{\mathrm{tot}}(\mathcal{X})\big). \tag{1.35}$$

If $\mathcal{X} = [0, 1]$, then for each $\varepsilon > 0$, there exist $K_1, K_2 \in \mathcal{K}_{\mathrm{tot}}(\mathcal{X})$ such that $d_{\mathrm{part}}(K_1, K_2) \le \varepsilon d_{\mathrm{tot}}(K_1, K_2)$.

We call the topology on $\mathcal{K}_{\mathrm{tot}}(\mathcal{X})$ generated by d_{part} or d_{tot} the *ordered Hausdorff topology*. Since $\mathcal{K}_{\mathrm{tot}}(\mathcal{X})$ is Polish, there exists a complete metric generating the topology, but as we will show in Lemma 3.10 below, if $(\mathcal{X}, d)$ is complete, then $\mathcal{K}_{\mathrm{tot}}(\mathcal{X})$ is not in general complete in the metrics d_{part} or d_{tot}.

We also have a compactness criterion for subsets of $\mathcal{K}_{\rm tot}(\mathcal{X})$. For $K \in \mathcal{K}_{\rm tot}(\mathcal{X})$ and $\varepsilon > 0$, we define the *mismatch modulus* $m_\varepsilon(K)$ as

$$m_\varepsilon(K) := \sup\big\{d(x_1,y_1) \vee d(x_2,y_2) : x_1,y_1,x_2,y_2 \in K \\ d(x_1,x_2) \vee d(y_1,y_2) \le \varepsilon,\ x_1 \preceq y_1,\ y_2 \preceq x_2\big\}. \tag{1.36}$$

The following theorem is a major ingredient in the proof of Theorems 1.11, 1.12, and 1.13.

Theorem 1.15 (Compact subsets)**.** *Let $(\mathcal{X},d)$ be a metric space and let $\mathcal{K}_{\rm tot}(\mathcal{X})$ be equipped with the ordered Hausdorff topology. Then a set $\mathcal{A} \subset \mathcal{K}_{\rm tot}(\mathcal{X})$ is precompact if and only if*

$$\text{(i) } \exists \text{ compact } C \subset \mathcal{X} \text{ s.t. } K \subset C\ \forall K \in \mathcal{A} \quad \text{and} \quad \text{(ii) } \lim_{\varepsilon\to 0} \sup_{K\in\mathcal{A}} m_\varepsilon(K) = 0. \tag{1.37}$$

We need one more result that will help us make the link between our definition of the Skorokhod topologies and the more traditional formulations. Recall the definition of the space $\mathcal{D}_{[0,1]}(\mathcal{X})$ of cadlag functions $f\colon [0,1] \to \mathcal{X}$ in (1.8). A *cadlag parametrisation* of an element $K \in \mathcal{K}_{\rm tot}(\mathcal{X})$ is a function $\gamma \in \mathcal{D}_{[0,1]}(\mathcal{X})$ such that

$$K = \big\{\gamma(t), \gamma(t-) : t \in [0,1]\big\} \quad \text{and} \quad \gamma(s) \prec \gamma(t)\ \forall 0 \le s < t \le 1. \tag{1.38}$$

Clearly, not every element of $\mathcal{K}_{\rm tot}(\mathcal{X})$ has a cadlag parametrisation. For those that do, the following proposition gives an expression for the metrics $d_{\rm H}$ and $d_{\rm tot}$ in terms of time changes. Let Λ be the space of all bijections $\lambda\colon [0,1] \to [0,1]$ and let Λ_+ be the subset consisting of all bijections λ that are monotone in the sense that $s \le t$ implies $\lambda(s) \le \lambda(t)$. Note that each $\lambda \in \Lambda_+$ is continuous and strictly increasing with $\lambda(0) = 0$ and $\lambda(1) = 1$.

Proposition 1.16 (Distance between cadlag curves)**.** *Let $(\mathcal{X},d)$ be a metric space, and assume that $K_1, K_2 \in \mathcal{K}_{\rm tot}(\mathcal{X})$ have cadlag parametrisations γ_1, γ_2, respectively. Then*

$$d_{\rm H}(K_1,K_2) = \inf_{\lambda\in\Lambda} \sup_{t\in[0,1]} d\big(\gamma_1(t), \gamma_2\big(\lambda(t)\big)\big), \\ d_{\rm tot}(K_1,K_2) = \inf_{\lambda\in\Lambda_+} \sup_{t\in[0,1]} d\big(\gamma_1(t), \gamma_2\big(\lambda(t)\big)\big). \tag{1.39}$$

1.7. *Discussion.* It is instructive to compare our approach with the traditional approach to the Skorokhod topologies. We believe that our approach simplifies several parts of the existing theory, which we outline herein.

For each $0 < t < \infty$, let $\Lambda_+[0,t]$ denote the space of continuous strictly increasing functions $\lambda\colon [0,t] \to [0,t]$ such that $\lambda(0) = 0$ and $\lambda(t) = t$. Then for any metric space $(\mathcal{X},d)$, setting

$$d_{\rm J1}(f_1,f_2) := \inf_{\lambda\in\Lambda_+[0,t]} \sup_{s\in[0,t]} \big[d\big(f_1(s), f_2(\lambda(s))\big) \vee |s - \lambda(s)|\big] \qquad \big(f_1,f_2 \in \mathcal{D}_{[0,t]}(\mathcal{X})\big) \tag{1.40}$$

defines a metric on $\mathcal{D}_{[0,t]}(\mathcal{X})$ that generates the J1 topology. The definition of the M1 topology is similar, but instead of time changes $\lambda \in \Lambda_+[0,t]$ one now has to work with parametrisations of the filled graphs of f_1 and f_2. For brevity, we omit the details.

There is a close connection between the metric $d_{\rm J1}$ defined in (1.40) and our metric $d_{\rm tot}$. Very loosely, $d_{\rm J1}$ uses time changes whilst $d_{\rm tot}$ uses correspondences. More precisely: equip the space $\mathcal{X} \times [0,t]$ with the metric $d'\big((x,s),(y,r)\big) := d(x,y) \vee |s-r|$. Fix $f_1, f_2 \in \mathcal{D}_{[0,t]}(\mathcal{X})$ and let π_1, π_2 denote their associated paths, viewed as totally ordered compact subsets of $\mathcal{X} \times [0,t]$. Then Proposition 1.16 says that $d_{\rm J1}(f_1,f_2) = d_{\rm tot}(\pi_1,\pi_2)$, where $d_{\rm tot}$ is the metric on $\mathcal{K}_{\rm tot}(\mathcal{X} \times [0,t])$ defined in terms of d' as in (1.34). By contrast, our metric $d_{\rm part}$ does not seem to correspond to any previously known metric.

In spite of the close connection between $d_{\rm J1}$ and $d_{\rm tot}$, the latter is more flexible since it can more easily deal with paths that are defined on (perhaps very) different time domains. For example, it allows discrete and continuous time to be cast within the same framework, and naturally handles paths whose domain is not an interval or paths that jump at the endpoints of an interval. In

relation to the last point, it should be noted that if the only aim is to allow paths defined on some fixed interval to jump at the endpoints of this interval, then this can be achieved by more simple methods as in Jakubowski (2018, Thm 4.6), in which this extension is called the *mJ1 topology*. An example of doing similarly within the M1 topology can be found in Dort et al. (2024, Section 2.2.1) (motivated by an application to the excursions of α-stable Lévy processes). The fact that d_{tot} allows jumps at the endpoints of intervals leads to simpler compactness criteria: compare condition (iii) of Theorem 1.13 that has no equivalent in Theorem 1.12.

The metric in (1.40) works only for compact time intervals. The extension to unbounded time intervals is not entirely straightforward as was first realised by Stone (1963). More complete treatments were given in Lindvall (1973) and Whitt (2002). The approach taken in the latter reference is based on the following lemma, which is well known but also follows from our own Lemma 4.8.

Lemma 1.17 (J1 topology on the half-line)**.** *Let $(\mathcal{X}, d)$ be a metric space and for each $0 < t < \infty$ let d_{J1} denote the metric on $\mathcal{D}_{[0,t]}(\mathcal{X})$ defined in (1.40). Then there exists a metric d_{J1} on $\mathcal{D}_{[0,\infty)}(\mathcal{X})$ such that for all $f_n, f \in \mathcal{D}_{[0,\infty)}(\mathcal{X})$, one has $d_{\mathrm{J1}}(f_n, f) \to 0$ if and only if*

$$d_{\mathrm{J1}}\big(f_n|_{[0,t]}, f|_{[0,t]}\big) \underset{n\to\infty}{\longrightarrow} 0 \quad \forall t > 0 \text{ s.t. } f(t-) = f(t). \tag{1.41}$$

Although the main idea of Lemma 1.17 is nice, to prove it one needs to explicitly construct a metric d_{J1} on $\mathcal{D}_{[0,\infty)}(\mathcal{X})$ satisfying (1.41), which is a bit involved. By contrast, our method to deal with unbounded intervals is based on the squeezed space, which cares less about spatial distances between points whose time coordinates are very large or very negative. In Lemma 4.8 below will show that our metrics d_{tot} and d_{part} restricted to $\mathcal{D}_{[0,\infty)}(\mathcal{X})$ generate the same topology as d_{J1} from (1.41).

An attractive aspect of our approach is that in many cases where discrete-time processes approximate a continuous-time process, it becomes unnecessary to interpolate. Let $\mathcal{X}$ be equipped with a betweenness and let $\pi \in \Pi_{\mathrm{c}}(\mathcal{X})$. Then we say that $\pi' \in \Pi_{\mathrm{c}}(\mathcal{X})$ is an *interpolation* of π if $I(\pi')$ is the convex hull of $I(\pi)$ and for each $s, u \in I(\pi)$ such that $s < u$ and $(s, u) \cap I(\pi) = \emptyset$, the function $[s, u] \ni t \mapsto \pi'(t)$ takes values in the segment $\langle \pi(s), \pi(u) \rangle$ and is monotone with respect to the total order on $\langle \pi(s), \pi(u) \rangle$. Note that an interpolation with respect to the trivial betweenness is a piecewise constant interpolation. We will prove the following simple lemmas.

Lemma 1.18 (Continuous interpolation)**.** *Let $\mathcal{X}$ be a metrisable space that is equipped with a proper betweenness. Let $\pi_n \in \Pi_{\mathrm{c}}(\mathcal{X})$ and $\pi \in \Pi^{|}_{\mathrm{c}}(\mathcal{X})$ and for each n, let π'_n be an interpolation of π_n. Assume that for each $t \in I(\pi)$, there exist $t_n \in I(\pi_n)$ such that $t_n \to t$. Then $\pi_n \to \pi$ in the topology on $\Pi_{\mathrm{c}}(\mathcal{X})$ if and only if $\pi'_n \to \pi$.*

Lemma 1.19 (Piecewise constant interpolation)**.** *Let $\mathcal{X}$ be a metrisable space that is equipped with the trivial betweenness. Let $\pi_n \in \Pi_{\mathrm{c}}(\mathcal{X})$ and $\pi \in \Pi^{|}(\mathcal{X})$ and for each n, let π'_n be an interpolation of π_n. Assume that for each $t \in I(\pi)$, there exist $t_n \in I(\pi_n)$ such that $t_n \to t$. Then $\pi_n \to \pi$ in the topology on $\Pi(\mathcal{X})$ if and only if $\pi'_n \to \pi$.*

Lemmas 1.18 and 1.19 show that for convergence to a continuous function or convergence in the J1 topology, it is not necessary to interpolate. For convergence in the M1 topology, it may sometimes still make sense to interpolate, however. We believe the way we are able to treat the J1 topology and various variants of the M1 topology in a unified way are an improvement on earlier attempts to do so because our work shows that both these cases are naturally distinguished within a more general framework.

By definition, a *laglad* function (from the French "limite à gauche, limite à droite") is a function that has both left- and right- limits in each point, but whose value in a point does not need to be equal to either the left or right limit at that point. In Chapter 15 of Whitt (2002) a topology is introduced on spaces of laglad functions. It seems it should be possible to generalise many of our results to laglad functions, where as a first step, one needs a variant of the split real line in

which each point of the real line is replaced by three, rather than two points. The details are not straightforward, however, so we leave this avenue open.

Apart from Skorokhod's topologies, other useful topologies on the space of cadlag functions on a fixed time interval have been introduced over the years. The weaker Meyer–Zheng topology (Meyer and Zheng, 1984), which is non-Polish, was motivated by highly singular diffusions and is based on convergence in measure for probability laws on $[0,\infty]\times\overline{\mathbb{R}}$. Jakubowski's S topology (Jakubowski, 1997, 2018) is also weaker than both J1 and M1, and is non-metrisable, but shares several useful features of the stronger framework including an equivalent of Prokhorov's Theorem.

1.8. *Outline of the proofs.* In Section 2 we collect some preliminary facts about cadlag functions, the Hausdorff topology, and betweenness. In Subsection 2.1 we develop the observation, originally due to Kolmogorov (1956), that there exists a topology on the split real line $\mathbb{R}_{\mathfrak{s}}$ introduced below Theorem 1.11 such that cadlag functions on $\mathbb{R}$ correspond to continuous functions on $\mathbb{R}_{\mathfrak{s}}$. Although we will not use this very often, it will occasionally be handy in our proofs. The facts in this subsection are of some independent interest as they can also be used to give a natural definition of cadlag functions of several variables, that seems simpler than the approach used by other authors such as Bickel and Wichura (1971) and Neuhaus (1971). Subsection 2.2 is devoted to the squeezed space $\mathcal{R}(\mathcal{X})$ and contains the proof of Lemmas 1.1 and 1.19. Subsection 2.3 collects some facts about the Hausdorff topology (such as Lemma 1.3) that will be needed many times later on. In Subsection 2.4 we study betweenness and prove Lemmas 1.5, 1.6, and 1.7. Subsection 2.5, finally, contains the proofs of Lemmas 1.2 and 1.8.

Section 3 is devoted to the ordered Hausdorff topology. Theorem 1.14 is proved in Subsections 3.2 and 3.3, Theorem 1.15 in Subsection 3.4, and Proposition 1.16 Subsection 3.5. These results lay the basis for the proofs of our main results, which can be found in Section 4. The basic statements about the metrics d_{part}, d_{tot}, and d_{H} from Theorems 1.4 and 1.9 are proved in Subsection 4.1, while the statements about Polishness are proved in Subsection 4.2. The compactness criteria Theorems 1.11, 1.12, and 1.13 are proved in Subsection 4.3, which also contains the proof of Lemmas 1.10 and 1.18. In Subsection 4.4, we complete the proofs of Theorems 1.4 and 1.9 by showing that our definitions of the J1, J2, M1, and M2 topologies on the space $\mathcal{D}_I(\mathcal{X})$ coincide with the classical definitions. This subsection also contains an alternative proof of Lemma 1.17, which is of course well known from the literature.

2. Paths and cadlag functions

2.1. *The split real line.* Basic notation for the split real line $\mathbb{R}_{\mathfrak{s}}$ has been introduced in Subsection 1.5 below Theorem 1.11. This space can be equipped with a natural topology that is the subject of this subsection. The split unit interval $[0,1]_{\mathfrak{s}}$ (defined as in (1.27)) as a topological space originates from the work of Alexandroff and Urysohn (1929). It is known as the *split interval*, *double arrow space*, or *two arrows space*. An account of many of its properties can be found in Chapter 41 of Fremlin (2006). The topology on the split unit interval $[0,1]_{\mathfrak{s}}$ with the points $0-$ and $1+$ removed is called the *weak parallel line topology* in Steen and Seebach (1978), where again several of its properties are proved. These references do not discuss cadlag functions, however. Below, we give an overview of the most important properties of the topology on the split real line. Proofs will be provided at the end of this subsection.

We use notation for intervals in $\mathbb{R}_{\mathfrak{s}}$ similar to the usual notation for the real line, i.e.,

$$\begin{aligned}(\!(\sigma,\rho)\!) &:= \{\tau\in\mathbb{R}_{\mathfrak{s}} : \sigma<\tau<\rho\}, \qquad [\![\sigma,\rho)\!) := \{\tau\in\mathbb{R}_{\mathfrak{s}} : \sigma\le\tau<\rho\},\\ (\!(\sigma,\rho]\!] &:= \{\tau\in\mathbb{R}_{\mathfrak{s}} : \sigma<\tau\le\rho\}, \qquad [\![\sigma,\rho]\!] := \{\tau\in\mathbb{R}_{\mathfrak{s}} : \sigma\le\tau\le\rho\}.\end{aligned} \tag{2.1}$$

Note that there is some redundancy in this notation: for example, $(\!(s-,t+)\!) = [\![s+,t-]\!]$.

We equip $\mathbb{R}_\mathfrak{s}$ with the *order topology*, which is the weakest topology to make all sets of the form $\{\tau \in \mathbb{R}_\mathfrak{s} : \tau > \sigma\}$ and $\{\tau \in \mathbb{R}_\mathfrak{s} : \tau < \sigma\}$ with $\sigma \in \mathbb{R}_\mathfrak{s}$ open. A consequence of this is that intervals of the form $(\!(\sigma, \rho)\!)$ are open, those of the form $[\![\sigma, \rho]\!]$ are closed, and $(\!(s-, t+)\!) = [\![s+, t-]\!]$ is both open and closed. The following lemma describes convergent sequences in this topology.

Lemma 2.1 (Convergence criterion)**.** *For $\tau_n \in \mathbb{R}_\mathfrak{s}$ and $t \in \mathbb{R}$, one has*

(i) *$\tau_n \to t+$ if and only if $\underline{\tau}_n \to t$ and $\tau_n \geq t+$ for all n sufficiently large,*
(ii) *$\tau_n \to t-$ if and only if $\underline{\tau}_n \to t$ and $\tau_n \leq t-$ for all n sufficiently large.*

The following lemma lists some elementary properties of $\mathbb{R}_\mathfrak{s}$.

Lemma 2.2 (The split real line)**.** *The space $\mathbb{R}_\mathfrak{s}$ is first countable, Hausdorff, and separable, but not second countable and not metrisable.*

Let $\mathcal{X}$ be a Hausdorff topological space. For any set $\mathcal{I} \subset \mathbb{R}_\mathfrak{s}$, we let $\mathcal{C}_\mathcal{I}(\mathcal{X})$ the space of continuous functions $f\colon \mathcal{I} \to \mathcal{X}$. Recall that in Subsection 1.2 we defined a cadlag function with values in $\mathcal{X}$ to be a triple (I, f^-, f^+) where $I \subset \mathbb{R}$ is closed and $f^-, f^+\colon I \to \mathcal{X}$ are left and right continuous functions that satisfy (1.6). The following lemma is a direct consequence of Lemma 2.1.

Lemma 2.3 (Cadlag functions)**.** *Let $\mathcal{X}$ be a Hausdorff topological space, let $I \subset \mathbb{R}$ be closed, let $f\colon I_\mathfrak{s} \to \mathcal{X}$ be a function, and let $f^\pm\colon I \to \mathcal{X}$ be defined by $f^\pm(t) := f(t\pm)$ $(t \in I)$. Then $f \in \mathcal{C}_{I_\mathfrak{s}}(\mathcal{X})$ if and only if (I, f^-, f^+) is a cadlag function.*

We equip the product space $\mathbb{R}_\mathfrak{s}^d$ with the product topology. By definition, a subset $A \subset \mathbb{R}_\mathfrak{s}^d$ is *bounded* if $A \subset [\![\sigma, \tau]\!]^d$ for some $\sigma, \tau \in \mathbb{R}_\mathfrak{s}$. The following proposition gives a characterisation of the compact subsets of $\mathbb{R}_\mathfrak{s}^d$, similar to the well-known characterisation of compact subsets of $\mathbb{R}^d$.

Proposition 2.4 (Compact sets)**.** *For a subset $C \subset \mathbb{R}_\mathfrak{s}^d$, the following three claims are equivalent: (i) C is compact, (ii) C is sequentially compact, and (iii) C is closed and bounded.*

We define the *extended split real line* by $\overline{\mathbb{R}}_\mathfrak{s} := \{t\star : t \in \overline{\mathbb{R}},\ \star \in \{-, +\}\}$ and equip it with a topology so that it is homeomorphic to $[\![0-, 1+]\!]$. We also set

$$\hat{\mathbb{R}}_\mathfrak{s} := [\![-\boldsymbol{\infty}, +\boldsymbol{\infty}]\!] \quad \text{with} \quad -\boldsymbol{\infty} := -\infty+ \ \text{ and } \ +\boldsymbol{\infty} := +\infty- . \tag{2.2}$$

Note that $\hat{\mathbb{R}}_\mathfrak{s}$ is homeomorphic to $[\![0+, 1-]\!]$, therefore compact by Proposition 2.4, and that $\mathbb{R}_\mathfrak{s}$ is dense in $\hat{\mathbb{R}}_\mathfrak{s}$. In other words, $\hat{\mathbb{R}}_\mathfrak{s}$ is a compactification of $\mathbb{R}_\mathfrak{s}$. The notation introduced in (2.2) provides us with a natural way to denote half infinite intervals in $\mathbb{R}_\mathfrak{s}$; for example, $[\![\sigma, \boldsymbol{\infty})\!) = \{\tau \in \mathbb{R}_\mathfrak{s} : \sigma \leq \tau\}$. We now provide proofs for Lemmas 2.1–2.3 and Proposition 2.4.

Proof of Lemma 2.1: By symmetry, it suffices to prove (i). A basis for the topology is formed by all intervals of the form $(\!(\sigma, \rho)\!)$ with $\sigma, \rho \in \mathbb{R}_\mathfrak{s}$. If $t+ \in (\!(\sigma, \rho)\!)$, then $\sigma < t+ < \rho$ and hence $(\!(t-, u+)\!) \subset (\!(\sigma, \rho)\!)$ for some $u > t$. It follows that the sets of the form $(\!(t-, u+)\!) = [\![t+, u-]\!]$ with $u \in \{t + n^{-1} : n \geq 1\}$ form a fundamental system of neighbourhoods of $t+$, which is easily seen to imply the claim. □

Proof of Lemma 2.2: It is easy to see that $\mathbb{R}_\mathfrak{s}$ has the Hausdorff property. In the proof of Lemma 2.1, we have already seen that each point has a countable fundamental system of neighbourhoods, so $\mathbb{R}_\mathfrak{s}$ is first countable. On the other hand, each basis of the topology must for each $t \in \mathbb{R}$ contain an open set O such that $t \in O \subset (\!(t-, (t+1)+)\!) = [\![t+, (t+1)-]\!]$. These open sets are all distinct, so $\mathbb{R}_\mathfrak{s}$ is not second countable. By Lemma 2.1, the set $\{t+ : t \in \mathbb{Q}\}$ is dense so $\mathbb{R}_\mathfrak{s}$ is separable. Since in metric spaces, separability implies second countability, we conclude that $\mathbb{R}_\mathfrak{s}$ is not metrisable. □

Proof of Lemma 2.3: If $f \in \mathcal{C}_{I_\mathfrak{s}}(\mathcal{X})$, then Lemma 2.1 implies that f^- and f^+ are left and right continuous, respectively, and satisfy (1.6). Conversely, if f^- and f^+ are left and right continuous and satisfy (1.6), then Lemma 2.1 implies that $f(\tau_n) \to f(\tau)$ for each $\tau_n, \tau \in I_\mathfrak{s}$ such that $\tau_n \to \tau$

in the topology on $\mathbb{R}_\mathfrak{s}$. Since $\mathbb{R}_\mathfrak{s}$ is first countable by Lemma 2.2, this implies that $f\colon I_\mathfrak{s} \to \mathcal{X}$ is continuous. □

The next lemma prepares for the proof of Proposition 2.4. Even though $\mathbb{R}_\mathfrak{s}$ is not second countable, it has a property that is almost as good. We cite the following property from Steen and Seebach (1978, Example 96, point 5).

Lemma 2.5 (Strong Lindelöf property)**.** *Every open cover of a subset of $\mathbb{R}_\mathfrak{s}$ has a countable subcover.*

Proof of Proposition 2.4: We first prove the statement for $\mathbb{R}_\mathfrak{s}$. In any first countable space, being sequentially compact is equivalent to being countably compact, which means that every countable open covering has a finite subcovering. By the strong Lindelöf property, a subset of $\mathbb{R}_\mathfrak{s}$ is compact if and only if it is countably compact, proving that (i) and (ii) are equivalent.

Since the map $\tau \mapsto \underline{\tau}$, that assigns to a split real number τ its real part, is continuous, and since the continuous image of a compact set is compact, (i) implies that $\underline{C} := \{\underline{\tau} : \tau \in C\}$ is closed and bounded. Since moreover a compact subset of a Hausdorff space is closed, (i) implies (iii).

Property (iii) implies that each sequence $\tau_n \in C$ has a subsequence τ_n' such that $\underline{\tau}_n'$ converges to a limit $t \in \underline{C}$. The τ_n' must then contain a further subsequence τ_n'' such that one of the following three cases occurs: 1. $\underline{\tau}_n'' < t$ for all n, 2. $\underline{\tau}_n'' > t$ for all n, or 3. $\underline{\tau}_n''$ is constant. In either case, the fact that C is closed implies that τ_n'' converges to a limit in C, proving the implication (iii)⇒(ii). This completes the proof for $\mathbb{R}_\mathfrak{s}$.

The strong Lindelöf property does not hold for $\mathbb{R}_\mathfrak{s}^d$ in dimensions $d \geq 2$ (as a consequence of e.g. Munkres, 2000, Examples 3 and 4, §30, Chapter 4) so to prove the statement for these spaces we have to proceed differently. Property (i) implies countable compactness which by the fact that $\mathbb{R}_\mathfrak{s}$ and hence also $\mathbb{R}_\mathfrak{s}^d$ are first countable is equivalent to (ii). The continuous image of a countably compact set is countably compact. Applying this to the coordinate projections and using what we already know for $\mathbb{R}_\mathfrak{s}$, we see that (ii) implies that C is bounded. Since, moreover, in any first countable Hausdorff space, being sequentially compact implies being closed, we see that (ii) implies (iii). By Tychonoff's theorem and what we already know for $\mathbb{R}_\mathfrak{s}$, the set $[\![s-,t+]\!]^d$ is compact for each $-\infty < s < t < \infty$. Since a closed subset of a compact set is compact, (iii) implies (i). □

2.2. *Squeezed space.* In this subsection we prove Lemmas 1.1 and 1.19. A metric generating the topology on $\mathcal{R}(\overline{\mathbb{R}})$ is given in Fontes et al. (2004, formula (3.4)) but this formula does not easily extend to more general metric spaces $(\mathcal{X}, d)$, so we will proceed a bit differently. Let $d_{\overline{\mathbb{R}}}$ be a metric that generates the topology on the extended real line $\overline{\mathbb{R}}$ and let $\phi\colon \overline{\mathbb{R}} \to [0,\infty)$ be a continuous function such that $\phi(\pm\infty) = 0$ and $\phi(t) > 0$ for all $t \in \mathbb{R}$. We define $d_{\mathrm{sqz}}\colon \mathcal{R}(\mathcal{X})^2 \to [0,\infty)$ by

$$d_{\mathrm{sqz}}\big((x,s),(y,t)\big) := \big(\phi(s) \wedge \phi(t)\big)\big(d(x,y) \wedge 1\big) + \big|\phi(s) - \phi(t)\big| + d_{\overline{\mathbb{R}}}(s,t), \tag{2.3}$$

where naturally the first term is zero if (x,s) or (y,t) are elements of $\big\{(*,-\infty),(*,\infty)\big\}$ (even though $d(x,y)$ is not defined in this case).

Lemma 2.6 (Squeezed space)**.** *Let $(\mathcal{X},d)$ be a metric space. Then d_{sqz} is a metric on $\mathcal{R}(\mathcal{X})$. One has $d_{\mathrm{sqz}}\big((x_n,t_n),(x,t)\big) \to 0$ if and only if:*

$$\text{(i)} \quad t_n \to t, \qquad \text{(ii)} \quad \text{if } t \in \mathbb{R}, \text{ then } x_n \to x.$$

Proof: We first prove that d_{sqz} is a metric on $\mathcal{R}(\mathcal{X})$. For brevity, we write $d'(x,y) := d(x,y) \wedge 1$. Then d' is a metric on $\mathcal{X}$. The only nontrivial statement that we have to prove is the triangle inequality, and it suffices to prove this for the function

$$\rho\big((x,s),(y,t)\big) := \big(\phi(s) \wedge \phi(t)\big)d'(x,y) + \big|\phi(s) - \phi(t)\big|.$$

We estimate

$$\rho\big((x,s),(z,u)\big) \leq \big(\phi(s) \wedge \phi(u)\big)\big(d'(x,y) + d'(y,z)\big) + \big|\phi(s) - \phi(u)\big|. \tag{2.4}$$

If $\phi(t) \geq \phi(s) \wedge \phi(u)$, then $\phi(s) \wedge \phi(u)$ is less than $\phi(s) \wedge \phi(t)$ and also less than $\phi(t) \wedge \phi(u)$, so we can simply estimate the expression in (2.4) from above by

$$\big(\phi(s) \wedge \phi(t)\big)d'(x,y) + \big(\phi(t) \wedge \phi(u)\big)d'(y,z)\big) + \big|\phi(s) - \phi(t)\big| + \big|\phi(t) - \phi(u)\big|$$

and we are done. On the other hand, if $\phi(t) < \phi(s) \wedge \phi(u)$, then

$$\big|\phi(s) - \phi(t)\big| + \big|\phi(t) - \phi(u)\big| = \big|\phi(s) - \phi(u)\big| + 2\big(\phi(s) \wedge \phi(u) - \phi(t)\big).$$

Using the fact that $d' \leq 1$, we can now estimate the right-hand side of (2.4) from above by

$$\begin{aligned}
&\phi(t)\big(d'(x,y) + d'(y,z)\big) + 2\big(\phi(s) \wedge \phi(u) - \phi(t)\big) + \big|\phi(s) - \phi(u)\big| \\
&\quad = \big(\phi(s) \wedge \phi(t)\big)d'(x,y) + \big(\phi(t) \wedge \phi(u)\big)d'(y,z) \\
&\qquad + \big|\phi(s) - \phi(t)\big| + \big|\phi(t) - \phi(u)\big|,
\end{aligned}$$

and again we are done. This completes the proof that d_{sqz} is a metric on $\mathcal{R}(\mathcal{X})$.

It remains to prove that

$$\big(\phi(t_n) \wedge \phi(t)\big)\big(d(x_n,x) \wedge 1\big) + \big|\phi(t_n) - \phi(t)\big| + d_{\overline{\mathbb{R}}}(t_n,t) \underset{n\to\infty}{\longrightarrow} 0 \tag{2.5}$$

if and only if conditions (i) and (ii) of the lemma are satisfied. Because of the third term on the left-hand side, a necessary condition for (2.3) is that $t_n \to t$, and this condition also guarantees that the second term tends to zero. If $t \in \{-\infty, +\infty\}$, then this is all one needs since the first term now tends to zero regardless of the values of x_n and x, but if $t \in \mathbb{R}$, then one needs in addition that $d(x_n,x) \to 0$. □

The following lemma shows that $\mathcal{R}(\mathcal{X})$ is Polish if $\mathcal{X}$ is.

Lemma 2.7 (Preservation of Polishness)**.**

(a) *If $(\mathcal{X},d)$ is separable, then so is $(\mathcal{R}(\mathcal{X}),d_{\mathrm{sqz}})$.*

(b) *If $(\mathcal{X},d)$ is complete, then so is $(\mathcal{R}(\mathcal{X}),d_{\mathrm{sqz}})$.*

Proof: If D is a countable dense subset of $(\mathcal{X},d)$, then $D \times \mathbb{Q}$ is a countable dense subset of $(\mathcal{R}(\mathcal{X}),d_{\mathrm{sqz}})$, proving (a).

To prove (b), let (x_n,t_n) be a Cauchy sequence in $(\mathcal{R}(\mathcal{X}),d_{\mathrm{sqz}})$. Then by (2.3) t_n is a Cauchy sequence in $\overline{\mathbb{R}}$ and hence $t_n \to t$ for some $t \in \overline{\mathbb{R}}$. If $t \in \mathbb{R}$, then by (2.3) x_n is a Cauchy sequence in $(\mathcal{X},d)$ so by the completeness of the latter, $x_n \to x$ for some $x \in \mathcal{X}$. By Lemma 2.6, it follows that (x_n,t_n) converges, proving the completeness of $(\mathcal{R}(\mathcal{X}),d_{\mathrm{sqz}})$. □

The following lemma identifies the compact subsets of $\mathcal{R}(\mathcal{X})$. In particular, the lemma shows that $\mathcal{R}(\mathcal{X})$ is compact if $\mathcal{X}$ is compact.

Lemma 2.8 (Compactness criterion)**.** *A set $A \subset \mathcal{R}(\mathcal{X})$ is precompact if and only if for each $T < \infty$, there exists a compact set $K \subset \mathcal{X}$ such that $\{x \in \mathcal{X} : \exists t \in [-T,T] \text{ s.t. } (x,t) \in A\} \subset K$.*

Proof: Assume that $A \subset \mathcal{R}(\mathcal{X})$ has the property that for each $T < \infty$, there exists a compact set $K \subset \mathcal{X}$ such that $\{x \in \mathcal{X} : (x,t) \in A,\ t \in [-T,T]\} \subset K$. To show that A is precompact, we will show that each sequence $(x_n,t_n) \in A$ has a convergent subsequence. By the compactness of $\overline{\mathbb{R}}$, we can select a subsequence (x'_n,t'_n) such that $t'_n \to t$ for some $t \in \overline{\mathbb{R}}$. If $t = \pm\infty$, then by Lemma 2.6 $(x'_n,t'_n) \to (*,\pm\infty)$ and we are done. Otherwise, there exists a $T < \infty$ such that $t'_n \in [-T,T]$ for all n large enough. By assumption, there then exists a compact set $K \subset \mathcal{X}$ such that $x'_n \in K$ for all n large enough, so we can select a further subsequence such that (x''_n,t''_n) converges to a limit $(x,t) \in \mathcal{X} \times \mathbb{R}$.

Assume, on the other hand, that $A \subset \mathcal{R}(\mathcal{X})$ has the property that for some $T < \infty$, there does not exist a compact set $K \subset \mathcal{X}$ such that $\{x \in \mathcal{X} : (x,t) \in A,\ t \in [-T,T]\} \subset K$. Set

$$B := \big\{x \in \mathcal{X} : (x,t) \in A \text{ for some } t \in [-T,T]\big\}$$

The closure of B cannot be compact, since this would contradict our assumption. It follows that there exists a sequence $x_n \in B$ that does not contain a convergent subsequence, and there exist $t_n \in [-T,T]$ such that $(x_n,t_n) \in A$. But then, in view of Lemma 2.6, the sequence (x_n,t_n) cannot contain a convergent subsequence either, proving that A is not precompact. □

Proof of Lemma 1.1: Immediate from Lemmas 2.6, 2.7, and 2.8. □

Proof of Lemma 1.19: Let d_{sqz} be defined as in (2.3) and let d_{tot} be defined as in (1.13). To prove the lemma, it suffices to show that $d_{\mathrm{tot}}(\pi_n,\pi'_n) \to 0$. Let us say that (s,u) is a *gap* of π_n if $s<u$, $s,u \in I(\pi_n)$, and $(s,u) \cap I(\pi_n) = \emptyset$. We call $(\pi_n(s),s)$ and $(\pi_n(u),u)$ the *endpoints* of the gap. Let C_n be the correspondence between π_n and π'_n consisting of all pairs (z,z') with $z \in \pi_n$ and $z' \in \pi'_n$ such that either $z=z'$ or there exists a gap (s,u) of π_n such that $z=(x,t)$ is an endpoint of the gap while $z'=(x',t')$ satisfies $s<t'<u$ and $x'=x$. The definition of an interpolation implies that this correspondence is monotone, so using (2.3) we obtain

$$d_{\mathrm{tot}}(\pi_n,\pi'_n) \leq \sup_{(z,z')\in C} d_{\mathrm{sqz}}(z,z') \leq \sup\bigl\{|\phi(s)-\phi(u)| + d_{\overline{\mathbb{R}}}(s,u) : (s,u) \text{ is a gap of } \pi_n\bigr\}. \tag{2.6}$$

Since $\pi \in \Pi^{|}(\mathcal{X})$ and for each $t \in I(\pi)$, there exist $t_n \in I(\pi_n)$ such that $t_n \to t$, the maximal length of gaps of π_n as measured by $d_{\overline{\mathbb{R}}}$ tends to zero completing the proof. □

2.3. *The Hausdorff metric.* If $(\mathcal{X},d)$ is a metric space, then the Hausdorff metric d_{H} on the space $\mathcal{K}_+(\mathcal{X})$ of nonempty compact subsets of $\mathcal{X}$ has been defined in (1.10). We have not actually given a proof or reference for the equality of the two formulas there. The following lemma fills this gap and shows that the infimum in the second formula is attained.

Lemma 2.9 (Hausdorff metric and correspondences)**.** *Let $(\mathcal{X},d)$ be a metric space. Then*

$$d_{\mathrm{H}}(K_1,K_2) := \sup_{z_1\in K_1} d(z_1,K_2) \vee \sup_{z_2\in K_2} d(z_2,K_1) = \inf_{R\in \mathrm{Cor}(K_1,K_2)} \sup_{(x_1,x_2)\in R} d(x_1,x_2). \tag{2.7}$$

Moreover, there exists an $R \in \mathrm{Cor}(K_1,K_2)$ such that $d_{\mathrm{H}}(K_1,K_2) = \max_{(x_1,x_2)\in R} d(x_1,x_2)$.

Proof: Let $R \in \mathrm{Cor}(K_1,K_2)$ and let $D := \sup_{(x_1,x_2)\in R} d(x_1,x_2)$. Then $d(x_1,K_2) \leq D$ and $d(x_2,K_1) \leq D$ for each $x_1 \in K_1$, $x_2 \in K_2$, and hence $d_{\mathrm{H}}(K_1,K_2) \leq D$. On the other hand, by the compactness of K_2 and the continuity of the function $d(x_1,\,\cdot\,)$, for each $x_1 \in K_1$, there exists an $x_2 \in K_2$ such that $d(x_1,K_2) = d(x_1,x_2)$. The same statement holds with the roles of K_1 and K_2 interchanged, so setting

$$R := \bigl\{(x_1,x_2) \in K_1 \times K_2 : d(x_1,x_2) \in \{d(x_1,K_2), d(x_2,K_1)\}\bigr\} \tag{2.8}$$

defines a correspondence between K_1 and K_2. By the compactness of K_1 and the continuity of the map $d(\,\cdot\,,K_2)$, there exists an $x'_1 \in K_1$ such that $d(x'_1,K_2) = \max_{x_1\in K_1} d(x_1,K_2)$, and similarly there exists an $x''_2 \in K_2$ such that $d(x''_2,K_1) = \max_{x_2\in K_2} d(x_2,K_1)$. By our earlier arguments, there exist $x'_2 \in K_2$ and $x''_1 \in K_1$ such that $d(x'_1,K_2) = d(x'_1,x'_2)$ and $d(x''_2,K_1) = d(x''_1,x''_2)$. Then

$$d_{\mathrm{H}}(K_1,K_2) = d(x'_1,x'_2) \vee d(x''_1,x''_2) = \max_{(x_1,x_2)\in R} d(x_1,x_2). \tag{2.9}$$

□

We will often need the following lemma which we cite from Schertzer et al. (2014, Lemma B.1). Note that this lemma implies that the topology generated by d_{H} does not depend on the choice of the metric d on $\mathcal{X}$.

Lemma 2.10 (Convergence criterion)**.** *Let $K_n, K \in \mathcal{K}_+(\mathcal{X})$ $(n \geq 1)$. Then $K_n \to K$ in the Hausdorff topology if and only if there exists a $C \in \mathcal{K}_+(\mathcal{X})$ such that $K_n \subset C$ for all $n \geq 1$ and*

$$\begin{aligned} K &= \{x \in \mathcal{X} : \exists x_n \in K_n \text{ s.t. } x_n \to x\} \\ &= \{x \in \mathcal{X} : \exists x_n \in K_n \text{ s.t. } x \text{ is a cluster point of } (x_n)_{n\in\mathbb{N}}\}. \end{aligned} \tag{2.10}$$

The following lemma is Schertzer et al. (2014, Lemma B.2). In particular, it shows that $\mathcal{K}_+(\mathcal{X})$ is Polish if $\mathcal{X}$ is.

Lemma 2.11 (Properties of the Hausdorff metric)**.**

(a) *If $(\mathcal{X}, d)$ is separable, then so is $(\mathcal{K}_+(\mathcal{X}), d_{\mathrm{H}})$.*

(b) *If $(\mathcal{X}, d)$ is complete, then so is $(\mathcal{K}_+(\mathcal{X}), d_{\mathrm{H}})$.*

Recall that a set is called precompact if its closure is compact. The following lemma is Schertzer et al. (2014, Lemma B.3). In particular, it shows that $\mathcal{K}_+(\mathcal{X})$ is compact if $\mathcal{X}$ is.

Lemma 2.12 (Compactness in the Hausdorff topology)**.** *A set $\mathcal{A} \subset \mathcal{K}_+(\mathcal{X})$ is precompact if and only if there exists a $C \in \mathcal{K}_+(\mathcal{X})$ such that $K \subset C$ for each $K \in \mathcal{A}$.*

The following lemma says connectedness is a property of compact sets that is preserved under limits.

Lemma 2.13 (Preservation of connectedness)**.** *The set $\mathcal{K}_{\mathrm{c}}(\mathcal{X})$ of all connected nonempty compact subsets of $\mathcal{X}$ is a closed subset of $\mathcal{K}_+(\mathcal{X})$.*

Proof: Imagine that $K_n, K \in \mathcal{K}_+(\mathcal{X})$ satisfy $K_n \to K$. If K is not connected, then there exist disjoint nonempty compact sets C_1, C_2 such that $K = C_1 \cup C_2$. Let $\varepsilon := d(C_1, C_2) = \inf\{d(x_1, x_2) : x_1 \in C_1,\ x_2 \in C_2\}$. By the compactness of C_1 and C_2, the infimum is attained and $\varepsilon > 0$. Let $U_i := \{x \in \mathcal{X} : d(x, C_i) \le \varepsilon/3\}$ $(i = 1, 2)$. Then U_1, U_2 are disjoint closed sets. For all n large enough such that $d_{\mathrm{H}}(K_n, K) \le \varepsilon/3$, one has $K_n \subset U_1 \cup U_2$ while $K_n \cap U_1$ and $K_n \cap U_2$ are both nonempty, which proves that K_n is not connected. □

We recall that the image of a compact set under a continuous map is compact. In what follows, we will need the following simple observation.

Lemma 2.14 (Continuous image)**.** *Let $\mathcal{X}, \mathcal{Y}$ be metrisable spaces and let $\psi\colon \mathcal{X} \to \mathcal{Y}$ be continuous. If $K_n, K \in \mathcal{K}_+(\mathcal{X})$ satisfy $K_n \to K$, then their images under ψ satisfy $\psi(K_n) \to \psi(K)$ in $\mathcal{K}_+(\mathcal{X})$.*

Proof: This follows easily from Lemma 2.10. Since $K_n \to K$, there exists a compact set $C \subset \mathcal{X}$ such that $K_n \subset C$ for all n, and now $\psi(C) \subset \mathcal{Y}$ is a compact set such that $\psi(K_n) \subset \psi(C)$ for all n. By (2.10), it now suffices to check that:

$$\begin{aligned} &\text{(i)} \quad \psi(K) \subset \{y \in \mathcal{Y} : \exists x_n \in K_n \text{ s.t. } \psi(x_n) \to y\}, \\ &\text{(ii)} \quad \{y \in \mathcal{Y} : \exists x_n \in K_n \text{ s.t. } y \text{ is a subsequential limit of } (\psi(x_n))_{n\in\mathbb{N}}\} \subset \psi(K). \end{aligned} \tag{2.11}$$

Here (i) follows from (2.10) and the continuity of ψ. To prove (ii), if $\psi(x'_n) \to y$ for some subsequence x'_n, then since $K_n \subset C$ for all n there exists a further subsequence x''_n such that $x''_n \to x$ for some $x \in \mathcal{X}$. Then $x \in K$ by (2.10) and hence $\psi(x) = y$ by the continuity of ψ. □

2.4. *Betweenness.* In this subsection we prove Lemmas 1.5, 1.6, and 1.7. We start by proving some elementary consequences of the axioms (i)–(iv).

Lemma 2.15 (Elementary properties)**.** *Each betweenness satisfies, for each $x, y, y', z \in \mathcal{X}$:*

(v) $\langle x, x\rangle = \{x\}$,
(vi) $y \in \langle x, z\rangle \ \Rightarrow\ \langle x, y\rangle \subset \langle x, z\rangle$,
(vii) $x \in \langle y, z\rangle$ *and* $y \in \langle x, z\rangle \ \Rightarrow\ x = y$.
(viii) $y, y' \in \langle x, z\rangle$ *and* $y' \in \langle x, y\rangle \ \Rightarrow\ y \in \langle y', z\rangle$.

Proof: Clearly, (ii) and (iii) imply (v) and (iv) implies (vi). To prove (vii), we first observe that $x \in \langle y, z\rangle$ and $y \in \langle x, z\rangle$ imply by (vi) $\langle x, z\rangle \subset \langle y, z\rangle \subset \langle x, z\rangle$ and hence $\langle x, z\rangle = \langle y, z\rangle$. Using this as well as the assumptions $y \in \langle x, z\rangle$ and $x \in \langle y, z\rangle$ we can conclude by (i) and (iii) that $\{y\} = \langle x, y\rangle \cap \langle y, z\rangle = \langle y, x\rangle \cap \langle x, z\rangle = \{x\}$ and hence $x = y$.

To prove (viii), assume that $y, y' \in \langle x, z\rangle$ and $y' \in \langle x, y\rangle$. The statement is trivial if $y = y'$ so without loss of generality we assume that $y \neq y'$. Since $y' \in \langle x, z\rangle$ we have by (iv) that $\langle x, z\rangle = \langle x, y'\rangle \cup \langle y', z\rangle$. Since also $y \in \langle x, z\rangle$ we must have either $y \in \langle x, y'\rangle$, or $y \in \langle y', z\rangle$, or both. The first possibility would by (i) and (vii) and the fact that $y' \in \langle x, y\rangle$ imply that $y = y'$, which contradicts our assumptions, so we conclude that $y \in \langle y', z\rangle$. □

Proof of Lemma 1.5: The first implication $\Rightarrow$ in (1.17) follow from the fact that $y \in \langle x, y\rangle$ by (i) and (ii), while the reverse implication follows from (vi). The second equivalence in (1.17) follows from (viii) and the third equivalence follows from the first one, by the symmetry (i). It is clear that (1.17) defines a partial order $\leq_{x,z}$ on $\langle x, z\rangle$. By (iv), if $y, y' \in \langle x, z\rangle$, then at least one of the conditions $y' \in \langle x, y\rangle$ and $y' \in \langle y, z\rangle$ must hold, which shows that $\leq_{x,z}$ is a total order. □

Lemma 1.5 has a useful consequence.

Lemma 2.16 (Subsegments)**.** *For any betweenness, if $y \in \langle x, z\rangle$, then $\langle y, z\rangle = \{y' \in \langle x, z\rangle : y \leq_{x,z} y'\}$, and $y', y'' \in \langle y, z\rangle$ satisfy $y' \leq_{y,z} y''$ if and only if $y' \leq_{x,z} y''$.*

Proof: By Lemma 1.5 $y' \in \langle x, z\rangle$ satisfies $y \leq_{x,z} y'$ if and only if $y' \in \langle y, z\rangle$. This proves that $\langle y, z\rangle = \{y' \in \langle x, z\rangle : y \leq_{x,z} y'\}$. Again by Lemma 1.5, $y', y'' \in \langle y, z\rangle$ satisfy $y' \leq_{y,z} y''$ if and only if $\langle y'', z\rangle \subset \langle y', z\rangle$, which in turn is equivalent to $y' \leq_{x,z} y''$. □

Proof of Lemma 1.6: We need to check that our definition satisfies axioms (i)–(iv) of a betweenness. Axioms (i) and (ii) are trivial. To prove (iii) and (iv), set $r := d(x, z)$ and let $\gamma\colon [0, r] \to \mathcal{X}$ be the unique isometry such that $\gamma(0) = x$ and $\gamma(r) = z$. Since an isometry is one-to-one, there exists a unique $p \in [0, r]$ such that $\gamma(p) = y$. Clearly, the restrictions of γ to $[0, p]$ and $[p, r]$ are isometries, so $\langle x, y\rangle = \{\gamma(t) : 0 \leq t \leq p\}$ and $\langle y, z\rangle = \{\gamma(t) : p \leq t \leq r\}$. From these observations, axioms (iii) and (iv) follow immediately. □

We say that a betweenness on a metrisable space $\mathcal{X}$ is *generated by an interpolation function* if there exists a continuous function $\varphi\colon \mathcal{X}^2 \times [0, 1] \to \mathcal{X}$ that satisfies $\varphi(x, z, 0) = x$, $\varphi(x, z, 1) = z$, and

$$\langle x, z\rangle = \bigl\{\varphi(x, z, p) : p \in [0, 1]\bigr\} \qquad (x, z \in \mathcal{X}). \tag{2.12}$$

The following lemma prepares for the proof of Lemma 1.7.

Lemma 2.17 (Proper betweennesses)**.** *If $\mathcal{X}$ is a metrisable space, then the trivial betweenness is proper. The same is true for any betweenness that is generated by an interpolation function. If $\mathcal{X}$ is a closed subset of $\mathbb{R}$, then the order betweenness on $\mathcal{X}$ is proper.*

Proof: For the trivial betweenness, $\langle x, z\rangle = \{x, z\}$ is clearly compact for each $x, z \in \mathcal{X}$, and the continuity of the map $(x, z) \mapsto \{x, z\}$ with respect to the Hausdorff topology follows immediately from Lemma 2.10.

If a betweenness is generated by an interpolation function, then $\langle x, z\rangle$, being the image of $[0, 1]$ under the continuous map $p \mapsto \varphi(x, z, p)$, is clearly compact for all $x, z \in \mathcal{X}$. Let $x_n \to x$ and $z_n \to z$. To show that $\langle x_n, z_n\rangle \to \langle x, z\rangle$ in the Hausdorff topology, we check the conditions of Lemma 2.10. Since $x_n \to x$ and $z_n \to z$, the sets $A := \{x\} \cup \{x_n : n \in \mathbb{N}\}$ and $B := \{z\} \cup \{z_n : n \in \mathbb{N}\}$ are compact. Let $\varphi(A \times B \times [0, 1])$ denote the image of $A \times B \times [0, 1]$ under φ, which is compact. Clearly $\langle x_n, z_n\rangle \subset \varphi(A \times B \times [0, 1])$ for all n. To complete the argument, it suffices to show that

$$\begin{aligned}\bigl\{y \in \mathcal{X} : \exists y_n \in \langle x_n, z_n\rangle \text{ s.t. } y \text{ is a cluster point of } (y_n)_{n\in\mathbb{N}}\bigr\}\\ \subset \langle x, z\rangle \subset \bigl\{y \in \mathcal{X} : \exists y_n \in \langle x_n, z_n\rangle \text{ s.t. } y_n \to y\bigr\}.\end{aligned} \tag{2.13}$$

For the first inclusion, assume that y is a cluster point of $y_n = \varphi(x_n, z_n, p_n)$. By going to a subsequence, we can assume that $p_n \to p$ for some $p \in [0, 1]$. Then $y = \varphi(x, z, p) \in \langle x, z\rangle$. For the second inclusion, assume that $y = \varphi(x, z, p)$ for some $p \in [0, 1]$. Then $y_n := \varphi(x_n, z_n, p) \in \langle x_n, z_n\rangle$

converge to y, completing the proof that each betweenness that is generated by an interpolation function is proper.

For the final statement of the lemma, assume that $\mathcal{X}$ is a closed subset of $\mathbb{R}$. Then clearly $\langle x,z\rangle := [x,z]\cap\mathcal{X}$ is compact for each $x,z\in\mathbb{R}$. Let $x_n\to x$ and $z_n\to z$. To show that $\langle x_n,z_n\rangle\to\langle x,z\rangle$ in the Hausdorff topology, we again check the conditions of Lemma 2.10. Clearly, $\langle x_n,z_n\rangle\subset[S,T]\cap\mathcal{X}$ for each n, where $S:=\inf_n x_n$ and $T:=\sup_n z_n$, so to complete the argument, it again suffices to check (2.13). For the first inclusion, assume that $y\in\mathcal{X}$ is a cluster point of $y_n\in\langle x_n,z_n\rangle$. Since $x_n\le y_n\le z_n$ for each n, taking the limit, we see that $x\le y\le z$ and hence $y\in\langle x,z\rangle$. For the second inclusion, assume that $y\in\langle x,z\rangle$. If $x<y<z$, then $x_n<y<z_n$ for all n large enough, so setting $y_n:=y$ for n large enough and $y_n:=x_n$ otherwise proves that y is al element of the set on the right-hand side of (2.13). If $y\in\{x,z\}$, then setting $y_n:=x_n$ or $:=z_n$ proves the same claim, so the proof is complete. □

To prove Lemma 1.7 we need one more result.

Lemma 2.18 (Interpolation functions)**.** *If $\mathcal{X}$ is a normed linear space, then the linear betweenness is generated by an interpolation function. If $\mathcal{X}$ is a proper metric space with unique geodesics, then the same is true for the geodesic betweenness.*

We first show how this implies Lemma 1.7 and then prove Lemma 2.18.

Proof of Lemma 1.7: Parts (i) and (iii) follow directly from Lemma 2.17 while parts (ii) and (iv) follow by combining Lemma 2.17 with Lemma 2.18. □

We next set out to prove Lemma 2.18. The statement about the linear betweenness is trivial, but before we can prove the statement about the geodesic betweenness, we first need a better understanding of metric spaces with unique geodesics, which is provided by Proposition 2.19 below. In any metric space $(\mathcal{X},d)$, for all $x,z\in\mathcal{X}$ and $\varepsilon\ge0$, we define

$$\eta_{x,z}(\varepsilon):=\sup\big\{d(y_1,y_2):\big[d(x,y_1)\vee d(x,y_2)\big]+\big[d(y_1,z)\vee d(y_2,z)\big]\le d(x,z)+\varepsilon\big\}. \tag{2.14}$$

In other words, this is the largest distance between two points $y_1,y_2\in\mathcal{X}$ for which there exist constants $r,r'\ge0$ with $r+r'\le d(x,z)+\varepsilon$ such that $d(x,y_i)\le r$ and $d(y_i,z)\le r'$ $(i=1,2)$.

Proposition 2.19 (Unique geodesics)**.** *Let $(\mathcal{X},d)$ be a metric space. Consider the following conditions.*

(i) *For all $x,z\in\mathcal{X}$ and $r,r'\ge0$ with $r+r'=d(x,z)$, there exists an $y\in\mathcal{X}$ such that $d(x,y)=r$ and $d(y,z)=r'$.*
(ii) *$\eta_{x,z}(0)=0$ for all $x,z\in\mathcal{X}$.*
(ii)' *$\lim_{\varepsilon\to0}\eta_{x,z}(\varepsilon)=0$ for all $x,z\in\mathcal{X}$.*

Then $(\mathcal{X},d)$ has unique geodesics if and only if (i) and (ii) hold. Moreover, (ii)' implies (ii), and if $(\mathcal{X},d)$ is a proper metric space, then (ii) implies (ii)'.

Proof: In any metric space $(\mathcal{X},d)$, let us introduce the notation

$$\langle x,z\rangle:=\big\{y\in\mathcal{X}:d(x,y)+d(y,z)=d(x,z)\big\}. \tag{2.15}$$

We claim that

$$y\in\langle x,z\rangle,\ y'\in\langle x,y\rangle,\ y''\in\langle y,z\rangle\quad\Rightarrow\quad y\in\langle y',y''\rangle. \tag{2.16}$$

To see this, we note that if the assumptions in (2.16) hold but the conclusion does not, then by the triangle inequality

$$\begin{aligned}d(x,z)=d(x,y)+d(y,z)&=d(x,y')+d(y',y)+d(y,y'')+d(y'',z)\\&>d(x,y')+d(y',y'')+d(y'',z),\end{aligned} \tag{2.17}$$

which contradicts the triangle inequality. Now let $(\mathcal{X},d)$ be a metric space with unique geodesics and let $\langle\langle x,z\rangle\rangle$ denote the unique geodesic with endpoints x,z. Clearly $\langle\langle x,z\rangle\rangle \subset \langle x,z\rangle$. We claim that

$$y \in \langle x,z\rangle \quad \Rightarrow \quad \langle\langle x,y\rangle\rangle \cup \langle\langle y,z\rangle\rangle = \langle\langle x,z\rangle\rangle. \tag{2.18}$$

To see this, let $r := d(x,y)$, $r' := d(y,z)$, and let $\gamma\colon [0,r] \to \mathcal{X}$ and $\gamma''\colon [r,r+r'] \to \mathcal{X}$ be the unique isometries with $\gamma(0)=x$, $\gamma(r)=\gamma'(r)=y$, and $\gamma'(r+r')=z$. We claim that $\gamma''\colon [0,r+r'] \to \mathcal{X}$ defined as $\gamma''(t)=\gamma(t)$ for $t\in[0,r]$ and $:=\gamma'(t)$ for $t\in[r,r+r']$ is an isometry. So see this, let $0\le t'<t''\le r+r'$. We need to show that $d\bigl(\gamma''(t'),\gamma''(t'')\bigr)=t''-t'$. This is clear when $t''\le r$ or $r\le t''$, while in the remaining case $t'<r<t''$ the claim follows from (2.16).

We now prove that if $(\mathcal{X},d)$ is a metric space with unique geodesics, then conditions (i) and (ii) are satisfied. Condition (i) is trivial. To prove (ii), let $x,z\in\mathcal{X}$, let $r,r'\ge 0$ satisfy $r+r':=d(x,z)$, and assume that $y_1,y_2\in\mathcal{X}$ satisfy $d(x,y_i)=r$, $d(y_i,z)=r'$ $(i=1,2)$. By (2.18), there exist isometries $\gamma_i\colon [0,r+r'] \to \mathcal{X}$ with $\gamma_i(0)=x$, $\gamma_i(r)=y_i$, and $\gamma_i(r+r')=z$, so by the assumption that $(\mathcal{X},d)$ has unique geodesics we conclude that $y_1=y_2$, proving (ii).

Conversely, if $(\mathcal{X},d)$ is a metric space for which (i) and (ii) hold, then for each $x,z\in\mathcal{X}$ with $r:=d(x,z)$, we can uniquely define $\gamma_{x,z}\colon [0,r]\to\mathcal{X}$ by

$$\gamma_{x,z}(t) := y \quad \text{with} \quad d(x,y)=t,\ d(y,z)=r-t. \tag{2.19}$$

Clearly, if $\gamma\colon [0,r]\to\mathcal{X}$ is an isometry with $\gamma(0)=x$ and $\gamma(r)=z$, then we must have $\gamma=\gamma_{x,z}$, so to prove that $(\mathcal{X},d)$ has unique geodesics, it suffices to show that $\gamma_{x,z}$ is an isometry. Let $0\le t_1\le t_2\le r$ and let $y_i:=\gamma_{x,z}(t_i)$ $(i=1,2)$. Set $y_1':=\gamma_{x,y_2}(t_1)$. Then $d(x,y_1')=t_1$ and $d(y_1',z)\le d(y_1',y_2)+d(y_2,z)=r-t_1$, which by the assumption (ii) implies $y_1'=y_1$. Since $d(y_1',y_2)=t_2-t_1$, this proves that $\gamma_{x,z}$ is an isometry. This completes the proof that a metric space $(\mathcal{X},d)$ has unique geodesics if and only if (i) and (ii) hold.

Trivially, (ii)' implies (ii), so to complete the proof of the proposition, it suffices to prove that for proper metric spaces, (ii) implies (ii)'. Assume that (ii)' does not hold for some $x,z\in\mathcal{X}$. Let $0<\varepsilon_n\le 1$ satisfy $\varepsilon_n\to 0$. Then for some $\delta>0$, we can find $y_1^n,y_2^n\in\mathcal{X}$ with $d(y_1^n,y_2^n)\ge\delta$, as well as $r_n,r_n'\ge 0$ with $r_n+r_n'\le d(x,z)+\varepsilon_n$, such that $d(x,y_i^n)\le r_n$ and $d(y_i^n,z)\le r_n'$ $(i=1,2)$. Since $d(x,y_i^n)\le d(x,z)+1$ for all n, by the properness assumption, we can select a subsequence such that $y_i^n\to y_i$ for some $y_1,y_2\in\mathcal{X}$. Since $r_n+r_n'\le d(x,z)+1$, by going to a further subsequence, we can assume that $r_n\to r$ and $r_n'\to r'$ for some $r,r'\ge 0$ with $r+r'=d(x,z)$. Then $d(x,y_i)\le r$ and $d(y_i,z)\le r'$ while $d(y_1,y_2)\ge\delta$ which shows that $\eta_{x,z}(0)\ge\delta$, violating (ii). □

If a metric space $(\mathcal{X},d)$ has unique geodesics, then by conditions (i) and (ii) of Proposition 2.19, we can uniquely define a function $\varphi\colon \mathcal{X}^2\times[0,1]\to\mathcal{X}$ by

$$\varphi(x,z,p) := y \quad \text{with} \quad d(x,y)=pd(x,z) \text{ and } d(y,z)=(1-p)d(x,z). \tag{2.20}$$

Lemma 2.18 is now implied by Proposition 2.19 and the following lemma (the statement in Lemma 2.18 about normed linear spaces being trivial).

Lemma 2.20 (Geodesic interpolation function)**.** *Let $(\mathcal{X},d)$ be a metric space with unique geodesics and let φ be defined as in (2.20). Then for each $x,z\in\mathcal{X}$, the unique geodesic with endpoints x,z is given by $\{\varphi(x,z,p) : p\in[0,1]\}$. If condition (ii)' of Proposition 2.19 is satisfied, then $\varphi\colon \mathcal{X}^2\times[0,1]\to\mathcal{X}$ is continuous.*

Proof: Let $x,z\in\mathcal{X}$ and $r:=d(x,z)$. We observe that $\{\varphi(x,z,p) : p\in[0,1]\} = \{\gamma_{x,z}(t) : t\in[0,r]\}$ where $\gamma_{x,z}$ is defined as in (2.19). It has already been shown in the proof of Proposition 2.19 that this is the unique geodesic with endpoints x,z. Therefore, to complete the proof, it suffices to show that condition (ii)' of Proposition 2.19 implies that φ is continuous.

Assume that $x_n, x, z_n, z \in \mathcal{X}$ and $p_n, p \in [0,1]$ satisfy $x_n \to x$, $z_n \to z$, and $p_n \to p$. Set $y_n := \varphi(x_n, z_n, p_n)$ and $y := \varphi(x, z, p)$. We have to show that $y_n \to y$. We observe that

$$\begin{aligned} d(x, y_n) + d(y_n, z) &\le d(x_n, y_n) + d(y_n, z_n) + d(x, x_n) + d(z, z_n) \\ &= d(x_n, z_n) + d(x, x_n) + d(z, z_n) \underset{n\to\infty}{\longrightarrow} d(x, z). \end{aligned} \tag{2.21}$$

Thus, for each $\varepsilon > 0$, we can find an m such that $d(x, y_n) + d(y_n, z) \le d(x, z) + \varepsilon$ for all $n \ge m$. Since moreover $d(x, y) + d(y, z) = d(x, z)$, it follows that $d(y_n, y) \le \eta_{x,z}(\varepsilon)$ for all $n \ge m$. Since $\varepsilon > 0$ is arbitrary, by (ii)', this implies $d(y_n, y) \to 0$. □

We conclude this subsection with the following lemma that will be needed in Subsection 2.5 below.

Lemma 2.21 (Segments as ordered sets)**.** *Let $\mathcal{X}$ be a metrisable space that is equipped with a proper betweenness. Then for each $x, z \in \mathcal{X}$, the segment $\langle x, z\rangle$ equipped with the total order $\le_{x,z}$ is an element of $\mathcal{K}_{\mathrm{tot}}(\mathcal{X})$, and the map $(x, z) \mapsto \langle x, z\rangle$ is continuous with respect to the product topology on $\mathcal{X}^2$ and the topology on $\mathcal{K}_{\mathrm{tot}}(\mathcal{X})$.*

Proof: To show that $\langle x, z\rangle$ is an element of $\mathcal{K}_{\mathrm{tot}}(\mathcal{X})$, we must show that the total order $\le_{x,z}$ is compatible with the induced topology on $\langle x, z\rangle$. Assume that $y_n, y_n', y, y' \in \langle x, z\rangle$ satisfy $y_n \to y$, $y_n' \to y'$, and $y_n \le_{x,z} y_n'$ for all n. Then $y_n \in \langle x, y_n'\rangle$ for all n. Since the betweenness is compatible with the topology, $\langle x, y_n'\rangle \to \langle x, y'\rangle$ in the Hausdorff topology, which by Lemma 2.10 implies that $y \in \langle x, y'\rangle$ and hence $y \le_{x,z} y'$. This shows that the total order $\le_{x,z}$ is compatible with the induced topology on $\langle x, z\rangle$.

To show that the map $(x, z) \mapsto \langle x, z\rangle$ is continuous with respect to the topology on $\mathcal{K}_{\mathrm{tot}}(\mathcal{X})$, assume that $x_n \to x$, $z_n \to z$. We will show that

$$d_{\mathrm{part}}\big(\langle x_n, z_n\rangle, \langle x, z\rangle\big) \underset{n\to\infty}{\longrightarrow} 0, \tag{2.22}$$

which is equivalent to the statement that $\langle x_n, z_n\rangle^{\langle 2\rangle}$ converges to $\langle x, z\rangle^{\langle 2\rangle}$ in the Hausdorff topology on $\mathcal{K}_+(\mathcal{X}^2)$. We apply Lemma 2.10. Since $\langle x_n, z_n\rangle \to \langle x, z\rangle$, there exists a compact $C \subset \mathcal{X}$ such that $\langle x_n, z_n\rangle \subset C$ for all n and hence $\langle x_n, z_n\rangle^{\langle 2\rangle} \subset C^2$ for all n. Thus, it suffices to check that (compare (2.13))

$$\begin{aligned} &\big\{(y, y') \in \mathcal{X}^2 : \exists y_n, y_n' \in \langle x_n, z_n\rangle \text{ with } y_n \le_{x_n,z_n} y_n' \text{ s.t. } (y, y') \text{ is a cluster point of } (y_n, y_n')_{n\in\mathbb{N}}\big\} \\ &\quad \subset \langle x, z\rangle^{\langle 2\rangle} \subset \big\{(y, y') \in \mathcal{X} : \exists y_n, y_n' \in \langle x_n, z_n\rangle \text{ with } y_n \le_{x_n,z_n} y_n' \text{ s.t. } (y_n, y_n') \to (y, y')\big\}. \end{aligned} \tag{2.23}$$

If (y, y') is an element of the set on the left-hand side of (2.23) and (y_n, y_n') fulfill the conditions of the definition of this set, then by going to a subsequence we may assume that $(y_n, y_n') \to (y, y')$. Then $y, y' \in \langle x, z\rangle$ since $\langle x_n, z_n\rangle \to \langle x, z\rangle$. Moreover $y_n \le_{x_n,z_n} y_n'$ means $y_n \in \langle x_n, y_n'\rangle$. Since $y_n \to y$ and $\langle x_n, y_n'\rangle \to \langle x, y'\rangle$, this implies $y \in \langle x, y'\rangle$ and hence $y \le_{x,z} y'$, proving that $(y, y') \in \langle x, z\rangle^{\langle 2\rangle}$.

To prove the second inclusion in (2.23), assume that $(y, y') \in \langle x, z\rangle^{\langle 2\rangle}$. Since $\langle x_n, z_n\rangle \to \langle x, z\rangle$, there exist $y_n, y_n' \in \langle x_n, z_n\rangle$ such that $y_n \to y$ and $y_n' \to y'$. We now distinguish two cases: $y \ne y'$ and $y = y'$. If $y \ne y'$, then we claim that $y_n \le_{x_n,z_n} y_n'$ for all n large enough. Indeed, in the opposite case, since $\le_{x_n,z_n}$ is a total order, by going to a subsequence, we can assume that $y_n' \le_{x_n,z_n} y_n$ for all n, which by the arguments we have already seen implies $y' \le_{x,z} y$, so that by the fact that $(y, y') \in \langle x, z\rangle^{\langle 2\rangle}$ we must have $y = y'$, contradicting our assumption. Since $y_n \le_{x_n,z_n} y_n'$ for all n large enough, changing the definitions of y_n, y_n' for finitely many n, we see that there exist $y_n, y_n' \in \langle x_n, z_n\rangle$ with $y_n \le_{x_n,z_n} y_n'$ such that s.t. $(y_n, y_n') \to (y, y')$. In the case $y = y'$ the argument is even simpler, since now $(y_n, y_n) \to (y, y')$ while obviously $y_n \le_{x_n,z_n} y_n$ for all n. □

2.5. *Cadlag functions.* In this subsection we prove Lemmas 1.2 and 1.8 as well as some related results that will be needed later. Let $\mathcal{X}$ be a metrisable space. In Subsection 1.2, we defined a cadlag function to be a triple (I, f^-, f^+), where $I \subset \mathbb{R}$ is a closed set and $f^-, f^+ \colon I \to \mathcal{X}$ are left and right continuous functions that satisfy (1.6). By Lemma 2.3, we can equivalently define a cadlag function to be a pair (I, f) such that $I \subset \mathbb{R}$ is closed and $f \colon I_{\mathfrak{s}} \to \mathcal{X}$ is continuous, where $I_{\mathfrak{s}}$ is the subset of the split real line defined in (1.27). We work with this equivalent definition from now on and moreover define an *extended cadlag function* to be a pair $(\hat{I}, f)$ such that $\hat{I}$ is a nonempty closed subset of $\overline{\mathbb{R}}$ and setting $I := \hat{I} \cap \mathbb{R}$, we have that (I, f) is a cadlag function. We define the *closed graph* $\mathcal{G}(\hat{I}, f)$ of an extended cadlag function $(\hat{I}, f)$ as

$$\mathcal{G}(\hat{I}, f) := \bigl\{(x,t) : t \in I,\ x \in \{f(t-), f(t+)\}\bigr\} \cup \bigl\{(*, \pm\infty) : \pm\infty \in \hat{I}\bigr\}. \tag{2.24}$$

If $\mathcal{X}$ is equipped with a betweenness (see Subsection 2.4), then we define the *filled graph*[1] of an extended cadlag function $(\hat{I}, f)$ as

$$\mathcal{G}_{\mathrm{f}}(\hat{I}, f) := \bigl\{(x,t) : t \in I,\ x \in \langle f(t-), f(t+)\rangle\bigr\} \cup \bigl\{(*, \pm\infty) : \pm\infty \in \hat{I}\bigr\}. \tag{2.25}$$

Note that for the trivial betweenness, the filled and closed graphs coincide. The filled graph is naturally equipped with a total order, which is defined by setting $(x,s) \prec (y,t)$ if either $s < t$ and x, y are arbitrary, or $s = t$ and $x <_{f(t-),f(t+)} y$, where $\leq_{f(t-),f(t+)}$ is the total order on the segment $\langle f(t-), f(t+)\rangle$ defined in Lemma 1.5.

Lemma 2.22 (Filled graphs)**.** *Assume that $\mathcal{X}$ is a metrisable space that is equipped with a proper betweenness. Then for any extended cadlag function $(\hat{I}, f)$, the filled graph $\mathcal{G}_{\mathrm{f}}(\hat{I}, f)$ is a compact subset of the squeezed space $\mathcal{R}(\mathcal{X})$, and the total order $\preceq$ is compatible with the induced topology on $\mathcal{G}_{\mathrm{f}}(\hat{I}, f)$.*

Proof: We will show that each sequence $(x_n, t_n) \in \mathcal{G}_{\mathrm{f}}(\hat{I}, f)$ has a subsequence that converges to a limit in $\mathcal{G}_{\mathrm{f}}(\hat{I}, f)$. Since $\hat{I}$ is compact, we can select a subsequence such that $t_n \to t$ for some $t \in \hat{I}$. If $t = \pm\infty$, then Lemma 2.6 tells us that $(x_n, t_n) \to (*, \pm\infty) \in \mathcal{G}_{\mathrm{f}}(\hat{I}, f)$ so we are done, so from now on we can assume that $t \in \mathbb{R}$. By going to a further subsequence, we can assume that we are in one of the following three cases: (i) $t_n < t$ for all n, (ii) $t_n > t$ for all n, and (iii) $t_n = t$ for all n. In case (i), we use the continuity of $f \colon I_{\mathfrak{s}} \to \mathcal{X}$ and the fact that the betweenness is compatible with the topology to see, using Lemma 2.15 (v), that

$$x_n \in \langle f(t_n-), f(t_n+)\rangle \underset{n\to\infty}{\longrightarrow} \langle f(t-), f(t-)\rangle = \{f(t-)\}, \tag{2.26}$$

where the convergence is in $\mathcal{K}_+(\mathcal{X})$. We conclude from this that (x_n, t_n) converges to $\bigl(f(t-), t\bigr) \in \mathcal{G}_{\mathrm{f}}(\hat{I}, f)$. In case (ii), the same argument shows that (x_n, t_n) converges to $\bigl(f(t+), t\bigr) \in \mathcal{G}_{\mathrm{f}}(\hat{I}, f)$. In case (iii), finally, using the compactness of $\langle f(t-), f(t+)\rangle$, we can select a further subsequence such that $(x_n, t) \to (x, t)$ for some $x \in \langle f(t-), f(t+)\rangle$. Since also in this case the limit (x, t) is an element of $\mathcal{G}_{\mathrm{f}}(\hat{I}, f)$, we are done.

To see that the total order $\preceq$ on $\mathcal{G}_{\mathrm{f}}(\hat{I}, f)$ is compatible with the (induced) topology on $\mathcal{G}_{\mathrm{f}}(\hat{I}, f)$, it suffices to show that

$$S := \bigl\{\bigl((x,s),(y,t)\bigr) \in \mathcal{G}_{\mathrm{f}}(\hat{I}, f)^2 : (x,s) \prec (y,t)\bigr\} \tag{2.27}$$

is an open subset of $\mathcal{G}_{\mathrm{f}}(\hat{I}, f)^2$. If $\bigl((x,s),(y,t)\bigr)$ is an element of S, then either: (i) $s < t$, or: (ii) $s = t \in \mathbb{R}$ and $x \neq y$. In case (i), we can choose $s < S < T < t$. Then

$$O := \bigl\{\bigl((x',s'),(y',t')\bigr) \in \mathcal{G}_{\mathrm{f}}(\hat{I}, f)^2 : s' < S,\ T < t'\bigr\} \tag{2.28}$$

is an open subset of $\mathcal{G}_{\mathrm{f}}(\hat{I}, f)^2$ such that $\bigl((x,s),(y,t)\bigr) \in O \subset S$. In case (ii), we recall that by definition $(x,t) \preceq (y,t)$ if $x \leq_{f(t-),f(t+)} y$, where by Lemma 2.21 the total order $\leq_{f(t-),f(t+)}$ on

[1] Except for the points at infinity, this is what Whitt (2002) calls the *completed graph*.

$\langle f(t-), f(t+)\rangle$ is compatible with the topology. It follows that we can choose $\varepsilon > 0$ small enough such that for $z \in \langle f(t-), f(t+)\rangle$, if $d(z,x) < \varepsilon$ then $(z,t) \prec (y,t)$, while if $d(z,y) < \varepsilon$ then $(x,t) \prec (z,t)$. Next, we use the continuity of $f\colon I_{\mathfrak{s}} \to \mathcal{X}$ to choose $\delta > 0$ small enough such that $d\big(f(s\pm), x\big) > \varepsilon$ for all $t < s < t + \delta$ and $d\big(f(s\pm), y\big) > \varepsilon$ for all $t - \delta < s < t$. Then

$$O := \big\{\big((x',s),(y',u)\big) \in \mathcal{G}_{\mathrm{f}}(\hat{I}, f) : |s-t| \vee |u-t| < \delta,\ d(x',x) \vee d(y',y) < \varepsilon\big\} \tag{2.29}$$

is an open subset of $\mathcal{G}_{\mathrm{f}}(\hat{I}, f)^2$ such that $\big((x,s),(y,t)\big) \in O \subset S$. Together, these observations prove that S is an open subset of $\mathcal{G}_{\mathrm{f}}(\hat{I}, f)^2$. □

Recall that for any metrisable space $\mathcal{X}$, in Subsection 1.6 we define $\mathcal{K}_{\mathrm{tot}}(\mathcal{X})$ to be the set of pairs $(K, \preceq)$ such that K is a compact subset of $\mathcal{X}$ and $\preceq$ is a total order on K that is compatible with the induced topology on K.

Lemma 2.23 (Characterisation of filled graphs)**.** *Let $\mathcal{X}$ be a metrisable space that is equipped with a betweenness that is compatible with the topology. Assume that $(G, \preceq) \in \mathcal{K}_{\mathrm{tot}}(\mathcal{R}(\mathcal{X}))$. Then $(G, \preceq)$ is the filled graph of an extended cadlag function $(\hat{I}, f)$ if and only if the following conditions are satisfied.*

(i) *$(x,s) \preceq (y,t)$ for all $(x,s),(y,t) \in G$ such that $s < t$.*
(ii) *For each $t \in \mathbb{R}$ and $(x_1,t),(x_2,t),(x_3,t) \in G$ with $(x_1,t) \preceq (x_2,t) \preceq (x_3,t)$, one has $x_2 \in \langle x_1, x_3\rangle$.*

Moreover, the extended cadlag function $(\hat{I}, f)$ is uniquely determined by its filled graph $(G, \preceq)$.

Proof: By Lemma 2.22, the filled graph of an extended cadlag function $(\hat{I}, f)$ corresponds to an element of $\mathcal{K}_{\mathrm{tot}}(\mathcal{R}(\mathcal{X}))$. Properties (i) and (ii) now follow from the definition of the total order $\preceq$ on $\mathcal{G}_{\mathrm{f}}(\hat{I}, f)$ and Lemma 2.16.

Assume that conversely, $(G, \preceq) \in \mathcal{K}_{\mathrm{tot}}(\mathcal{X})$ satisfies (i) and (ii). We define

$$I := \hat{I} \cap \mathbb{R} \quad \text{with} \quad \hat{I} := \big\{t \in \overline{\mathbb{R}} : \exists x \in \mathcal{X} \cup \{*\} \text{ s.t. } (x,t) \in G\big\}. \tag{2.30}$$

Since $\hat{I}$ is the image of G under the continuous map $\mathcal{R}(\mathcal{X}) \ni (x,t) \mapsto t \in \overline{\mathbb{R}}$, we see that $\hat{I}$ is compact. We claim that for each $t \in I$, there exist unique $f(t-), f(t+) \in \mathcal{X}$ such that $f(t-) \preceq f(t+)$ and

$$S_t := \big\{x \in \mathcal{X} : (x,t) \in G\big\} = \langle f(t-), f(t+)\rangle. \tag{2.31}$$

Indeed, S_t is a compact metrisable set, so we can choose a countable dense set $\{x_n : n \in \mathbb{N}\} \subset S_t$. Set $y_0 := x_0$ and define y_n as the maximum of x_n and y_{n-1} in the total order $\preceq$ $(n \geq 1)$. By the compactness of S_t, by going to a subsequence, we can assume that $y_n \to f(t+)$ for some $f(t+) \in S_t$. Then $y' \preceq y$ for all $y' \in D$ and hence also for all $y' \in S_t$ since the order is compatible with the topology. In the same way, we see that S_t has a (necessarily unique) minimal element $f(t-)$. By (ii), we conclude that $S_t = \langle f(t-), f(t+)\rangle$.

We can now use the claim we have just proved to define $f\colon I_{\mathfrak{s}} \to \mathcal{X}$ by

$$\langle f(t-), f(t+)\rangle := \{x \in \mathcal{X} : (x,t) \in G\} \quad \text{with} \quad f(t-) \preceq f(t+) \qquad \big(t \in I\big) \tag{2.32}$$

which has the result that $(G, \preceq)$ is the filled graph of $(\hat{I}, f)$. To show that $(\hat{I}, f)$ is an extended cadlag function, we need to show that $f\colon I_{\mathfrak{s}} \to \mathcal{X}$ is continuous. By symmetry, it suffices to show that if $\tau_n \in I_{\mathfrak{s}}$ and $t \in I$ satisfy $\underline{\tau}_n > t$ for all n and $\underline{\tau}_n \to t$ as $n \to \infty$, then $f(\tau_n) \to f(t+)$. It suffices to show that $\{f(\tau_n) : n \in \mathbb{N}\}$ is precompact and its only cluster point is $f(t+)$. Equivalently, we may show that each subsequence of $f(\tau_n)$ contains a further subsequence that converges to $f(t+)$. By the compactness of G, for any subsequence, we can select a further subsequence such that $f(\tau_n) \to x$ for some $x \in \mathcal{X}$ such that $(x,t) \in G$. By (i), we have $\big(f(t+), t\big) \preceq \big(f(\tau_n), \underline{\tau}_n\big)$ for all n, so using the fact that the total order is compatible with the topology, we see that $\big(f(t+), t\big) \preceq (x,t)$, which using the fact that $f(t+)$ is the maximal element of $\langle f(t-), f(t+)\rangle$ with respect to the order $\preceq$ identifies x as $f(t+)$. □

Lemma 2.24 (Characterisation of paths)**.** *Let $\mathcal{X}$ be a metrisable space, let π be a subset of $\mathcal{R}(\mathcal{X})$, and let $\preceq$ be a total order on π. Then $(\pi,\preceq)\in\Pi(\mathcal{X})$ if and only if $(\pi,\preceq)$ is the closed graph of an extended cadlag function $(\hat{I},f)$.*

Proof: In Section 1.2, we defined $\Pi(\mathcal{X})$ to be the set of $(\pi,\preceq)\in\mathcal{K}_{\rm tot}(\mathcal{R}(\mathcal{X}))$ such that:

(i) $(x,s)\preceq(y,t)$ for all $(x,s),(y,t)\in\pi$ such that $s<t$,
(ii)' for each $t\in\mathbb{R}$, the set $\{x\in\mathcal{X}:(x,t)\in\pi\}$ has at most two elements.

We observe that (ii)' is equivalent to condition (ii) of Lemma 2.23 for the trivial betweenness. In view of this, the claim follows from Lemma 2.23. □

Lemma 2.25 (Characterisation of filled paths)**.** *Let $\mathcal{X}$ be a metrisable space that is equipped with a proper betweenness, let $\overline{\pi}$ be a subset of $\mathcal{R}(\mathcal{X})$, and let $\preceq$ be a total order on $\overline{\pi}$. Then $\overline{\pi}$ is the filled path associated with a path $\pi\in\Pi(\mathcal{X})$ if and only if $(\overline{\pi},\preceq)$ is the filled graph of an extended cadlag function $(\hat{I},f)$.*

Proof: By Lemma 2.24, there is a one-to-one correspondence between paths $\pi\in\Pi(\mathcal{X})$ and extended cadlag functions $(\hat{I},f)$. The claim now follows by comparing the definition of the filled path $\overline{\pi}$ associated with a path π in (1.20) with the definition of the filled graph of an extended cadlag function $(\hat{I},f)$ in (2.25). □

Proof of Lemma 1.2: Immediate from Lemma 2.24 and the definition of the closed graph of an extended cadlag function $(\hat{I},f)$. □

Proof of Lemma 1.8: Immediate from Lemmas 2.23 and 2.25. □

By definition, an *extended continuous function* is a pair $(\hat{I},f)$ such that $\hat{I}$ is a nonempty closed subset of $\overline{\mathbb{R}}$ and $f\colon I\to\mathcal{X}$ is a continuous function, where $I:=\hat{I}\cap\mathbb{R}$. Equivalently, this is an extended cadlag function such that $f(t-)=f(t+)$ for all $t\in I$. The *closed graph* of an extended continuous function is the set

$$\mathcal{G}(\hat{I},f):=\big\{(f(t),t):t\in I\big\}\cup\big\{(*,\pm\infty):\pm\infty\in\hat{I}\big\},\tag{2.33}$$

which coincides with the closed graph of $(\hat{I},f)$, viewed as an extended cadlag function. We equip $\mathcal{G}(\hat{I},f)$ with a total order $\preceq$ by setting $(x,s)\prec(y,t)$ if $s<t$. Then Lemma 2.22 tells us that $\mathcal{G}(\hat{I},f)$ is compact and that the total order on $\mathcal{G}(\hat{I},f)$ is compatible with the induced topology on $\mathcal{G}(\hat{I},f)$. The following lemma is similar to Lemma 2.23.

Lemma 2.26 (Characterisation of continuous graphs)**.** *Let $\mathcal{X}$ be a metrisable space. Assume that $G\in\mathcal{K}_+(\mathcal{R}(\mathcal{X}))$. Then G is the closed graph of an extended continuous function $(\hat{I},f)$ if and only if for each $t\in\mathbb{R}$, the set $\{x\in\mathcal{X}:(x,t)\in G\}$ has at most one element.*

Proof: We have just observed that if $G=\mathcal{G}(\hat{I},f)$ is the graph of an extended continuous function, then G is compact. Clearly, the set $\{x\in\mathcal{X}:(x,t)\in G\}$ has at most one element for each $t\in\mathbb{R}$.

Conversely, if $G\in\mathcal{K}_+(\mathcal{R}(\mathcal{X}))$ has this property, then we define

$$I:=\hat{I}\cap\mathbb{R}\quad\text{with}\quad\hat{I}:=\big\{t\in\overline{\mathbb{R}}:\exists x\in\mathcal{X}\cup\{*\}\text{ s.t. }(x,t)\in G\big\}.\tag{2.34}$$

Then $\hat{I}$ is compact since it is the continuous image of a compact set. We use the fact that G contains at each time at most one point to define $f\colon I\to\mathcal{X}$ by

$$\big\{f(t)\big\}:=\{x\in\mathcal{X}:(x,t)\in G\}\qquad(t\in I).\tag{2.35}$$

To show that f is continuous, assume that $t_n,t\in I$ satisfy $t_n\to t$. By the same argument as in the proof of Lemma 2.22, using the compactness of G, it suffices to show that if $f(t_n)\to x$ along a subsequence then $x=f(t)$. This follows from the assumption that $\{x\in\mathcal{X}:(x,t)\in G\}$ has at most one element. □

Continuous paths whose domain is an interval have the property that viewed as a subset of the squeezed space, they are connected. The following lemma says that this is an if and only if.

Lemma 2.27 (Connected graphs)**.** *A path $\pi \in \Pi(\mathcal{X})$ is connected if and only if $\pi \in \Pi_{\mathrm{c}}^{|}(\mathcal{X})$.*

Proof: If $\pi \in \Pi_{\mathrm{c}}(\mathcal{X})$, then π is the image of the compact set $\hat{I}(\pi)$ under the continuous map from $\overline{\mathbb{R}}$ to $\mathcal{R}(\mathcal{X})$ given by $t \mapsto \big(\pi(t), t\big)$ (with $\pm\infty \mapsto (*, \pm\infty)$). If $\pi \in \Pi_{\mathrm{c}}^{|}(\mathcal{X})$, then $\hat{I}(\pi)$ is connected, so by the fact that the continuous image of a connected set is connected, we conclude that π is connected.

Conversely, if π is connected, then $\hat{I}(\pi)$ must be connected and hence $\pi \in \Pi^{|}(\mathcal{X})$. To see that π is moreover continuous, assume that conversely, $\pi(t-) \neq \pi(t+)$ for some $t \in I(\pi)$. Then we can define new paths π', π'' with extended domains $\hat{I}(\pi') := [-\infty, t] \cap \hat{I}(\pi)$ and $\hat{I}(\pi'') := [t, \infty] \cap \hat{I}(\pi)$, by setting $\pi'(s\pm) := \pi(s\pm)$ and $\pi''(s\pm) := \pi(s\pm)$ for $s < t$ and $s > t$, respectively, and $\pi'(t\pm) := \pi(t-)$ and $\pi''(t\pm) := \pi(t+)$. By Lemma 2.22, π' and π'' are compact sets. Since $\pi' \cap \pi'' = \emptyset$ and $\pi' \cup \pi'' = \pi$, this proves that π is not connected. □

3. The ordered Hausdorff metric

3.1. *The ordered Hausdorff metric.* In this subsection, we study the metrics d_{part} and d_{tot} defined in (1.33) and (1.34), preparing for the proofs of Theorems 1.14 and 1.15. Let $(\mathcal{X}, d)$ be a metric space. Generalising the definition in (1.31), for each $m \geq 1$ and $K \in \mathcal{K}_{\mathrm{part}}(\mathcal{X})$, we set

$$K^{\langle m \rangle} := \big\{(x_1, \dots, x_m) \in K^m : x_1 \preceq \cdots \preceq x_m\big\}. \tag{3.1}$$

It is straightforward to check that $K^{\langle m \rangle}$ is a closed subset of K^m and hence a compact subset of $\mathcal{X}^m$. Generalising the definition in (1.32), we equip $\mathcal{X}^m$ with the metric

$$d^m\big((x_1, \dots, x_m), (y_1, \dots, y_m)\big) := \bigvee_{k=1}^{m} d(x_k, y_k), \tag{3.2}$$

and we equip $\mathcal{K}_+(\mathcal{X}^m)$ with the associated Hausdorff metric d_{H}^m. Generalising the definition in (1.33), for each $m \geq 1$, we define a function $d^{\langle m \rangle}$ on $\mathcal{K}_{\mathrm{part}}(\mathcal{X})^2$ by

$$d^{\langle m \rangle}(K_1, K_2) := d_{\mathrm{H}}^m(K_1^{\langle m \rangle}, K_2^{\langle m \rangle}) \qquad \big(K_1, K_2 \in \mathcal{K}_{\mathrm{part}}(\mathcal{X})\big). \tag{3.3}$$

In particular, when $m \geq 2$, this is a metric on $\mathcal{K}_{\mathrm{part}}(\mathcal{X})$ since $(K, \preceq)$ is uniquely characterised by $K^{\langle m \rangle}$ for $m \geq 2$. On the other hand, $d^{\langle 1 \rangle}(K_1, K_2)$ is simply the Hausdorff distance between K_1 and K_2 as sets, which gives no information about the partial order. The following lemma describes a simple property of the metric $d^{\langle 2 \rangle}$.

Lemma 3.1 (Ordered limit)**.** *Let $\mathcal{X}$ be a metrisable space. Assume that $K_n, K \in \mathcal{K}_{\mathrm{part}}(\mathcal{X})$ satisfy $d^{\langle 2 \rangle}(K_n, K) \to 0$ and that $x_n, y_n \in K_n$ satisfy $x_n \to x$, $y_n \to y$, and $x_n \preceq y_n$ for all n. Then $x, y \in K$ satisfy $x \preceq y$.*

Proof: Since $d^{\langle 2 \rangle}(K_n, K) \to 0$, we have $K_n^{\langle 2 \rangle} \to K^{\langle 2 \rangle}$ and hence by Lemma 2.10 $x, y \in K^{\langle 2 \rangle}$, which proves that $x \preceq y$. □

The following lemma gives a one-sided bound between metrics of the form $d^{\langle m \rangle}$ for different values of m.

Lemma 3.2 (One-sided bound)**.** *One has*

$$d^{\langle m \rangle}(K_1, K_2) \leq d^{\langle m+1 \rangle}(K_1, K_2) \qquad \big(m \geq 1,\ K_1, K_2 \in \mathcal{K}_{\mathrm{part}}(\mathcal{X})\big). \tag{3.4}$$

Proof: By Lemma 2.9, there exists a correspondence R between $K_1^{\langle m+1\rangle}$ and $K_2^{\langle m+1\rangle}$ such that $d^{m+1}(x,y) \leq d_{\rm H}^{m+1}(K_1^{\langle m+1\rangle}, K_2^{\langle m+1\rangle})$ for all $(x,y) \in R$. Let $\psi\colon \mathcal{X}^{m+1} \to \mathcal{X}$ denote the projection $\psi(x_1,\ldots,x_{m+1}) := (x_1,\ldots,x_m)$. Then (3.2) implies that

$$d^m\big(\psi(x),\psi(y)\big) \leq d^{m+1}(x,y) \qquad (x,y \in \mathcal{X}^{m+1}). \tag{3.5}$$

Since $\psi(K_i^{\langle m+1\rangle}) = K_i^{\langle m\rangle}$ $(i=1,2)$, it follows that $R' := \big\{\big(\psi(x),\psi(y)\big) : (x,y) \in R\big\}$ is a correspondence between $K_1^{\langle m\rangle}$ and $K_2^{\langle m\rangle}$ such that $d^m(x',y') \leq d_{\rm H}^{m+1}(K_1^{\langle m+1\rangle}, K_2^{\langle m+1\rangle})$ for all $(x',y') \in R'$. By Lemma 2.9, this proves that

$$d_{\rm H}^m(K_1^{\langle m\rangle}, K_2^{\langle m\rangle}) \leq d_{\rm H}^{m+1}(K_1^{\langle m+1\rangle}, K_2^{\langle m+1\rangle}), \tag{3.6}$$

which in view of (3.3) implies the claim. □

The following lemmas show that in general, the metrics $d^{\langle m\rangle}$ for different values of m are not comparable. More precisely, the one-sided bound in Lemma 3.2 is not matched by an opposite inequality of the form $d^{\langle m+1\rangle}(K_1,K_2) \leq C d^{\langle m\rangle}(K_1,K_2)$ for any finite constant C, and convergence in $d^{\langle m\rangle}$ does not imply convergence in $d^{\langle m+1\rangle}$.

Lemma 3.3 (No opposite inequality). *Let $\mathcal{X} = [0,1]$, equipped with the usual distance. Then for each $m \geq 1$ and $0 < \varepsilon \leq 1/4$, there exist $K_1, K_2 \in \mathcal{K}_{\rm tot}(\mathcal{X})$ such that $d^{\langle m\rangle}(K_1,K_2) \leq \varepsilon$ and $d^{\langle m+1\rangle}(K_1,K_2) \geq 1/2$.*

Proof: We choose $K_1 = \{x_1,\ldots,x_{m+1}\}$ with $x_k \in [0,\varepsilon]$ when k is even and $x_k \in [1-\varepsilon,1]$ if k is odd, and we choose $K_2 = \{y_1,\ldots,y_{m+1}\}$ with $y_k \in [0,\varepsilon]$ when k is odd and $y_k \in [1-\varepsilon,1]$ if k is even. We equip K_1 and K_2 with total orders such that $x_1 \prec \cdots \prec x_{m+1}$ and $y_1 \prec \cdots \prec y_{m+1}$. It is easy to see that

$$d\big((x_1,\ldots,x_{m+1}), K_2^{\langle m+1\rangle}\big) \geq 1/2, \tag{3.7}$$

and hence $d^{\langle m+1\rangle}(K_1,K_2) \geq 1/2$. On the other hand, it is easy to see that for each $(z_1,\ldots,z_m) \in K_1^{\langle m\rangle}$, there exists a $(z_1',\ldots,z_m') \in K_2^{\langle m\rangle}$ such that $|z_k - z_k'| \leq \varepsilon$ for all k, and vice versa, so $d^{\langle m\rangle}(K_1,K_2) \leq \varepsilon$. □

We note that $d^{\langle m\rangle}(K_1,K_2) \leq \sup_{(x_1,x_2)\in K_1\times K_2} d(x_1,x_2)$, which is finite by the continuity of d and the compactness of $K_1 \times K_2$. We use this and Lemma 3.2 to define $d^{\langle\infty\rangle}$ as the increasing limit

$$d^{\langle\infty\rangle}(K_1,K_2) := \lim_{m\to\infty} d^{\langle m\rangle}(K_1,K_2) \qquad \big(K_1,K_2 \in \mathcal{K}_{\rm part}(\mathcal{X})\big). \tag{3.8}$$

It is straightforward to check that $d^{\langle\infty\rangle}$ is a metric on $\mathcal{K}_{\rm part}(\mathcal{X})$: symmetry and the triangle inequality follow by taking the limit in the corresponding properties of the metrics $d^{\langle m\rangle}$, and $d^{\langle\infty\rangle}(K_1,K_2) = 0$ clearly implies $d^{\langle m\rangle}(K_1,K_2) = 0$ for all $m \geq 1$ and hence equality of K_1 and K_2 as partially ordered spaces. In the special case that K_1 and K_2 are totally ordered, the following proposition identifies $d^{\langle\infty\rangle}(K_1,K_2)$ as the metric $d_{\rm tot}$ defined in (1.34).

Proposition 3.4 (Monotone correspondences). *Let $(\mathcal{X},d)$ be a metric space. Then, for all $K_1,K_2 \in \mathcal{K}_{\rm tot}(\mathcal{X})$, one has $d^{\langle\infty\rangle}(K_1,K_2) = d_{\rm tot}(K_1,K_2)$.*

The proof of Proposition 3.4 uses the following simple lemma.

Lemma 3.5 (Eventually ordered sequences). *Let K be a compact metrisable set that is equipped with a total order that is compatible with the topology. Assume that $x \prec y$ and that $x_n, y_n \in K$ satisfy $x_n \to x$ and $y_n \to y$. Then $x_n \prec y_n$ for all n sufficiently large.*

Proof: Since $\preceq$ is a total order, if the statement is not true, then $y_n \preceq x_n$ for infinitely many n, so we can select a subsequence such that $y_n \preceq x_n$ for all n. Taking the limit, using the fact that the total order that is compatible with the topology, we find that $y \preceq x$, which contradicts $x \prec y$. □

Proof of Proposition 3.4: We first prove the inequality $d^{\langle\infty\rangle}(K_1,K_2) \le d_{\mathrm{tot}}(K_1,K_2)$. Let R be a monotone correspondence between K_1 and K_2. Let $x_1,\dots,x_m \in K_1$ satisfy $x_1 \preceq \cdots \preceq x_m$. Then we can choose $x'_1,\dots,x'_m \in K_2$ such that $(x_k,x'_k)\in R$ for all $1\le k\le m$, and moreover $x'_k = x'_{k+1}$ whenever $x_k = x_{k+1}$ $(1\le k<m)$. Since R is monotone and K_2 is totally ordered, we must have $x'_1 \preceq \cdots \preceq x'_m$. This shows that

$$d^m\big((x_1,\dots,x_m),K_2^{\langle m\rangle}\big) \le \sup_{(x,x')\in R} d(x,x') \qquad \big((x_1,\dots,x_m)\in K_1^{\langle m\rangle}\big). \tag{3.9}$$

The same is true with the roles of K_1 and K_2 interchanged, so we conclude that

$$d^{\langle m\rangle}(K_1,K_2) = d^m_{\mathrm{H}}(K_1^{\langle m\rangle},K_2^{\langle m\rangle}) \le \sup_{(x,x')\in R} d(x,x'). \tag{3.10}$$

Taking the infimum over all monotone correspondences between K_1 and K_2 and letting $m\to\infty$ we see that $d^{\langle\infty\rangle}(K_1,K_2)\le d_{\mathrm{tot}}(K_1,K_2)$.

To prove the opposite inequality, let ε_n be positive constants, tending to zero. Since K_1 is totally bounded, for each n, we can find an $m(n)\ge 1$ and $x^n_1,\dots,x^n_{m(n)}\in K_1$ such that

$$d\big(x,\{x^n_1,\dots,x^n_{m(n)}\}\big)\le\varepsilon_n \quad \forall x\in K_1. \tag{3.11}$$

Since K_1 is totally ordered, we can assume without loss of generality that $x^n_1\preceq\cdots\preceq x^n_{m(n)}$. Since

$$d^m_{\mathrm{H}}(K_1^{\langle m\rangle},K_2^{\langle m\rangle}) = d^{\langle m\rangle}(K_1,K_2)\le d^{\langle\infty\rangle}(K_1,K_2), \tag{3.12}$$

we can find $y^n_1,\dots,y^n_{m(n)}\in K_2$ with $y^n_1\preceq\cdots\preceq y^n_{m(n)}$ such that

$$d(x^n_k,y^n_k)\le d^{\langle\infty\rangle}(K_1,K_2) \qquad (1\le k\le m(n)). \tag{3.13}$$

Using the fact that K_2 is totally bounded, adding points to $\{y^n_1,\dots,y^n_{m(n)}\}$ and making $m(n)$ larger if necessary, we can arrange things so that also

$$d\big(y,\{y^n_1,\dots,y^n_{m(n)}\}\big)\le\varepsilon \quad \forall y\in K_2. \tag{3.14}$$

Now using again (3.12) we can add corresponding points in K_1 for the new points we have added to K_2 so that (3.13) remains true. Adding points will not spoil (3.11) so we can arrange things such that (3.11), (3.12), and (3.14) are satisfied simultaneously.

Let $R_n\subset K_1\times K_2$ be the set

$$R_n := \big\{(x^n_k,y^n_k) : 1\le k\le m(n)\big\}. \tag{3.15}$$

We claim that R_n is monotone in the sense that

$$\text{there are no } (x^n_k,y^n_k),(x^n_l,y^n_l)\in R_n \text{ such that } x^n_k\prec x^n_l \text{ and } y^n_l\prec y^n_k. \tag{3.16}$$

Indeed, $x^n_k\prec x^n_l$ implies $k<l$ and $y^n_l\prec y^n_k$ implies $l<k$, which is a contradiction.

Since $K_1\times K_2$ is compact, by Lemma 2.12, we can select a subsequence such that $R_n\to R$ in the Hausdorff topology on $\mathcal{K}_+(K_1\times K_2)$, for some compact set $R\subset K_1\times K_2$. We claim that R is a correspondence between K_1 and K_2. Indeed, by (3.11), for each $x\in K_1$, we can choose $k(n)$ such that $x^n_{k(n)}\to x$. By the compactness of $K_1\times K_2$, the sequence $(x^n_{k(n)},y^n_{k(n)})$ has at least one cluster point (x,y), and by Lemma 2.10 $(x,y)\in R$. Similarly, for each $y\in K_2$ there exists an $x\in K_1$ such that $(x,y)\in R$.

We next claim that R is monotone. Assume that conversely, there exist $(x,y),(x',y')\in R$ such that $x\prec x'$ and $y'\prec y$. Then by Lemma 2.10, there exist $k(n),k'(n)$ such that $(x^n_{k(n)},y^n_{k(n)})\to(x,y)$ and $(x^n_{k'(n)},y^n_{k'(n)})\to(x',y')$. By Lemma 3.5, $x^n_{k(n)}\prec y^n_{k(n)}$ and $y^n_{k'(n)}\prec x^n_{k'(n)}$ for all n large enough, which contradicts (3.16).

Taking the limit in (3.13), using Lemma 2.10, we see that $d(x,y)\le d^{\langle\infty\rangle}(K_1,K_2)$ for all $(x,y)\in R$ and hence by (1.34) $d_{\mathrm{tot}}(K_1,K_2)\le d^{\langle\infty\rangle}(K_1,K_2)$. □

3.2. *The mismatch modulus.* In this subsection, we prove Theorem 1.14, except for the statement about Polishness. Generalising the definition in (1.36) for any $K_1, K_2 \in \mathcal{K}_{\rm tot}(\mathcal{X})$ and $\varepsilon > 0$, we define the *mismatch modulus* $m_\varepsilon(K_1, K_2)$ by

$$m_\varepsilon(K_1, K_2) := \sup\big\{ d(x_1, y_1) \vee d(x_2, y_2) : x_1, y_1 \in K_1,\ x_2, y_2 \in K_2, \\ d(x_1, x_2) \vee d(y_1, y_2) \leq \varepsilon,\ x_1 \preceq y_1,\ y_2 \preceq x_2\big\}. \tag{3.17}$$

Lemma 3.6 (Convergence of the mismatch modulus)**.** *Let $\mathcal{X}$ be a metrisable space. Assume that $K_n, K \in \mathcal{K}_{\rm part}(\mathcal{X})$ satisfy $d^{\langle 2\rangle}(K_n, K) \to 0$. Then*

$$m_{\varepsilon_n}(K_n, K) \underset{n\to\infty}{\longrightarrow} 0 \quad \textit{with} \quad \varepsilon_n := d^{\langle 1\rangle}(K_n, K). \tag{3.18}$$

Proof: If (3.18) does not hold, then, by going to a subsequence, we can assume that there exists a $\delta > 0$ such that $m_{\varepsilon_n}(K_n, K) \geq \delta$ for all n. It follows that for each n, we can find $x_1(n), y_1(n) \in K_n$ and $x_2(n), y_2(n) \in K$ with $d(x_1(n), y_1(n)) \vee d(x_2(n), y_2(n)) \geq \delta$ and $d(x_1(n), x_2(n)) \vee d(y_1(n), y_2(n)) \leq \varepsilon_n$ such that $x_1(n) \preceq y_1(n)$ and $y_2(n) \preceq x_2(n)$. By Lemma 3.2, our assumption $d^{\langle 2\rangle}(K_n, K) \to 0$ implies $\varepsilon_n = d^{\langle 1\rangle}(K_n, K) \to 0$ and hence $K_n \to K$. Therefore, by Lemma 2.10, there exists a compact set $C \subset \mathcal{X}$ such that $K_n \subset C$ for all n, so by going to a subsequence, we can assume that $x_1(n), x_2(n), y_1(n), y_2(n)$ converge to limits x_1, x_2, y_1, y_2 in $\mathcal{X}$. Since $d(x_1(n), x_2(n)) \vee d(y_1(n), y_2(n)) \leq \varepsilon_n \to 0$, we have $x := x_1 = x_2$ and $y := y_1 = y_2$. On the other hand, our assumption that $d(x_1(n), y_1(n)) \vee d(y_2(n), x_2(n)) \geq \delta$ implies that $d(x, y) \geq \delta$. This leads to a contradiction, since by the assumption that $d^{\langle 2\rangle}(K_n, K) \to 0$ and Lemma 3.1, $x_1(n) \preceq y_1(n)$ and $y_2(n) \preceq x_2(n)$ imply $x \preceq y$ and $y \preceq x$ and hence $x = y$. □

The following estimate essentially uses that the spaces are totally ordered.

Lemma 3.7 (Estimate in terms of mismatch modulus)**.** *Let $\mathcal{X}$ be a metrisable space and let $K_1, K_2 \in \mathcal{K}_{\rm tot}(\mathcal{X})$ satisfy $d^{\langle 1\rangle}(K_1, K_2) \leq \varepsilon$. Then*

$$d^{\langle m\rangle}(K_1, K_2) \leq m_\varepsilon(K_1, K_2) + \varepsilon \qquad (m \geq 1). \tag{3.19}$$

Proof: By symmetry, it suffices to show that for each $x_1 = (x_1^1, \ldots, x_1^m) \in K_1^{\langle m\rangle}$, there exists an $x_2 = (x_2^1, \ldots, x_2^m) \in K_2^{\langle m\rangle}$ such that $d^m(x_1, x_2) \leq m_\varepsilon(K_1, K_2) + \varepsilon$. In view of (3.2), the latter means that $d(x_1^k, x_2^k) \leq m_\varepsilon(K_1, K_2) + \varepsilon$ for all $1 \leq k \leq m$. Since $d^{\langle 1\rangle}(K_1, K_2) \leq \varepsilon$, there exists a $z(1) \in K_2$ such that $d(x_1^1, z(1)) \leq \varepsilon$. We set $x_2^i = z(1)$ for all $1 \leq i < I(1)$, where $I(1) := \inf\{i > 1 : d(x_1^i, z(1)) > m_\varepsilon(K_1, K_2) + \varepsilon\}$. Using again that $d^{\langle 1\rangle}(K_1, K_2) \leq \varepsilon$, there exists a $z(2) \in K_2$ such that $d(x_1^{I(1)}, z(2)) \leq \varepsilon$. Then

$$d\big(z(1), z(2)\big) \geq d\big(z(1), x_1^{I(1)}\big) - d\big(x_1^{I(1)}, z(2)\big) > m_\varepsilon(K_1, K_2) \tag{3.20}$$

and hence by the definition of $m_\varepsilon(K_1, K_2)$ and the fact that $x_1^1 \preceq x_1^{I(1)}$ we cannot have $z(2) \preceq z(1)$. Since K_2 is totally ordered, we conclude that $z(1) \prec z(2)$. This allows us to set $x_2^i = z(2)$ for all $I(1) \leq i < I(2)$, where $I(2) := \inf\{i > I(1) : d(x_1^i, z(2)) > m_\varepsilon(K_1, K_2) + \varepsilon\}$. Continuing inductively, we obtain $(x_2^1, \ldots, x_2^m) \in K_2^{\langle m\rangle}$ such that $d(x_1^k, x_2^k) \leq m_\varepsilon(K_1, K_2) + \varepsilon$ for all $1 \leq k \leq m$. □

As a consequence of Lemmas 3.6 and 3.7, we can prove that the metrics $d^{\langle m\rangle}$ with $2 \leq m \leq \infty$ all generate the same topology on $\mathcal{K}_{\rm tot}(\mathcal{X})$. The restriction to totally ordered sets is essential here: one can prove that these metrics generate different topologies on $\mathcal{K}_{\rm part}([0, 1])$.

Lemma 3.8 (Convergence of totally ordered sets)**.** *Let $\mathcal{X}$ be a metrisable space. Then $K_n, K \in \mathcal{K}_{\rm tot}(\mathcal{X})$ satisfy $d^{\langle 2\rangle}(K_n, K) \to 0$ if and only if $d^{\langle \infty\rangle}(K_n, K) \to 0$.*

Proof: Since $d^{\langle m\rangle}(K_n,K) \le d^{\langle m+1\rangle}(K_n,K)$ by Lemma 3.2 and since $d^{\langle\infty\rangle}(K_n,K)$ is defined as the increasing limit of $d^{\langle m\rangle}(K_n,K)$ as $m\to\infty$, it is clear that $d^{\langle\infty\rangle}(K_n,K)\to 0$ implies $d^{\langle 2\rangle}(K_n,K)\to 0$. To prove the opposite implication, it suffices to show that $d^{\langle 2\rangle}(K_n,K)\to 0$ implies

$$\sup_{m\ge 1} d^{\langle m\rangle}(K_n,K) \underset{n\to\infty}{\longrightarrow} 0. \tag{3.21}$$

By Lemma 3.2, $d^{\langle 2\rangle}(K_n,K)\to 0$ implies $\varepsilon_n := d^{\langle 1\rangle}(K_n,K)\to 0$. Lemmas 3.6 and 3.7 now imply that

$$\sup_{m\ge 1} d^{\langle m\rangle}(K_n,K) \le m_{\varepsilon_n}(K_n,K) + \varepsilon_n \underset{n\to\infty}{\longrightarrow} 0. \tag{3.22}$$

□

Proof of Theorem 1.14 (in part): By the definition of d_{part} and Proposition 3.4 we have $d_{\mathrm{part}} = d^{\langle 2\rangle}$ and $d_{\mathrm{tot}} = d^{\langle\infty\rangle}$. By Lemma 3.8 both metrics generate the same topology on $\mathcal{K}_{\mathrm{tot}}(\mathcal{X})$. If d and d' generate the same topology on $\mathcal{X}$ and d_{part} and d'_{part} are defined in terms of d and d' as in (1.33), then by Lemmas 2.11 and 2.12, d_{part} and d'_{part} generate the same topology on $\mathcal{K}_{\mathrm{tot}}(\mathcal{X})$. The inequalities (1.35) follow from the fact that $d^{\langle 1\rangle}(K_1,K_2) \le d^{\langle 2\rangle}(K_1,K_2) \le d^{\langle\infty\rangle}(K_1,K_2)$ by Lemma 3.2. On the other hand, Lemma 3.3 shows that if $\mathcal{X}=[0,1]$, then for each $\varepsilon>0$ we can find $K_1,K_2\in\mathcal{K}_{\mathrm{tot}}(\mathcal{X})$ such that

$$d^{\langle 2\rangle}(K_1,K_2)\le\varepsilon \quad \text{while} \quad 1/2 \le d^{\langle 3\rangle}(K_1,K_2) \le d^{\langle\infty\rangle}(K_1,K_2), \tag{3.23}$$

proving the final claim of the theorem. This proves all claims of the theorem except for the claim that $\mathcal{K}_{\mathrm{tot}}(\mathcal{X})$ is Polish if $\mathcal{X}$ is Polish, which will be proved in the next subsection. □

3.3. *Polishness.* In this subsection, we complete the proof of Theorem 1.14 by proving the following proposition.

Proposition 3.9 (Preservation of Polishness)**.** *If $\mathcal{X}$ is a Polish space, then so is $\mathcal{K}_{\mathrm{tot}}(\mathcal{X})$, equipped with the ordered Hausdorff topology.*

We start with the following lemma, announced in the introduction, that shows that even when $(\mathcal{X},d)$ is complete, it is in general not true that the metrics d_{part} and d_{tot} are complete on $\mathcal{K}_{\mathrm{tot}}(\mathcal{X})$.

Lemma 3.10 (Metric not complete)**.** *Let $\mathcal{X}=[0,1]$, equipped with the usual distance. Then the metrics $d^{\langle m\rangle}$ with $2\le m\le\infty$ are not complete on $\mathcal{K}_{\mathrm{tot}}(\mathcal{X})$.*

Proof: It suffices to construct a Cauchy sequence that does not converge. In view of Lemma 3.2, it suffices to construct a Cauchy sequence in the metric $d^{\langle\infty\rangle}$, which by Proposition 3.4 equals d_{tot}. Let ε_n be positive constants converging to zero, and let $K_n := \{0,1,\varepsilon_n\}$ equipped with a total order such that $0\prec 1\prec\varepsilon_n$. For each n,m, we define a monotone correspondence $R_{n,m}$ between K_n and K_m by $R_{n,m} := \{(0,0),(1,1),(\varepsilon_n,\varepsilon_m)\}$. Then

$$d_{\mathrm{tot}}(K_n,K_m) \le \sup_{(x_1,x_2)\in R_{n,m}} |x_1-x_2| = |\varepsilon_n-\varepsilon_m|, \tag{3.24}$$

so K_n is clearly a Cauchy sequence in d_{tot}. However, the sequence K_n does not converge in the ordered Hausdorff topology. If it had a limit K, then (in view of Lemma 3.1) this would have to be the set $K=\{0,1\}$ equipped with a total order such that $0\preceq 1$ and $1\preceq 0$, but such a totally ordered set does not exist. □

For each $L\in\mathcal{K}_+(\mathcal{X}^2)$ and $\varepsilon\ge 0$, we set

$$m_\varepsilon^{\langle 2\rangle}(L) := \sup\big\{d(x_1,y_1)\vee d(x_2,y_2) : (x_1,y_1),(y_2,x_2)\in L,\ d(x_1,x_2)\vee d(y_1,y_2)\le\varepsilon\big\}. \tag{3.25}$$

In particular, this implies $m_\varepsilon^{\langle 2\rangle}(K^{\langle 2\rangle}) = m_\varepsilon(K)$ $(K\in\mathcal{K}_{\mathrm{tot}}(\mathcal{X}))$.

Lemma 3.11 (Right continuity). *For any metric space $\mathcal{X}$ and $L \in \mathcal{K}_+(\mathcal{X}^2)$, the function $[0,\infty) \ni \varepsilon \to m_\varepsilon^{\langle 2\rangle}(L)$ is right continuous.*

Proof: The function $\varepsilon \mapsto m_\varepsilon^{\langle 2\rangle}(L)$ is clearly nondecreasing, so it suffices to prove that

$$m_\varepsilon^{\langle 2\rangle}(L) \geq \lim_{\eta\downarrow\varepsilon} m_\eta^{\langle 2\rangle}(L) \quad (\varepsilon \geq 0) \tag{3.26}$$

where the limit exist by monotonicity. Fix $\varepsilon_n > \varepsilon$ such that $\varepsilon_n \to \varepsilon$. Then for each $\delta > 0$ and for each n, we can choose $(x_1^n, y_1^n), (y_2^n, x_2^n) \in L$ such that $d(x_1^n, x_2^n) \vee d(y_1^n, y_2^n) \leq \varepsilon_n$ and $d(x_1^n, y_1^n) \vee d(x_2^n, y_2^n) \geq m_{\varepsilon_n}^{\langle 2\rangle}(L) - \delta$. Since L is compact, by going to a subsequence, we can assume that $(x_1^n, y_1^n) \to (x_1, y_1)$ and $(y_2^n, x_2^n) \to (y_2, x_2)$ for some $(x_1, y_1), (y_2, x_2) \in L$. Then $d(x_1, x_2) \vee d(y_1, y_2) \leq \varepsilon$ and $d(x_1, y_1) \vee d(x_2, y_2) \geq \lim_{\eta\downarrow\varepsilon} m_\eta^{\langle 2\rangle}(L) - \delta$, which proves that $m_\varepsilon^{\langle 2\rangle}(L) \geq \lim_{\eta\downarrow\varepsilon} m_\eta^{\langle 2\rangle}(L) - \delta$. Since $\delta > 0$ is arbitrary, this implies (3.26). □

Lemma 3.12 (Upper semi-continuity). *Let $\mathcal{X}$ be a metric space and let $L_n, L \in \mathcal{K}_+(\mathcal{X}^2)$ satisfy $L_n \to L$. Then*

$$m_\varepsilon^{\langle 2\rangle}(L) \geq \limsup_{n\to\infty} m_\varepsilon^{\langle 2\rangle}(L_n) \quad (\varepsilon \geq 0). \tag{3.27}$$

Proof: By the compactness of $[0,\infty]$ we can select a subsequence for which $\lim_{n\to\infty} m_\varepsilon^{\langle 2\rangle}(L_n)$ exists and is equal to the limit superior of the original sequence. Let $\delta_n > 0$ converge to zero and pick $(x_1^n, y_1^n), (y_2^n, x_2^n) \in L_n$ such that $d(x_1^n, x_2^n) \vee d(y_1^n, y_2^n) \leq \varepsilon$ and $d(x_1^n, y_1^n) \vee d(x_2^n, y_2^n) \geq m_\varepsilon^{\langle 2\rangle}(L_n) - \delta_n$. By Lemma 2.10, there exists a compact $C \subset \mathcal{X}^2$ such that $L_n \subset C$ for all n, so by going to a further subsequence we can assume that $(x_1^n, y_1^n) \to (x_1, y_1)$ and $(y_2^n, x_2^n) \to (y_2, x_2)$ for some $(x_1, y_1), (y_2, x_2) \in \mathcal{X}^2$. Then $(x_1, y_1), (y_2, x_2) \in L$ by Lemma 2.10, $d(x_1, x_2) \vee d(y_1, y_2) \leq \varepsilon$, and hence

$$m_\varepsilon^{\langle 2\rangle}(L) \geq d(x_1, y_1) \vee d(x_2, y_2) \geq \lim_{n\to\infty} \bigl(m_\varepsilon^{\langle 2\rangle}(L_n) - \delta_n\bigr). \tag{3.28}$$

Since $\delta_n \to 0$, this proves (3.27). □

Before we can prove Proposition 3.9 we need one more lemma. For any metric space $(\mathcal{X}, d)$, we define $\mathcal{L}(\mathcal{X}) \subset \mathcal{K}_+(\mathcal{X}^2)$ by

$$\mathcal{L}(\mathcal{X}) := \bigl\{K^{\langle 2\rangle} : K \in \mathcal{K}_{\mathrm{tot}}(\mathcal{X})\bigr\}, \tag{3.29}$$

and we let $\overline{\mathcal{L}(\mathcal{X})}$ denote the closure of $\mathcal{L}(\mathcal{X})$ in the metric space $\bigl(\mathcal{K}_+(\mathcal{X}^2), d_{\mathrm{H}}^2\bigr)$.

Lemma 3.13 (Totally ordered sets). *For any metric space $\mathcal{X}$, one has*

$$\mathcal{L}(\mathcal{X}) = \bigl\{L \in \overline{\mathcal{L}(\mathcal{X})} : m_0^{\langle 2\rangle}(L) = 0\bigr\}. \tag{3.30}$$

Proof: To prove the inclusion $\subset$ in (3.30), it suffices to observe that

$$m_0^{\langle 2\rangle}(K^{\langle 2\rangle}) = \sup\bigl\{d(x,y) : (x,y), (y,x) \in K^{\langle 2\rangle}\bigr\} = 0 \tag{3.31}$$

for all $K \in \mathcal{K}_{\mathrm{tot}}(\mathcal{X})$, since $x \preceq y$ and $y \preceq x$ imply $x = y$.

We next prove the inclusion $\supset$ in (3.30). Assume that $L \in \overline{\mathcal{L}(\mathcal{X})}$ satisfies $m_0^{\langle 2\rangle}(L) = 0$. Since $L \in \overline{\mathcal{L}(\mathcal{X})}$, there exist $K_n \in \mathcal{K}_{\mathrm{tot}}(\mathcal{X})$ such that $K_n^{\langle 2\rangle} \to L$ in the topology on $\mathcal{K}_+(\mathcal{X}^2)$. Let $\pi_i(x_1, x_2) := x_i$ $(i = 1, 2)$ denote the coordinate projections $\pi_i \colon \mathcal{X}^2 \to \mathcal{X}$. Since $\pi_1(K_n^{\langle 2\rangle}) = K_n = \pi_2(K_n^{\langle 2\rangle})$ for each n, using Lemma 2.14, we see that $K_n \to K$ in the Hausdorff topology on $\mathcal{K}_+(\mathcal{X})$, where $K := \pi_1(L) = \pi_2(L)$. We define a relation $\preceq$ on K by setting $x \preceq y$ if and only if $(x, y) \in L$. To complete the proof, it suffices to show that $\preceq$ is a total order on K, i.e.,

(i) for each $x, y \in K$, either $x \preceq y$ or $y \preceq x$, or both,
(ii) $x \preceq y$ and $y \preceq x$ imply $x = y$,
(iii) $x \preceq y \preceq z$ imply $x \preceq z$.

To prove (i), let $x, y \in K$. Since $K_n \to K$ in the Hausdorff topology on $\mathcal{K}_+(\mathcal{X})$, by Lemma 2.10, there exist $x_n, y_n \in K_n$ such that $x_n \to x$ and $y_n \to y$. Since K_n is totally ordered, either $x_n \preceq y_n$ happens for infinitely many n, or $y_n \preceq x_n$ happens for infinitely many n, or both. Since $K_n^{\langle 2\rangle} \to L$ in the Hausdorff topology on $\mathcal{K}_+(\mathcal{X}^2)$, by Lemma 2.10, it folows that either $x \preceq y$ or $y \preceq x$, or both. Property (ii) follows immediately from the fact that $\sup\{d(x,y) : (x,y),(y,x) \in L\} = m_0^{\langle 2\rangle}(L) = 0$. To prove (iii), assume that $x, y, z \in K$ satisfy $x \preceq y \preceq z$. If $x = y$ or $y = z$ then trivially $x \preceq z$, so without loss of generality we may assume that $x \neq y \neq z$. Since $K_n \to K$ in the Hausdorff topology on $\mathcal{K}_+(\mathcal{X})$, by Lemma 2.10, there exist $x_n, y_n, z_n \in K_n$ such that $x_n \to x$, $y_n \to y$, and $z_n \to z$. Since K_n is totally ordered, for each n either $x_n \preceq y_n$, or $y_n \preceq x_n$, or both. But $y_n \preceq x_n$ can happen only for finitely many n since otherwise the fact that $K_n^{\langle 2\rangle} \to L$ in the Hausdorff topology on $\mathcal{K}_+(\mathcal{X}^2)$ and Lemma 2.10 would imply that $y \preceq x$, which together with our assumptions $x \preceq y$ and $x \neq y$ contradicts (ii). We conclude that $x_n \preceq y_n$ for all n sufficiently large and by the same argument also $y_n \preceq z_n$ for all n sufficiently large. It follows that $x_n \preceq z_n$ for all n sufficiently large. Since $K_n^{\langle 2\rangle} \to L$ in the Hausdorff topology on $\mathcal{K}_+(\mathcal{X}^2)$, Lemma 2.10 shows that $(x,z) \in L$ and hence $x \preceq z$. □

We are now ready to prove Proposition 3.9. We need to recall one well-known fact. A subset $A \subset \mathcal{X}$ of a topological space $\mathcal{X}$ is called a *G_δ-set* if A is a countable intersection of open sets. Our proof of Proposition 3.9 makes use of the following fact, that we cite from Bourbaki (1958, §6 No. 1, Thm. 1) (see also Oxtoby, 1980, Thms 12.1 and 12.3).

Lemma 3.14 (Subsets of Polish spaces)**.** *A subset $\mathcal{Y} \subset \mathcal{X}$ of a Polish space $\mathcal{X}$ is Polish in the induced topology if and only if $\mathcal{Y}$ is a G_δ-subset of $\mathcal{X}$.*

Proof of Proposition 3.9: We first observe that if $\mathcal{X}$ is Polish, then so is $\mathcal{X}^2$, equipped with the product topology, and hence, by Lemma 2.11, also $\mathcal{K}_+(\mathcal{X}^2)$. Therefore, since $\big(\mathcal{K}_{\mathrm{tot}}(\mathcal{X}), d_{\mathrm{part}}\big)$ is isometric to $\big(\mathcal{L}(\mathcal{X}), d_{\mathrm{H}}^2\big)$ defined in (3.29), in view of Lemma 3.14, it suffices to show that $\mathcal{L}(\mathcal{X})$ is a G_δ-subset of $\mathcal{K}_+(\mathcal{X}^2)$.

It follows from Lemma 3.12 that for each $\varepsilon, \delta > 0$, the set

$$\mathcal{A}_{\delta,\varepsilon} := \big\{L \in \mathcal{K}_+(\mathcal{X}^2) : m_\varepsilon^{\langle 2\rangle}(L) \geq \delta\big\} \tag{3.32}$$

is a closed subset of $\mathcal{K}_+(\mathcal{X}^2)$ and hence its complement $\mathcal{A}_{\varepsilon,\delta}^{\mathrm{c}}$ is open. As a consequence,

$$\mathcal{G} := \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \mathcal{A}_{1/n,1/m}^{\mathrm{c}} \tag{3.33}$$

is a G_δ-set. Since each closed set is a G_δ-set, and the intersection of two G_δ-sets is a G_δ-set, using Lemmas 3.11 and 3.13, we conclude that

$$\begin{aligned} \overline{\mathcal{L}(\mathcal{X})} \cap \mathcal{G} &= \big\{L \in \overline{\mathcal{L}(\mathcal{X})} : \forall \delta > 0\ \exists \varepsilon > 0 \text{ s.t. } m_\varepsilon^{\langle 2\rangle}(L) < \delta\big\} \\ &= \big\{L \in \overline{\mathcal{L}(\mathcal{X})} : \lim_{\varepsilon \to 0} m_\varepsilon^{\langle 2\rangle}(L) = 0\big\} = \big\{L \in \overline{\mathcal{L}(\mathcal{X})} : m_0^{\langle 2\rangle}(L) = 0\big\} = \mathcal{L}(\mathcal{X}) \end{aligned} \tag{3.34}$$

is a G_δ-set. □

3.4. *Compactness criterion.*

Proof of Theorem 1.15: Since the map $K \mapsto K^{\langle 2\rangle}$ is a homeomorphism from $\mathcal{K}_{\mathrm{tot}}(\mathcal{X})$ to $\mathcal{L}(\mathcal{X})$, equipped with the induced topology from $\mathcal{K}_+(\mathcal{X}^2)$, a set $\mathcal{A} \subset K^{\langle 2\rangle}$ is precompact if and only if $\mathcal{B} := \{K^{\langle 2\rangle} : K \in \mathcal{A}\}$ is a precompact subset of $\mathcal{K}_+(\mathcal{X}^2)$ and its closure $\overline{\mathcal{B}}$ is contained in $\mathcal{L}(\mathcal{X})$. We will show that $\mathcal{B}$ is a precompact subset of $\mathcal{K}_+(\mathcal{X}^2)$ if and only if (1.37) (i) holds. Moreover, if (1.37) (i) holds, then $\overline{\mathcal{B}}$ is contained in $\mathcal{L}(\mathcal{X})$ if and only if (1.37) (ii) holds.

If (1.37) (i) holds, then C^2 is a compact subset of $\mathcal{X}^2$ and $K^{\langle 2\rangle} \subset C^2$ for all $K \in \mathcal{A}$, so $\mathcal{B}$ is a precompact subset of $\mathcal{K}_+(\mathcal{X}^2)$ by Lemma 2.12. Conversely, if $\mathcal{B}$ is a precompact subset of $\mathcal{K}_+(\mathcal{X}^2)$, then by Lemma 2.12 there exists a compact subset $D \subset \mathcal{X}^2$ such that $K^{\langle 2\rangle} \subset D$ for all $K \in \mathcal{A}$. Without loss of generality, we may assume that D is of the form $D = C^2$ for some compact subset C of $\mathcal{X}$; for example, we may take for C the union of the two coordinate projections of D. Then $K \subset C$ for all $K \in \mathcal{A}$, proving that (1.37) (i) holds.

To complete the proof, assume that (1.37) (i) holds. We must show that $\overline{\mathcal{B}}$ is contained in $\mathcal{L}(\mathcal{X})$ if and only if (1.37) (ii) holds. Assume, first, that (1.37) (ii) does not hold. Then we can find a $\delta > 0$ and $\varepsilon_n > 0$ tending to zero, as well as $K_n \in \mathcal{A}$, such that $m^{\langle 2\rangle}_{\varepsilon_n}(K_n) \geq \delta$ for each n. By (1.37) (i), going to a subsequence if necessary, we can assume that $K_n^{\langle 2\rangle} \to L$ for some $L \in \mathcal{K}_+(\mathcal{X}^2)$. Now Lemma 3.12 implies that

$$m^{\langle 2\rangle}_{\varepsilon}(L) \geq \limsup_{n\to\infty} m^{\langle 2\rangle}_{\varepsilon}(K_n^{\langle 2\rangle}) \geq \limsup_{n\to\infty} m^{\langle 2\rangle}_{\varepsilon_n}(K_n^{\langle 2\rangle}) \geq \delta \tag{3.35}$$

for each $\varepsilon > 0$, so letting $\varepsilon \downarrow 0$, using Lemma 3.11, we conclude that $m^{\langle 2\rangle}_0(L) \geq \delta$. By Lemma 3.13, we conclude that $L \notin \mathcal{L}(X)$ and hence $\overline{\mathcal{B}}$ is not contained in $\mathcal{L}(\mathcal{X})$.

Assume now that (1.37) (ii) holds. We must show that $\overline{\mathcal{B}}$ is contained in $\mathcal{L}(\mathcal{X})$. Assume that $K_n^{\langle 2\rangle} \to L$ for some $K_n \in \mathcal{A}$. Then clearly $L \in \overline{\mathcal{L}(\mathcal{X})}$ so by Lemma 3.13 it suffices to prove that $m^{\langle 2\rangle}_0(L) = 0$. Assume that, conversely, there exist $x, y \in \mathcal{X}$ with $x \neq y$ such that $(x, y), (y, x) \in L$. Then by Lemma 2.10, there exist $x_1^n, x_2^n, y_1^n, y_2^n \in K_n$ with $x_1^n \preceq y_1^n$ and $y_2^n \preceq x_2^n$ such that $x_i^n \to x$ and $y_i^n \to y$ as $n \to \infty$ $(i = 1, 2)$. Then for each $\varepsilon > 0$, we can choose n large enough such that $d(x_i^n, x) \leq \varepsilon/2$ $(i = 1, 2)$. It follows that $d(x_1^n, y_1^n) \vee d(x_2^n, y_2^n) \geq d(x, y) - \varepsilon$ and $d(x_1^n, x_2^n) \vee d(y_1^n, y_2^n) \leq \varepsilon$ so that $m_\varepsilon(K_n) \geq d(x, y) - \varepsilon$. This clearly contradicts (1.37) (ii), so we conclude that $m^{\langle 2\rangle}_0(L) = 0$ as required. □

3.5. *Cadlag curves.* In this subsection, we prove Proposition 1.16. Let $(\mathcal{X}, d)$ be a metric space. If R is any subset of $\mathcal{X}^2$, then let us call

$$\mathrm{dist}(R) := \sup_{(x_1,x_2)\in R} d(x_1, x_2) \tag{3.36}$$

the *distortion* of R. Then (2.7) and (1.34) say that

$$d_{\mathrm{H}}(K_1, K_2) = \inf_{R\in \mathrm{Corr}(K_1,K_2)} \mathrm{dist}(R) \quad \text{and} \quad d_{\mathrm{tot}}(K_1, K_2) = \inf_{R\in \mathrm{Corr}_+(K_1,K_2)} \mathrm{dist}(R), \tag{3.37}$$

where $\mathrm{Corr}(K_1, K_2)$ and $\mathrm{Corr}_+(K_1, K_2)$ denote the sets of all (monotone) correspondences between K_1 and K_2. Let $\overline{R}$ denote the closure of a set $R \subset \mathcal{X}^2$. Then

$$\mathrm{dist}(R) = \mathrm{dist}(\overline{R}) \qquad (R \subset \mathcal{X}^2). \tag{3.38}$$

Indeed, the inequality $\leq$ is trivial, while the opposite inequality follows from the fact that for each $(x_1, x_2) \in \overline{R}$, there exist $(x_1^n, x_2^n) \in R$ such that $(x_1^n, x_2^n) \to (x_1, x_2)$ and hence $d(x_1^n, x_2^n) \to d(x_1, x_2)$.

We need one preparatory lemma.

Lemma 3.15 (Fine partition)**.** *Let $(\mathcal{X}, d)$ be a metric space. Then for each $\gamma \in \mathcal{D}_{[0,1]}(\mathcal{X})$ and $\varepsilon > 0$, there exist $t_0 < 0 < t_1 < \cdots < t_{n-1} < 1 < t_n$ such that*

$$\sup\bigl\{d\bigl(\gamma(s), \gamma(s')\bigr) : 1 \leq k \leq n,\ s, t \in [0, 1] \cap [t_{k-1}, t_k)\bigr\} < \varepsilon. \tag{3.39}$$

Proof: By Lemma 2.3, writing $\gamma(t+) := \gamma(t)$, we can view γ as a continuous function on the split real interval $[\![0-, 1+]\!]$, that moreover satisfies $\gamma(0-) = \gamma(0+)$ and $\gamma(1-) = \gamma(1+)$. Fix $\varepsilon > 0$ and let

$$R := \bigl\{(s, t) \in \mathbb{R}^2 : s < t,\ s \neq 0,\ t \neq 1,\ d\bigl(\gamma(\sigma), \gamma(\tau)\bigr) < \varepsilon\ \forall \sigma, \tau \in [\![s+, t-]\!] \cap [\![0-, 1+]\!]\bigr\}. \tag{3.40}$$

Using the properties of γ, it is easy to see that

$$\bigcup_{(s,t)\in R} [\![s+,t-]\!] \supset [\![0-,1+]\!]. \tag{3.41}$$

Since the intervals $[\![s+,t-]\!]$ are open in the topology of the split real line and since $[\![0-,1+]\!]$ is compact by Proposition 2.4, there exists a finite subcover, i.e. a finite set $S \subset R$ such that (3.41) holds with R replaced by S. Let $T := \{s : (s,t) \in S\} \cup \{t : (s,t) \in S\}$. Then, letting t_0 denote the largest element of $T \cap (-\infty, 0)$, ordering the elements of $T \cap (0,1)$ as $t_1 < \cdots < t_{n-1}$, and letting t_n denote the smallest element of $T \cap (1,\infty)$, we obtain times $t_0 < \cdots < t_n$ as in (3.39). □

Proof of Proposition 1.16: If γ_1, γ_2 are cadlag parametrisations of K_1, K_2, and $\lambda \in \Lambda$, then let us set

$$R_\lambda := \big\{\big(\gamma_1(t), \gamma_2\big(\lambda(t)\big)\big) : t \in [0,1]\big\} = \big\{\big(\gamma_1\big(\lambda^{-1}(t)\big), \gamma_2(t)\big) : t \in [0,1]\big\}, \tag{3.42}$$

and let $\overline{R}_\lambda$ denote its closure. We claim that $\overline{R}_\lambda$ is a correspondence between K_1 and K_2. Indeed, by the definition of a cadlag parametrisation, each element $x_1 \in K_1$ is of the form $x_1 = \gamma_1(t)$ or $= \gamma_1(t-)$ for some $t \in [0,1]$. If $x_1 = \gamma_1(t)$, then clearly there exists an $x_2 \in K_2$ such that $(x_1, x_2) \in R_\lambda$, namely $x_2 := \gamma_2(\lambda(t))$. If $x_1 = \gamma_1(t-)$, then we can choose $t_n \uparrow t$ and set $x_1^n := \gamma_1(t_n)$. Then $x_1^n \to x_1$ by the left continuity of $t \mapsto \gamma_1(t-)$. We have already seen that there exist $x_2^n \in K_2$ such that $(x_1^n, x_2^n) \in R_\lambda$. Since K_2 is compact, by going to a subsequence, we can assume that $x_2^n \to x_2$ for some $x_2 \in K_2$. Then $(x_1, x_2) \in \overline{R}_\lambda$. In the same way, we see that for each $x_2 \in K_2$, there exists an $x_1 \in K_1$ such that $(x_1, x_2) \in \overline{R}_\lambda$. This completes the proof that $\overline{R}_\lambda$ is a correspondence. Using the fact that the total orders on K_1 and K_2 are compatible with the topology, it is easy to see that $\overline{R}_\lambda$ is monotone if $\lambda \in \Lambda_+$.

Using these facts as well as (3.37) and (3.38), we see that

$$\begin{aligned} d_{\mathrm{H}}(K_1,K_2) &\le \inf_{\lambda\in\Lambda} \operatorname{dist}(\overline{R}_\lambda) = \inf_{\lambda\in\Lambda} \operatorname{dist}(R_\lambda) = \inf_{\lambda\in\Lambda} \sup_{t\in[0,1]} d\big(\gamma_1(t), \gamma_2\big(\lambda(t)\big)\big), \\ d_{\mathrm{tot}}(K_1,K_2) &\le \inf_{\lambda\in\Lambda_+} \operatorname{dist}(\overline{R}_\lambda) = \inf_{\lambda\in\Lambda_+} \operatorname{dist}(R_\lambda) = \inf_{\lambda\in\Lambda_+} \sup_{t\in[0,1]} d\big(\gamma_1(t), \gamma_2\big(\lambda(t)\big)\big). \end{aligned} \tag{3.43}$$

To complete the proof, we must show that:

(i) For each $R \in \operatorname{Corr}(K_1,K_2)$ and $\varepsilon > 0$, there exists a $\lambda \in \Lambda$ such that $\operatorname{dist}(R_\lambda) \le \operatorname{dist}(R) + \varepsilon$.
(ii) For each $R \in \operatorname{Corr}_+(K_1,K_2)$ and $\varepsilon > 0$, there exists a $\lambda \in \Lambda_+$ such that $\operatorname{dist}(R_\lambda) \le \operatorname{dist}(R) + \varepsilon$.

We first prove (i). Fix $R \in \operatorname{Corr}(K_1,K_2)$ and $\varepsilon > 0$. By Lemma 3.15, for $i = 1,2$, there exist $t_0^i < 0 < t_1^i < \cdots < t_{n_i-1}^i < 1 < t_{n_i}^i$ such that

$$\sup\big\{d\big(\gamma_i(s), \gamma_i(s')\big) : 1 \le k \le n_i,\ s,t \in [0,1] \cap [t_{k-1}^i, t_k^i)\big\} < \varepsilon/2 \quad (i = 1,2). \tag{3.44}$$

For $1 \le k \le n_i$, let us write $K_k^i := \{\gamma_i(t) : t \in [0,1] \cap [t_{k-1}^i, t_k^i)\}$ $(i = 1,2)$. We can choose a correspondence S between $\{1,\dots,n_1\}$ and $\{1,\dots,n_2\}$ such that for each $(k_1,k_2) \in S$, there exists an $(x_1,x_2) \in R$ with $x_i \in K_{k_i}^i$ $(i = 1,2)$. Then

$$\sup_{(k_1,k_2)\in S} \sup_{\substack{x_1\in K_1\\ x_2\in K_2}} d(x_1,x_2) \le \operatorname{dist}(R) + \varepsilon. \tag{3.45}$$

By refining the partitions $t_0^i, \dots, t_{n_i}^i$, we can make sure that for each $k_1 \in \{1,\dots,n_1\}$, there is a unique $k_2 \in \{1,\dots,n_2\}$ such that $(k_1,k_2) \in S$, and vice versa. We can then construct a bijection $\lambda\colon [0,1] \to [0,1]$ such that for each $(k_1,k_2) \in S$, the restriction of λ to $[0,1] \cap [t_{k_1-1}^1, t_{k_1}^1)$ is a bijection to $[0,1] \cap [t_{k_2-1}^2, t_{k_2}^2)$. Then (3.45) implies that $\operatorname{dist}(R_\lambda) \le \operatorname{dist}(R) + \varepsilon$, completing the proof of (i).

To also prove (ii), we observe that if R is a monotone correspondence, then S as we initially constructed it will be a monotone correspondence between $\{1,\dots,n_1\}$ and $\{1,\dots,n_2\}$, and monotonicity will be preserved after we refine the partitions so that they have the same size and S is a bijection. Now λ can be chosen monotone too, completing the proof of (ii). □

4. Proofs of the main results

4.1. *Metrics on path space.* In this section we prove Theorems 1.4 and 1.9. We fix a metrisable space $\mathcal{X}$ on which there is defined a betweenness that is compatible with the topology. Recall that $\overline{\pi}$ denotes the filled path associated with a path $\pi \in \Pi(\mathcal{X})$. By Lemmas 2.22, 2.23, and 2.25,

$$\begin{aligned}\overline{\Pi}(\mathcal{X}) &:= \bigl\{\overline{\pi} : \pi \in \Pi(\mathcal{X})\bigr\}\\ &= \bigl\{K \in \mathcal{K}_{\mathrm{tot}}(\mathcal{R}(\mathcal{X})) : K \text{ satisfies conditions (i) and (ii) of Lemma 2.23}\bigr\}.\end{aligned} \tag{4.1}$$

By Lemma 2.23, a path π is uniquely characterised by its filled graph $\overline{\pi}$, so given a metric d_{sqz} on the squeezed space $\mathcal{R}(\mathcal{X})$, we can define metrics d_{tot} and d_{part} on $\Pi(\mathcal{X})$ by

$$d_{\mathrm{tot}}(\pi_1,\pi_2) := d_{\mathrm{tot}}(\overline{\pi}_1,\overline{\pi}_2) \quad \text{and} \quad d_{\mathrm{part}}(\pi_1,\pi_2) := d_{\mathrm{part}}(\overline{\pi}_1,\overline{\pi}_2), \tag{4.2}$$

where the metrics on the right-hand side of each equation are those on $\mathcal{K}_{\mathrm{tot}}(\mathcal{R}(\mathcal{X}))$ defined in (1.33) and (1.34), and those on the left-hand side of each equation coincide with those on $\Pi(\mathcal{X})$ already defined in (1.22). Similarly, $d_{\mathrm{H}}(\pi_1,\pi_2)$ defined in (1.21) is the Hausdorff distance between $\overline{\pi}_1$ and $\overline{\pi}_2$, which is in general only a pseudometric. Since the total order $\preceq$ on a continuous path $\pi \in \Pi_{\mathrm{c}}(\mathcal{X})$ is uniquely determined by the compact set $\pi \subset \mathcal{R}(\mathcal{X})$, however, d_{H} is a metric on $\Pi_{\mathrm{c}}(\mathcal{X})$. For the same reason, if $I \subset \mathbb{R}$ is a closed interval of positive length, then d_{H} is a metric on $\mathcal{D}_I(\mathcal{X})$, where we identify the latter with a subset of $\Pi(\mathcal{X})$ as in (1.7). The following proposition summarises what we already know, as an immediate consequence of Theorem 1.14.

Proposition 4.1 (Basic properties of the metrics)**.** *Assume that $\mathcal{X}$ is a metrisable space that is equipped with a proper betweenness. Let d_{H}, d_{tot}, and d_{part} be defined as in (1.21) and (1.22). Then*

$$d_{\mathrm{H}}(\pi_1,\pi_2) \le d_{\mathrm{part}}(\pi_1,\pi_2) \le d_{\mathrm{tot}}(\pi_1,\pi_2) \qquad \bigl(\pi_1,\pi_2 \in \Pi(\mathcal{X})\bigr). \tag{4.3}$$

The functions d_{tot} and d_{part} are metrics on $\Pi(\mathcal{X})$ that generate the same topology. This topology does not depend on the choice of the metric d_{sqz} on the squeezed space $\mathcal{R}(\mathcal{X})$.

Proposition 4.2 (Space of continuous paths)**.** *Assume that $\mathcal{X}$ is a metrisable space that is equipped with a proper betweenness. Let d_{H}, d_{tot}, and d_{part} be defined as in (1.21) and (1.22). Then for paths $\pi_n \in \Pi(\mathcal{X})$ and $\pi \in \Pi_{\mathrm{c}}(\mathcal{X})$, the following statements are equivalent:*

$$\text{(i) } d_{\mathrm{part}}(\pi_n,\pi) \underset{n\to\infty}{\longrightarrow} 0, \quad \text{(ii) } d_{\mathrm{tot}}(\pi_n,\pi) \underset{n\to\infty}{\longrightarrow} 0, \quad \text{(iii) } d_{\mathrm{H}}(\pi_n,\pi) \underset{n\to\infty}{\longrightarrow} 0.$$

Hence, these metrics all generate the same topology on $\Pi_{\mathrm{c}}(\mathcal{X})$.

Proof: By Theorem 1.14, d_{part} and d_{tot} generate the same topology on $\Pi(\mathcal{X})$, and by (4.3) convergence in any of these two metrics implies convergence in d_{H}. Therefore, to show that the conditions (i)–(iii) are equivalent, it suffices to show that for $\pi_n \in \Pi(\mathcal{X})$ and $\pi \in \Pi_{\mathrm{c}}(\mathcal{X})$, condition (iii) implies (i). More precisely, we will show that for the filled paths defined in (1.20), $\overline{\pi}_n \to \overline{\pi}$ in the Hausdorff topology implies $\overline{\pi}_n^{\langle 2\rangle} \to \overline{\pi}^{\langle 2\rangle}$ in the Hausdorff topology. By Lemma 2.10, convergence of $\overline{\pi}_n$ implies the existence of a compact set $C \subset \mathcal{R}(\mathcal{X})$ such that $\overline{\pi}_n \subset C$ for all n, which implies $\overline{\pi}_n^{\langle 2\rangle} \subset C^2$. To complete the proof, by Lemma 2.10, we need to prove the following two statements.

(i) For every $\bigl((x,s),(y,t)\bigr) \in \overline{\pi}^{\langle 2\rangle}$, there exist $\bigl((x_n,s_n),(y_n,t_n)\bigr) \in \overline{\pi}_n^{\langle 2\rangle}$ such that $\bigl((x_n,s_n),(y_n,t_n)\bigr) \to \bigl((x,s),(y,t)\bigr)$.

(ii) If a sequence $\bigl((x_n,s_n),(y_n,t_n)\bigr) \in \overline{\pi}_n^{\langle 2\rangle}$ has a cluster point $\bigl((x,s),(y,t)\bigr) \in \mathcal{R}(\mathcal{X})^2$, then $\bigl((x,s),(y,t)\bigr) \in \overline{\pi}^{\langle 2\rangle}$.

To prove (i), we use the fact that $\overline{\pi}_n \to \overline{\pi}$ to find $(x_n, s_n), (y_n, t_n) \in \overline{\pi}_n$ such that $(x_n, s_n) \to (x, s)$ and $(y_n, t_n) \to (y, t)$. If $s < t$, then $\big((x_n, s_n), (y_n, t_n)\big) \in \overline{\pi}_n^{\langle 2\rangle}$ for n large enough, so (i) follows. On the other hand, if $s = t$, then $\big((x_n, s_n), (x_n, s_n)\big) \in \overline{\pi}_n^{\langle 2\rangle}$ so (i) also holds in this case.

To prove (ii), we use the fact that $\overline{\pi}_n \to \overline{\pi}$ to conclude that any cluster point $\big((x, s), (y, t)\big)$ satisfies $(x, s), (y, t) \in \overline{\pi}$ with $s \le t$, and hence by the continuity of π either $s < t$ or $(x, s) = (y, t)$, from which we conclude that $\big((x, s), (y, t)\big) \in \overline{\pi}^{\langle 2\rangle}$. □

4.2. *Polishness.* In this subsection we prove that if $\mathcal{X}$ is Polish and equipped with a betweenness that is compatible with the topology, then so are $\Pi(\mathcal{X})$, $\Pi_{\mathrm{c}}(\mathcal{X})$, and $\mathcal{D}_I(\mathcal{X})$ equipped with the Skorokhod topology corresponding to the betweenness on $\mathcal{X}$. Recall the definition of the modulus of continuity $m_{T,\delta}(\pi)$ in (1.25). By a slight abuse of notation, for any $K \in \mathcal{K}_+(\mathcal{R}(\mathcal{X}))$, we set

$$m_{T,\delta}(K) := \sup\big\{d(x_1, x_2) : (x_i, t_i) \in K,\ -T \le t_1 \le t_2 \le T,\ t_2 - t_1 \le \delta\big\}. \tag{4.4}$$

Lemma 4.3 (Upper semi-continuity)**.** *Let $\mathcal{X}$ be a metrisable space and assume that $K_n, K \in \mathcal{K}_+(\mathcal{R}(\mathcal{X}))$ satisfy $K_n \to K$. Then, for each $T < \infty$ and $\delta > 0$,*

$$m_{T,\delta}(K) \ge \limsup_{n\to\infty} m_{T,\delta}(K_n). \tag{4.5}$$

Proof: By the compactness of $[0, \infty]$ we can select a subsequence for which $\lim_{n\to\infty} m_{T,\delta}(K_n)$ exists and is equal to the limit superior of the original sequence. Let $\varepsilon_n > 0$ converge to zero and pick $(x_i^n, t_i^n) \in K_n$ with $-T \le t_1^n \le t_2^n \le T$ and $t_2^n - t_1^n \le \delta$, such that $d(x_1^n, x_2^n) \ge m_{T,\delta}(K_n) - \varepsilon_n$. Since $K_n \to K$, by Lemma 2.10 there exists a compact $C \subset \mathcal{R}(\mathcal{X})$ such that $K_n \subset C$ for all n. It follows that we can select a subsequence such that $(x_i^n, t_i^n) \to (x_i, t_i)$ for some $(x_i, t_i) \in K$ $(i = 1, 2)$. Then $-T \le t_1 \le t_2 \le T$ and $t_2 - t_1 \le \delta$, so

$$m_{T,\delta}(K) \ge d(x_1, x_2) \ge \lim_{n\to\infty} \big(m_{T,\delta}(K_n) - \varepsilon_n\big) = \lim_{n\to\infty} m_{T,\delta}(K_n). \tag{4.6}$$

Since we have chosen our subsequence such that the right-hand side is equal to the limit superior of the original sequence, this proves the claim. □

Proposition 4.4 (The space of continuous paths is Polish)**.** *If $\mathcal{X}$ is a Polish space, then so is $\Pi_{\mathrm{c}}(\mathcal{X})$, equipped with the topology generated by d_{H}.*

Proof: By Lemma 2.26 (using also Lemma 2.24), we may identify $\Pi_{\mathrm{c}}(\mathcal{X})$ with the set of all $G \in \mathcal{K}_+(\mathcal{R}(\mathcal{X}))$ such that

$$\big|\{x \in \mathcal{X} : (x, t) \in G\}\big| \le 1 \quad \forall t \in \mathbb{R}. \tag{4.7}$$

We claim that this condition is equivalent to

$$\lim_{\delta\to 0} m_{T,\delta}(G) = 0\ \forall T < \infty. \tag{4.8}$$

If for some $t \in [-T, T]$ there exist $x_1 \ne x_2$ such that $(x_1, t), (x_2, t) \in G$, then $m_{T,\delta}(G) \ge d(x_1, x_2)$ for all $\delta > 0$ so (4.8) is violated. Conversely, if (4.8) does not hold then for some $\varepsilon > 0$ we can choose $(x_i^n, t_i^n) \in G$ with $t_i^n \in [-T, T]$ $(i = 1, 2)$ such that $d(x_1^n, x_2^n) \ge \varepsilon$ for all n. Taking the limit, using the compactness of G, we see that (4.7) is violated.

By Lemma 4.3, for each $T < \infty$ and $\varepsilon, \delta > 0$, the set $\mathcal{G}_{T,\varepsilon,\delta} := \{G \in \mathcal{K}_+(\mathcal{R}(\mathcal{X})) : m_{T,\delta}(G) \ge \varepsilon\}$ is a closed subset of $\mathcal{K}_+(\mathcal{X}^2)$ and hence its complement $\mathcal{G}^{\mathrm{c}}_{T,\varepsilon,\delta}$ is open. As a consequence, using (4.8), we see that

$$\Pi_{\mathrm{c}}(\mathcal{X}) = \bigcap_{N=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \mathcal{G}^{\mathrm{c}}_{N,1/n,1/m} \tag{4.9}$$

is a G_δ-subset of $\mathcal{K}_+(\mathcal{R}(\mathcal{X}))$ and hence Polish in the induced topology by Lemma 3.14. □

Recall the definition of the Skorokhod modulus of continuity $m^{\mathrm{S}}_{T,\delta}(\pi)$ in (1.28). By a slight abuse of notation, for any $K \in \mathcal{K}_{\mathrm{tot}}(\mathcal{R}(\mathcal{X}))$, we set

$$m^{\mathrm{S}}_{T,\delta}(K) := \sup\big\{d\big(x_2, \langle x_1, x_3\rangle\big) : (x_i, t_i) \in K \text{ and } -T \le t_i \le T \ \forall i = 1,2,3, \\ (x_1,t_1) \preceq (x_2,t_2) \preceq (x_3,t_3),\ t_3 - t_1 \le \delta\big\}. \tag{4.10}$$

With this definitions, if $\overline{\pi}$ is the filled path associated with a path $\pi \in \Pi(\mathcal{X})$, then $m^{\mathrm{S}}_{T,\delta}(\overline{\pi})$ as defined in (4.10) corresponds to $m^{\mathrm{S}}_{T,\delta}(\pi)$ as defined (1.28). The following lemma is similar to Lemmas 3.12 and 4.3.

Lemma 4.5 (Upper semi-continuity)**.** *Let $\mathcal{X}$ be a metrisable space that is equipped with a proper betweenness, and assume that $K_n, K \in \mathcal{K}_{\mathrm{tot}}(\mathcal{R}(\mathcal{X}))$ satisfy $K_n \to K$. Then, for each $T < \infty$ and $\delta > 0$,*

$$m^{\mathrm{S}}_{T,\delta}(K) \ge \limsup_{n\to\infty} m^{\mathrm{S}}_{T,\delta}(K_n). \tag{4.11}$$

Proof: The proof will be very similar to the proof of Lemma 3.12. By the compactness of $[0,\infty]$ we can select a subsequence for which $\lim_{n\to\infty} m^{\mathrm{S}}_{T,\delta}(K_n)$ exists and is equal to the limit superior of the original sequence. Let $\varepsilon_n > 0$ converge to zero and pick $(x_i^n, t_i^n) \in K_n$ with $-T \le t_i^n \le T$ $(i = 1,2,3)$, $(x_1^n,t_1^n) \preceq (x_2^n,t_2^n) \preceq (x_3^n,t_3^n)$, and $t_3^n - t_1^n \le \delta$, such that $d\big(x_2^n, \langle x_1^n, x_3^n\rangle\big) \ge m^{\mathrm{S}}_{T,\delta}(K_n) - \varepsilon_n$. Since $K_n \to K$ in the topology on $\mathcal{K}_{\mathrm{tot}}(\mathcal{R}(\mathcal{X}))$, by the first inequality in (1.35) they also converge in the topology on $\mathcal{K}_+(\mathcal{R}(\mathcal{X}))$, so by Lemma 2.10 there exists a compact $C \subset \mathcal{R}(\mathcal{X})$ such that $K_n \subset C$ for all n. It follows that we can select a subsequence such that $(x_i^n, t_i^n) \to (x_i, t_i)$ for some $(x_i,t_i) \in K$ $(i = 1,2,3)$. Recall from Proposition 3.4 that $d_{\mathrm{tot}} = d^{\langle\infty\rangle}$ so by Lemma 3.2, convergence in $\mathcal{K}_{\mathrm{tot}}(\mathcal{R}(\mathcal{X}))$ implies that $K_n^{\langle m\rangle} \to K^{\langle m\rangle}$ in the Hausdorff topology for any $1 \le m \le \infty$. In particular, $K_n^{\langle 3\rangle} \to K^{\langle 3\rangle}$, which by Lemma 2.10 implies $(x_1,t_1) \preceq (x_2,t_2) \preceq (x_3,t_3)$. Moreover $-T \le t_i \le T$ $(i = 1,2,3)$ and $t_3 - t_1 \le \delta$, so

$$m^{\mathrm{S}}_{T,\delta}(K) \ge d\big(x_2, \langle x_1, x_3\rangle\big) \ge \lim_{n\to\infty} \big(m^{\mathrm{S}}_{T,\delta}(K_n) - \varepsilon_n\big) = \lim_{n\to\infty} m^{\mathrm{S}}_{T,\delta}(K_n), \tag{4.12}$$

where we have used that $d\big(x_2^n, \langle x_1^n, x_3^n\rangle\big) \to d\big(x_2, \langle x_1, x_3\rangle\big)$ since the betweenness is compatible with the topology. Since we have chosen our subsequence such that the right-hand side is equal to the limit superior of the original sequence, this proves the claim. □

Proposition 4.6 (Skorokhod topologies are Polish)**.** *Let $\mathcal{X}$ be a metrisable space equipped with a betweenness that is compatible with the topology. Then if $\mathcal{X}$ is Polish, so is $\Pi(\mathcal{X})$, equipped with the Skorokhod topology corresponding to the betweenness.*

Proof: We observe that if $\mathcal{X}$ is Polish, then, by Lemma 2.7, so is $\mathcal{R}(\mathcal{X})$ and hence, by Proposition 3.9, also $\mathcal{K}_{\mathrm{tot}}(\mathcal{R}(\mathcal{X}))$. By identifying a path with its filled path, we can as in (4.1) identity $\Pi(\mathcal{X})$ with a subset of $\mathcal{K}_{\mathrm{tot}}(\mathcal{R}(\mathcal{X}))$. The Skorokhod topology on $\Pi(\mathcal{X})$ is then the induced topology from $\mathcal{K}_{\mathrm{tot}}(\mathcal{R}(\mathcal{X}))$. Therefore, in view of Lemma 3.14, it suffices to show that the set $\overline{\Pi}(\mathcal{X})$ defined in (4.1) is a G_δ-subset of $\mathcal{K}_{\mathrm{tot}}(\mathcal{R}(\mathcal{X}))$.

We start by showing that condition (ii) of Lemma 2.23 can be replaced by

(ii)' $\lim_{\delta\to 0} m^{\mathrm{S}}_{T,\delta}(G) = 0 \ \forall T < \infty$.

To see this, we argue as follows. If (ii) does not hold, then for some $t \in \mathbb{R}$, there exist $(x_i,t_i) \in G$ $(i = 1,2,3)$ with $(x_1,t_1) \preceq (x_2,t_2) \preceq (x_3,t_3)$ and $x_2 \notin \langle x_1, x_3\rangle$, which implies that $m^{\mathrm{S}}_{T,\delta}(G) \ge d\big(x_2, \langle x_1, x_3\rangle\big) > 0$ for all $\delta > 0$ and $T < \infty$ such that $-T \le t \le T$, so (ii)' clearly does not hold. Conversely, if (ii)' does not hold, then for some $T < \infty$ and $\varepsilon > 0$ we can choose $\delta_n > 0$ tending to zero and $(x_i^n, t_i^n) \in G$ $(i = 1,2,3)$ with $(x_1^n,t_1^n) \preceq (x_2^n,t_2^n) \preceq (x_3^n,t_3^n)$ such that $d\big(x_2^n, \langle x_1^n, x_3^n\rangle\big) \ge \varepsilon$. By the compactness of G, we can select a subsequence such that $(x_i^n, t_i^n) \to (x_i, t_i)$ $(i = 1,2,3)$. Then clearly $t_1 = t_2 = t_3 =: t$ for some $-T \le t \le T$. Since the order is compatible with the

topology moreover $(x_1,t_1) \preceq (x_2,t_2) \preceq (x_3,t_3)$. The fact that the betweenness is compatible with the topology allows us to conclude that $d\big(x_2, \langle x_1, x_3\rangle\big) = \lim_{n\to\infty} d\big(x_2^n, \langle x_1^n, x_3^n\rangle\big) \geq \varepsilon$. This shows that $x_2 \notin \langle x_1, x_3\rangle$ and hence (ii) does not hold.

Let $\mathcal{H}$ denote the set of all elements of $\mathcal{K}_{\mathrm{tot}}(\mathcal{R}(\mathcal{X}))$ that satisfy condition (i) of Lemma 2.23. By what we have just proved,

$$\overline{\Pi}(\mathcal{X}) = \big\{G \in \mathcal{H} : \lim_{\delta\to 0} m^{\mathrm{S}}_{T,\delta}(G) = 0 \ \forall T < \infty\big\}. \tag{4.13}$$

It follows from Lemma 4.5 that for each $T < \infty$ and $\varepsilon, \delta > 0$, the set $\mathcal{G}_{T,\varepsilon,\delta} := \{G \in \mathcal{K}_{\mathrm{tot}}(\mathcal{R}(\mathcal{X})) : m^{\mathrm{S}}_{T,\delta}(G) \geq \varepsilon\}$ is a closed subset of $\mathcal{K}_{\mathrm{tot}}(\mathcal{R}(\mathcal{X}))$ and hence its complement $\mathcal{G}^{\mathrm{c}}_{T,\varepsilon,\delta}$ is open. As a consequence,

$$\mathcal{G} := \bigcap_{N=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \mathcal{G}^{\mathrm{c}}_{N,1/n,1/m} \tag{4.14}$$

is a G_δ-set. Formula (4.13) says that $\overline{\Pi}(\mathcal{X}) = \mathcal{G} \cap \mathcal{H}$. It is easy to see that $\mathcal{H}$ is a closed subset of $\mathcal{K}_{\mathrm{tot}}(\mathcal{R}(\mathcal{X}))$, and hence a G_δ-set. Since the intersection of two G_δ-sets is a G_δ-set, this yields the statement we wanted to prove. □

Proposition 4.7 (Polishness of the space of functions on a fixed interval)**.** *Let $\mathcal{X}$ be a metrisable space, equipped with a betweenness that is compatible with the topology, let $I \subset \mathbb{R}$ be a closed real interval of positive length, and let $\mathcal{D}_I(\mathcal{X})$ be the space in (1.7), equipped with the Skorokhod topology corresponding to the betweenness on $\mathcal{X}$. If $\mathcal{X}$ is Polish, then so is $\mathcal{D}_I(\mathcal{X})$.*

Proof: It follows from Lemma 2.14 that the set

$$\overline{\mathcal{D}}_I(\mathcal{X}) := \big\{\pi \in \Pi(\mathcal{X}) : \hat{I}(\pi) = I \cup \{-\infty, \infty\}\big\} \tag{4.15}$$

is a closed subset of $\Pi(\mathcal{X})$, and hence Polish by Lemma 3.14 and Proposition 4.6. We observe that for each $t \in I$, the set

$$\begin{aligned} &\big\{\pi \in \overline{\mathcal{D}}_I(\mathcal{X}) : \pi(t-) = \pi(t+)\big\} \\ &\quad = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \big\{\pi \in \overline{\mathcal{D}}_I(\mathcal{X}) : d\big(\pi(s), \pi(t)\big) < \tfrac{1}{n} \ \forall s \in [t - \tfrac{1}{m}, t + \tfrac{1}{m}] \cap I\big\} \end{aligned} \tag{4.16}$$

is a G_δ-subset of $\overline{\mathcal{D}}_I(\mathcal{X})$, and hence by (1.7), so is $\mathcal{D}_I(\mathcal{X})$. By Lemma 3.14, it follows that $\mathcal{D}_I(\mathcal{X})$ is Polish. □

4.3. *Compactness criteria.* In this subsection we prove Theorems 1.11, 1.12, and 1.13, based upon which we then prove Lemmas 1.10 and 1.18.

Proof of Theorem 1.11: As in the proof of Proposition 4.4 we identify $\Pi_{\mathrm{c}}(\mathcal{X})$ with the subset of $\mathcal{K}_+(\mathcal{R}(\mathcal{X}))$ consisting of all G that satisfy (4.7) or equivalently (4.8). Let $\mathcal{A} \subset \Pi_{\mathrm{c}}(\mathcal{X})$. Then $\mathcal{A}$ is precompact if and only if its closure $\overline{\mathcal{A}}$ in $\mathcal{K}_+(\mathcal{R}(\mathcal{X}))$ is compact and $\overline{\mathcal{A}} \subset \Pi_{\mathrm{c}}(\mathcal{X})$. By Lemmas 2.8 and 2.12, $\overline{\mathcal{A}}$ is a compact subset of $\mathcal{K}_+(\mathcal{R}(\mathcal{X}))$ if and only if $\mathcal{A}$ satisfies the compact containment condition. Therefore, to complete the proof, it suffices to show that if $\mathcal{A} \subset \Pi_{\mathrm{c}}(\mathcal{X})$ satisfies the compact containment condition, then $\overline{\mathcal{A}} \subset \Pi_{\mathrm{c}}(\mathcal{X})$ if and only if $\mathcal{A}$ is equicontinuous.

Assume that $\mathcal{A}$ satisfies the compact containment condition and is equicontinuous, and that $G_n \in \mathcal{A}$ satisfy $G_n \to G$ for some $G \in \mathcal{K}_+(\mathcal{R}(\mathcal{X}))$. We need to show that $G \in \Pi_{\mathrm{c}}(\mathcal{X})$. If this is not the case, then there exist $(x_1,t), (x_2,t) \in G$ with $t \in \mathbb{R}$ and $x_1 \neq x_2$. Then by Lemma 2.10, there exist $(x_1^n, t_1^n), (x_2^n, t_2^n) \in G_n$ such that $(x_i^n, t_i^n) \to (x_i, t)$ as $n \to \infty$ $(i = 1, 2)$. Choose $T < \infty$ such that $-T < t < T$. Then for each $\delta > 0$ we have for n large enough that $-T < t_1^n, t_2^n < T$, $|t_1^n - t_2^n| \leq \delta$, and $d(x_1^n, x_2^n) \geq d(x_1, x_2)/2$. This proves that

$$\sup_{\pi\in\mathcal{A}} m_{T,\delta}(\pi) \geq d(x_1, x_2)/2 \qquad \forall \delta > 0, \tag{4.17}$$

contradicting the equicontinuity of $\mathcal{A}$.

Assume, on the other hand, that $\mathcal{A}$ satisfies the compact containment condition and is not equicontinuous. Let δ_n be positive constants tending to zero. Since $\mathcal{A}$ is not equicontinuous, for some $T < \infty$ and $\varepsilon > 0$ we can find $G_n \in \mathcal{A}$ such that $m_{T,\delta_n}(G_n) \geq \varepsilon$ for all n. Since $\mathcal{A}$ satisfies the compact containment condition, $\overline{\mathcal{A}}$ is a compact subset of $\mathcal{K}_+(\mathcal{R}(\mathcal{X}))$, so by going to a subsequence we may assume that $G_n \to G$ for some $G \in \mathcal{K}_+(\mathcal{R}(\mathcal{X}))$. Since $m_{T,\delta_n}(G_n) \geq \varepsilon$ we can find $(x_1^n, t_1^n), (x_2^n, t_2^n) \in G_n$ such that $-T \leq t_1^n < t_2^n \leq T$, $t_2^n - t_1^n \leq \delta_n$, and $d(x_1^n, x_2^n) \geq \varepsilon$. By Lemma 2.10, going to a further subsequence if necessary, we can assume that $(x_i^n, t_i^n) \to (x_i, t)$ as $n \to \infty$ $(i = 1, 2)$ for some $(x_1, t), (x_2, t) \in G$. Then $-T \leq t \leq T$ and $d(x_1, x_2) \geq \varepsilon$, which shows that $\{x \in \mathcal{X} : (x, t) \in G\}$ has more than one element and hence $\overline{\mathcal{A}}$ is not contained in $\Pi_{\mathrm{c}}(\mathcal{X})$. □

Proof of Theorem 1.12: We identify $\Pi(\mathcal{X})$ with $\overline{\Pi}(\mathcal{X})$ which as in (4.1) we view as a subset of $\mathcal{K}_{\mathrm{tot}}(\mathcal{R}(\mathcal{X}))$. In this identification, the metrics d_{part} and d_{tot} on $\mathcal{K}_{\mathrm{tot}}(\mathcal{R}(\mathcal{X}))$ induce metrics d_{part} and d_{tot} on $\Pi(\mathcal{X})$ that both generate the Skorokhod topology.

Let $\mathcal{A} \subset \Pi(\mathcal{X})$. Then $\mathcal{A}$ is precompact in the Skorokhod topology if and only if its closure $\overline{\mathcal{A}}$ in $\mathcal{K}_{\mathrm{tot}}(\mathcal{R}(\mathcal{X}))$ is compact and $\overline{\mathcal{A}} \subset \Pi(\mathcal{X})$. By Theorem 1.15 and Lemma 2.8, $\overline{\mathcal{A}}$ is a compact subset of $\mathcal{K}_{\mathrm{tot}}(\mathcal{R}(\mathcal{X}))$ if and only if $\mathcal{A}$ satisfies the compact containment condition and

$$\lim_{\varepsilon \to 0} \sup_{G \in \mathcal{A}} m_\varepsilon(G) = 0, \tag{4.18}$$

where $m_\varepsilon(G)$ denotes the mismatch modulus of G. To complete the proof we will prove the following three statements.

I Assume that $\mathcal{A}$ is satisfies the compact containment condition and is Skorokhod-equicontinuous. Then $\mathcal{A}$ satisfies (4.18).

II Assume that $\mathcal{A}$ is Skorokhod-equicontinuous. Then $\overline{\mathcal{A}} \subset \Pi(\mathcal{X})$.

III Assume that $\overline{\mathcal{A}}$ is a compact subset of $\mathcal{K}_{\mathrm{tot}}(\mathcal{R}(\mathcal{X}))$ and that $\overline{\mathcal{A}} \subset \Pi(\mathcal{X})$. Then $\mathcal{A}$ is Skorokhod-equicontinuous.

Now if $\mathcal{A}$ satisfies the compact containment condition and is Skorokhod-equicontinuous, then by our earlier remarks I implies that $\overline{\mathcal{A}}$ is a compact subset of $\mathcal{K}_{\mathrm{tot}}(\mathcal{R}(\mathcal{X}))$ and II implies that $\overline{\mathcal{A}} \subset \Pi(\mathcal{X})$, so $\mathcal{A}$ is precompact in the Skorokhod topology. Conversely, if $\mathcal{A}$ is precompact in the Skorokhod topology, then $\overline{\mathcal{A}}$ is a compact subset of $\mathcal{K}_{\mathrm{tot}}(\mathcal{R}(\mathcal{X}))$ and hence by our earlier remarks $\mathcal{A}$ satisfies the compact containment condition, and moreover $\overline{\mathcal{A}} \subset \Pi(\mathcal{X})$ which by III implies that $\mathcal{A}$ is Skorokhod-equicontinuous.

We start by proving I. Since $\mathcal{A}$ satisfies the compact containment condition, by Lemma 2.8, we see that there exists a compact $C \subset \mathcal{R}(\mathcal{X})$ such that $G \subset C$ for all $G \in \mathcal{A}$. Since $\sup_{G \in \mathcal{A}} m_\varepsilon(G)$ is nondecreasing as a function of ε, the limit in (4.18) always exists. Let ε_n be positive constants, tending to zero. If the limit in (4.18) is positive, then there exists a $\delta > 0$ such that for each n we can find a $G \in \mathcal{A}$ and $(x_i^n, s_i^n), (y_i^n, t_i^n) \in G$ $(i = 1, 2)$ with $(x_1^n, s_1^n) \preceq (y_1^n, t_1^n)$, $(y_2^n, t_2^n) \preceq (x_2^n, s_2^n)$ such that

$$d_{\mathrm{sqz}}\bigl((x_1^n, s_1^n), (x_2^n, s_2^n)\bigr) \vee d_{\mathrm{sqz}}\bigl((y_1^n, t_1^n), (y_2^n, t_2^n)\bigr) \leq \varepsilon_n,$$
$$d_{\mathrm{sqz}}\bigl((x_i^n, s_i^n), (y_i^n, t_i^n)\bigr) \geq \delta \quad (i = 1, 2).$$

Since $G \subset C$ for all $G \in \mathcal{A}$, by going to a subsequence, we may assume that $(x_i^n, s_i^n) \to (x, s)$ and $(y_i^n, t_i^n) \to (y, t)$ $(i = 1, 2)$ for some $(x, s), (y, t) \in \mathcal{R}(\mathcal{X})$. Then $d_{\mathrm{sqz}}\bigl((x, s), (y, t)\bigr) \geq \delta$ and hence $(x, s) \neq (y, t)$. Since $(x_1^n, s_1^n) \preceq (y_1^n, t_1^n)$ and $(y_2^n, t_2^n) \preceq (x_2^n, s_2^n)$ we have $s_1^n \leq t_1^n$ and $t_2^n \leq s_2^n$ for all n which implies $s = t$ and hence $x \neq y$, since $(x, t) \neq (y, t)$. By the structure of $\mathcal{R}(\mathcal{X})$, this implies $t \in \mathbb{R}$. Let (x_-^n, s_-^n) (resp. (x_+^n, s_+^n)) be the smallest (resp. largest) of the points (x_i^n, s_i^n) $(i = 1, 2)$ with respect to the order $\preceq$, and define $(y_\pm^n, t_\pm^n)$ similarly. Since G is totally ordered, by going to a subsequence, we can assume that we are in one of the following two cases. 1. $(x_-^n, s_-^n) \preceq (y_-^n, t_-^n)$ for all n, or 2. $(y_-^n, t_-^n) \preceq (x_-^n, s_-^n)$ for all n. Let us assume that we are in

case 1. Then $(x_-^n, s_-^n) \preceq (y_-^n, t_-^n) \preceq (x_+^n, s_+^n)$ for all n. Since the betweenness is compatible with the topology,

$$d\big(y_-^n, \langle x_-^n, x_+^n\rangle\big) \underset{n\to\infty}{\longrightarrow} d(y,x) > 0, \tag{4.19}$$

which contradicts the Skorokhod-equicontinuity. Case 2 is completely the same, exchanging the roles of x and y.

We next prove II. Assume that $\pi_n \in \mathcal{A}$ converge in $\mathcal{K}_{\rm tot}(\mathcal{R}(\mathcal{X}))$ to a limit G. Recall from Subsection 3.1 that $d_{\rm part} = d^{\langle 2\rangle} \le d^{\langle m\rangle} \le d^{\langle\infty\rangle} = d_{\rm tot}$ for all $m \ge 2$. In particular, $\pi_n \to G$ in $\mathcal{K}_{\rm tot}(\mathcal{X})$ implies that $\pi_n^{\langle m\rangle} \to G^{\langle m\rangle}$ in the Hausdorff topology for all $m \ge 2$. It suffices to check that G satisfies conditions (i) and (ii) of Lemma 2.23. Condition (i) easily follows from the fact that $\pi_n^{\langle 2\rangle} \to G^{\langle 2\rangle}$ in the Hausdorff topology, using Lemma 2.10. It remains to prove that G satisfies condition (ii) of Lemma 2.23. Assume that conversely, for some $t \in \mathbb{R}$, there exist $(x_1,t),(x_2,t),(x_3,t) \in G$ with $(x_1,t) \preceq (x_2,t) \preceq (x_3,t)$ such that $x_2 \notin \langle x_1, x_3\rangle$. Since $\pi_n^{\langle 3\rangle} \to G^{\langle 3\rangle}$ in the Hausdorff topology, by Lemma 2.10 there exist $\tau_i^n \in I_{\mathfrak{s}}(\pi_n) \cup \{\pm\boldsymbol{\infty}\}$ $(i = 1,2,3)$ with $\tau_1^n \le \tau_2^n \le \tau_3^n$ such that $\underline{\tau}_i^n \to t$ and $\pi_n(\tau_i^n) \to x_i$ $(i = 1,2,3)$. Using the fact that the betweenness is compatible with the topology, we see that

$$d\big(\pi_n(\tau_2^n), \langle \pi_n(\tau_1^n), \pi_n(\tau_3^n)\rangle\big) \underset{n\to\infty}{\longrightarrow} d\big(x_2, \langle x_1, x_3\rangle\big), \tag{4.20}$$

which is easily seen to contradict Skorokhod-equicontinuity, completing the proof of II.

To prove III, finally, we will show that if $\overline{\mathcal{A}}$ is a compact subset of $\mathcal{K}_{\rm tot}(\mathcal{R}(\mathcal{X}))$ and $\mathcal{A}$ is not Skorokhod-equicontinuous, then $\overline{\mathcal{A}}$ is not contained in $\Pi(\mathcal{X})$. Let δ_n be positive constants, tending to zero. Since $\mathcal{A}$ is not Skorokhod-equicontinuous, there exists a $\varepsilon > 0$, $T < \infty$, $\pi_n \in \mathcal{A}$, and $\tau_i^n \in I_{\mathfrak{s}}(\pi_n)$ $(i = 1,2,3)$ such that $\tau_1^n \le \tau_2^n \le \tau_3^n$, $-T \le \underline{\tau}_1^n, \underline{\tau}_3^n \le T$, $\underline{\tau}_3^n - \underline{\tau}_1^n \le \delta_n$, and $d\big(\pi_n(\tau_2^n), \langle \pi_n(\tau_1^n), \pi_n(\tau_3^n)\rangle\big) \ge \varepsilon$. Since $\overline{\mathcal{A}}$ is a compact subset of $\mathcal{K}_{\rm tot}(\mathcal{R}(\mathcal{X}))$, we can select a subsequence such that $\pi_n \to G$ for some $G \in \mathcal{K}_{\rm tot}(\mathcal{R}(\mathcal{X}))$. Then $\pi_n^{\langle 3\rangle} \to G^{\langle 3\rangle}$ in the Hausdorff topology, so by Lemma 2.10 there exist $C \subset \mathcal{R}(\mathcal{X})^3$ such that $\pi_n^{\langle 3\rangle} \subset C$ for all n. It follows that by going to a further subsequence we can assume that $\big(\pi_n(\tau_i^n), \underline{\tau}_i^n\big) \to (x_i, t_i)$ as $n \to \infty$ for some $(x_i,t) \in G$ $(i = 1,2,3)$ with $(x_1,t_1) \preceq (x_2,t_2) \preceq (x_3,t_3)$ and $-T \le t \le T$. Using the fact that the betweenness is compatible with the topology, we see that $d(x_2, \langle x_1, x_3\rangle) \ge \varepsilon$, which shows that G is not the filled graph of a path $\pi \in \Pi(\mathcal{X})$, hence $\overline{\mathcal{A}}$ is not contained in $\Pi(\mathcal{X})$. □

Proof of Theorem 1.13: We view $\mathcal{D}_I(\mathcal{X})$ as a subset of $\Pi(\mathcal{X})$ as in (1.7). Then $\mathcal{F}$ is compact as a subset of $\mathcal{D}_I(\mathcal{X})$ if and only if its closure $\overline{\mathcal{F}}$ in the larger space $\Pi(\mathcal{X})$ is compact and satisfies $\overline{\mathcal{F}} \subset \mathcal{D}_I(\mathcal{X})$. By Theorem 1.12, $\overline{\mathcal{F}}$ is a compact subset of $\Pi(\mathcal{X})$ if an only if conditions (i) and (ii) hold. To complete the proof, we will show that, assuming (i) and (ii), one has $\overline{\mathcal{F}} \subset \mathcal{D}_I(\mathcal{X})$ if and only if (iii) holds.

We first show that (iii) implies that $\overline{\mathcal{F}} \subset \mathcal{D}_I(\mathcal{X})$. Assume that (iii) holds and let $f_n \in \mathcal{D}_I(\mathcal{X})$ and $\pi \in \Pi(\mathcal{X})$ satisfy $f_n \to \pi$ in the Skorokhod topology associated with the given betweenness. Then clearly $I(\pi) = I$. To show that $\pi \in \mathcal{D}_I(\mathcal{X})$ assume that conversely $\pi(t-) \ne \pi(t+)$ for some $t \in \partial I$. Then, by the fact that $d^{\rm S}_{\rm part}(f_n, \pi) \to 0$, there exist $s_n, t_n \in I$ with $s_n < t_n$ such that $f_n(s_n) \to \pi(t-)$ and $f_n(t_n) \to \pi(t+)$, which is easily seen to contradict (iii).

Assume, on the other hand, that (iii) does not hold for some $t \in \partial I$. Let δ_n be positive constants, tending to zero. Then there exists an $\varepsilon > 0$ such that for each n, we can find $f_n \in \mathcal{F}$ and $s_n \in I$ with $|s_n - t| \le \delta_n$ and $d\big(f_n(s_n), f_n(t)\big) \ge \varepsilon$. By (i) and (ii), $\overline{\mathcal{F}}$ is compact in $\Pi(\mathcal{X})$ so by going to a subsequence we can assume that $f_n \to \pi$ for some $\pi \in \Pi(\mathcal{X})$. By (i), we can moreover assume that $f_n(s_n) \to x$ and $f_n(t) \to y$ for some $x, y \in \mathcal{X}$. Then $d(x,y) \ge \varepsilon$ and $(x,t),(y,t) \in \mathcal{G}_{\rm f}(\pi)$, which by Lemma 2.15 (v) shows that $\pi(t-) \ne \pi(t+)$ and hence $\overline{\mathcal{F}}$ is not contained in $\mathcal{D}_I(\mathcal{X})$. □

Proof of Lemma 1.10: We start by noting that for each $x, z \in \mathcal{X}$, the total order on $\langle x, z\rangle$ coincides with the induced order from its embedding in $\langle x, z\rangle'$. Indeed, $y, y' \in \langle x, z\rangle$ satisfy $y \le_{x,z} y'$ if and

only if $y' \in \langle y, z\rangle$, which implies $y' \in \langle y, z\rangle'$ and hence $y \leq'_{x,z} y'$, where $\leq'_{x,z}$ denotes the total order on $\langle x, z\rangle'$.

For any path π, let $\overline{\pi}$ and $\overline{\pi}'$ denote the filled paths defined in terms of the betweennesses $\langle\,\cdot\,,\cdot\,\rangle$ and $\langle\,\cdot\,,\cdot\,\rangle'$, respectively. The condition $\langle x, z\rangle \subset \langle x, z\rangle'$ $(x, z \in \mathcal{X})$ implies that $\overline{\pi} \subset \overline{\pi}'$. By our earlier remarks, the total order on $\overline{\pi}$ coincides with the induced order from $\overline{\pi}'$. Using notation as in (1.31), it follows that $\overline{\pi}^{\langle 2\rangle} \subset \overline{\pi}'^{\langle 2\rangle}$ for all $\pi \in \Pi(\mathcal{X})$.

Let S and S′ denote the Skorokhod topologies defined by the betweennesses $\langle\,\cdot\,,\cdot\,\rangle$ and $\langle\,\cdot\,,\cdot\,\rangle'$. By Theorem 1.14, one has $\pi_n \to \pi$ in the topology S if and only if $\overline{\pi}_n^{\langle 2\rangle} \to \overline{\pi}^{\langle 2\rangle}$ in the Hausdorff topology on $\mathcal{K}_+(\mathcal{R}(\mathcal{X})^2)$, and an analogous statement holds for for convergence in the topology S′. Let $m^{\mathrm{S}}_{T,\delta}$ and $m^{\mathrm{S}'}_{T,\delta}$ denote the Skorokhod moduli of continuity associated with both betweennesses. The condition $\langle x, z\rangle \subset \langle x, z\rangle'$ $(x, z \in \mathcal{X})$ implies $d(y, \langle x, z\rangle) \geq d(y, \langle x, z\rangle')$ $(x, y, z \in \mathcal{X})$ and hence $m^{\mathrm{S}}_{T,\delta}(\pi) \geq m^{\mathrm{S}'}_{T,\delta}(\pi)$ for all $\pi \in \Pi(\mathcal{X})$, $T < \infty$, and $\delta > 0$.

Assume that $\pi_n, \pi \in \Pi(\mathcal{X})$ satisfy $\pi_n \to \pi$ in the topology S. Then by Theorem 1.12, the set $\{\pi_n : n \in \mathbb{N}\}$ satisfies the compact containment condition and is Skorokhod equicontinuous with respect to $m^{\mathrm{S}}_{T,\delta}$ and hence also with respect to $m^{\mathrm{S}'}_{T,\delta}$, which by Theorem 1.12 implies that the set $\{\pi_n : n \in \mathbb{N}\}$ is precompact in the topology S′. Therefore, to complete the proof, it suffices to show that if π' is a cluster point of the sequence π_n in the topology S′, then $\pi' = \pi$.

By going to a subsequence, we may assume that $\pi_n \to \pi$ in the topology S and $\pi_n \to \pi'$ in the topology S′. By our earlier remarks, $\overline{\pi}_n^{\langle 2\rangle} \to \overline{\pi}^{\langle 2\rangle}$ in the Hausdorff topology and similarly $\overline{\pi}'^{\langle 2\rangle}_n \to \overline{\pi}'^{\langle 2\rangle}$. It follows that $I(\pi') = I(\pi)$ and $\overline{\pi}^{\langle 2\rangle} \subset \overline{\pi}'^{\langle 2\rangle}$, which implies $\pi' = \pi$, as required. □

Proof of Lemma 1.18: If $\pi'_n \to \pi$, then by Theorem 1.11 the π'_n are equicontinuous and satisfy the compact containment condition. Since $\pi_n \subset \pi'_n$ the same is then true for the π_n so by Theorem 1.11, going to a subsequence if necessary, we can assume that $\pi_n \to \tilde{\pi}$ for some $\tilde{\pi} \subset \pi$. Since for each $t \in I(\pi)$, there exist $t_n \in I(\pi_n)$ such that $t_n \to t$, each such subsequential limit must satisfy $\tilde{\pi} = \pi$, which allows us to conclude that $\pi_n \to \pi$.

Assume, conversely, that $\pi_n \to \pi$. We claim that for each $z_n = (x_n, t_n) \in \pi'_n$, we can select a subsequence such that $z_n \to z \in \pi$. To see this, let $s_n := \sup\{t \in I(\pi_n) : t \leq t_n\}$ and $u_n := \inf\{t \in I(\pi_n) : t \geq t_n\}$. By going to a subsequence, we can assume that s_n, t_n, and u_n converge to limits $s, t, u \in \overline{\mathbb{R}}$ with $s \leq t \leq u$. Since $\pi \in \Pi^{|}_{\mathrm{c}}(\mathcal{X})$ and $\pi_n \to \pi$ we must have $s = u$. Since $\pi_n \to \pi$ both $(\pi_n(s_n), s_n)$ and $(\pi_n(u_n), u_n)$ converge to $(\pi(t), t)$. If $t = \pm\infty$, then $z_n \to (*, \pm\infty) = (\pi(t), t)$ and we are done. If $t \in \mathbb{R}$ then we use that $\pi'_n(t_n) \in \langle \pi_n(s_n), \pi_n(u_n)\rangle$ and the betweenness is compatible with the topology to conclude $(\pi'_n(t_n), t_n) \to (\pi(t), t)$.

By the claim we have just proved, there exists a compact set $C \subset \mathcal{R}(\mathcal{X})$ such that $\pi'_n \subset C$ for each n. By Lemma 2.12, it follows that $\{\pi'_n : n \in \mathbb{N}\}$ is precompact in the topology on $\mathcal{K}_+(\mathcal{R}(\mathcal{X}))$. By the claim we have just proved, each subsequential limit π^* satisfies $\pi^* \subset \pi$. Since $\pi_n \subset \pi'_n$ and $\pi_n \to \pi$ we conclude $\pi^* = \pi$. It follows that $\pi'_n \to \pi$. □

4.4. *Identification of the J1, J2, M1, and M2 topologies.* In this subsection we complete the proofs of Theorems 1.4 and 1.9 by showing that for the trivial betweenness, the topology we have introduced on the path space $\Pi(\mathcal{X})$ induces on the space $\mathcal{D}_I(\mathcal{X})$ from (1.7) Skorokhod's J1 topology while the metric d_{H} generates the J2 topology, and similarly, if $\mathcal{X} = \mathbb{R}$ equipped with the linear betweenness, then we obtain the M1 and M2 topologies. In the process we also establish Lemma 1.17.

We view a function $f \in \mathcal{D}_{[0,t]}(\mathcal{X})$ as an extended cadlag function $(\hat{I}, f)$ with $\hat{I} = [0, t] \cup \{-\infty, \infty\}$, and similarly we view a function $f \in \mathcal{D}_{[0,\infty)}(\mathcal{X})$ as an extended cadlag function $(\hat{I}, f)$ with $\hat{I} = [0, \infty] \cup \{-\infty\}$. We let $\mathcal{G}_{\mathrm{f}}(f) := \mathcal{G}_{\mathrm{f}}(\hat{I}, f)$ denote its filled graph, defined in (2.25). For $f \in \mathcal{D}_{[0,\infty)}(\mathcal{X})$ we let $f\big|_{[0,t]}$ denote the restriction of f to the interval $[0, t]$.

Lemma 4.8 (Convergence of restricted functions)**.** *Let $(\mathcal{X}, d)$ be a metric space that is equipped with a proper betweenness and let d_{H} and d_{tot} be defined as in (1.21) and (1.22). Then for all $f_n, f \in \mathcal{D}_{[0,\infty)}(\mathcal{X})$, one has*

$$\begin{aligned} d_{\mathrm{H}}(f_n, f) \underset{n\to\infty}{\longrightarrow} 0 \quad &\Leftrightarrow \quad d_{\mathrm{H}}\big(f_n|_{[0,t]}, f|_{[0,t]}\big) \underset{n\to\infty}{\longrightarrow} 0 \qquad \forall t > 0 \text{ s.t. } f(t-) = f(t), \\ d_{\mathrm{tot}}(f_n, f) \underset{n\to\infty}{\longrightarrow} 0 \quad &\Leftrightarrow \quad d_{\mathrm{tot}}\big(f_n|_{[0,t]}, f|_{[0,t]}\big) \underset{n\to\infty}{\longrightarrow} 0 \qquad \forall t > 0 \text{ s.t. } f(t-) = f(t). \end{aligned} \tag{4.21}$$

Proof: We start by proving the statement for d_{H}. Let $T := \{t > 0 : f(t-) = f(t)\}$. It is well-known that the complement of T is countable (see Ethier and Kurtz, 1986, Lemma 3.5.1), so T is dense in $[0, \infty)$. Let $G := \mathcal{G}_{\mathrm{f}}(f)$, $G_n := \mathcal{G}_{\mathrm{f}}(f_n)$, $G^t := \mathcal{G}_{\mathrm{f}}(f|_{[0,t]})$, and $G_n^t := \mathcal{G}_{\mathrm{f}}(f_n|_{[0,t]})$.

We first prove the implication $\Rightarrow$. Let $t \in T$. Since $G_n \to G$, by Lemma 2.10, there exists a compact set C such that $G_n \subset C$ for all n, and hence also $G_n^t \subset C$ for all n. By Lemma 2.12, it follows that $\{G_n^t : n \in \mathbb{N}\}$ is compact, so to prove that $G_n^t \to G^t$, it suffices to show that G^t is the only subsequential limit of the G_n^t. Let G^* be such a subsequential limit. Since $G_n^t \subset G_n$ it is clear from Lemma 2.10 that $G^* \subset G$. Let ψ denote the projection $\psi(x, s) := s$ and let $\psi(G^*)$ denote the image of G^* under ψ. By Lemma 2.14, $\psi(G^*) = [0, t]$. It follows that $G^* \subset G^t$. To prove the opposite inclusion, assume that $(y, s) \in G^t$. If $s < t$, then we use that by Lemma 2.10, there exist $(x_n, s_n) \in G_n$ such that $(x_n, s_n) \to (x, s)$. Since $s < t$, we have $(x_n, s_n) \in G_n^t$ for n large enough and hence $(x, s) \in G^*$. If $s = t$, then we use that $\psi(G^*) = [0, t]$ to conclude that there must be at least one $y' \in \mathcal{X}$ such that $(y', t) \in G^*$. Since $G^* \subset G^t$ we must have $y, y' \in \langle f(t-), f(t)\rangle = \{f(t)\}$, where we have used that $t \in T$, so we conclude that $y' = y$, concluding the proof that $G^* = G^t$.

We next prove the implication $\Leftarrow$. Since $G_n^t \to G^t$ for each $t \in T$, using Lemmas 2.10 and 2.8, we see that there exists a compact set C such that $G_n \subset C$ for all n, so by Lemma 2.12 it suffices to show that if G_* is a subsequential limit of the G_n, then $G_* = G$. Since $G_n^t \subset G_n$ for each n it is clear from Lemma 2.10 that $G^t \subset G_*$ for each $t \in T$. We claim that conversely, for each $(x, s) \in G_*$ and $s < t \in T$, we have $(x, s) \in G^t$. Indeed, by Lemma 2.10 for some subsequence there exist $(x_n, s_n) \in G_n$ such that $(x_n, s_n) \to (x, s)$. Since $s_n < t$ for n large enough, it follows that $(x, s) \in G^t$. These arguments show that $\{(x, t) \in G_* : t < \infty\} = \{(x, t) \in G : t < \infty\}$, which is enough to conclude $G_* = G$.

We next prove the statement for d_{tot}. Let $\pi := f$, $\pi_n := f_n$, $\pi^t := f|_{[0,t]}$, and $\pi_n^t := f_n|_{[0,t]}$, which we view as elements of the path space $\Pi(\mathcal{X})$. We first prove the implication $\Rightarrow$. By Theorem 1.12, $d_{\mathrm{tot}}(\pi_n, \pi) \to 0$ implies that $\{\pi_n : n \in \mathbb{N}\}$ is Skorokhod-equicontinuous and satisfies the compact containment condition, which implies the same is true for $\{\pi_n^t : n \in \mathbb{N}\}$ for any $t \in T$. Therefore, by Theorem 1.12, it suffices to show that all subsequential limits of the π_n^t are equal to π^t. Since by Theorem 1.14, convergence in d_{tot} implies convergence in d_{H}, we can use what we have already proved for d_{H} to draw the desired conclusion. The implication $\Leftarrow$ follows in the same way; now we use that if $\{\pi_n^t : n \in \mathbb{N}\}$ is Skorokhod-equicontinuous and satisfies the compact containment condition for any $t \in T$, then the same is true for $\{\pi_n : n \in \mathbb{N}\}$. □

Proof of Theorems 1.4 and 1.9: Most statements of the theorems have already been proved in Propositions 4.1 and 4.2, while the statements about Polishness follow from Propositions 4.4, 4.6, and 4.7. Let I be a closed real interval of positive length. The fact that d_{H} is a metric on $\mathcal{D}_I(\mathcal{X})$ has already been observed at the beginning of Subsection 4.1. If I is a compact interval, then as discussed below Lemma 1.17 it follows from Proposition 1.16 that for the trivial betweenness d_{tot} generates Skorokhod's J1 topology on $\mathcal{D}_I(\mathcal{X})$ as defined in Skorokhod (1956, Def 2.2.2) while d_{H} generates the J2 topology as defined in Skorokhod (1956, Def 2.2.3). For the J1 case in Theorem 1.4, the fact that $\Pi_{\mathrm{c}}^{\uparrow}(\mathcal{X})$ is a closed subset of $\Pi(\mathcal{X})$ follows from Lemmas 2.13 and 2.27. If $\mathcal{X} = \mathbb{R}$ equipped with the linear betweenness, then similarly one obtains the M1 and M2 topologies from Skorokhod (1956, Defs 2.2.4 and 2.2.6). On half infinite intervals I one moreover needs Lemma 4.8

to see that our approach leads to the same topology as the classical definitions. If $I = \mathbb{R}$, then the argument is basically the same. □

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